Computability, Complexity and Programming with Ordinary Differential Equations

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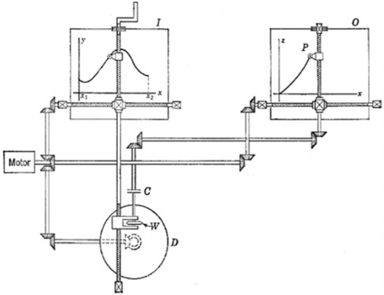
Menu

How to compute an integral?



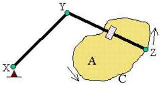
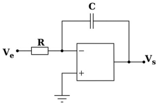
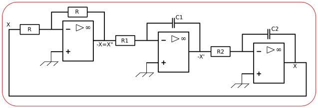
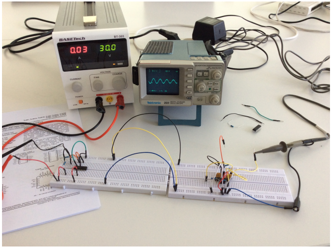


Figure 1. A simple planimeter.







Our actual motivation

- Understand how analog models compare to classical digital models of computation.
  - At **computability** level
  - At **complexity** level.
- Continuous time analog models correspond to various classes of ordinary differential equations.
- Discussing hardness of solving IVP according to various classes of dynamics is basically discussing the **computational power** of various classes of analog models.

Take home message

- **Turing machines $\sim$ polynomial Ordinary Differential Equations**
  I.e.

$$\begin{array}{rcl} \mathbf{y}' & = & \mathbf{p}(t, \mathbf{y}) \\ \mathbf{y}(t_0) & = & \mathbf{y}_0 \end{array}$$

  where $\mathbf{p}$ is a (vector of) polynomials.

  in a very very strong sense.

- Programming with/Solving ODEs is **simple** and **fun**.

Take home message

- **Turing machines $\sim$ polynomial Ordinary Differential Equations**
  I.e.

$$\begin{aligned} \mathbf{y}' &= \mathbf{p}(t, \mathbf{y}) \\ \mathbf{y}(t_0) &= \mathbf{y}_0 \end{aligned}$$

  where **p** is a (vector of) polynomials.

  in a very very strong sense.

- Programming with/Solving ODEs is **simple** and **fun**.

- Analog's world: Many concepts from **computer science** can be defined using polynomial ODEs
  - Computable functions.
  - Polynomial Time Computable Functions
  - *NP*, *PSPACE*, ...?
  - Revisiting computation theory with pODEs ...
  - Bioinformatics (proteins) computations $\geq$ Turing machines $=$ Classical computers.
  - ...

Some basics concepts/remarks

- $f : \mathbb{R}^d \to \mathbb{R}^d$
- f can be continuous, derivable, $\mathcal{C}^\infty$, ..., $\mathcal{C}^\infty$, analytic, generable, ...
- Dynamical system:
  - Discrete time: $\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t)$
  - Continuous time: $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$
- Computability $\neq$ Complexity

Some basics concepts/remarks

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- Dynamical system:
  - Discrete time: $\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t)$
    - AKA: $\frac{\delta \mathbf{x}}{\delta t}(t) = \frac{\mathbf{x}_{t+1} - \mathbf{x}_t}{1} = \overline{\mathbf{f}}(\mathbf{x}_t)$, for $\overline{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{x}$
  - Continuous time: $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x}(t))$
- Computability $\neq$ Complexity

Menu

Analog Computations: Our actual motivation

How to Compute with Iterations (dODEs)

How to Compute with Ordinary Differential Equations

Some computability results based on these ideas

Conclusion

Sub-menu

Turing Machines

- Let M be some one tape Turing machine, with m states and 10 symbols.
- If

$$\ldots B\ B\ B\ a_{-k}\ a_{-k+1}\ldots\ a_{-1}\ a_0\ a_1\ldots\ a_n\ B\ B\ B\ldots$$

  is the tape content of M, it can be seen as

$$\begin{aligned} y_1 &= 0.a_0a_1\ldots a_n \\ y_2 &= 0.a_{-1}a_{-2}\ldots a_{-k} \end{aligned} \qquad (1)$$

- The configuration of M is then given by three values: its internal state s, y_1 and y_2.

