Embedding Turing machines in the dynamics of smooth maps

Alonso H. Núñez

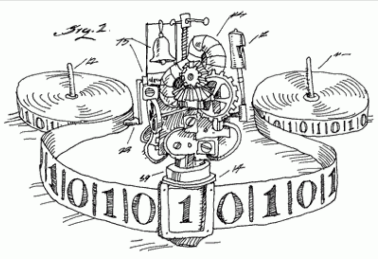
Joint work with Cristóbal Rojas (PUC)

SDA2, March 31th, 2023

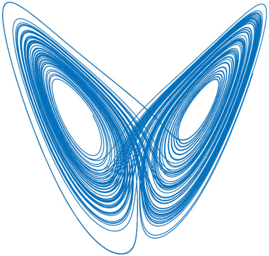
UNIVERSITÉ TOULOUSE III PAUL SABATIER Université de Toulouse

INSTITUT DE MATHÉMATIQUES de TOULOUSE

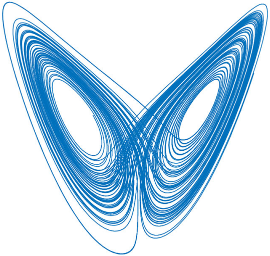
Turing machines simulate dynamical systems



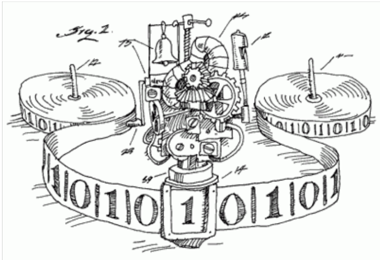
$\longrightarrow$



Dynamical systems simulate Turing machines



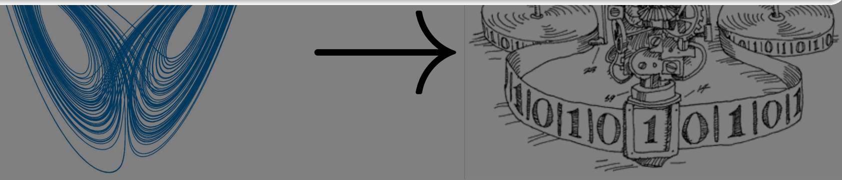
→



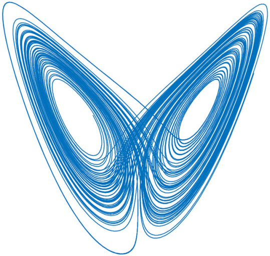
Dynamical systems simulate Turing machines

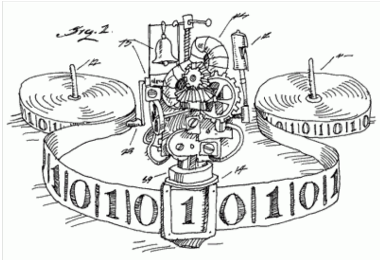
Some references

- Christopher Moore in [4], 1990.
- Daniel Graça et al. in [2], 2005.
- Terrence Tao in [5], 2017.
- Robert Cardona et al. in [1], 2021.



Dynamical systems simulate Turing machines





Main results

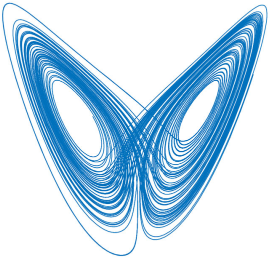
Theorem A

The sets of all Turing universal smooth maps and diffeomorphisms of the disk is an uncountable and dense set in $\mathcal{C}^\infty(\mathcal{D})$ and Diff$(\mathcal{D})$, respectively.

Main results

Theorem A

The sets of all Turing universal smooth maps and diffeomorphisms of the disk is an uncountable and dense set in $\mathcal{C}^\infty(\mathcal{D})$ and $\mathrm{Diff}(\mathcal{D})$, respectively.



Main results

Theorem A

The sets of all Turing universal smooth maps and diffeomorphisms of the disk is an uncountable and dense set in $\mathcal{C}^\infty(\mathcal{D})$ and $\mathrm{Diff}(\mathcal{D})$, respectively.

Theorem B

The set of non-statistical diffeomorphisms of the disk is an uncountable and dense set in $\mathrm{Diff}(\mathcal{D})$.

Outline

What comes next?

Outline

What comes next?

- Moore's construction: Generalized shifts and piece-wise linear maps

Outline

What comes next?

- Moore's construction: Generalized shifts and piece-wise linear maps
- Theorem A's proof sketch

Outline

What comes next?

- Moore's construction: Generalized shifts and piece-wise linear maps
- Theorem A's proof sketch
- Theorem B's proof idea

Generalized shifts

x :	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

Generalized shifts

x :	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

$\sigma(x)$:	$\cdots$	x_{-6}	x_{-5}	x_{-5}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\cdots$
									0								

Generalized shifts

x :	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

$I \subset \mathbb{Z}$ finite

$F : \mathcal{A}^I \to \mathbb{Z}$

Generalized shifts

x :

$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
								0								

Suppose $I = \{-1, 0, 1\}$

$$F : \mathcal{A}^I \to \mathbb{Z}$$

Generalized shifts

x :

$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
								0								

Suppose $I = \{-1, 0, 1\}$

$$x_I = x_{-1}x_0x_1$$

$$F : \mathcal{A}^I \to \mathbb{Z}$$

Generalized shifts

$x:$	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

Suppose $I = \{-1, 0, 1\}$

$$x_I = x_{-1}x_0x_1$$

$$F : \mathcal{A}^I \to \mathbb{Z}$$

$$F(x_I) = n$$

Generalized shifts

$x:$	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

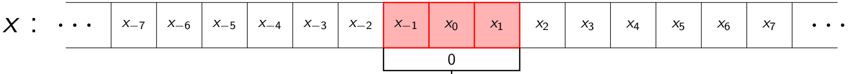
Suppose $I = \{-1, 0, 1\}$

$$x_I = x_{-1}x_0x_1$$

$$F : \mathcal{A}^I \to \mathbb{Z}$$

$$F(x_I) = 3$$

Generalized shifts

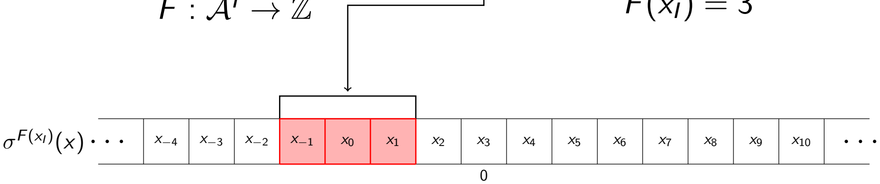


Suppose $I = \{-1, 0, 1\}$

$$x_I = x_{-1}x_0x_1$$

$$F : \mathcal{A}^I \to \mathbb{Z}$$

$$F(x_I) = 3$$



Generalized shifts

x :	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

Generalized shifts

x :

···	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	···
								0								

$J \subset \mathbb{Z}$ finite

$G : \mathcal{A}^J \to \mathcal{A}^J$

Generalized shifts

x :

$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
								0								

Suppose $J = \{-1, 0, 1\}$

$$x_J = x_{-1}x_0x_1$$

$$G : \mathcal{A}^J \to \mathcal{A}^J$$

Generalized shifts

x :	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

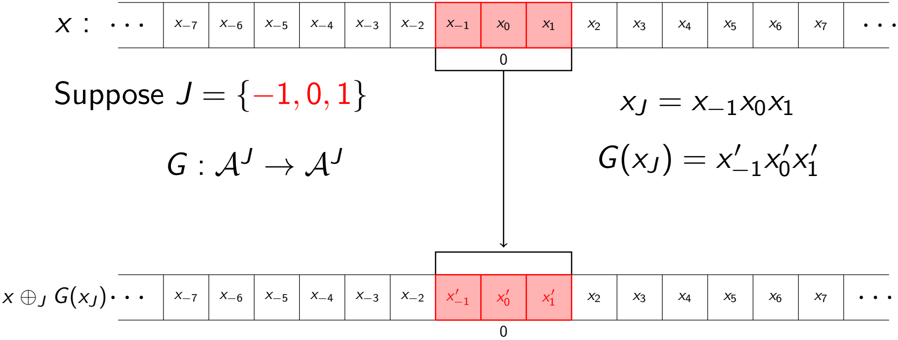
Suppose $J = \{-1, 0, 1\}$

$$x_J = x_{-1}x_0x_1$$

$$G : \mathcal{A}^J \to \mathcal{A}^J$$

$$G(x_J) = x'_{-1}x'_0x'_1$$

Generalized shifts



Suppose $J = \{-1, 0, 1\}$

$$G : \mathcal{A}^J \to \mathcal{A}^J$$

$$x_J = x_{-1}x_0x_1$$

$$G(x_J) = x'_{-1}x'_0x'_1$$

Generalized shifts

x :	$\cdots$	x_{-7}	x_{-6}	x_{-5}	x_{-4}	x_{-3}	x_{-2}	x_{-1}	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
									0								

Suppose $J = \{-1, 0, 1\} = I$

$$x_J = x_{-1}x_0x_1 = x_I$$

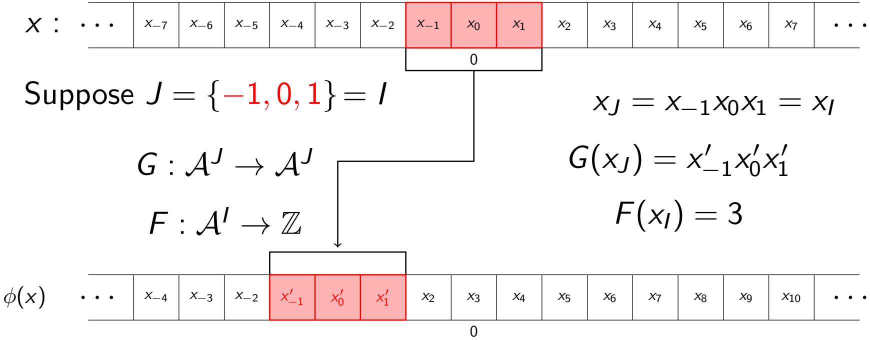
$$G : \mathcal{A}^J \to \mathcal{A}^J$$

$$G(x_J) = x'_{-1}x'_0x'_1$$

$$F : \mathcal{A}^I \to \mathbb{Z}$$

$$F(x_I) = 3$$

Generalized shifts



Suppose $J = \{-1, 0, 1\} = I$

$$x_J = x_{-1}x_0x_1 = x_I$$

$$G : \mathcal{A}^J \to \mathcal{A}^J$$

$$G(x_J) = x'_{-1}x'_0x'_1$$

$$F : \mathcal{A}^I \to \mathbb{Z}$$

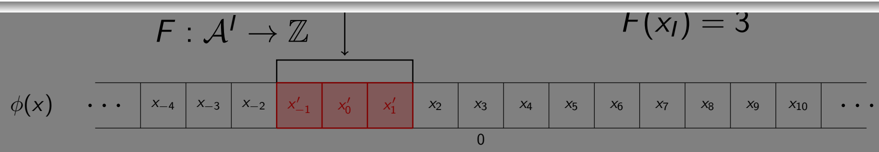
$$F(x_I) = 3$$

Generalized shifts

Definition (Moore, 1990)

A *generalized shift* is a map $\Phi = (F, G)$ from $\mathcal{A}^{\mathbb{Z}}$ to itself defined by

$$x \mapsto \sigma^{F(x_I)}(x \oplus_J G(x_J))$$



GS as Turing machines

Let $T = (\Sigma, Q, q_0, q_h, \delta)$ be a Turing machine and let $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

GS as Turing machines

Let $T = (\Sigma, Q, q_0, q_h, \delta)$ be a Turing machine and let $x = (x_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

$$\delta = (\delta_\Sigma, \delta_Q, \delta_\mathbb{Z}) \text{ with } \begin{aligned} \delta_\Sigma(q, x_0) &= x_0' \\ \delta_Q(q, x_0) &= q' \\ \delta_\mathbb{Z}(q, x_0) &= -1, 0 \textit{ or } 1 \end{aligned}$$

GS as Turing machines

Let $T = (\Sigma, Q, q_0, q_h, \delta)$ be a Turing machine and let $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

$$\delta = (\delta_\Sigma, \delta_Q, \delta_\mathbb{Z}) \text{ with } \begin{aligned} \delta_\Sigma(q, x_0) &= x_0' \\ \delta_Q(q, x_0) &= q' \\ \delta_\mathbb{Z}(q, x_0) &= -1, 0 \textit{ or } 1 \end{aligned}$$

Set $I = J = \{-1, 0, 1\}$.

