Totalization of ODEs

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Ordinary differential equations

Let $E \subset \mathbb{R}$ compact. Let $y : [a, b] \to E$ be the unique solution of:

$$\begin{cases} y' = f(y(t)) \\ y(a) = y_0 \end{cases}$$

- Obtain y: if f is continuous, Peano's theorem, limit of sequence of continuous functions
- Compute y: if f is continuous, Ten thousand monkeys [CG09]

Question 1:

Relaxing continuity for f, when can we obtain y from f ?

Question 2:

What is the set theoretical complexity of y relative to f?

Antiderivative

- Let $F : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable on $[a, b]$
- Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be such that $F'(x) = f(x)$ for all $x \in [a, b]$

Question 1:
When can we obtain F from f?

Question 2:
What is the set theoretical complexity of F relative to f?

- Question 1 investigates methods related to given conditions on f
- Question 2 investigates the complexity of such methods for set descriptive theory

Conditions on the derivative

If f satisfies (A) then by (B) we get:

$$F(b) = F(a) + \int_a^b f(x)$$

- (A) f continuous
  (B) Fundamental theorem of calculus
  $F \in C^1([a,b])$
- (A) f bounded, continuous almost everywhere $(\mu_L(D_f) = 0)$
  (B) Lebesgue-Vitali theorem
  $F \in C^1([a,b])$ almost everywhere
- (A) f Lebesgue integrable
  (B) Lebesgue differentiation theorem
  $F \in AC$, Absolutely continuous

Absolutely continuous AC and bounded variation BV

- If $F : [a, b] \to \mathbb{R} \in \text{AC}$ then $F \in \text{BV}$

Definition 1 (BV)

Let $F : [a, b] \to \mathbb{R}$, define the quantity $V(F) = \sup_{P \in \mathcal{P}} \sum_k |F(x_{k+1}) - F(x_k)|$.

Then, F is of *bounded variation* if $V(F) < +\infty$

- $F \notin \text{BV}$ on $[a, b] \Rightarrow F(b) \neq F(a) + \int_a^b f(x)$
- Bounded variation for $F \longleftrightarrow$ bounded length for y

Non-integrable derivative

Goal:

Investigate complex, non-integrable derivative

We need to have:

- Baire category theorem $\Rightarrow f$ continuous on a dense subset of $[a, b]$
- Darboux theorem $\Rightarrow f$ has the Darboux property on $[a, b]$

Theorem 2 (Darboux)

Let $F : [a, b] \to \mathbb{R}$ be differentiable, and let f be its derivative. Then, for every $f(a) < c < f(b)$ there is a point $x \in (a, b)$ such that $c = f(x)$.

Dirichlet's function

$$f(x) = \begin{cases} 1 & \text{if } \ x \in [a, b] \cap \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Darboux functions

- Darboux functions corresponds to continuous functions?
- Topologist's sine curve, $f(0) = 0$, $f(x) = \sin(\frac{1}{x})$
- Conway base 13, strongly Darboux, nowhere continuous

Question:
Can $F \notin \mathrm{BV}$ while being an antiderivative?

Function Ω

$$F(x) \equiv \Omega(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0 \end{cases}$$

$$f(x) \equiv \Omega'(x) = \begin{cases} 2x \sin(\frac{1}{x^2}) - \frac{2}{x} \cos(\frac{1}{x^2}) & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0 \end{cases}$$

Graphics

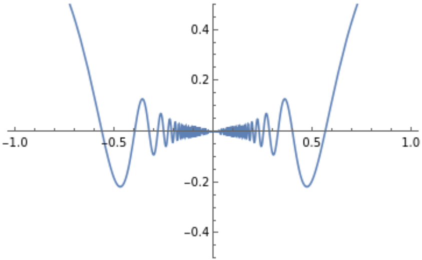


Figure: Function Ω

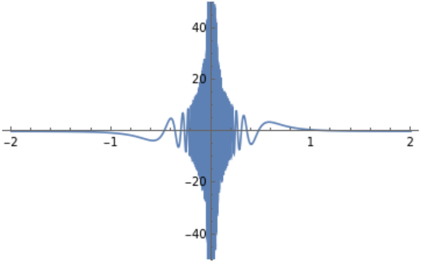
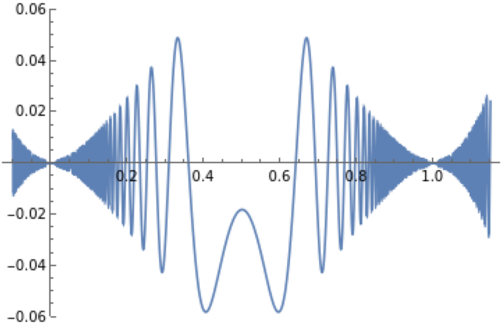