Alternative View of a Turing Machine

$$
\begin{aligned}
y_1 &= a_0 10^{-1} + a_1 10^{-2} + \ldots + a_n 10^{-n-1} \\
y_2 &= a_{-1} 10^{-1} + a_{-2} 10^{-2} + \ldots + a_{-k} 10^{-k}.
\end{aligned} \tag{2}
$$

$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t))$$

Turing Machine	
State Space $\{q_0, q_1, \cdots, q_{m-1}\} \times \Sigma^*$	
State $(q_i, a_{-m} \ldots a_{-1}, a_0 \ldots a_n)$	

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$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t))$$

Turing Machine	
State Space $\{q_0, q_1, \cdots, q_{m-1}\} \times \Sigma^*$	State Space $[0, m] \times [0, 1]$
State $(q_i, a_{-m} \ldots a_{-1}, a_0 \ldots a_n)$	State $\mathbf{x} = (x = s + y_2, y = y_1)$

Alternative View of a Turing Machine

$$\begin{aligned} y_1 &= a_0 10^{-1} + a_1 10^{-2} + ... + a_n 10^{-n-1} \\ y_2 &= a_{-1} 10^{-1} + a_{-2} 10^{-2} + ... + a_{-k} 10^{-k}. \end{aligned} \quad (2)$$

$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t))$$

Turing Machine	
State Space $\{q_0, q_1, \cdots, q_{m-1}\} \times \Sigma^*$	State Space $[0, m] \times [0, 1]$
State $(q_i, a_{-m}...a_{-1}, a_0...a_n)$	State $\mathbf{x} = (x = s + y_2, y = y_1)$
q_1: if 2 is read, then write 4; goto q_2	

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q_1: if 2 is read, then write 4; goto q_2	$\begin{cases} x := x + 1 \\ y := y + \frac{2}{10} \end{cases}$ if $\begin{cases} 1 \leq x < 2 \\ \frac{2}{10} \leq y < \frac{3}{10} \end{cases}$

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q_5: if 3 is read, then move right; goto q_1	$\begin{cases} x := \frac{x-5}{10} + \frac{3}{10} - 4 \\ y := 10y - 3 \end{cases}$ if $\begin{cases} 5 \leq x < 6 \\ \frac{3}{10} \leq y < \frac{4}{10} \end{cases}$

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Alternative View of a Turing Machine

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Turing Machine	PAM
State Space $\{q_0, q_1, \cdots, q_{m-1}\} \times \Sigma^*$	State Space $[0, m] \times [0, 1]$
State $(q_i, a_{-m} \ldots a_{-1}, a_0 \ldots a_n)$	State $\mathbf{x} = (x = s + y_2, y = y_1)$
q_1: if 2 is read, then write 4; goto q_2	$\begin{cases} x := x + 1 \\ y := y + \frac{2}{10} \end{cases}$ if $\begin{cases} 1 \le x < 2 \\ \frac{2}{10} \le y < \frac{3}{10} \end{cases}$
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q_3: if 5 is read, then move left; goto q_7	$\begin{cases} x := 10(x-3) - j + 4 \\ y := \frac{y}{10} + \frac{j}{10} \end{cases}$ if $\begin{cases} 3 + \frac{j}{10} \le x < 3 + \frac{j+1}{10} \\ \frac{5}{10} \le y < \frac{6}{10} \end{cases}$ for $j \in \{0, 1, \ldots, 9\}$.

Key remark: f (and $\overline{f}$) is piecewise affine

Morality

- If you prefer, a Turing Machine can be seen as a **piecewise affine function**
- It remains to simulate

$$\mathbf{x}(t+1) := \mathbf{f}(\mathbf{x}(t))$$

  for $t = 1, 2, \ldots$.