Define $c : \Sigma^{\mathbb{Z}} \times Q \to \mathcal{A}^{\mathbb{Z}}$ by $((x_n)_{n\in\mathbb{Z}}, q) \mapsto \ldots x_{-1}.qx_0x_1 \ldots$

GS as Turing machines

Let $T = (\Sigma, Q, q_0, q_h, \delta)$ be a Turing machine and let $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

$$\delta = (\delta_\Sigma, \delta_Q, \delta_\mathbb{Z}) \text{ with } \begin{aligned} \delta_\Sigma(q, x_0) &= x_0' \\ \delta_Q(q, x_0) &= q' \\ \delta_\mathbb{Z}(q, x_0) &= -1, 0 \textit{ or } 1 \end{aligned}$$

Set $I = J = \{-1, 0, 1\}$.

Define $c : \Sigma^{\mathbb{Z}} \times Q \to \mathcal{A}^{\mathbb{Z}}$ by $((x_n)_{n\in\mathbb{Z}}, q) \mapsto \ldots x_{-1}.qx_0x_1\ldots$

Let $x \in A$ be s.t. $x_{[-1,1]} = aqb$, with $a, b \in \Sigma$ and $q \in Q$.

GS as Turing machines

Let $T = (\Sigma, Q, q_0, q_h, \delta)$ be a Turing machine and let $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

$$\delta = (\delta_\Sigma, \delta_Q, \delta_\mathbb{Z}) \text{ with } \begin{array}{l} \delta_\Sigma(q, x_0) = x_0' \\ \delta_Q(q, x_0) = q' \\ \delta_\mathbb{Z}(q, x_0) = -1, 0 \textit{ or } 1 \end{array}$$

Set $I = J = \{-1, 0, 1\}$.

Define $c : \Sigma^{\mathbb{Z}} \times Q \to \mathcal{A}^{\mathbb{Z}}$ by $((x_n)_{n\in\mathbb{Z}}, q) \mapsto \ldots x_{-1}.qx_0x_1\ldots$

Let $x \in A$ be s.t. $x_{[-1,1]} = aqb$, with $a, b \in \Sigma$ and $q \in Q$.

Define $F \equiv \delta_\mathbb{Z}$ and

$$G(x_I) = \begin{cases} x_{-1}.x_0'q' & \text{if } \delta_\mathbb{Z}(x_0, q) = 1 \\ x_{-1}.q'x_0' & \text{if } \delta_\mathbb{Z}(x_0, q) = 0 \\ q'.x_{-1}x_0' & \text{if } \delta_\mathbb{Z}(x_0, q) = -1, \end{cases}$$

GS as Turing machines

Let $T = (\Sigma, Q, q_0, q_h, \delta)$ be a Turing machine and let $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

$$\delta = (\delta_\Sigma, \delta_Q, \delta_{\mathbb{Z}}) \text{ with } \begin{array}{l} \delta_\Sigma(q, x_0) = x_0' \\ \delta_Q(q, x_0) = q' \\ \delta_{\mathbb{Z}}(q, x_0) = -1, 0 \textit{ or } 1 \end{array}$$

Set $I = J = \{-1, 0, 1\}$.

Define $c : \Sigma^{\mathbb{Z}} \times Q \to \mathcal{A}^{\mathbb{Z}}$ by $((x_n)_{n\in\mathbb{Z}}, q) \mapsto \ldots x_{-1}.qx_0x_1\ldots$

Let $x \in A$ be s.t. $x_{[-1,1]} = aqb$, with $a, b \in \Sigma$ and $q \in Q$.

Define $F \equiv \delta_{\mathbb{Z}}$ and

$$G(x_I) = \begin{cases} x_{-1}.x_0'q' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = 1 \\ x_{-1}.q'x_0' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = 0 \\ q'.x_{-1}x_0' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = -1, \end{cases}$$

For x s.t. $x_{[-1,1]} \neq aqb$ define $F(x_J) = 0$ and $G(x_I) = id$.

GS as Turing machines

Let $T = (\Sigma, Q, q_0, q_h, \delta)$ be a Turing machine and let $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$.

$$\delta = (\delta_\Sigma, \delta_Q, \delta_{\mathbb{Z}}) \text{ with } \begin{aligned} \delta_\Sigma(q, x_0) &= x_0' \\ \delta_Q(q, x_0) &= q' \\ \delta_{\mathbb{Z}}(q, x_0) &= -1, 0 \textit{ or } 1 \end{aligned}$$

Set $I = J = \{-1, 0, 1\}$.

Define $c : \Sigma^{\mathbb{Z}} \times Q \to \mathcal{A}^{\mathbb{Z}}$ by $((x_n)_{n\in\mathbb{Z}}, q) \mapsto \ldots x_{-1}.qx_0x_1\ldots$

Let $x \in A$ be s.t. $x_{[-1,1]} = aqb$, with $a, b \in \Sigma$ and $q \in Q$.

Define $F \equiv \delta_{\mathbb{Z}}$ and

$$G(x_I) = \begin{cases} x_{-1}.x_0'q' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = 1 \\ x_{-1}.q'x_0' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = 0 \\ q'.x_{-1}x_0' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = -1, \end{cases}$$

For x s.t. $x_{[-1,1]} \neq aqb$ define $F(x_J) = 0$ and $G(x_I) = id$. **We have** $T = c^{-1}\Phi c$

GS as Turing machines

Theorem (Moore, 1990. [4])

There exists a generalized shift Φ conjugate to T.

$$\delta_{\mathbb{Z}}(q, x_0) = -1, 0 \text{ or } 1$$

Set $I = J = \{-1, 0, 1\}$.

Define $c : \Sigma^{\mathbb{Z}} \times Q \to \mathcal{A}^{\mathbb{Z}}$ by $((x_n)_{n\in\mathbb{Z}}, q) \mapsto \ldots x_{-1}.qx_0x_1\ldots$

Let $x \in A$ be s.t. $x_{[-1,1]} = aqb$, with $a, b \in \Sigma$ and $q \in Q$.

Define $F \equiv \delta_{\mathbb{Z}}$ and

$$G(x_I) = \begin{cases} x_{-1}.x_0'q' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = 1 \\ x_{-1}.q'x_0' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = 0 \\ q'.x_{-1}x_0' & \text{if } \delta_{\mathbb{Z}}(x_0, q) = -1, \end{cases}$$

For x s.t. $x_{[-1,1]} \neq aqb$ define $F(x_J) = 0$ and $G(x_I) = id$. **We have** $T = c^{-1}\Phi c$

Square Cantor set

Square Cantor set

Cantor set $\mathcal{C}$

Square Cantor set

Cantor set $\mathcal{C}$

Square Cantor set

Cantor set $\mathcal{C}$

Square Cantor set

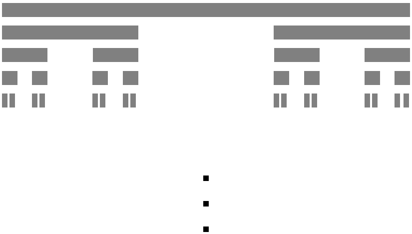
Cantor set $\mathcal{C}$

Square Cantor set

Cantor set $\mathcal{C}$

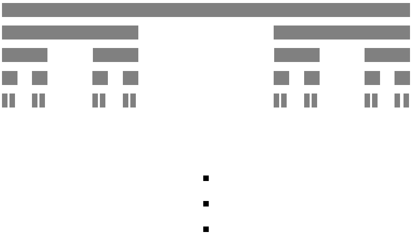
Square Cantor set

Cantor set $\mathcal{C}$



Square Cantor set

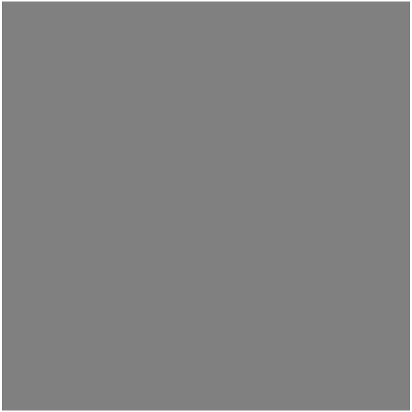
Cantor set $\mathcal{C}$



Square Cantor set

Cantor set $\mathcal{C}$

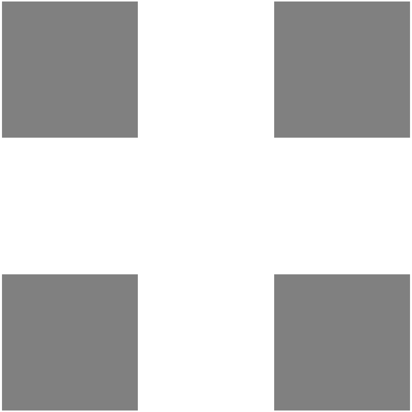
square Cantor set $\mathcal{C}^2$



Square Cantor set

Cantor set $\mathcal{C}$

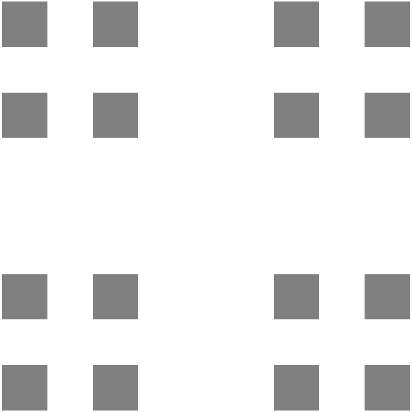
square Cantor set $\mathcal{C}^2$



Square Cantor set

Cantor set $\mathcal{C}$

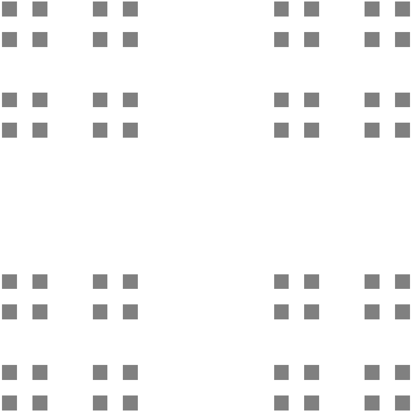
square Cantor set $\mathcal{C}^2$



Square Cantor set

Cantor set $\mathcal{C}$

square Cantor set $\mathcal{C}^2$



Square Cantor set

Cantor set $\mathcal{C}$

square Cantor set $\mathcal{C}^2$

Square Cantor set

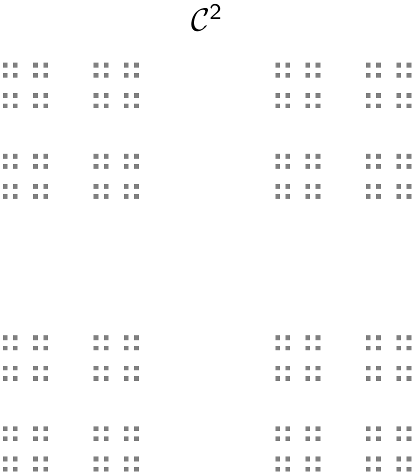
Cantor set $\mathcal{C}$

$\vdots$

square Cantor set $\mathcal{C}^2$

and so on...

