FO-model checking on tame classes of graphs

A survey

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∂IFFERENCE

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Properties of Graph Classes

Properties of Graph Classes

Bounded tree-width

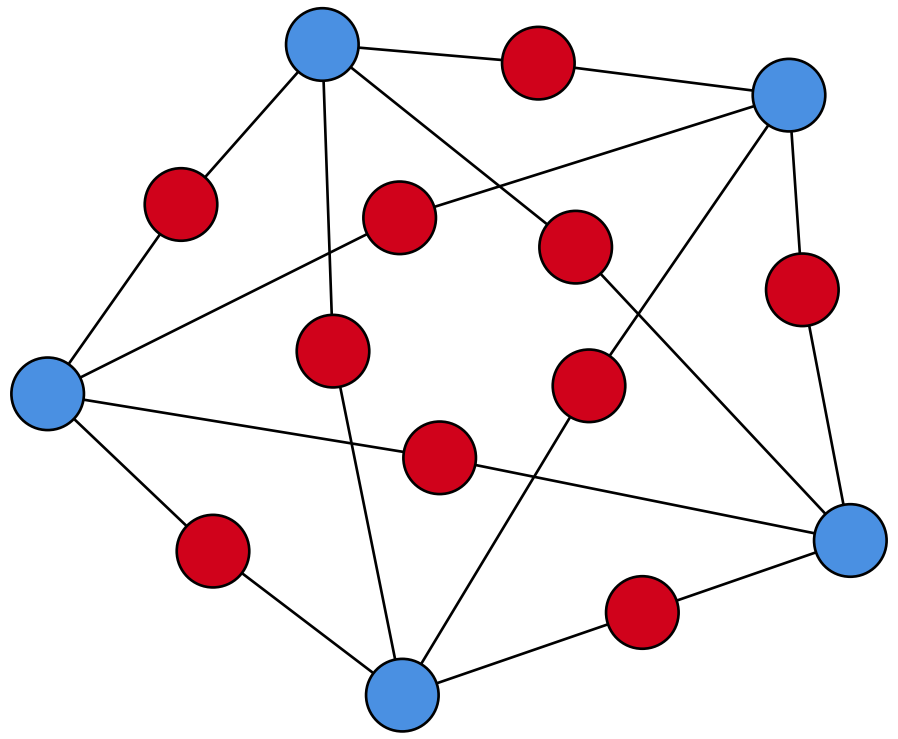
Properties of Graph Classes

Bounded tree-width ⊆ Nowhere dense

⊆

Nowhere dense

A class $\mathscr{C}$ is nowhere dense if for all $r \in \mathbb{N}$ there exists $t \in \mathbb{N}$ such that all graphs of $\mathscr{C}$ exclude r-subdivided K_t as a subgraph.



1-subdivided K_5

Properties of Graph Classes

Bounded tree-width

⊆ Nowhere dense

Properties of Graph Classes

Bounded tree-width ⊆ Excluding minors ⊆ Nowhere dense

Bounded tree-width $\subseteq$ Excluding minors $\subseteq$ Nowhere dense

A class $\mathscr{C}$ excludes a minor when there exists a graph H such that for all graphs $G \in \mathscr{C}$, G does not have H as a minor.

Example:

- Graph with tree-width less than 2 (trees) exclude cycles.
- Planar graphs exclude $K_{3,3}$ and K_5 as minors.
- Graphs with tree-width equal to 2 exclude K_4 as minor.

Properties of Graph Classes

Bounded tree-width ⊆ Excluding minors ⊆ Nowhere dense

Properties of Graph Classes

Bounded tree-width $\subseteq$ Excluding minors $\subseteq$ Bounded expansion $\subseteq$ Nowhere dense

Properties of Graph Classes

Bounded tree-depth ⊆ Bounded path-width ⊆ Bounded tree-width ⊆ Excluding minors ⊆ Bounded expansion ⊆ Nowhere dense

FO-model checking on a class $\mathscr{C}$ of graphs

Input : A first order formula φ of quantifier rank q and a graph $G \in \mathscr{C}$.
Output: Deciding whether $G \models \varphi$.

Bounded tree-depth	$\subseteq$	Bounded path-width	$\subseteq$	Bounded tree-width	$\subseteq$	Excluding minors	$\subseteq$	Bounded expansion	$\subseteq$	Nowhere dense

Can we FO-model check efficiently?

