Computability of the attractors

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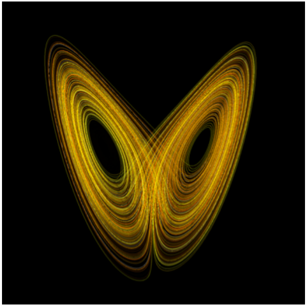
Dynamical Systems

Discrete dynamical Systems: $F : X \longrightarrow X$ where X is compact and F continuous

Continuous dynamical Systems: $\varphi : \mathbb{R} \times \mathcal{X} \longrightarrow X$ where X is compact and F continuous

It is possible to compute the asymptotic behavior (i.e. attractor)?



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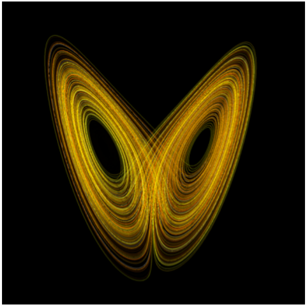

D. Graça, C. Rojas, and N. Zhong. Computing geometric lorenz attractors with arbitrary precision, 2018.

Algorithmical complexity of entropy

Entropy is the most popular dynamical invariant, it measures the quantity of information to encode a trajectory. Two point of view to link with computability:

- Natural question of *Milnor-2002:*

  "given an explicit (finitely described) dynamical system and given $\epsilon > 0$, is it possible to compute the associated topological entropy with a maximum error of ϵ?"

  - Uncomputabilitty of the entropy for different classes of dynamical systems (cellular automata *(Hurd-Kari-Culik-92)*, iterated piecewise affine maps *(Koiran-01)*...)
  - Upper and lower bounds for computable dynamical systems*(Gangloff-Herrera-Rojas-S-2019)*

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  - Uncomputabilitty of the entropy for different classes of dynamical systems (cellular automata *(Hurd-Kari-Culik-92)*, iterated piecewise affine maps *(Koiran-01)*...)
  - Upper and lower bounds for computable dynamical systems*(Gangloff-Herrera-Rojas-S-2019)*
- Characterization of the possible entropies of a countable class of dynamical systems
  - Characterization of the possible entropy of multidimensional SFT *(Hochaman-Meyerovitch-2009)*
  - Characterization of the possible entropy of multidimensional SFT under some dynamical conditions *(Gangloff-20)*

Computable space

- $(X, d, \mathcal{S})$ is a *Computable space* if
  - (X, d) metric space;
  - $\mathcal{S} = \{s_i : i \in \mathbb{N}\}$ countable dense set of X;
  - there exists an algorithm $\mathcal{T} : \mathbb{N}^3 \longrightarrow \mathbb{Q}$ such that

$$|d(s_i, s_j) - \mathcal{T}(i, j, n)| \leq 2^{-n}$$

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  Denote $(B_n)_{n \in \mathbb{N}}$ an enumeration of the ideal balls

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Examples:

- $(\mathcal{A}^{\mathbb{N}}, d, \mathcal{S})$ where $\mathcal{A}$ is a finite alphabet, $d(x, y) = 2^{\{-\min n : x_n \neq y_n\}}$ and $\mathcal{S} = \{ua^{\infty} : u \in \mathcal{A}^*\}$

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- $([0, 1], d, [0, 1] \cap \mathbb{Q})$ where $d(x, y) = |x - y|$

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- $d(x, y) \leq \mathcal{T}_1(n) \Longrightarrow d(F(x), F(y)) \leq n$
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- $F : [0,1] \longrightarrow [0,1]$, $F(x) = \begin{cases} 2x & \text{if } x < \frac{1}{2} \\ 2-2x & \text{if } x \geq \frac{1}{2} \end{cases}$

Different notions of attractors (following *Milnor-85*)

Accumulation points of an orbit:

$$\omega(x) = \bigcap_N \overline{\bigcup_{n \geq N} F^n(x)}$$

Realm of topological attraction of the closed set $A \subset X$:

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- a *statistical attractor* if there exists μ a F-invariant measure such that
  - $\mathrm{supp}(\mu) = A$
  - $\mu = \lim_N \sum_{n \leq N} \delta_{F^n(x)}$ for μ-almost $x \in X$.

Computability of closed sets

Let $K \subset X$ be a compact set, define the *inner* and *outer collection of ideal balls*:

$$\mathcal{B}_{in}(K) = \{i \in \mathbb{N} : B_i \cap K \neq \emptyset\} \qquad \text{and} \qquad \overline{\mathcal{B}_{out}}(K) = \left\{i \in \mathbb{N} : \overline{B_i} \cap K \neq \emptyset\right\}$$

- K is *Σ_n-computable* if $\mathcal{B}_{in}(K)$ is Σ_n-computable
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- K is *Π_n-complete* if $\mathcal{B}_{out}(K)$ is Π_n-complete
- K is *Δ_n-computable* if K is Σ_n-computable and Π_n-computable

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Remarque: If X is the Cantor space then $\mathcal{B}_{in}(K) = \overline{\mathcal{B}_{out}}(K)$.