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$



Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

Let $x = \ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\ldots$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

Let $x = \dots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\dots$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

Let $x = \ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\ldots$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

Let $x = \ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\ldots$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

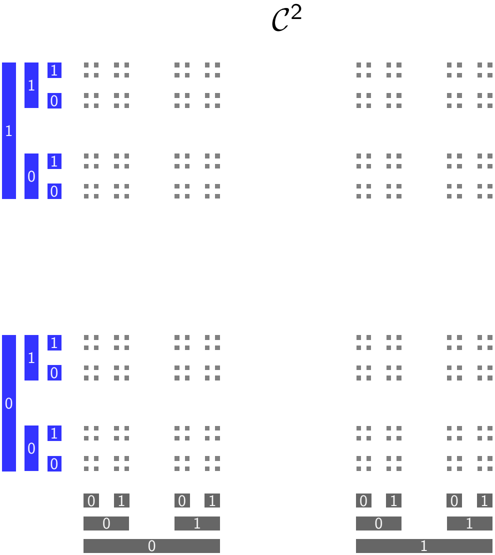
Let $x = \ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\ldots$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

Let $x = \ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\ldots$

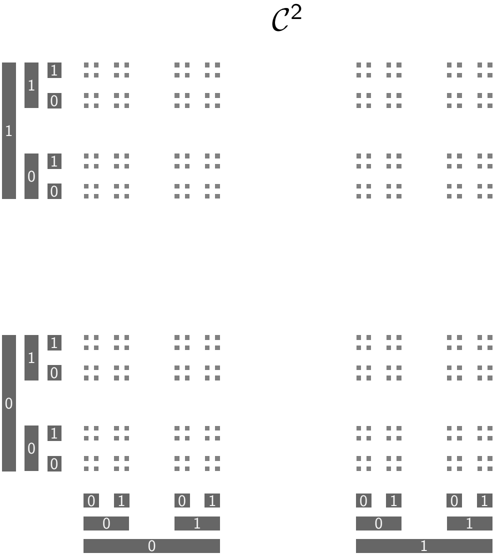
$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$



Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

Let $x = \dots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\dots$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$



Let Φ be a generalized shift on $\{0,1\}^{\mathbb{Z}}$

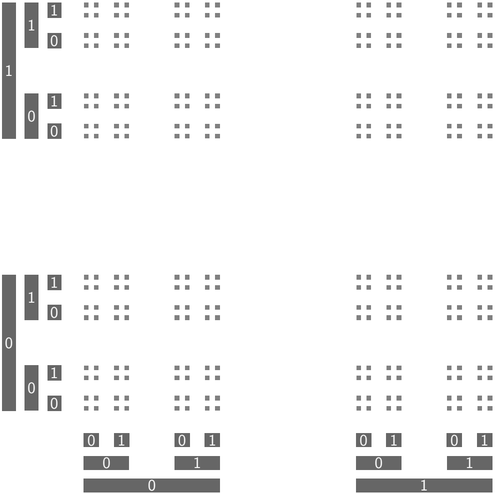
Let $x = \ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2\ldots$

What about Φ?

Is there a map in $\mathcal{C}^2$ conjugate to Φ?

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

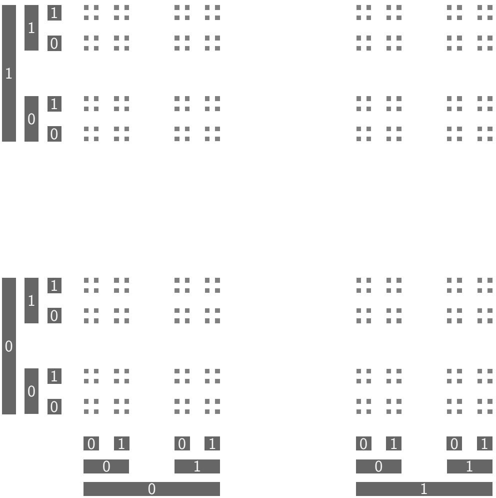


Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \dots x_{-2} 0.10 x_2 \dots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

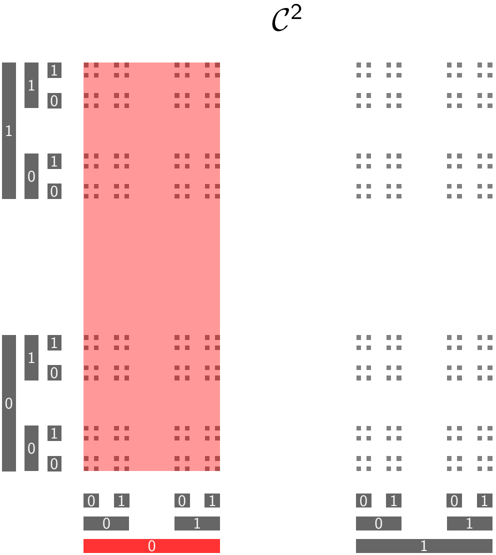
$\mathcal{C}^2$



Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \dots x_{-2}\textcolor{red}{0}.10x_2\dots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

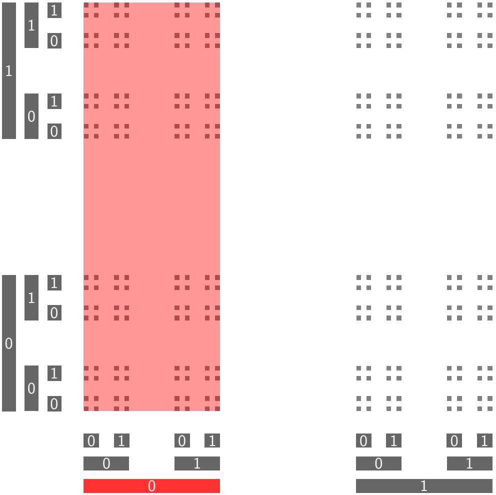


Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \dots x_{-2}0.10x_2\dots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

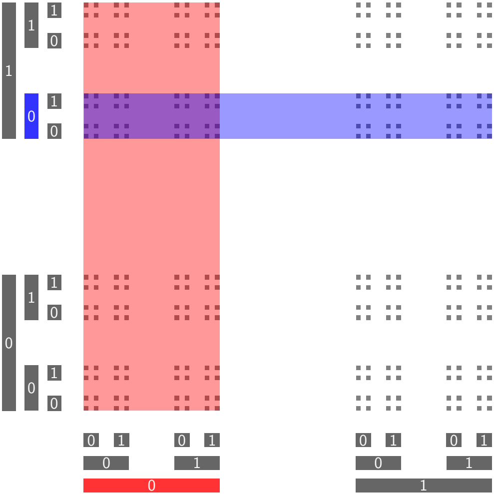


Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \ldots x_{-2}{\color{red}0}.{\color{blue}10}x_2\ldots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

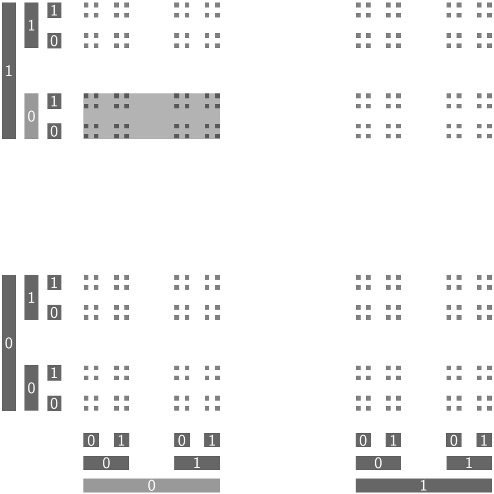


Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \dots x_{-2}0.10x_2\dots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

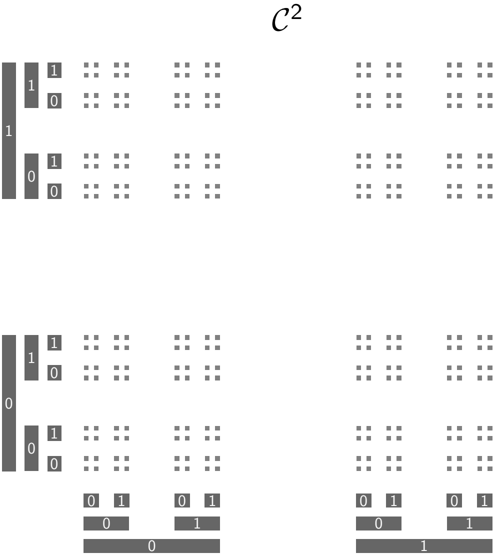
$\mathcal{C}^2$



Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \ldots x_{-2} 0.10 x_2 \ldots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$



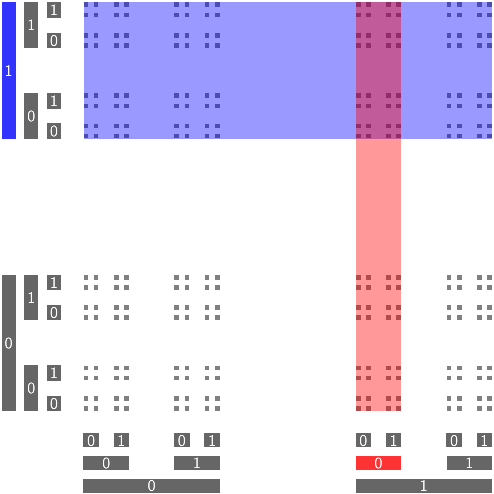
Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \ldots x_{-2} 0.10 x_2 \ldots$$

$$\Phi(x) = \quad \ldots 01.1 x_2 \ldots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

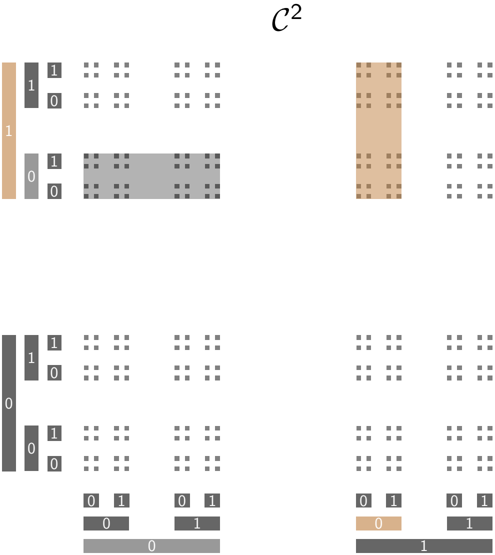


Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \dots x_{-2}0.10x_2 \dots$$

$$\Phi(x) = \quad \dots 01.1x_2 \dots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

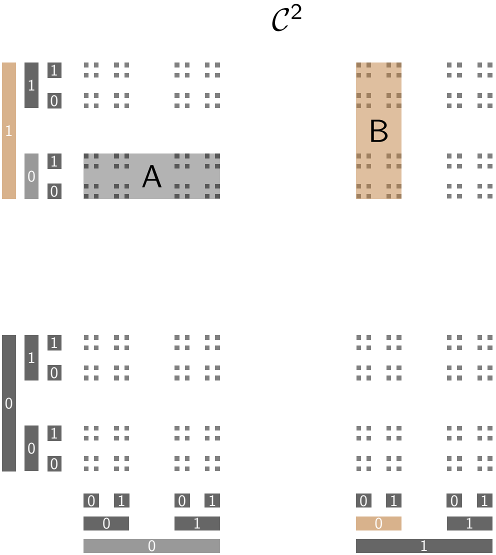


Suppose $I = J = \{-1, 0, 1\}$. Consider the point

$$x = \dots x_{-2} 0.10 x_2 \dots$$

$$\Phi(x) = \quad \dots 01.1 x_2 \dots$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$