- ... to compute ...

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- If you prefer, a Turing Machine can be seen as a **piecewise affine function**
- It remains to simulate

$$\mathbf{x}(t+1) := \mathbf{f}(\mathbf{x}(t))$$

for $t = 1, 2, \ldots$.

- . . . to compute . . .
  - Improvement:
    - We don't care about $\mathbf{f}$ on points not encoding a configuration.
    - Hence, with a slight modification, we can even assume $\mathbf{f}$ continuous, and even $\mathcal{C}^\infty$.

Sub-menu

Turing Machines

- Let M be some one tape Turing machine, with m states and 10 symbols.
- If

$$\ldots B\ B\ B\ a_{-k}\ a_{-k+1}\ldots\ a_{-1}\ a_0\ a_1\ldots\ a_n\ B\ B\ B\ldots$$

  is the tape content of M, it can be seen as

$$\begin{aligned} y_1 &= a_n \ldots a_1 a_0 \\ y_2 &= a_{-k} \ldots a_{-2} a_{-1} \end{aligned} \quad (3)$$

- The configuration of M is then given by three values: its internal state s, y_1 and y_2.

Alternative View of a Turing Machine

$$\begin{aligned} y_1 &= a_0 10^0 + a_1 10^1 + \ldots + a_n 10^n \\ y_2 &= a_{-1} 10^0 + a_{-2} 10^1 + \ldots + a_{-k} 10^{k-1} \end{aligned} \quad (4)$$

$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t))$$

Turing Machine	
State Space $\{q_0, q_1, \cdots, q_{m-1}\} \times \Sigma^*$	
State $(q_i, a_{-m} \ldots a_{-1}, a_0 \ldots a_n)$	

Key remark: f (and $\overline{f}$) is (KM-)elementary

Alternative View of a Turing Machine

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$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t))$$

Turing Machine	
State Space $\{q_0, q_1, \cdots, q_{m-1}\} \times \Sigma^*$	State Space $\mathbb{N}^2$
State $(q_i, a_{-m}\ldots a_{-1}, a_0 \ldots a_n)$	State $\mathbf{x} = (x = s + my_2, y = y_1)$

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q_1: if 2 is read, then write 4; goto q_2	$\begin{cases} x := x+1 \\ y := y+2 \end{cases}$ if $\begin{cases} mod_m(x) = 1 \\ mod_{10}(y) = 2 \end{cases}$

Key remark: f **(and** $\overline{f}$**) is (KM-)elementary**

Alternative View of a Turing Machine

$$y_1 = a_0 10^0 + a_1 10^1 + \ldots + a_n 10^n$$
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q_5: if 3 is read, then move right; goto q_1	$\begin{cases} x := 10(x-5) + 3 * m - 4 \\ y := (y-3)/10 \end{cases}$ if $\begin{cases} mod_m(x) = 5 \\ mod_{10}(y) = 3 \end{cases}$

Key remark: f **(and** $\overline{f}$**) is (KM-)elementary**

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q_3: if 5 is read, then move left; goto q_7	$\begin{cases} x := (x - j*m - 3)/10 + 4 \\ y := 10y + j \end{cases}$ if $\begin{cases} mod_m(x) = 3 \\ mod_{10}(\frac{x-3}{10}) = j \\ mod_{10}(y) = 5 \end{cases}$

Key remark: f **(and** $\overline{f}$**) is (KM-)elementary**

Alternative View of a Turing Machine

$$\begin{aligned} y_1 &= a_0 10^0 + a_1 10^1 + \ldots + a_n 10^n \\ y_2 &= a_{-1} 10^0 + a_{-2} 10^1 + \ldots + a_{-k} 10^{k-1} \end{aligned} \quad (4)$$

$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t))$$

Turing Machine	
State Space $\{q_0, q_1, \cdots, q_{m-1}\} \times \Sigma^*$	State Space $\mathbb{N}^2$
State $(q_i, a_{-m} \ldots a_{-1}, a_0 \ldots a_n)$	State $\mathbf{x} = (x = s + my_2, y = y_1)$
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Key remark: f (and $\overline{f}$) is (KM-)elementary

Morality

- If you prefer, a Turing Machine can be seen as a **(KM-)elementary map**
- It remains to simulate

$$\mathbf{x}(t+1) := f(\mathbf{x}(t))$$

for $t = 1, 2, \ldots$.