Summary

Theorem *Rojas-S-23*

Let $F : X \longrightarrow X$ be a computable map, λ a computable reference measure and A be a closed subset of X which is invariant by T.

- If A is a topological attractor, then it is a Π_2 set.
- If A is a metric attractor, then it is a Π_2 set.
- If A is a statistical attractor, then it is Σ_2.

Moreover the bounds are tight in the class of the action on a Cantor set.

Proposition *Rojas-S-23*

Let A be a metric or a topological attractor

- If A is strongly attracting, then it is Π_1.
- If the action of T is minimal on A and A is Π_i computable then it is also Σ_i.

Strongly attracting set: Upper bound

A is *strongly attracting* if there exists a neighborhood U of A such that

$$F(\overline{U}) \subset U \text{ and } A = \bigcap_n F^n(A)$$

Proposition

Let A be a topological or metric attractor, if A is strongly attracting, then it is Π_1.

Since A is compact, there exists a finite set I such that

$$A \subset \bigcup_{i \in I} B_i \subset \bigcup_{i \in I} \overline{B_i} \subset U$$

Thus

$$B \cap A \neq \varnothing \iff \forall n, \quad B \cap F^n\left(\bigcup_{i \in I} \overline{B_i}\right) \neq \varnothing$$

Strongly attracting set: the bound is tight

Theorem

There exists a computable map $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ and $A \subset X$ such that:

i) A is Π_1-complete.

ii) A is the topological and metric attractor of F which are strongly attracting.

iii) A is the statistical attractor of F. In particular, the unique physical measure of the system is non computable.

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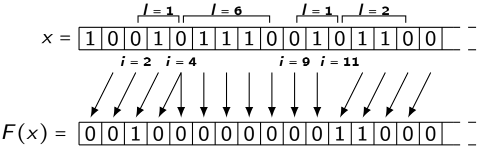

F shift the block 01^l0 at position i iff $l > i$ or $\mathcal{M}_l$ does not halt in less than i steps.

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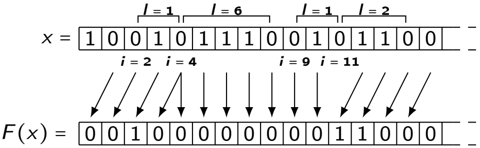

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The topological and metric attractor strongly attracting of F is

$$A = \bigcap_n F^n(X) = \left\{x \in \{0,1\}^{\mathbb{N}} : 01^l0 \not\sqsubset x \text{ iff } \mathcal{M}_l \text{ halts}\right\}$$

In particular

$$A \cap [01^n0]_0 \neq \varnothing \iff n \in \overline{\mathcal{H}alt}.$$

Upper bound: topological attractor

Proposition

If A is a topological attractor, then it is a Π_2 set.

Let B_i be an ideal ball, denote $\rho(B_i) = \bigcap_n \bigcup_{t \geq n} F^{-t}(B_i)$

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- Reciprocally if $\rho(B_i)$ is not dense, there exists a ball B such that $B \cap \rho(B_i) = \varnothing$. But $B \cap \rho(A)$ is not meager so by minimality of A one has $\omega(B \cap \rho(A)) = A$. One deduces the $B_i \cap A = \varnothing$ since element of B visit B_i finitely many time.

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Thus

$$\begin{aligned}
\overline{B} \cap A \neq \varnothing \quad & \Longleftrightarrow \quad \forall i, \quad \overline{B} \subset B_i \text{ and } B_i \cap A \neq \varnothing \\
& \Longleftrightarrow \quad \forall i \quad \overline{B} \subset B_i \text{ and } \rho(B_i) \text{ dense} \\
& \Longleftrightarrow \quad \forall i, j \quad \overline{B} \subset B_i \text{ and } \rho(B_i) \cap B_j \neq \varnothing \\
& \Longleftrightarrow \quad \forall i, j, n \exists m \quad \overline{B} \subset B_i \text{ and } \bigcup_{t=n}^{m} F^{-t}(B_i) \cap B_j \neq \varnothing
\end{aligned}$$

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- if $B_i \cap A \neq \varnothing$, since A is the only metric attractor $\forall_\mu x, \omega(x) = A$ and visit B_i, so $\mu(\rho(Bi)) > 0$

Thus

$$\begin{aligned} \overline{B} \cap A \neq \varnothing \quad & \iff \quad \forall i, \quad \overline{B} \subset B_i \text{ and } B_i \cap A \neq \varnothing \\ & \iff \quad \forall i, \quad \overline{B} \subset B_i \text{ and } \mu(\rho(B_i)) > 0 \\ & \iff \quad \forall i, j, n, \exists m, \quad \overline{B} \subset B_i \text{ and } \mu\left(\bigcup_{t=n}^{m} F^{-t}(B_i) \cap B_j\right) > 0 \end{aligned}$$

These bounds are tight for Cantor

Theorem

There exists a computable map $F : X \to X$ with a Π_2-complete invariant subset A which is the unique attractor of F in both, the topological and metric sense.