Suppose $I = J = \{-1,0,1\}$. Consider the point

$$x = \dots x_{-2}0.10x_2\dots$$

$$\Phi(x) = \quad \dots 01.1x_2\dots$$

$$\phi(A) = B$$

$\{0,1\}^{\mathbb{Z}}$ in $\mathcal{C}^2$

$\mathcal{C}^2$

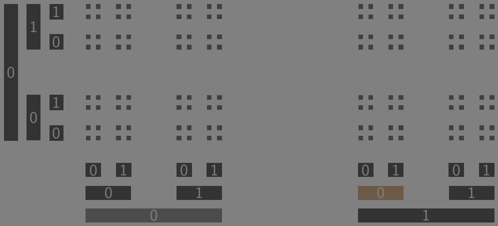
Theorem (Moore, 1990. [4])

Any generalized shift on n symbols is conjugate to a piecewise linear map ϕ of the square Cantor set into itself, this map will have a finite number of linear components

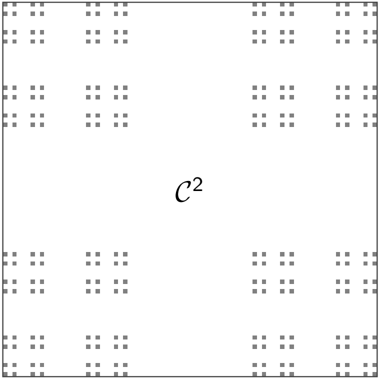
$$k \leq n^{|DoD \cup DoE| + \max|F|}.$$

$$\phi(x) = \quad \ldots 01.1x_2 \ldots$$

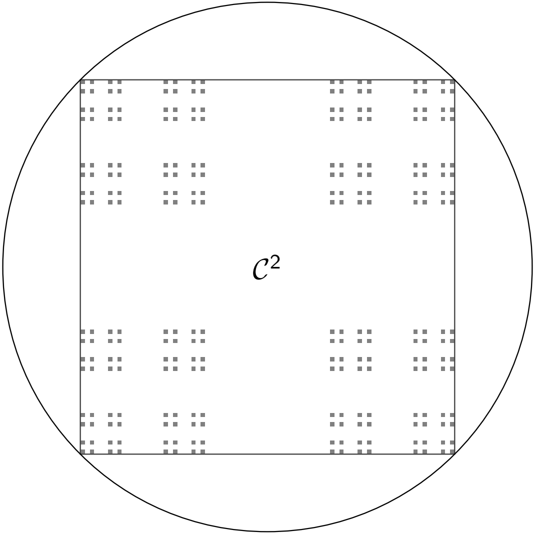
$$\phi(A) = B$$



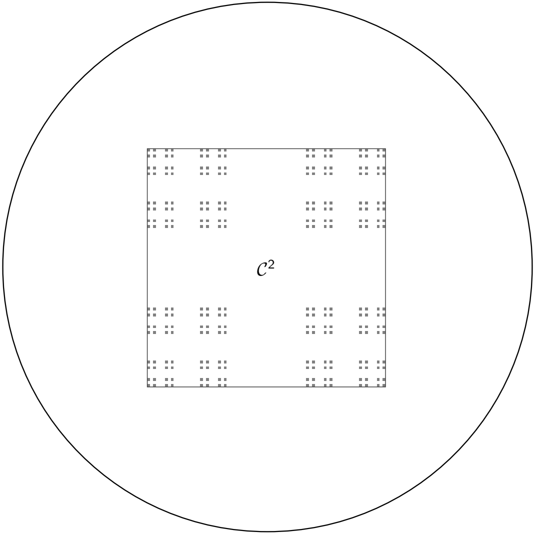
Shrinking and moving the Cantor



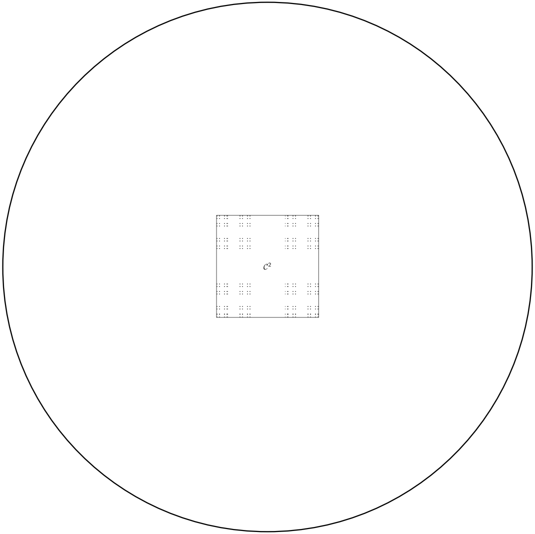
Shrinking and moving the Cantor



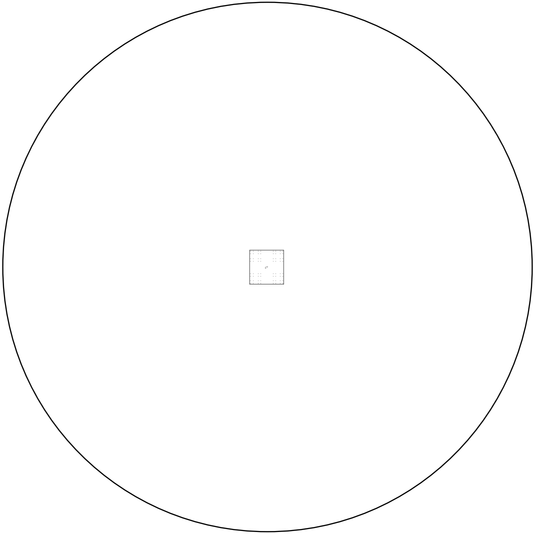
Shrinking and moving the Cantor



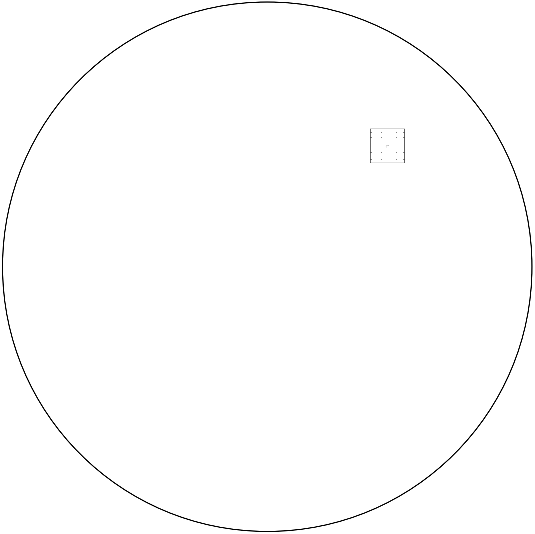
Shrinking and moving the Cantor



Shrinking and moving the Cantor



Shrinking and moving the Cantor

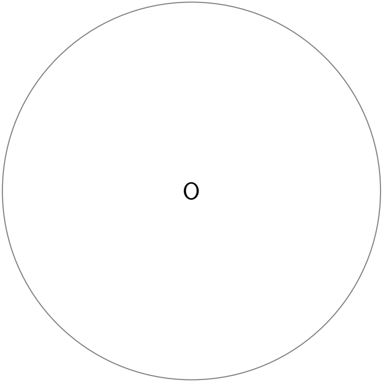


Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.

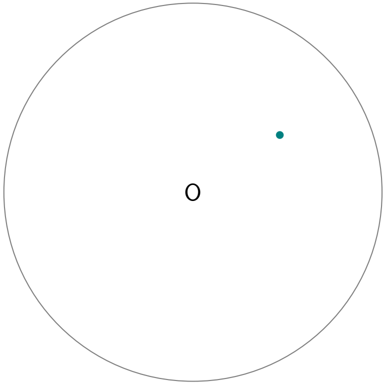
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



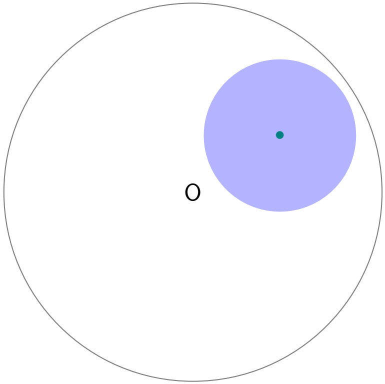
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



Embedding Turing machines in smooth dynamics

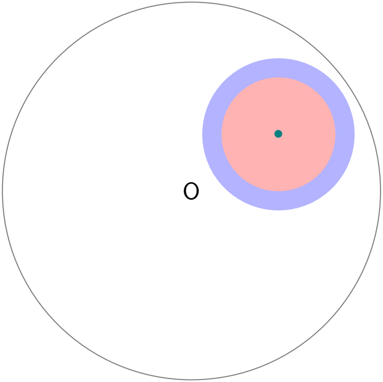
Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



$\mathcal{D}_{\epsilon/2} = \{x \in \mathcal{D} \mid |x - x_0| \leq \epsilon/2\}$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.

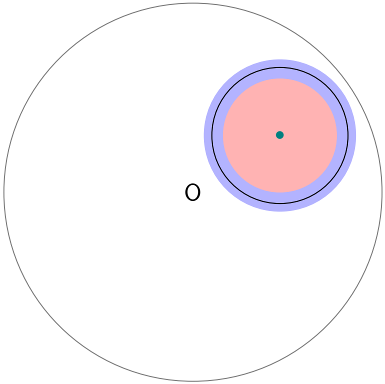


$$\mathcal{D}_{\epsilon/2} = \{x \in \mathcal{D} \mid |x - x_0| \leq \epsilon/2\}$$

$$\mathcal{D}_{\gamma} = \{x \in \mathcal{D} \mid |x - x_0| \leq \gamma\}$$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



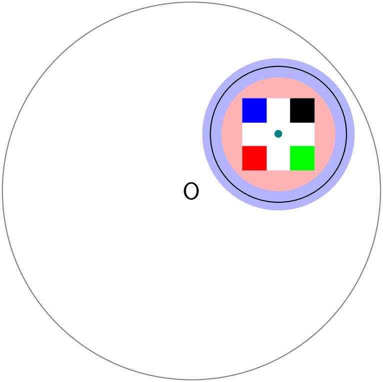
$$\mathcal{D}_{\epsilon/2} = \{x \in \mathcal{D} \mid |x - x_0| \leq \epsilon/2\}$$

$$\mathcal{D}_{\gamma} = \{x \in \mathcal{D} \mid |x - x_0| \leq \gamma\}$$

$$S_{\delta} = \{x \in \mathcal{D} \mid |x - x_0| = \gamma + \delta\}$$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



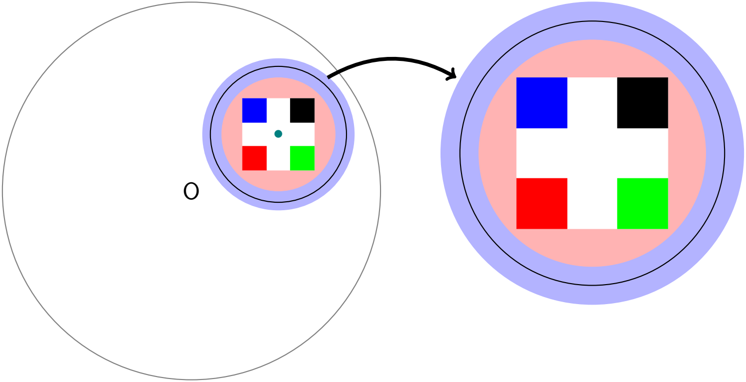
$$\mathcal{D}_{\epsilon/2} = \{x \in \mathcal{D} \mid |x - x_0| \leq \epsilon/2\}$$

$$\mathcal{D}_{\gamma} = \{x \in \mathcal{D} \mid |x - x_0| \leq \gamma\}$$

$$S_{\delta} = \{x \in \mathcal{D} \mid |x - x_0| = \gamma + \delta\}$$

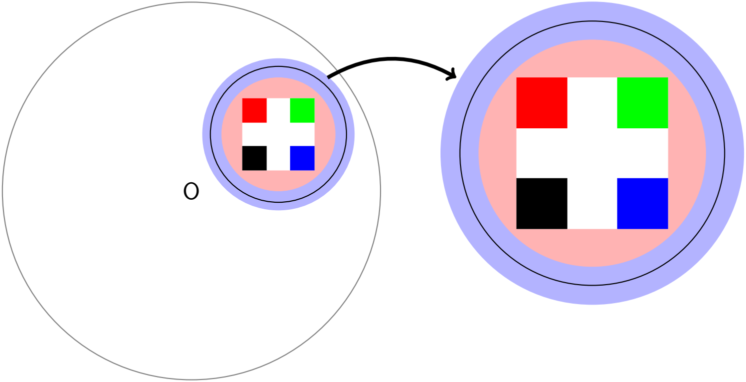
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