Fixed parameter tractable: A parameterized problem L is fixed-parameter tractable if the question "$(x, k) \in L$" can be decided in running time $f(k) \cdot |x|^{O(1)}$, where f is an arbitrary function depending only on k.

Example: Consider the complexity of database queries.

Bounded tree-depth ⊆ Bounded path-width ⊆ Bounded tree-width ⊆ Excluding minors ⊆ Bounded expansion ⊆ Nowhere dense

ed expansion ⊆ Nowhere dense

ed expansion $\subseteq$ Nowhere dense

Grohe, Kreutzer and Siebertz, 2017:

- FO-model checking on nowhere dense classes of graphs can be done in $f(|\varphi|, \epsilon) \cdot |G|^{1+\epsilon}$ for any $\epsilon > 0$.
- For classes that are closed under taking subgraph, nowhere denseness is the barrier for FO-model checking efficiently.

Properties of Graph Classes

Bounded tree-depth ⊆ Bounded path-width ⊆ Bounded tree-width ⊆ Excluding minors ⊆ Bounded expansion ⊆ Nowhere dense

Properties of Graph Classes

Bounded tree-depth $\subseteq$ Bounded path-width $\subseteq$ Bounded tree-width $\subseteq$ Excluding minors $\subseteq$ Bounded expansion $\subseteq$ Nowhere dense