- ... to compute ...

Koiran-Moore's 99 result

- **Theorem [KM99]:** For any Turing machine M and any input w, there is an elementary function $\mathbf{f}$ on two variables and constants a and b such that M halts on input w after t steps if and only if $f^{[t]}(a + bw, 0) = (0, 0)$.

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  - $f:\mathbb{R}^n \to \mathbb{R}^n$ is (KM-)elementary if its n components are in U_n, where U_n is the smallest class of functions $f:\mathbb{R}^n \to \mathbb{R}$ containing rational constants, π, the n projections $x \mapsto x_i$ and satisfying the following closure properties:
    - if $f, g \in U_n$ then $f \oplus g \in U_n$, where $\oplus \in \{+,-,\times\}$
    - if $f \in U_n$ then $\sin(f) \in U_n$
  - Using the trick that $\bmod_2(x)$ is basically $\sin(\pi x)^2$, etc.

Graça-Campagnolo-Buescu'2005

- Remarks:
  - Key point: With this encoding, integers are sent to integers...
  - $\sigma(x) = x - 0.2 \sin(2\pi x)$ is basically a contraction on the vicinity of integers.

(static error correction:)

- **Theorem [GCB'05]:** Let $0 < \delta < \epsilon < 1/2$. The transition function θ of a TM admits an analytic extension $f_M : \mathbb{R}^3 \to \mathbb{R}^3$, robust to perturbations.

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$$\left\| f^{[j]}(\overline{x_0}) - \theta^{[j]}(x_0) \right\| \leq \epsilon \text{ for all } j \in \mathbb{N}.$$

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    (static error correction:)

  - For
    $$l_2(x, y) = \frac{1}{\pi}\arctan(4y(x - 1/2)) + \frac{1}{2},$$
    we have $|a - l_2(\bar{a}, y)| < 1/y$ for $\bar{a}$ close to $a \in \{0, 1\}$, $y > 0$

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Menu

Sub-menu

Some ODEs...

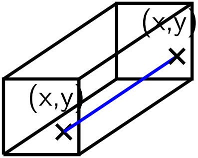


Figure: A linear path.

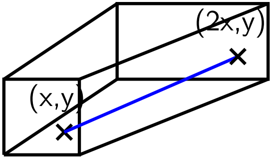


Figure: A dilation (acting on x of factor 2).

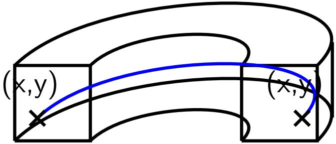


Figure: A U-turn.

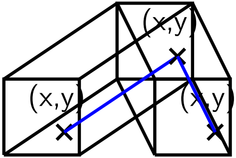
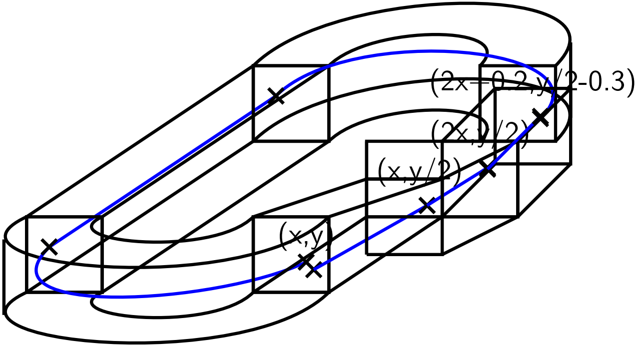
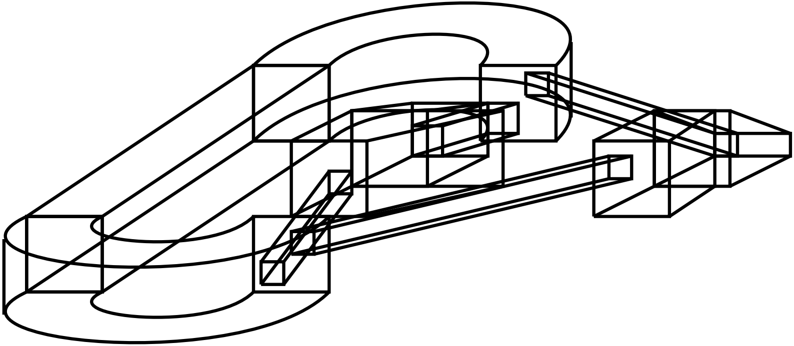