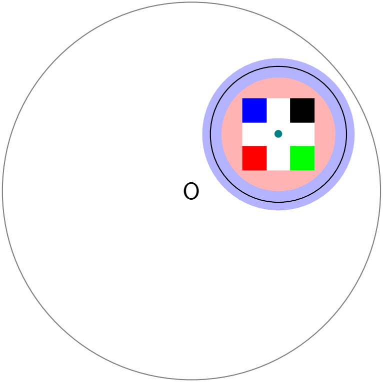
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



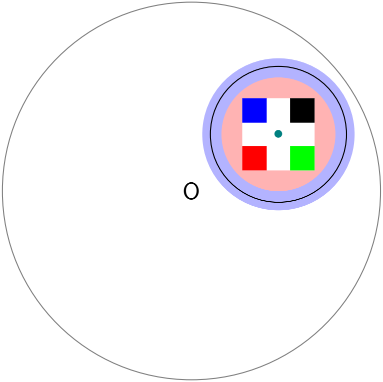
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



Embedding Turing machines in smooth dynamics

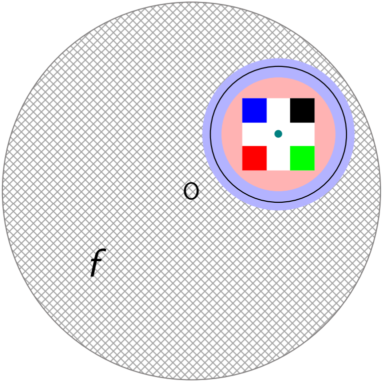
Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



φ (Moore) defined in $\mathcal{D}_{\gamma+\delta}$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.

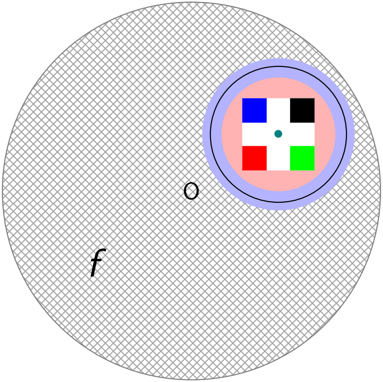


φ (Moore) defined in $\mathcal{D}_{\gamma+\delta}$

f defined in $\mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta} = \mathcal{D}'$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



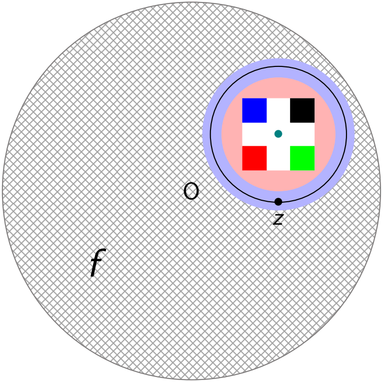
φ (Moore) defined in $\mathcal{D}_{\gamma+\delta}$

f defined in $\mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta} = \mathcal{D}'$

Gluing zone

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



φ (Moore) defined in $\mathcal{D}_{\gamma+\delta}$

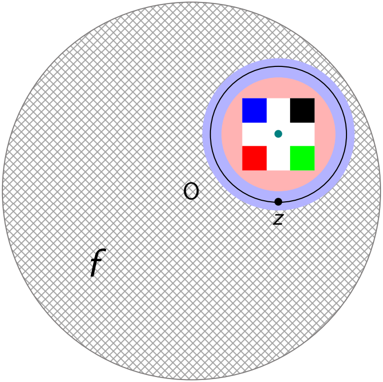
f defined in $\mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta} = \mathcal{D}'$

Gluing zone

$z \in \mathcal{D}$ such that $|z - x_0| = \gamma + \delta$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



φ (Moore) defined in $\mathcal{D}_{\gamma+\delta}$

f defined in $\mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta} = \mathcal{D}'$

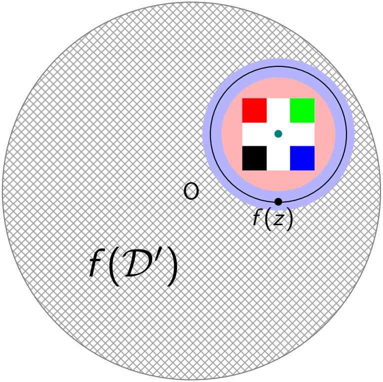
Gluing zone

$z \in \mathcal{D}$ such that $|z - x_0| = \gamma + \delta$

$m : \mathcal{D} \to \mathcal{D}$ a Möbius transformation
we require $m(f(z)) = 0$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



φ (Moore) defined in $\mathcal{D}_{\gamma+\delta}$

f defined in $\mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta} = \mathcal{D}'$

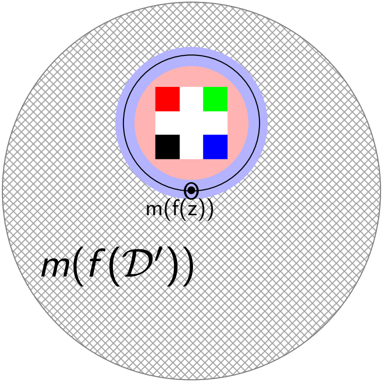
Gluing zone

$z \in \mathcal{D}$ such that $|z - x_0| = \gamma + \delta$

$m : \mathcal{D} \to \mathcal{D}$ a Möbius transformation
we require $m(f(z)) = 0$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



φ *(Moore) defined in* $\mathcal{D}_{\gamma+\delta}$

f *defined in* $\mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta} = \mathcal{D}'$

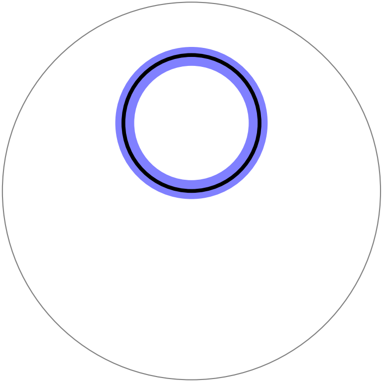
Gluing zone

$z \in \mathcal{D}$ such that $|z - x_0| = \gamma + \delta$

$m : \mathcal{D} \to \mathcal{D}$ a Möbius transformation
we require $m(f(z)) = 0$

Embedding Turing machines in smooth dynamics

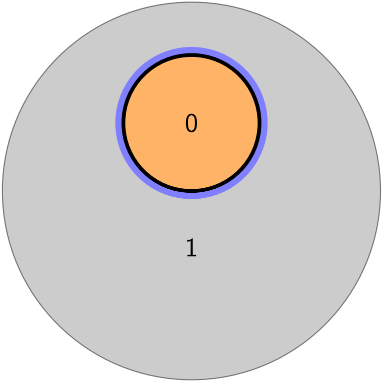
Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



Gluing zone

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.

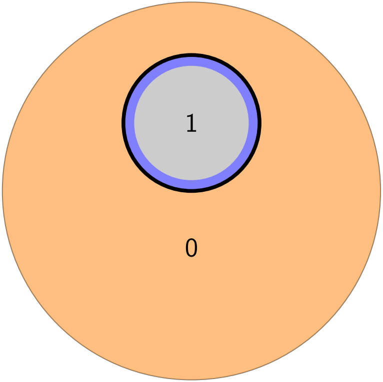


Gluing zone

$$\beta_f := \begin{cases} 0 & \text{in } \mathcal{D}_{\gamma+\delta} \\ (0,1) & \text{in } \mathcal{D}_{\epsilon/2} \setminus \mathcal{D}_{\epsilon/2-\delta} \\ 1 & \text{in } \mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta)}. \end{cases}$$

Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.

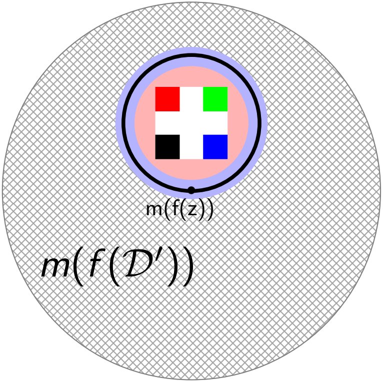


Gluing zone

$$\beta_\varphi := \begin{cases} 1 & \text{in } \mathcal{D}_\gamma \\ (0,1) & \text{in } \mathcal{D}_{\epsilon/2-\delta} \setminus \mathcal{D}_\gamma \\ 0 & \text{in } \mathcal{D} \setminus \mathcal{D}_{\epsilon/2-\delta)}. \end{cases}$$

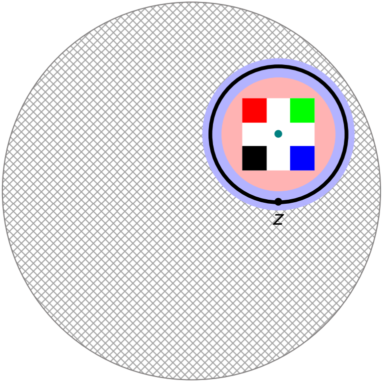
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



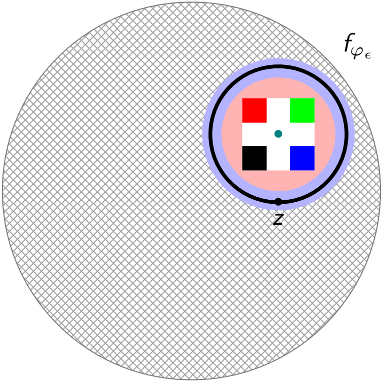
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem, there exists $x_0 \in \mathcal{D}$ such that $f(x_0) = x_0$. Let $\gamma = \epsilon/2 - 2\delta$ for small enough δ.



$$f_{\varphi_\epsilon} := \begin{cases} m^{-1} \circ (\beta_f (m \circ f)) & \text{in } \mathcal{D} \setminus \mathcal{B}(x_0, \gamma + \delta) \\ m^{-1} \circ (\beta_\varphi (m \circ \varphi)) & \text{in } \mathcal{B}(x_0, \gamma + \delta). \end{cases}$$

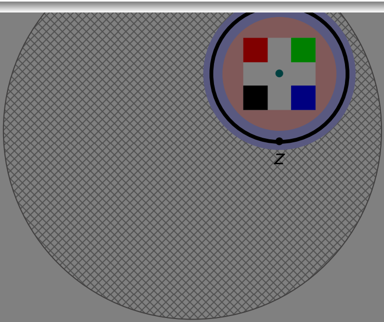
Embedding Turing machines in smooth dynamics

Let f be a smooth map of the disk $\mathcal{D}$ and $\epsilon > 0$. By Brower's Fixed point theorem,

Theorem (N. - Rojas, 2022 but still in progress)

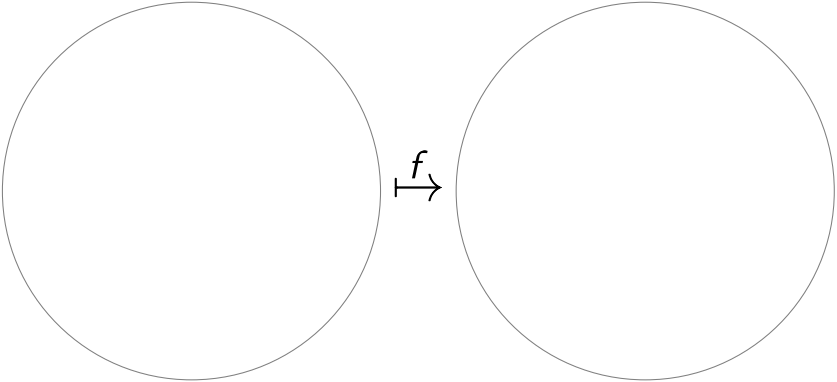
Let $f : \mathcal{D} \to \mathcal{D}$ be a smooth map and let $\epsilon > 0$. There exists a Turing universal smooth map $f_{\varphi_\epsilon} : \mathcal{D} \to \mathcal{D}$ such that

$$||f - f_{\varphi_\epsilon}||_\infty \leq \epsilon$$



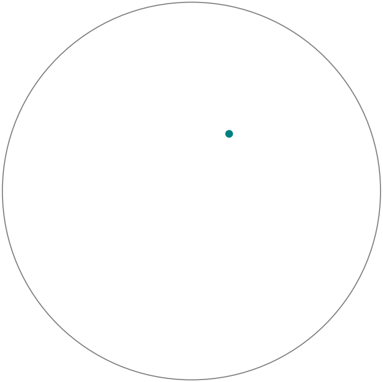
What about diffeos?