Figure: Function Ω'

More complex derivatives: two problematic discontinuities

- Function discontinuous in two points

$$F(x) \equiv \Omega_2(x) = \begin{cases} x^2(1-x)^2 \sin(\frac{1}{x^2(1-x)^2}) & \text{if } 0 < x < 1 \\ 0 \text{ if } x = 0, 1 \end{cases}$$



More complex: uncountable prblematic discontinuities

- Extension Cantor middle third:

Cantor set of discontinuities

Define $\Omega_{2,n}$ as the scaled $\Omega_{2,n}(x) = 4^{-n}\Omega_2(x)$

Define $I_{m,n}$: the mth (of 2^n) removed intervals from the nth step

Define $f_{m,n}(x) = \Omega_2(x)$ for $x \in I_{m,n}$; $f_{m,n}(x) = 0$ otherwise

Define $F(x) = \sum_{n=0}^{\infty} \sum_{m=1}^{2^n} f_{m,n}(x)$

- Since all $f_{m,n}$ are differentiable, F converges and F' converges uniformly, F is differentiable
- Using fat Cantor set, D_f with full measure

Extension of Lebesgue integral

Question:

Is there a way to compute such antiderivatives?

- Need for another formalization of integration
- Denjoy, 1912, [Den12], iterative process, transfinite induction
- Luzin, 1915, variation absolute continuity
- Perron, 1914, [Per14] , equivalent to Denjoy
- Kurzveil, 1957, [Kur57], gauge integral, similar to Riemann
- Henstock, 1957, [Hen57], equivalent to Kurzveil

Denjoy totalization

- Condition on $f : [a, b] \to \mathbb{R}$: f is Lebesgue measurable
- Lebesgue measurable $\not\Rightarrow$ Lebesgue integrable; ex: $f(x) = \frac{1}{x}$ if $x \neq 0$, $f(0) = 0$

Definition 3 (Nonsummable points of f)

Let E be a closed set $E \subseteq [a, b]$ and f a Lebesgue measurable function. A point $x \in E$ is a *point of nonsummability of* f *on* E if f is not Lebesgue integrable in every $I \in E$, I an open interval containing x.

Definition 4 (Divergence points of F)

Let F be a continuous function on $[a, b]$, let $E \subseteq [a, b]$ be closed, and let $\{(a_i, b_i)\}$ be the contiguous intervals of E in $[a, b]$. A point $x \in E$ is a point of divergence of F on E if $\sum_I |F(b_i) - F(a_i)| = \infty$ for all open intervals I containing x, where $\sum_I$ indicates that we include only the (a_i, b_i) contained in I.

Theorem 5

If F is a differentiable function on $[a, b]$ and $E \subseteq [a, b]$ is closed, then the nonsummable points of f on E and the divergence points of F on E form a closed nowhere dense set in E.

Definition 6 (Nowhere dense)

A subset A of a topological space X is nowhere dense in X if the closure of A has empty interior.

- Bad behaved points are few for derivatives
- Outside of them we can simply integrate

Theorem 7

Let $E \subseteq [a, b]$ be closed and $\{(a_i, b_i)\}$ the intervals contiguous to E in $[a, b]$. Let F be differentiable on $[a, b]$ and assume $f = F'$ is Lebesgue integrable on E and $\sum_i |F(b_i) - F(a_i)| < \infty$. Then:

$$F(b) - F(a) = \int_E f(x)dx + \sum_i [F(b_i) - F(a_i)].$$

Iterative process

Intuition:

We can iteratively compute $F(d) - F(c)$ for all $c, d \in [a, b]$, $c < d$

- Let $E_1 = \{x \in [a, b]$ such that x is a nonsummable point of f on $[a, b]\}$, and let $\{(a_i^1, b_i^1)\}$ be its contiguous intervals
- Obtain $F(d) - F(c)$ for all $c, d \in [a, b]$ such that $[c, d] \cap E_1 = \emptyset$
- Since F is continuous, take limits to obtain $F(b_i^1) - F(a_i^1)$ for all i.