Figure: A merge (symbolic view: this can exist only in dimension 4) .

Some ODEs. . .



Some ODEs. . .



Sub-menu

Branicky's clock (1995): with non-analytic functions

- We basically need to do $\mathbf{x} := \mathbf{f}(\mathbf{x})$ repeatedly

Branicky's clock (1995): with non-analytic functions

- Doing $\mathbf{x}_2 := \mathbf{f}(\mathbf{x}_1)$; $\mathbf{x}_1 := \mathbf{x}_2$ repeatedly is fine.

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- Key observation: the solution of

$$y' = c(g - y)^3 \varphi(t)$$

converges at $t = 1/2$ close to the goal g with some arbitrary precision, independently from initial condition at $t = 0$

for any function φ of positive integral if c is sufficiently big.

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  - **If you prefer, this roughly does $y(1/2) := g$.**

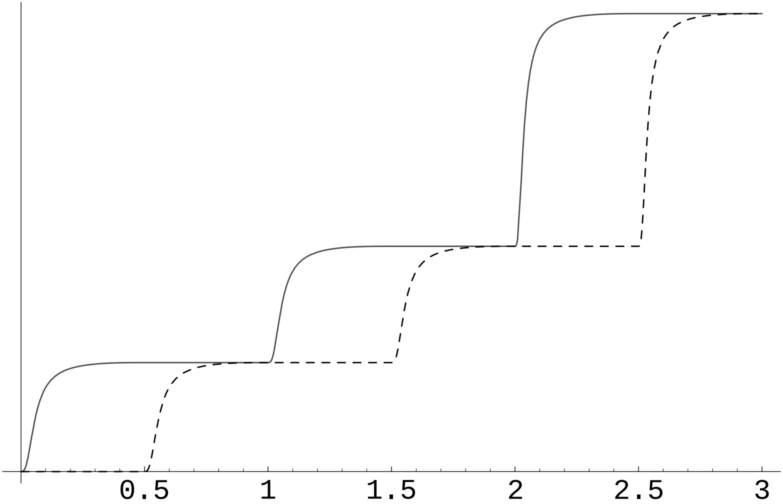
- The following system is then a solution

$$\begin{cases} \mathbf{x}_1' = c_1(\mathbf{r}(\mathbf{x}_2) - \mathbf{x}_1)^3 \theta(-\sin(2\pi t)) \\ \mathbf{x}_2' = c_2(\mathbf{f}(\mathbf{r}(\mathbf{x}_1)) - \mathbf{x}_2)^3 \theta(\sin(2\pi t)) \end{cases} \quad \begin{cases} \mathbf{x}_1(0) = \mathbf{x}_0 \\ \mathbf{x}_2(0) = \mathbf{x}_0 \end{cases}$$

considering functions:

  - θ such that $\theta(x) = 0$ if $x \leq 0$, $\theta(x) = x^2$ if $x \geq 0$.
  - $r(x) = j$ whenever $x \in [j - 1/4, j + 1/4]$, for $j \in \mathbb{Z}$.

Example: $y(t+1) := 2 * y(t)$



Simulation of iterations of $h(n) = 2^n$ by ODEs.