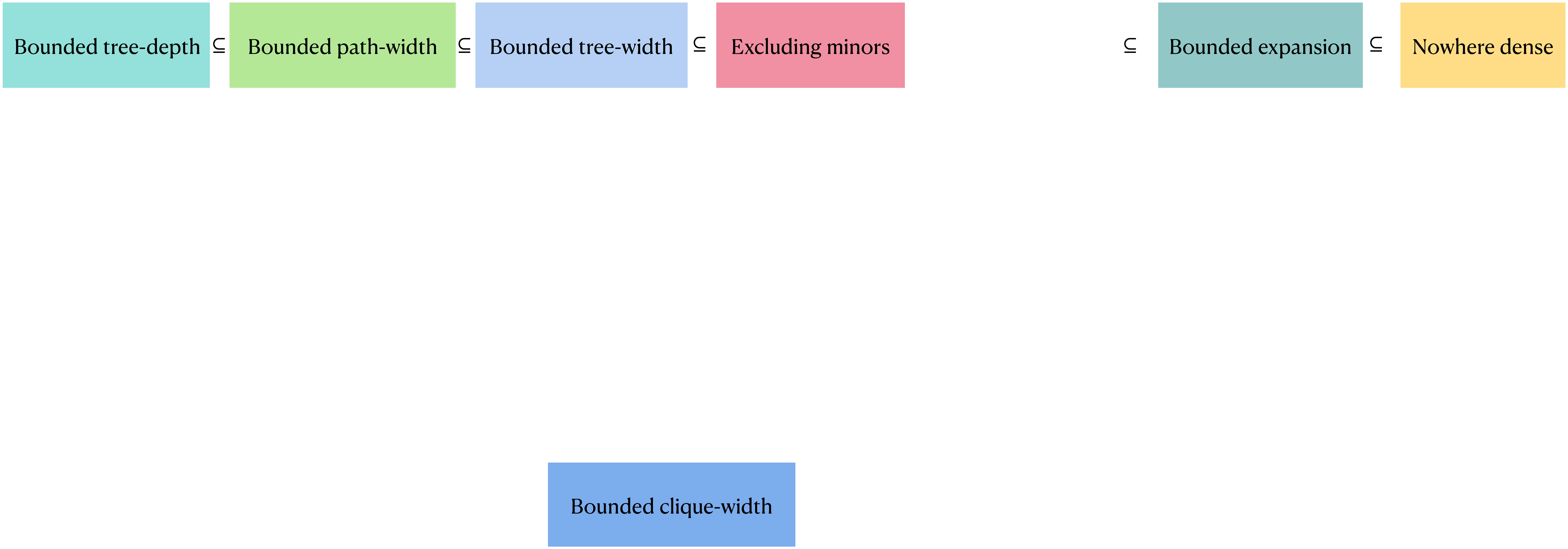
Bounded clique-width

Bounded clique-width

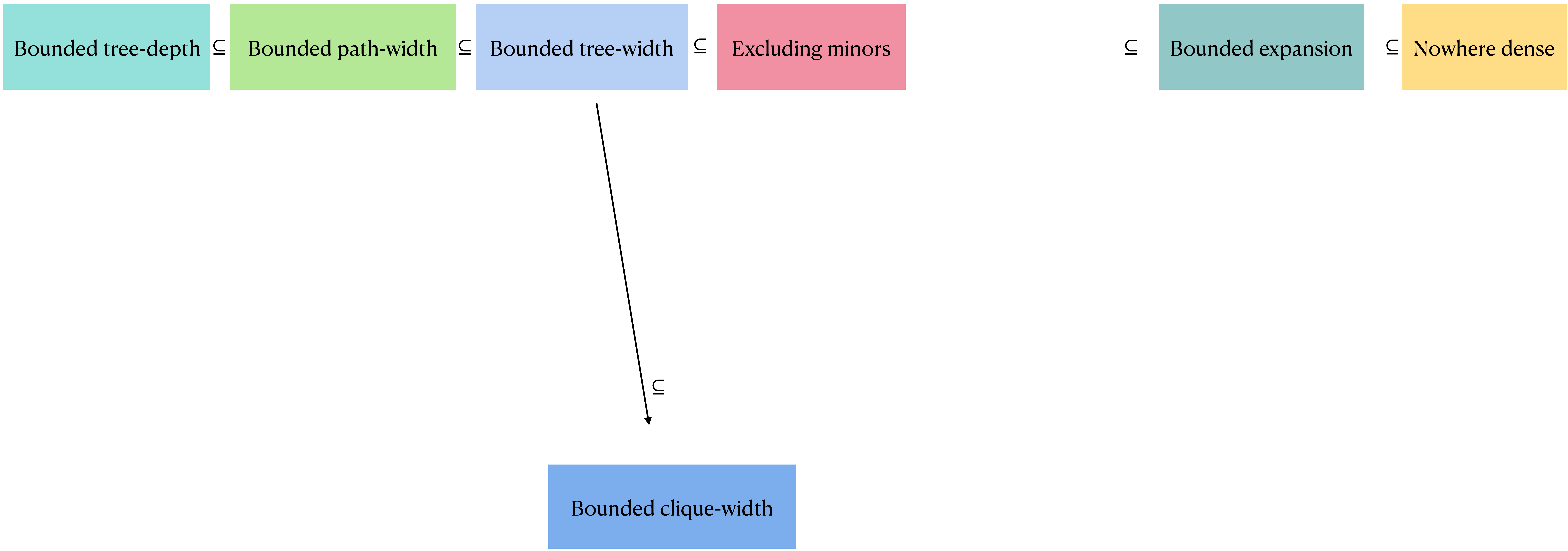
- Good parameter to treat dense classes of graphs.

Courcelle et al., 2000: FO-model checking can be done in $f(|\varphi|) \cdot |G|^2$.

- Clique-width of a clique is one.

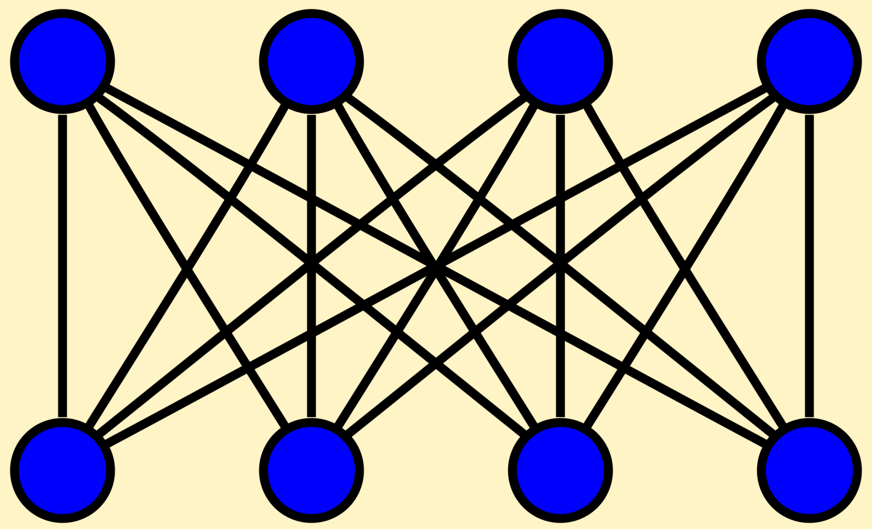

Bounded tree-depth
⊆
Bounded path-width
⊆
Bounded tree-width
⊆
Excluding minors
⊆
Bounded expansion
⊆
Nowhere dense
Bounded clique-width