Let f be in $\mathrm{Diff}(\mathcal{D})$,



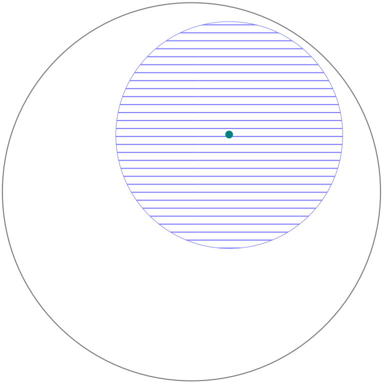
Placing the square Cantor

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



Placing the square Cantor

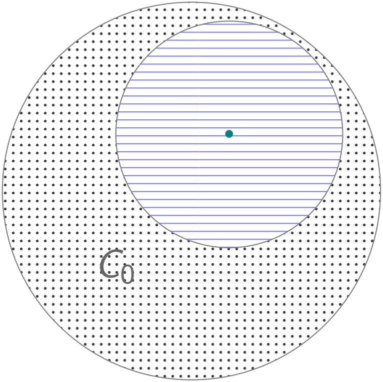
Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.



$$\mathcal{B}_{\epsilon/2}(x_0) = \{x \in \mathcal{D} \mid |x - x_0| < \epsilon/2\}$$

Placing the square Cantor

Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.



$$\mathcal{B}_{\epsilon/2}(x_0) = \{x \in \mathcal{D} \mid |x - x_0| < \epsilon/2\}$$

Set $C_0 = \mathcal{D} \setminus \mathcal{B}_{\epsilon/2}$

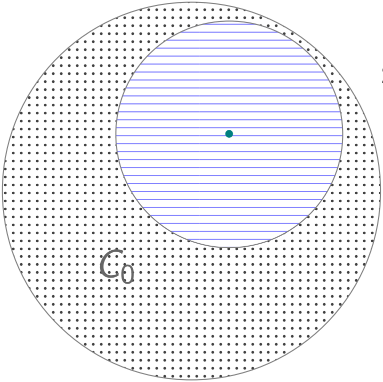
Placing the square Cantor

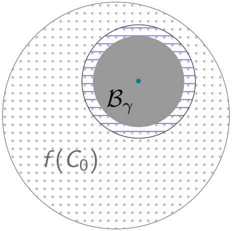
Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.

$$\mathcal{B}_{\epsilon/2}(x_0) = \{x \in \mathcal{D} \mid |x - x_0| < \epsilon/2\}$$

Set $C_0 = \mathcal{D} \setminus \mathcal{B}_{\epsilon/2}$

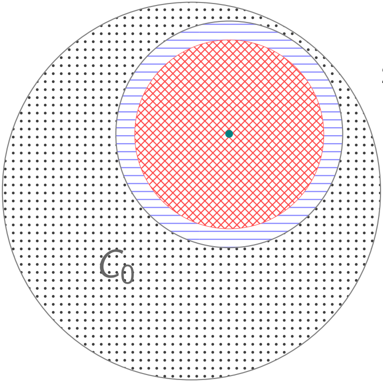
Set γ s.t. $\mathcal{B}_\gamma(x_0) \cap f(C_0) = \emptyset$.





Placing the square Cantor

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



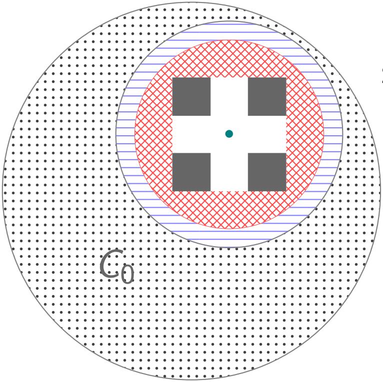
$$\mathcal{B}_{\epsilon/2}(x_0) = \{x \in \mathcal{D} \mid |x - x_0| < \epsilon/2\}$$

Set $C_0 = \mathcal{D} \setminus \mathcal{B}_{\epsilon/2}$

Set γ s.t. $\mathcal{B}_\gamma(x_0) \cap f(C_0) = \emptyset$. Take $\delta < \gamma$

Placing the square Cantor

Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.



$$\mathcal{B}_{\epsilon/2}(x_0) = \{x \in \mathcal{D} \mid |x - x_0| < \epsilon/2\}$$

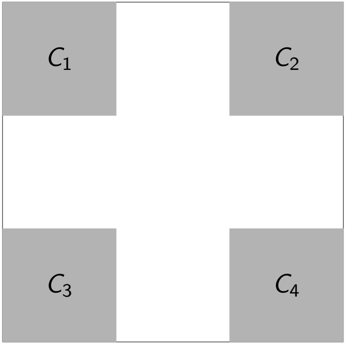
Set $C_0 = \mathcal{D} \setminus \mathcal{B}_{\epsilon/2}$

Set γ s.t. $\mathcal{B}_\gamma(x_0) \cap f(C_0) = \emptyset$. Take $\delta < \gamma$

Let $(C_i)_{1 \leq i \leq k}$ be Cantor blocks.

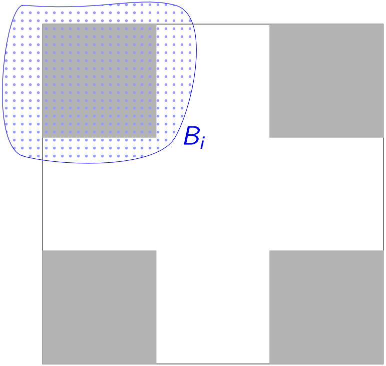
Defining the isotopy

Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.



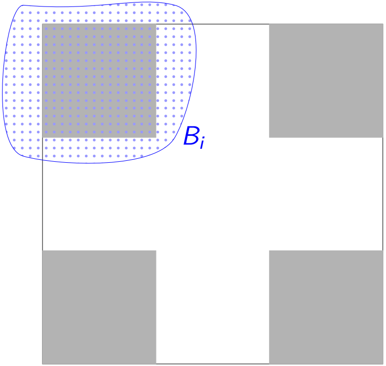
Defining the isotopy

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



Defining the isotopy

Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.

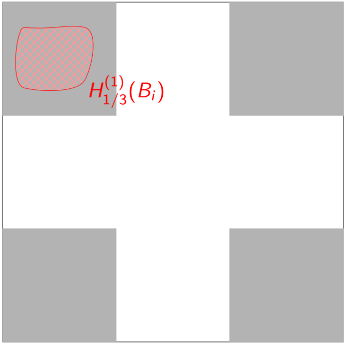


For $t \in [0, 1/3]$ define:

$H_t^{(1)}$ s.t. it shrinks B_i to a α-ngbhd.

Defining the isotopy

Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.

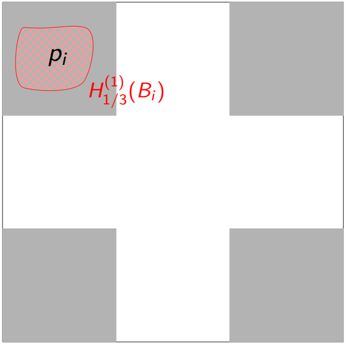


For $t \in [0, 1/3]$ define:

$H^{(1)}_t$ s.t. it shrinks B_i to a α-ngbhd.

Defining the isotopy

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



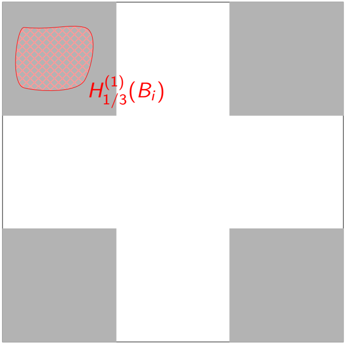
For $t \in [0, 1/3]$ define:

$H^{(1)}_t$ s.t. it shrinks B_i to a α-ngbhd.

$$H^{(1)}_t(x) = p_i + (1 - 3t(1-\alpha))(x - p_i)$$

Defining the isotopy

Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.



For $t \in [0, 1/3]$ define:

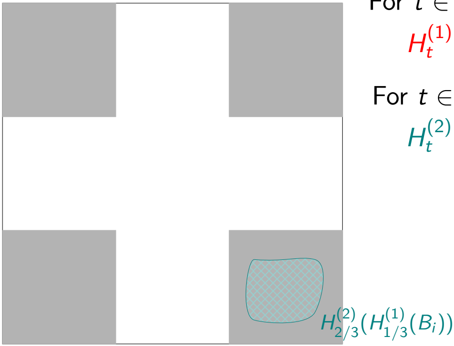
$H^{(1)}_t$ s.t. it shrinks B_i to a α-ngbhd.

For $t \in [1/3, 2/3]$ define:

$H^{(2)}_t$ s.t. it moves $H^{(1)}_t(B_i)$ inside $\phi(B_i)$

Defining the isotopy

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



For $t \in [0, 1/3]$ define:

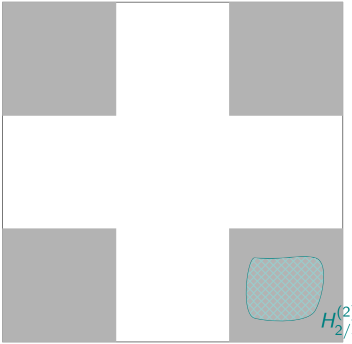
$H^{(1)}_t$ s.t. it shrinks B_i to a α-ngbhd.

For $t \in [1/3, 2/3]$ define:

$H^{(2)}_t$ s.t. it moves $H^{(1)}_t(B_i)$ inside $\phi(B_i)$

Defining the isotopy

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



For $t \in [0, 1/3]$ define:

$H_t^{(1)}$ s.t. it shrinks B_i to a α-ngbhd.

For $t \in [1/3, 2/3]$ define:

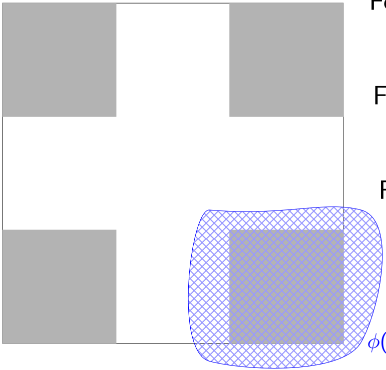
$H_t^{(2)}$ s.t. it moves $H_t^{(1)}(B_i)$ inside $\phi(B_i)$

For $t \in [2/3, 1]$ define:

$H_t^{(3)}$ s.t. it fills $\phi(B_i)$

Defining the isotopy

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



For $t \in [0, 1/3]$ define:

$H_t^{(1)}$ s.t. it shrinks B_i to a α-ngbhd.