Iterative step

Let $E_2 = \{x \in [a, b]$ such that x is a nonsummable point of f on E_1 or x is a divergence point of F on $E_1\}$, and let $\{(a_i^2, b_i^2)\}$ be its contiguous intervals

- Repeat the above for E_2
- Proceed by transfinite induction, taking intersections at limit ordinals

Theorem 8 ([Hob21])

Let P_n be a sequence of closed subsets of $\mathbb{R}$. If $P_{m+1} \subseteq P_m$ for all indexes m of the sequence, then, either P_α vanishes for some countable ordinal α, or else there is a countable ordinal α, from and after which all the sets are identical.

- If $P_{m+1} \subseteq P_m$ with P_m nowhere dense in P_{m+1}, $\Rightarrow P_\alpha = \emptyset$
- The iterative process converges, $E_\alpha = \emptyset \Rightarrow F(d) - F(c)$ for all $c, d \in [a, b]$, $c < d$; Totalization
- Denjoy and Lebesgue, application of Cantor's transfinite set theory to analysis.
- Denjoy, arbitrary number of countable steps

Question: Luzin

Totality of countable ordinals for antiderivatives?

Theorem 9 ([DK91])

The operation of antidifferentiation is not Borel.

Theorem 10 ([DK91])

Let $x \in \mathbb{R}$. Let f be the derivative of F, with f recursive. Then the following are equivalent:

- *x is hyperarithmetic,*
- $x = F(b) - F(a)$

Definition 11

A set $A \subset \mathbb{N}$ is hyperarithmetic if it is definable by a formula of second-order arithmetic with only existential set quantifiers or with only universal set quantifiers. A number $x \in \mathbb{R}$ is hyperarithmetic if the set $\{q \in \mathbb{Q}$ such that $q < x\}$ is hyperarithmetic.

Back to ordinary differential equations

Let $E \subset \mathbb{R}$ compact. Let $y : [a, b] \to E$ be the unique solution of:

$$\begin{cases} y' = f(y(t)) \\ y(0) = y_0 \end{cases}$$

Question 1:

When can we obtain y from f with a totalization?

Question 2:

Can we have hyperarithmetic solutions?

- f is continuous $\Rightarrow$ y: Peano's theorem, Ten thousand monkeys [CG09]

Lebesgue integration $\longrightarrow$ Ten Thousand Monkeys

Nonsummability points in $[a, b]$ $\longrightarrow$ Discontinuity points for f on E

- Let $E_1 = \{x$ such that x is a discontinuity point of f on $E\}$, and let $\{(a_i^1, b_i^1)\}$ be its contiguous intervals
- Obtain $y(d) - y(c)$ for all $c, d \in [a, b]$ such that $[c, d] \cap E_1 = \emptyset$
- Since y is continuous, take limits to obtain $y(b_i^1) - y(a_i^1)$ for all i.

Intuition:

We can iteratively compute $y(d) - y(c)$ for all $c, d \in [a, b]$, $c < d$

Main difference with integration:

We are not given the derivative, $f \circ y$, but f.
Nonsummability of $f \circ y$ in $[a, b]$ $\not\Rightarrow$ discontinuity of f on E.

Challenges

Problem 1: Induced topology

- $E_2 = \{x$ such that x is a discontinuity point of $f|_{E_1}$ on $E_1\}$
- Continuity and derivatives are in subspace topology of E_1

Problem 2: Convergence of iterations

Which conditions on f such that:

- We need E_m closed $\forall m$
- We need $E_{m+1} \subset E_m$, $\forall m$

Problem 3: How can f be given?

- Depending on solution of problem 2 above
- Identifying set descriptive complexity for f

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Thank you!