Properties of Graph Classes




Weakly sparse
Bounded tree-depth
⊆
Bounded path-width
⊆
Bounded tree-width
⊆
Excluding minors
⊆
Bounded expansion
⊆
Nowhere dense
⊆
Bounded clique-width

A class $\mathscr{C}$ is weakly sparse if there exists $t \in \mathbb{N}$ such that for all graphs $G \in \mathscr{C}$, $K_{t,t}$ is not a subgraph of G.



$K_{4,4}$

Weakly sparse

-width $\subseteq$ Bounded tree-width $\subseteq$ Excluding minors $\subseteq$ Bounded expansion $\subseteq$ Nowhere dense


Weakly sparse
Bounded tree-depth
⊆
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Excluding minors
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Bounded clique-width


Weakly sparse
Bounded tree-depth
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Bounded shrub-depth
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Bounded linear clique-width
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Bounded clique-width

Weakly sparse
Bounded tree-depth
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Bounded tree-width
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Excluding minors
⊆
Bounded sparse twin-width
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Bounded expansion
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Nowhere dense
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Bounded shrub-depth
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Bounded linear clique-width
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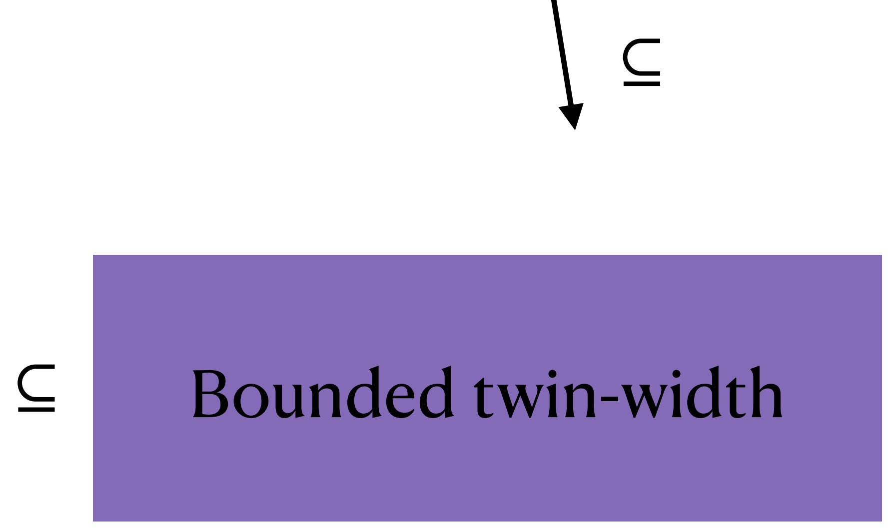
Bounded clique-width
⊆
Bounded twin-width



Bonnet et al., 2020: FO-model checking is fixed parameter tractable on classes of graphs with bounded twin-width if a suitable decomposition is given as additional input.


Weakly sparse
Bounded tree-depth ⊆ Bounded path-width ⊆ Bounded tree-width ⊆ Excluding minors ⊆ Bounded sparse twin-width ⊆ Bounded expansion ⊆ Nowhere dense
⊆
⊆
⊆
⊆
Bounded shrub-depth ⊆ Bounded linear clique-width ⊆ Bounded clique-width ⊆ Bounded twin-width

FO-Interpretation: Let $\varphi(x, y)$ be a symmetric FO-formula. For a graph G, $\varphi(G)$ is the graph on the vertex set of G, where the edge relation is interpreted as φ indicates.

$$\varphi(G) = (V(G), \{uv \mid G \models \varphi(u, v)\})$$

Example: Let $\varphi(u, v) = \neg E(u, v)$, then $\varphi(G) = \overline{G}$.

Example: Let $\varphi(u, v) = \mathsf{dist}_{\leq 2}(u, v)$, then $\varphi(G) = G^2$.

