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The Countably Infinite Boolean Vector Space and Constraint Satisfaction Problems

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Abstract/Résumé

English

Given a relational structure Γ, the problem CSP(Γ) takes as an argument a primitive positive sentence Ψ and asks whether $\Gamma \models \Psi$. Let $(V;+)$ be the countably infinite vector space over the two-element field. A reduct of $(V;+)$ is a relational structure with domain V whose relations are first-order definable over $(V;+)$. CSPs of reducts of $(V;+)$ are called Bit Vector CSPs. This thesis uses a method combining universal algebra, model theory and Ramsey theory in order to classify the complexity of Bit Vector CSPs

Besides establishing a P/NP-complete dichotomy for the complexity of nearly every Bit Vector CSP, this approach yields on the way an algebraic description of the lattices of automorphism groups and monoids of self-embeddings of reducts of $(V;+)$. The most interesting part of the lattice of endomorphism monoids of reducts of $(V;+)$, which correspond exactly to the locally closed monoids containing the automorphisms of $(V;+)$, is then described. Lastly, endomorphism monoids of model-complete cores of reducts of $(V;+)$ are fully classified up to existential positive interdefinability, foraying toward a dichotomy classification result for CSPs.

Français

Étant donnée une structure relationnelle Γ, le problème CSP(Γ) prend en entrée un énoncé primitif positif Ψ et répond à la question $\Gamma \models \Psi$? Soit $(V;+)$ l'espace vectoriel dénombrable sur le corps à deux éléments. On appelle *réduit* de $(V;+)$ toute structure relationnelle de domaine V définissable au premier ordre sur $(V;+)$. Dans cette thèse, on utilise une méthode alliant algèbre universelle, théorie des modèles et de Ramsey, afin de classifier la complexité des CSP sur les réduits de $(V;+)$.

En plus d'établir des résultats de dichotomie P/NP-complete pour la complexité des CSP sur presque tous les réduits, l'approche permet d'obtenir en passant des descriptions algébriques du treillis des groupes d'automorphismes et des monoïdes de plongements des réduits $(V;+)$. La partie la plus intéressante du treillis des monoïdes d'endomorphismes des réduits de $(V;+)$, qui correspondent exactement aux monoïdes locallement clos contenant les automorphismes de $(V;+)$, est également décrite. Pour finir, les monoïdes d'endomorphismes de noyaux modèle-complets de réduits de $(V;+)$ sont entièrement classifiés, ce qui ouvre la voie aux résultats sur les CSP.

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Contents

Chapter 1

Introduction

English

Constraint Satisfaction Problems show up frequently in many fields of Computer Science: computational linguistics, computational biology, scheduling, artificial intelligence, verification, etc. Some of the most well-known computational problems can be expressed as CSPs, such as the 3-colorability problem for graphs, the acyclicity problem for directed graphs, or 3-SAT. There are several ways to formalise Constraint Satisfaction Problems; the most fitting one uses notions of model theory and first-order logic. Given a structure Γ with a finite relational signature $\{R_1, \ldots, R_n\}$ over a finite or infinite domain, the problem CSP(Γ) is the computational problem to decide whether a given primitive positive sentence Ψ (i.e., a sentence of the form $\exists \overline{x}. \bigwedge R_i(\overline{x})$) is satisfied by Γ.

CSPs over finite and infinite domain are often considered separately. The former are the subject of active research for thirty years. One of the first explicit reference can be found in [DP87], though various great results were already known at the time. For instance, in 1978, Schaefer [Sch78] proved a famous theorem which can be reformulated as follows: CSPs over a two-element domain are either NP-complete or polynomial-time tractable. In 2006, Bulatov [Bul02a] goes further by proving that this dichotomy also holds for three-element domains. Meanwhile in 1999, Feder and Vardi [FV99] state their famous conjecture that this P/NP-complete dichotomy holds for every CSP over a finite domain. This conjecture is still open today, though several particular cases have already been proven such as the case for undirected graphs (see [HN90]).