Sub-menu

If analytic functions are forbidden...

- We want polynomial ODEs
  - non analytics functions (e.g. θ, r) are forbidden.
- Requires to program with ODEs.
  - and to deal with errors ...

Errors...

- We would dream to do:

$$y' = c(g - y)^3 \varphi(t)$$

- We do at best something like

$$z' = c(\bar{g}(t) - z)^3 \varphi(t) + E(t)$$

where $|\bar{g}(t) - g| \leq \rho$ and $|E(t)| \leq \delta$.

  - not so bad if ρ, δ small, and c big enough.

What do we get at the end?

- We would dream to do:

$$\left\{ \begin{array}{lcl} \mathbf{x}_1' & = & c_1(\mathbf{r}(\mathbf{x}_2) - \mathbf{x}_1)^3 \theta(-\sin(2\pi t)) \\ \mathbf{x}_2' & = & c_2(\mathbf{f}(\mathbf{r}(\mathbf{x}_1)) - \mathbf{x}_2)^3 \theta(\sin(2\pi t)) \end{array} \right.$$

- We do something like

$$\mathbf{x}_1' = c_1 \left(\sigma^{[n]}(\mathbf{x}_2) - \mathbf{x}_1 \right)^3 \zeta_{\epsilon_1}(t)$$
$$\mathbf{x}_2' = c_2 \left(\mathbf{f} \circ \sigma^{[m]}(\mathbf{x}_1) - \mathbf{x}_2 \right)^3 \zeta_{\epsilon_2}(-t)$$

Considering

$$\zeta_\epsilon(t) = l_2(\vartheta(t), 1/\epsilon),$$
$$\vartheta(t) = \frac{1}{2} \left(\sin^2(2\pi t) + \sin(2\pi t) \right)$$
$$l_2(x, y) = \frac{1}{\pi} \arctan(4y(x - 1/2)) + \frac{1}{2}.$$
$$\sigma(x) = x - 0.2 \sin(2\pi x).$$

$+$ dynamic error control on $\mathbf{f}$

Dynamic error control on $\mathbf{f}$

ω of period 10, $\bar{y} = \omega(\bar{y}_1)$

$$f_M(\overline{y}_1, \overline{y}_2, \bar{q}) = (\overline{y}_1^{next}, \overline{y}_2^{next}, \overline{q}_{next})$$

$$\overline{y}_1^{next} = \overline{P}_1 \frac{1}{2}(1-H)(2-H) + \overline{P}_2 H(2-H) + \overline{P}_3(-\frac{1}{2})H(1-H), \qquad (5)$$

with (move left, don't move, move right:)

$$\begin{aligned}
\overline{P}_1 &= 10(\sigma^{[j]}(\overline{y}_1) + \sigma^{[j]}(\overline{s}_{next}) - \sigma^{[j]}(\overline{y}) + \sigma^{[j]} \circ \omega \circ \sigma^{[j]}(\overline{y}_2) \\
\overline{P}_2 &= \sigma^{[j]}(\overline{y}_1) + \sigma^{[j]}(\overline{s}_{next}) - \sigma^{[j]}(\overline{y}) \\
\overline{P}_3 &= \frac{\sigma^{[j]}(\overline{y}_1) - \sigma^{[j]}(\overline{y})}{10},
\end{aligned}$$

$$H = l_3(\overline{h}_3, 10000(\overline{y}_1 + 1/2) + 2).$$

$$q_{next} = \sum_{i=0}^{9} \sum_{j=1}^{m} \left(\prod_{r=0, r\neq i}^{9} \frac{(\sigma^{[n]}(\overline{y}) - r)}{(i-r)} \right) \left(\prod_{s=1, s\neq j}^{m} \frac{(\sigma^{[n]}(\overline{q}) - s)}{(j-s)} \right) q_{i,j}, \qquad (6)$$

$s =$ interpolation of same type

Some philosophical comments

- We are basically considering ODEs which are obtained by simulating dODEs (iterations)