FO-**Transduction:** Let $\mathscr{L}$ be a finite set of unary predicates. An FO-transduction is an FO-interpretation from $\mathscr{L}$-coloured graphs.

$T(G)$: all graphs obtainable by

1. Colour G by colours from $\mathscr{L}$,
2. Apply φ on the coloured graph,
3. and take any induced subgraph.

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Bounded twin-width

Weakly sparse
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Bounded sparse twin-width
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Bounded expansion
⊆
Nowhere dense
All classes that are FO-transducible
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Stable bounded shrub-depth
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Stable bounded linear clique-width
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Structurally nowhere dense
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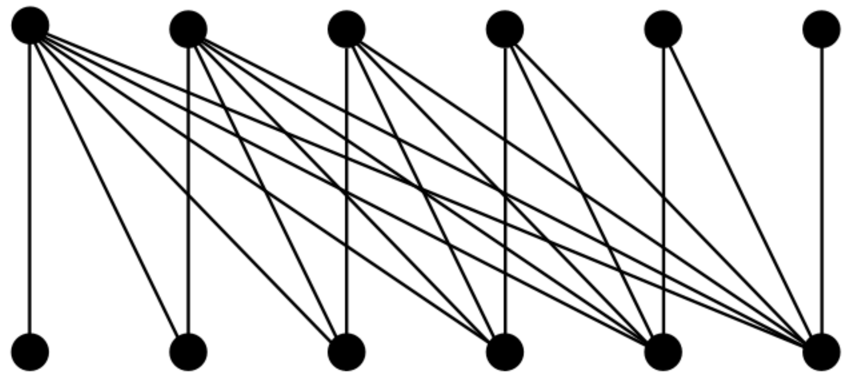
Bounded clique-width
⊆
Bounded twin-width

Stable bounded tree-width $\subseteq$ Stable Bounded sparse twin-width

$\subseteq$

A class $\mathscr{C}$ is stable if for all graphs $G \in \mathscr{C}$, no FO-transduction of arbitrary large ladders is an induced subgraph of G.



Ladder of length 6

Bounded clique-width $\subseteq$ Bounded twin-width


Weakly sparse
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Stable bounded shrub-depth ⊆ Stable bounded linear clique-width ⊆ Stable bounded tree-width ⊆ Stable Bounded sparse twin-width ⊆ Structurally nowhere dense
Bounded shrub-depth ⊆ Bounded linear clique-width ⊆ Bounded clique-width ⊆ Bounded twin-width
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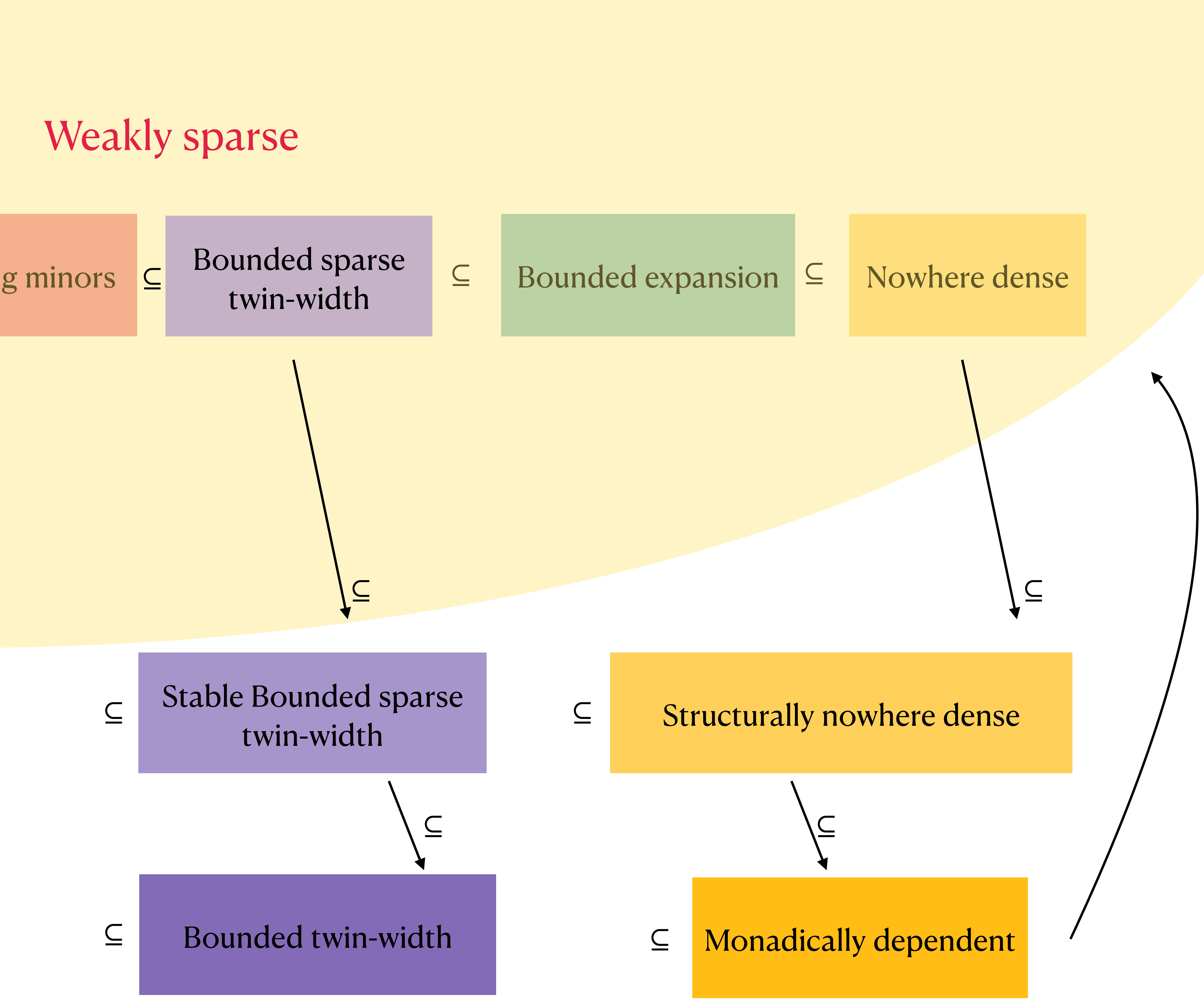
Bounded twin-width
⊆
Monadically dependent