Let $X = \{0, 1, S\}^{\mathbb{N}}$ and $F : X \to X$ which acts on $x = u_0 S u_1 S u_2 S u_3 S u_4 S u_5 \dots$ by

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- if there is no 1 in u_0, the first S let a block of 1 pass;

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- u_0 is shifted to the left;
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- if there is no 1 in u_0, the first S let a block of 1 pass;
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Consider $G \subset X$ be the set of configurations which contain infinitely many symbols S, and infinitely many blocks of 1's of all lengths. For all $x \in G$ one has

$$\omega(x) = A_{\mathcal{T}ot} = \left\{ y \in \{0, 1\}^{\mathbb{N}} : y \text{ contains at most one block } 01^l0 \text{ iff } l \in \mathcal{T}ot \right\}$$

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In particular

$$A_{\mathcal{T}ot} \cap [01^n0]_0 \neq \varnothing \iff n \in \overline{\mathcal{T}ot}.$$

Corollary

There exist computable maps $F' : X \longrightarrow X$ and $F'' : X \longrightarrow X$ such that:

- F': computable metric attractor and Π_2-complete topological attractor.
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Wild cantor attractors exist

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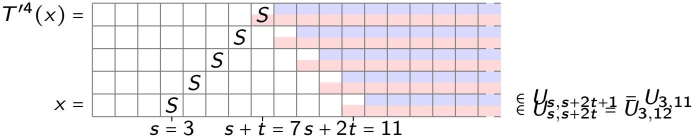

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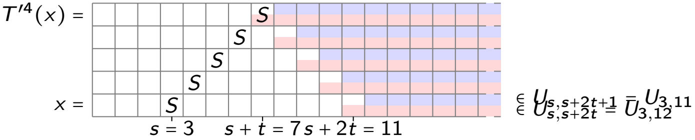

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One has

$$\lambda(U_{s,t}) \leq \frac{1}{3} \times \left(\frac{2}{3}\right)^{s-1} \times \sum_{t' \geq t} \left(\frac{1}{2}\right)^{t'+s} = \left(\frac{2}{3}\right)^{s} \times \frac{1}{2^{t+s}} \leq \left(\frac{2}{3}\right)^{s} \times \frac{1}{2^{t}}.$$

So $\lambda(U_t) \leq \frac{3}{2^t}$ and clearly $G' = \bigcap_{t \in \mathbb{N}} U_t$ is a dense G_δ of measure 0.

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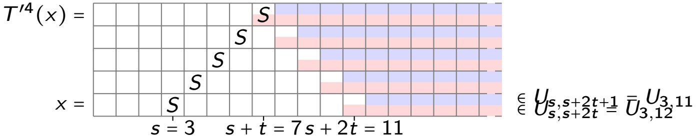

$$\forall x \in G, \qquad \omega(x) = A_{\mathcal{T}ot} \times \{a, b\}^{\mathbb{N}}$$
$$\forall_\mu x \in G \qquad \omega(x) = \{0^\infty\} \times \{a, b\}^{\mathbb{N}}$$

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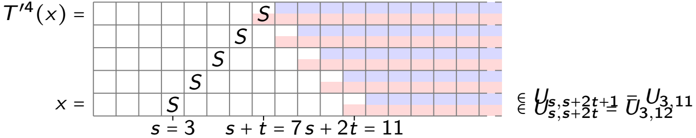
Consider F'' similar to the previous F on the first coordinate and the shift on the second but the first S at position i produces all blocks 01^l0 with $l \leq i$ if there is a^{2i} on the second.



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Physical attractor

A *statistical attractor* if there exists μ a F-invariant measure such that

- $\mathrm{supp}(\mu) = A$
- $\mu = \lim_N \sum_{n \leq N} \delta_{F^n(x)}$ for μ-almost $x \in X$.

Proposition

A physical attractor of a computable map is Σ_2-computable.

Theorem

There exists a computable map $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ with a Σ_2-complete statistical attractor.

Interval maps with computationally complex attractors

Theorem

Let $F : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ computable. Then there exists a computable cantor set $\mathcal{C} \subset [0,1]$ and a computable map $f : [0,1] \to [0,1]$ preserving $\mathcal{C}$ such that:

- $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{C}$ computable conjugacy F to f over $\mathcal{C}$,
- for Lebesgue almost every $x \in [0,1]$, $\omega_f(x) \subset \mathcal{C}$,
- if F has a metric or statistical attractor, then $\mathcal{A} = \phi(A) \subset \mathcal{C}$ is an attractor for f of the same type.

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There exist a computable map $f : [0,1] \to [0,1]$ and $A \subset [0,1]$ such that:

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Characterization for the Cellular automaton

Let F be a cellular automaton which admits X as topological attractor and Y as metric attractor and F acts as the shift on X and Y then:

- X and Y are Π_2-computable;
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Theorem *(Herrera-Törmä-S-23)*

Let $Y \subset X \subset \mathcal{A}^{\mathbb{Z}}$ such that X, Y are Π_2-subshift chain-mixing.
There exists $\mathcal{B} \supset \mathcal{A}$ and a cellular automaton $F : \mathcal{B}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$ such that

- Y is a metric attractor of F.
- X is a metric attractor of F.