For $t \in [1/3, 2/3]$ define:

$H_t^{(2)}$ s.t. it moves $H_t^{(1)}(B_i)$ inside $\phi(B_i)$

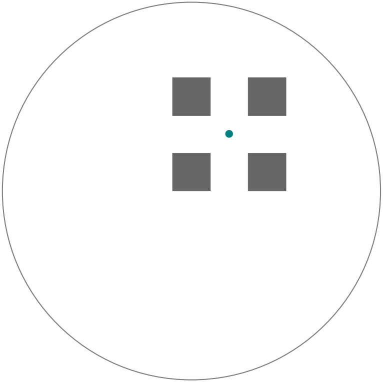
For $t \in [2/3, 1]$ define:

$H_t^{(3)}$ s.t. it fills $\phi(B_i)$

$\phi(B_i) = H_1^{(3)}(H_{2/3}^{(2)}(H_{1/3}^{(1)}(B_i)))$

Defining the isotopy

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



For $t \in [0, 1/3]$ define:

$H_t^{(1)}$ s.t. it shrinks B_i to a α-ngbhd.

For $t \in [1/3, 2/3]$ define:

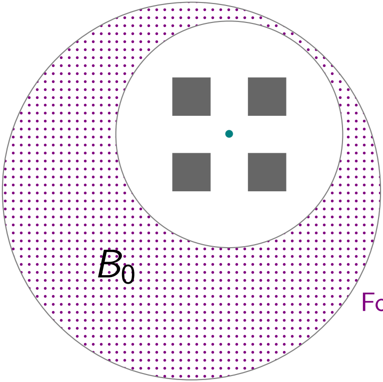
$H_t^{(2)}$ s.t. it moves $H_t^{(1)}(B_i)$ inside $\phi(B_i)$

For $t \in [2/3, 1]$ define:

$H_t^{(3)}$ s.t. it fills $\phi(B_i)$

Defining the isotopy

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



For $t \in [0, 1/3]$ define:

$H_t^{(1)}$ s.t. it shrinks B_i to a α-ngbhd.

For $t \in [1/3, 2/3]$ define:

$H_t^{(2)}$ s.t. it moves $H_t^{(1)}(B_i)$ inside $\phi(B_i)$

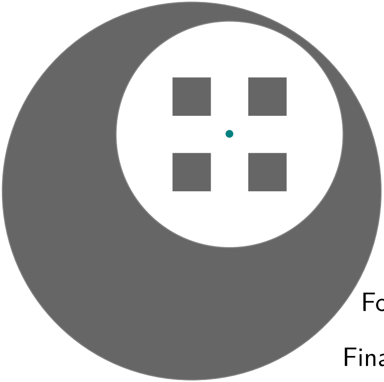
For $t \in [2/3, 1]$ define:

$H_t^{(3)}$ s.t. it fills $\phi(B_i)$

For $t \in [0, 1]$, define $G_t(x) = tf(x) + (1 - t)x$.

Defining the isotopy

Let f be in $\text{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.



For $t \in [0, 1/3]$ define:

$H_t^{(1)}$ s.t. it shrinks B_i to a α-ngbhd.

For $t \in [1/3, 2/3]$ define:

$H_t^{(2)}$ s.t. it moves $H_t^{(1)}(B_i)$ inside $\phi(B_i)$

For $t \in [2/3, 1]$ define:

$H_t^{(3)}$ s.t. it fills $\phi(B_i)$

For $t \in [0, 1]$, define $G_t(x) = tf(x) + (1 - t)x$.

Finally, define, for $t \in [0, 1]$:

$$F_t := \begin{cases} H_t & \text{in } \cup_i B_i \\ G_t & \text{in } B_0. \end{cases}$$

Help from the differential topology department

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.

Theorem (Theorem 1.4 in [3])

Let $U \subset \mathcal{D}$ be an open set, and $A \subset U$ a compact set. Let $F : U \times I \to \mathcal{D}$ be an isotopy such that $\hat{F}(U \times I) \subset \mathcal{D} \times I$ is open. Then, there exists a diffeotopy of $\mathcal{D}$ having compact support, which agrees with F on a neighbourhood of $A \times I$.

For $t \in [1/3, 2/3]$ define:

In our notation

$$U = \cup_{0 \leq i \leq k} B_i$$

$$A = \cup_{0 \leq i \leq k} C_i$$

$$F : U \times I \to \mathcal{D} \text{ is given by } F_t(x) = F(x, t)$$

Finally, define, for $t \in [0, 1]$:

$$F_t := \begin{cases} H_t & \text{in } \cup_i B_i \\ G_t & \text{in } B_0. \end{cases}$$

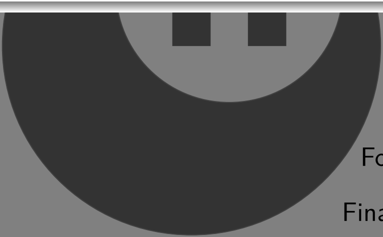
The result

Let f be in $\mathrm{Diff}(\mathcal{D})$, let $\epsilon > 0$, and let x_0 be a fixed point for f.

Theorem (N. - Rojas, 2022 but still in progress)

Let $f : \mathcal{D} \to \mathcal{D}$ be a diffeomorphism from the disk to itself. Given $\epsilon > 0$ there exists a diffeomorphism $f_{\varphi_\epsilon} : \mathcal{D} \to \mathcal{D}$ such that its restriction to a δ-square Cantor set is conjugate to any Turing machine and

$$||f - f_{\varphi_\epsilon}||_\infty \leq \epsilon.$$



For $t \in [2/3, 1]$ define:

$H_t^{(3)}$ s.t. it fills $\phi(B_i)$

For $t \in [0, 1]$, define $G_t(x) = tf(x) + (1 - t)x$.

Finally, define, for $t \in [0, 1]$:

$$F_t := \begin{cases} H_t & \text{in } \cup_i B_i \\ G_t & \text{in } B_0. \end{cases}$$

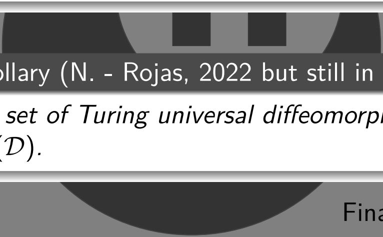
The result

Let f be in Diff($\mathcal{D}$), let $\epsilon > 0$, and let x_0 be a fixed point for f.

Theorem (N. - Rojas, 2022 but still in progress)

Let $f : \mathcal{D} \to \mathcal{D}$ be a diffeomorphism from the disk to itself. Given $\epsilon > 0$ there exists a diffeomorphism $f_{\varphi_\epsilon} : \mathcal{D} \to \mathcal{D}$ such that its restriction to a δ-square Cantor set is conjugate to any Turing machine and

$$||f - f_{\varphi_\epsilon}||_\infty \leq \epsilon.$$



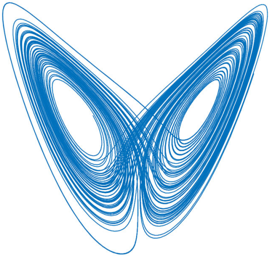
Corollary (N. - Rojas, 2022 but still in progress)

The set of Turing universal diffeomorphisms is an uncountable and dense set in Diff($\mathcal{D}$).

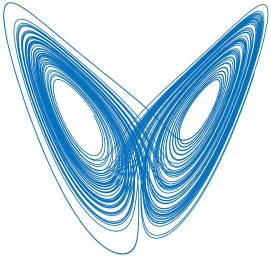
Finally, define, for $t \in [0, 1]$:

$$F_t := \begin{cases} H_t & \text{in } \cup_i B_i \\ G_t & \text{in } B_0. \end{cases}$$

The modern paradigm for chaos

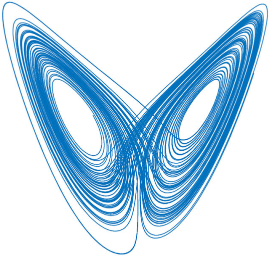


The modern paradigm for chaos



Chaotic systems challenge all computations of individual orbits.

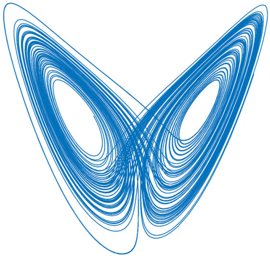
The modern paradigm for chaos



Chaotic systems challenge all computations of individual orbits.

Paradigm: the probability of finding the orbit in a given set.

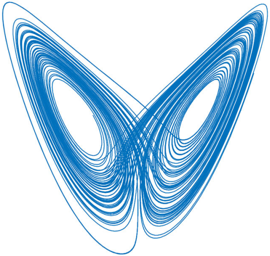
The modern paradigm for chaos



Chaotic systems challenge all computations of individual orbits.

Paradigm: the probability of finding the orbit in a given set. There are statistics

The modern paradigm for chaos

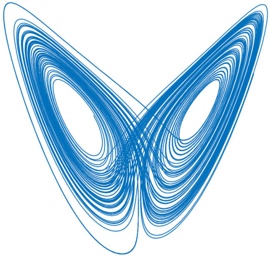


Chaotic systems challenge all computations of individual orbits.

Paradigm: the probability of finding the orbit in a given set. There are statistics

For this probability to be well defined, a typical orbit must "converge", otherwise we're blind!

The modern paradigm for chaos



Chaotic systems challenge all computations of individual orbits.

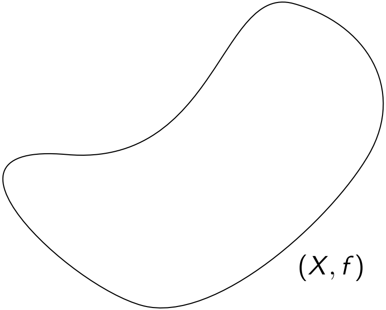
Paradigm: the probability of finding the orbit in a given set. There are statistics

For this probability to be well defined, a typical orbit must "converge", otherwise we're blind!

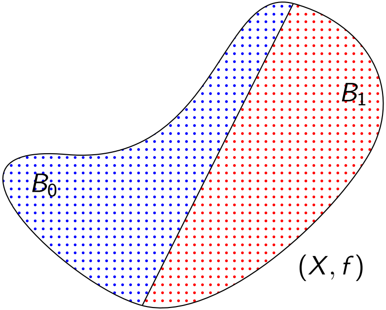
"Palis' Conjecture" (sort of)

In a typical system, we have statistics

Empirical measures



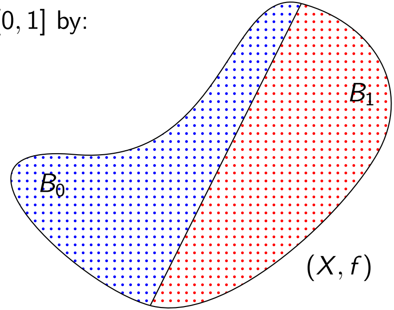
Empirical measures



Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

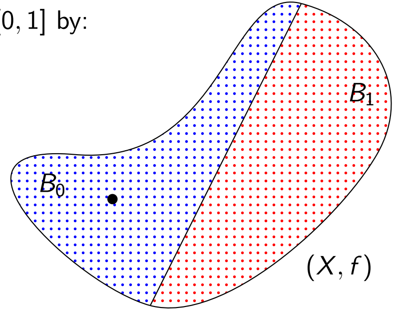
$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$



Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

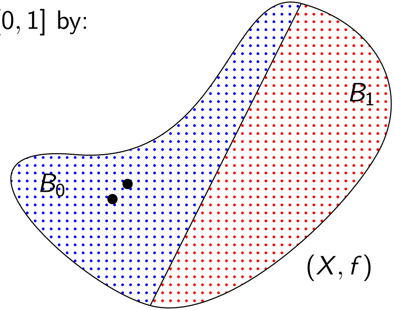


Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$e_1^f(x)(B_0) = \frac{1}{1}(1) = 1$$