- Basically,

  - we usually have in mind some discrete time/space model/reasonning

    (errorless)

  - That we simulate with some discete time/real space model

    (if we want analycity, should take care of errors)

  - That we simulate in turn continuous time models

    (introduces even more errors)

Menu

Analog Computations: Our actual motivation

How to Compute with Iterations (dODEs)

How to Compute with Ordinary Differential Equations

Some computability results based on these ideas

Conclusion

Sub-menu

- Some computability results based on these ideas
  - Doing stuffs with polynomial ODEs
  - Computable Analysis with GPACs

Discrete time simulation with a GPAC

Proposition ([**?**, **?**])

There is a computable polynomial p *and some computable value* $\alpha \in \mathbb{R}^n$

$$z' = p(z,t), \qquad z(0) = (\tilde{x}_0, \alpha)$$

such that for all $\tilde{x}_0 \in \mathbb{R}^{2l+1}$ *satisfying* $\|\tilde{x}_0 - x_0\|_\infty \leq \varepsilon$*, one has*

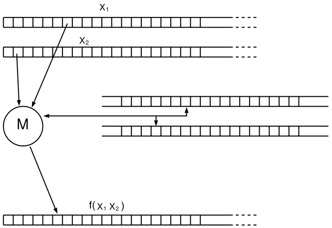
$$\left\| z_1(t) - \psi_M^{[j]}(x_0) \right\|_\infty \leq \delta.$$

for all $j \in \mathbb{N}$ *and for all* $t \in [j, j+1/2]$*.*

Sub-menu

Computable Analysis

Due to Turing, Grzegorczyk, Lacombe. Here presentation from Weihrauch.



A tape represents a real number

Each real number x is represented via an infinite sequence $(x_n)_n \in \mathbb{Q}$,

$$||x_n - x|| \leq 2^{-n}$$

M behaves like a Turing Machine

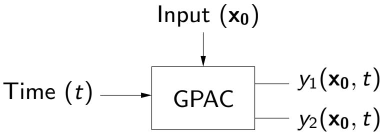
Read-only one-way input tapes
Write-only one-way output tape.
M outputs a representation of $f(x_1, x_2)$ from representations of x_1, x_2.

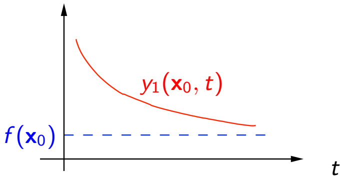
GPAC Computability vs GPAC Generation

Definition

A function $f : [a, b] \to \mathbb{R}$ is GPAC-computable iff there exist some computable polynomials $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $p_0 : \mathbb{R} \to \mathbb{R}$, and $n-1$ computable real values $\alpha_1, \ldots, \alpha_{n-1}$ such that:

1. $(y_1, \ldots, y_n)$ is the solution of the Cauchy problem $y' = p(y, t)$ with initial condition $(\alpha_1, \ldots, \alpha_{n-1}, p_0(x))$ set at time $t_0 = 0$
2. $\lim_{t\to\infty} y_2(t) = 0$
3. $|f(x) - y_1(t)| \leq y_2(t)$ for all $x \in [a, b]$ and all $t \in [0, +\infty)$.





Simulating Type-2 machines with a GPAC

Theorem ([?])

Let $f : [a, b] \to \mathbb{R}$ *be a computable function. Then there exists a GPAC and some index* i *such that if we set the initial conditions* $(x, \bar{n}) \in [a, b] \times \mathbb{R}$*, where* $|\bar{n} - n| \leq \varepsilon < 1/2$*, with* $n \in \mathbb{N}$*, there exists some* $T \geq 0$ *such that the output* y_i *of the GPAC satisfies* $|y_i(t) - f(x)| \leq 2^{-n}$ *for all* $t \geq T$*.*

Theorem (Bournez-Campagnolo-Graça-Hainry's Theorem [?])