Dvorak, 2018: For weakly sparse classes of graphs, monadically dependent classes collapse to nowhere denseness.

Weakly sparse
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Monadically dependent

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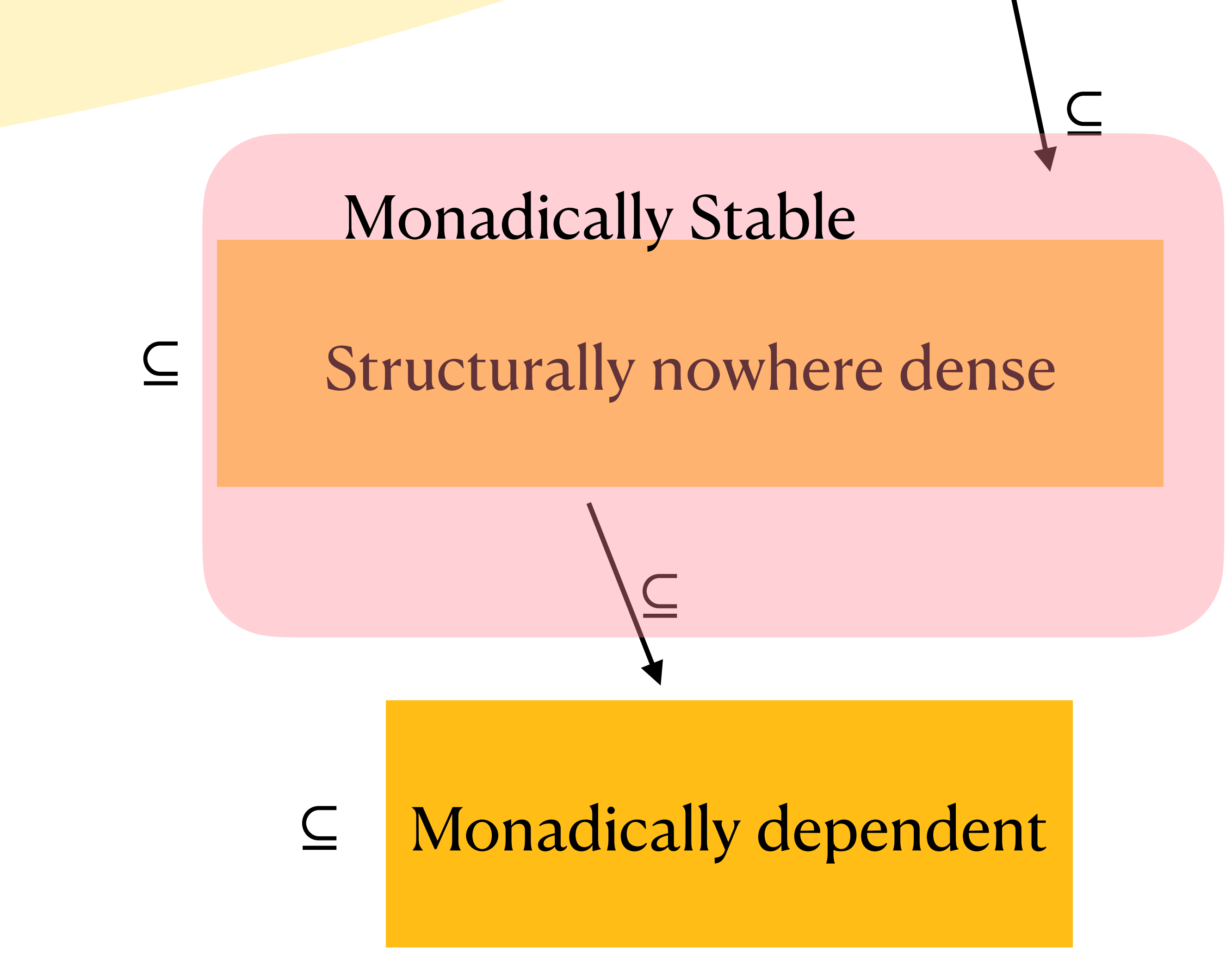