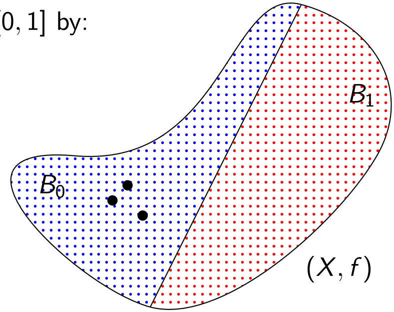


Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$e_2^f(x)(B_0) = \frac{1}{2}(1+1) = 1$$

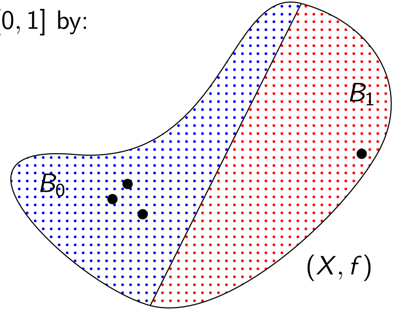


Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^n \delta_{f^i(x)}$$

$$e_3^f(x)(B_0) = \frac{1}{3}(1+1+0) = \frac{2}{3}$$

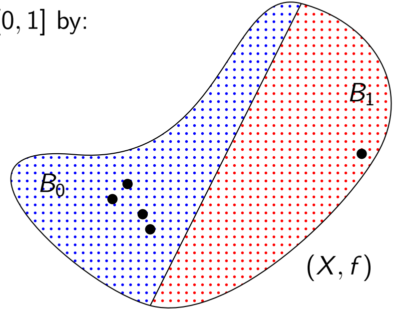


Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$e_4^f(x)(B_0) = \frac{1}{4}(1+1+0+1) = \frac{3}{4}$$

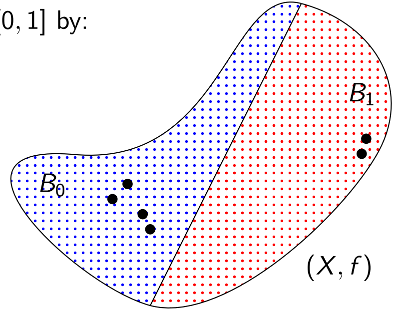


Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n} \sum_{i=1}^n \delta_{f^i(x)}$$

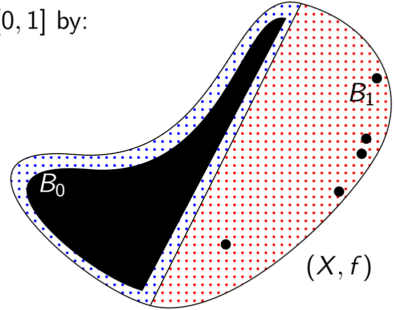
$$e_5^f(x)(B_0) = \frac{3}{5}$$



Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{f^i(x)}$$



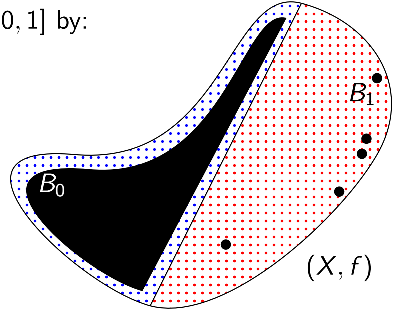
Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{f^i(x)}$$

$$e_n^f(x)(B_0) \longrightarrow 1$$

$$e_n^f(x)(B_1) \longrightarrow 0$$



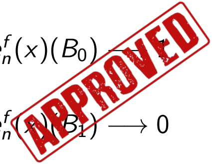
Empirical measures

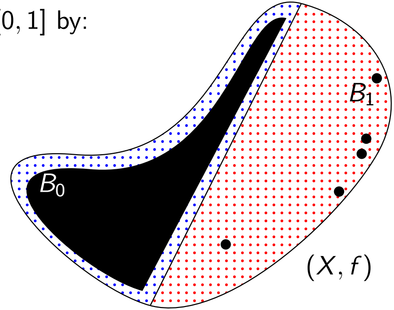
Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{f^i(x)}$$

$$e_n^f(x)(B_0) \longrightarrow 1$$

$$e_n^f(x)(B_1) \longrightarrow 0$$

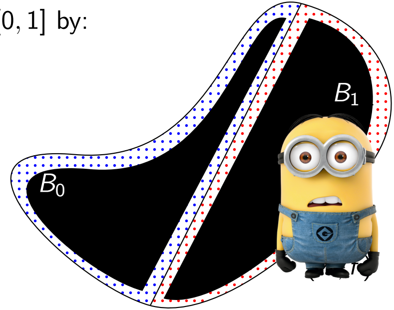




Empirical measures

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{f^i(x)}$$



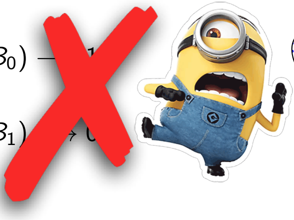
Empirical measures

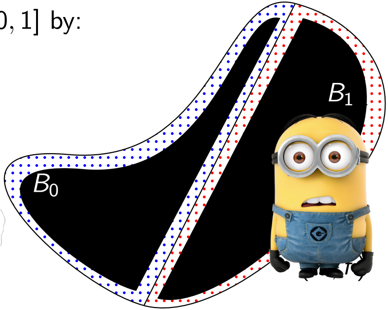
Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$e_n^f(x)(B_0) \to 1$$

$$e_n^f(x)(B_1) \to 0$$





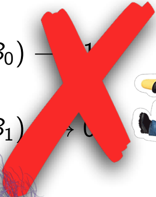
Empirical measures

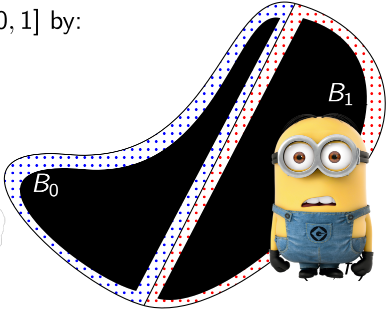
Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{f^i(x)}$$

$$e_n^f(x)(B_0) \to 1$$

$$e_n^f(x)(B_1) \to 0$$





Then f is non-statistical!!!

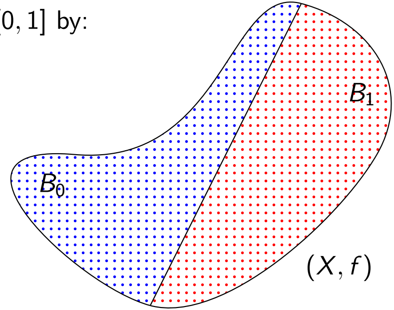
We're blind!

Constructing a non-statistical map

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{f^i(x)}$$

$$n_i = 2^{2^i}$$



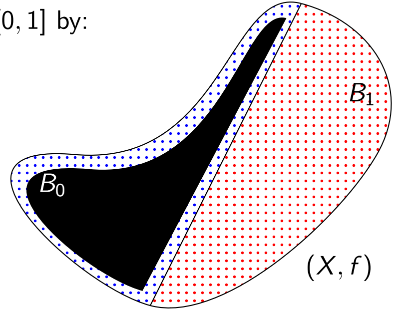
Constructing a non-statistical map

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$n_i = 2^{2^i}$$

$$f^n(x) \in B_0 \text{ for } n \in [n_i, n_{i+1}), \ i \text{ odd}$$



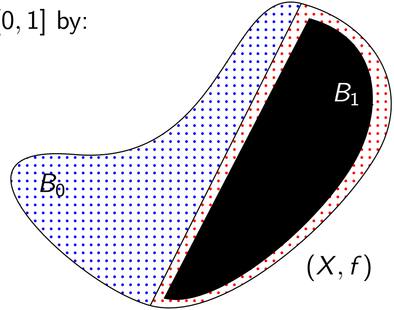
Constructing a non-statistical map

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$n_i = 2^{2^i}$$

$$f^n(x) \in B_1 \text{ for } n \in [n_i, n_{i+1}),\ \ i \text{ even}$$



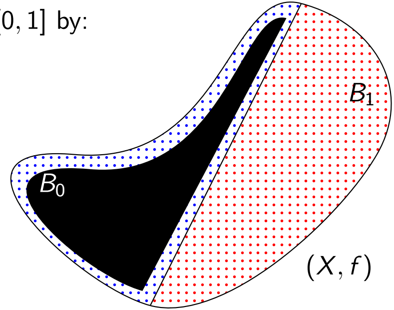
Constructing a non-statistical map

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$n_i = 2^{2^i}$$

$$e_{n_i}^f(x)(B_0) \longrightarrow 1 \text{ for } i \text{ odd}$$



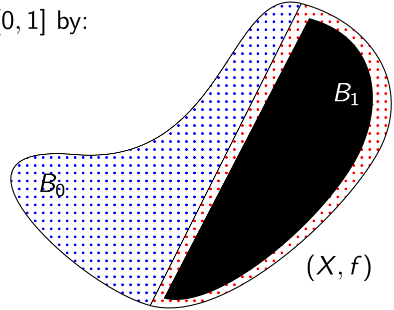
Constructing a non-statistical map

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$n_i = 2^{2^i}$$

$$e_{n_i}^f(x)(B_0) \longrightarrow 0 \text{ for } i \text{ even}$$



Constructing a non-statistical map

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

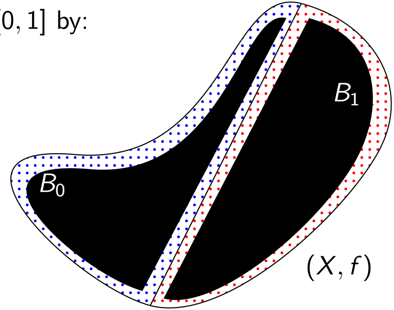
$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

$$n_i = 2^{2^i}$$

$$e_{n_i}^f(x)(B_0) \longrightarrow 1 \text{ for } i \text{ odd}$$

$$e_{n_i}^f(x)(B_0) \longrightarrow 0 \text{ for } i \text{ even}$$

Then f is non-statistical!!!



They are everywhere

Define the empirical measure $e_n^f(x) : X \to [0,1]$ by:

$$e_n^f(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{f^i(x)}$$

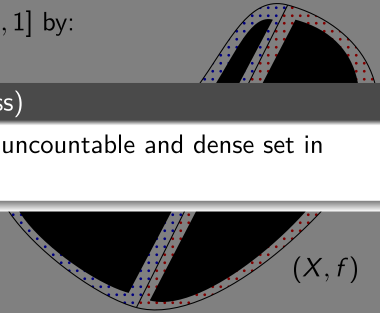
Theorem B (N. - Rojas, 2022 but still in progress)

The set of non-statistical diffeomorphisms is an uncountable and dense set in Diff($\mathcal{D}$).

$$e_{n_i}^f(x)(B_0) \longrightarrow 1 \text{ for } i \text{ odd}$$

$$e_{n_i}^f(x)(B_0) \longrightarrow 0 \text{ for } i \text{ even}$$

Then f is non-statistical!!!



(X, f)


That's all Folks!

Robert Cardona, Eva Miranda, Daniel Peralta-Salas, and Francisco Presas.
Constructing turing complete euler flows in dimension 3.
Proceedings of the National Academy of Sciences, 118(19), 2021.

Daniel S Graça, Manuel L Campagnolo, and Jorge Buescu.
Robust simulations of turing machines with analytic maps and flows.
In *Conference on Computability in Europe*, pages 169–179. Springer, 2005.

Morris W Hirsch.
Differential topology, volume 33.
Springer Science & Business Media, 2012.

Cristopher Moore.
Unpredictability and undecidability in dynamical systems.
Physical Review Letters, 20(64), 1990.

Terence Tao.
On the universality of potential well dynamics.
arXiv preprint arXiv:1707.02389, 2017.