Let a *and* b *be computable reals. A function* $f : [a, b] \to \mathbb{R}$ *is computable iff it is GPAC-computable.*

Idea of the construction

Proposition (e.g. Ko 91)

A real function $f : [a, b] \to \mathbb{R}$ *is computable iff there exist three computable functions* $m : \mathbb{N} \to \mathbb{N}$, $sgn, abs : \mathbb{N}^4 \to \mathbb{N}$ *such that:*

1. m *is a modulus of continuity for* f, *i.e. for all* $n \in \mathbb{N}$ *and all* $x, y \in [a, b]$, *one has*

$$|x - y| \leq 2^{-m(n)} \quad \Longrightarrow \quad |f(x) - f(y)| \leq 2^{-n}$$

2. *For all* $(i, j, k) \in \mathbb{N}^3$ *such that* $(-1)^i j/2^k \in [a, b]$, *and all* $n \in \mathbb{N}$,

$$\left|(-1)^{sgn(i,j,k,n)} \frac{abs(i, j, k, n)}{2^n} - f\left((-1)^i \frac{j}{2^k}\right)\right| \leq 2^{-n}.$$

General Idea

Idea

- From $n \in \mathbb{N}$, $x \in \mathbb{R}$,
  - compute integers i, j, k such that that

$$|(-1)^i j/2^k - x| \leq 2^{-m(n)}$$

  - compute values $sgn(i,j,k,n)$ and $abs(i,j,k,n)$.
  - output

$$(-1)^{sgn(i,j,k,n)} \frac{abs(i,j,k,n)}{2^n},$$

    guaranteed to be at 2^{-n} of $f(x)$

- Then increase n, and restart.

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Problems / Solutions.

1. Since we are on a compact, w.l.o.g. we can assume $x \geq 0$, and take $i = 1$ constant.

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Problems / Solutions.

1. One can take $k = m(n)$.

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- Then increase n, and restart.

Problems / Solutions.

1. Taking $j = \lceil x 2^{m(n)} \rceil$ would be great, but integer part is not analytic.

General Idea

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Problems / Solutions.

1. Taking $j = x2^{m(n)}$ yields valid output y_1 only when $x2^{m(n)}$ is close to an integer.

General Idea

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  guaranteed to be at 2^{-n} of $f(x)$

- Then increase n, and restart.

Problems / Solutions.

1. Taking $j = x2^{m(n)} + 1/2$ yields valid output y_1' only when $x2^{m(n)} - 1/2$ is close to an integer.

Getting a Valid Output For All x

Actually, according to the value of $\overline{k}_1 = x2^{m(n)}$ either y_1 or y_1' is valid.
We consider

$$\bar{y} = \frac{\omega_1(\overline{k}_1)y_1 + \omega_2(\overline{k}_1)y_1'}{\omega_1(\overline{k}_1) + \omega_2(\overline{k}_1)}. \tag{7}$$

where ω_1 (respectively ω_2) is ≥ 0 and close to 0 iff y_2 is valid (resp. y_2' is valid).

Last Ingredients (no details)

- We need to know when a computation is terminated.
- We use error-free external clocks
- We need to be able to switch dynamics
- We need to be able to reset machines

Menu

Take home message

- **Turing machines $\sim$ polynomial Ordinary Differential Equations**
  I.e.

$$\begin{aligned} \mathbf{y}' &= \mathbf{p}(t, \mathbf{y}) \\ \mathbf{y}(t_0) &= \mathbf{y}_0 \end{aligned}$$

where $\mathbf{p}$ is a (vector of) polynomials.

in a very very strong sense.

- Programming with/Solving ODEs is **simple** and **fun**.

Take home message

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- Programming with/Solving ODEs is **simple** and **fun**.

- Analog's world: Many concepts from **computer science** can be defined using polynomial ODEs
  - Computable functions.
  - Polynomial Time Computable Functions
  - *NP*, *PSPACE*, ...?
  - Revisiting computation theory with pODEs ...
  - Bioinformatics (proteins) computations $\geq$ Turing machines $=$ Classical computers.
  - ...