Dreier et al., 2023: FO-model checking on structurally nowhere dense classes of graphs can be done in $f(|\varphi|) \cdot |G|^{11}$.

Dreier et al., 2023: FO-model checking on monadically stable classes can be done in $f(|\varphi|, \epsilon) \cdot |G|^{5+\epsilon}$.


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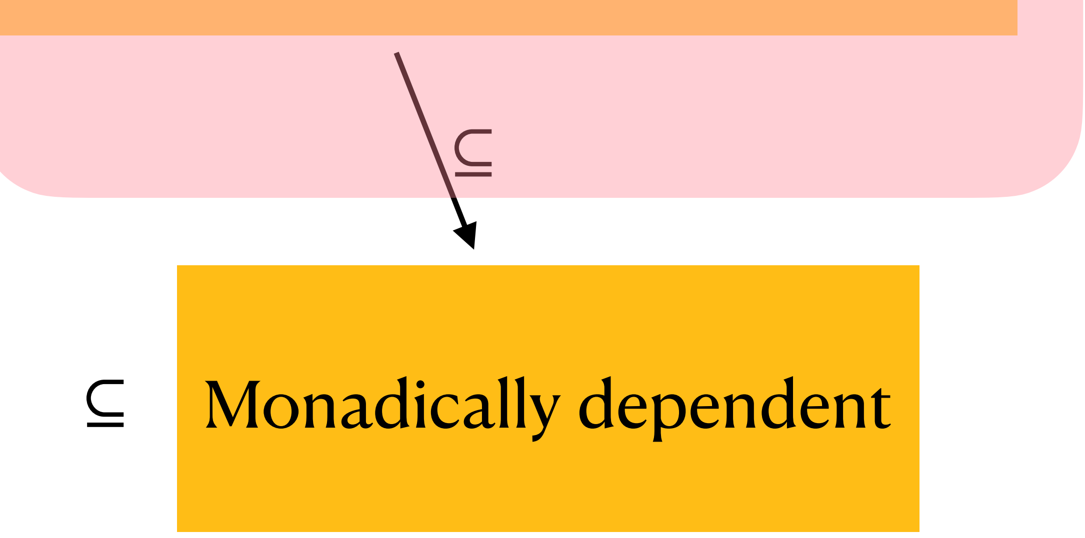

A class $\mathscr{C}$ is monadically dependent if the class of all graphs can not be transduced from $\mathscr{C}$.

Thank you!