The universal algebraic method using polymorphisms is one of the most powerful tools to prove such dichotomies. A polymorphism is a generalisation of the notion of homomorphism for any finite arity. In 1968, Geiger [Gei68] established that a relation R has a primitive positive definition in a finite relational structure Γ if and only if R is preserved by all polymorphisms of Γ. Bodnarcuk, Kaluznin, Kotov, and Romov [BKKR69] independently proved the same result. Given a set of operations $\mathcal{F}$ and a set of relations $\mathcal{R}$ over a domain D, we denote by Inv($\mathcal{F}$) the set of relations preserved by all operations of $\mathcal{F}$, and by Pol($\mathcal{R}$) the set of all polymorphisms of $\mathcal{R}$. With this notation, the previous theorem can be reformulated as follows: $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle \Gamma \rangle_{\mathrm{pp}}$. Consequently, since the complexity of CSP(Γ) remains the same when primitive positive definable relations in Γ are added to Γ, the CSPs of two structures which have the same polymorphisms can be reduced to each other, in polynomial time. In other words, the complexity of CSP(Γ) is entirely determined by Pol(Γ). In fact, some polymorphisms of Γ can be used to obtain algorithms to solve CSP(Γ).

A *weak near unanimity operation* is an operation which satisfies the following:

$$\forall x, y. f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, x, \ldots, x, y)$$

The CSP of a finite relational structure that does not possess such polymorphisms is necessarily NP-complete ([MM08] and [BKJ05]), and Bulatov, Jeavons, and Krokhin [BKJ05] conjectured (in different, but equivalent form) that the CSP is in P otherwise. A *near unanimity operation* is a weak near unanimity operation satisfying $f(y, x, \ldots, x) = x$ for all x, y. For instance, the *majority* function is a near unanimity operation. The Bulatov-Jeavons-Krokhin conjecture has already been confirmed for the particular case of near unanimity operations [JCC98], as well as for binary operations that are at once commutative, associative, and idempotent. The case of Mal'tsev operations (i.e., ternary operations satisfying $f(x, y, y) = f(y, y, x) = x$ for all x, y) has been dealt with as well [Bul02b].

The study of CSPs over infinite domains is justified by their expressive power, and by the fact that while CSPs over a finite domain cannot model many well-known computational problems, CSPs over infinite domains can, for instance: the directed graph acyclicity problem. Indeed, Bodirsky and Grohe [BG08] established that every computational problem is polynomially equivalent to a CSP over an infinite domain. How are the universal algebraic methods using polymorphisms adaptable to infinite structures? The equality $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle \Gamma \rangle_{\mathrm{pp}}$ can naturally be transposed to structures with infinite domains if these structures are ω-categorical.

A structure is *ω-categorical* if its first-order theory has only one countable model up to isomorphism. By the Ryll-Nardzewski theorem, a countable structure Δ is ω-categorical whenever for all $n \geq 1$, there are finitely many inequivalent formulas with n free variables over the first-order theory of Δ. A great number of well-known structures share this property of ω-categoricity, such as $(\mathbb{Q}, <)$, the vector space $(\mathbb{F}^n; +)$ with $\mathbb{F}$ being a finite field, the atomless boolean algebra, and the countably infinite vector space over $\mathbb{F}_2$.

While CSP($\mathbb{Q}; <$) is polynomial-time tractable, giving the complexity of CSP(Γ) for every first-order reduct of $(\mathbb{Q}, <)$ is non-trivial. Here a *first-order reduct* of a structure Δ is a first-order definable structure over Δ with same domain as Δ. The class of CSPs of reducts of $(\mathbb{Q}; <)$, called *Temporal Constraint Satisfaction Problems*, contains many interesting examples of well-known computational problems. For instance, the *Betweenness problem* listed in the book of Garey and Johnson [GJ78] can be modelled as $\mathrm{CSP}(\mathbb{Q}; \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z) \vee (z < y < x)\})$. We can also mention the *network consistency problem of the Point Algebra* in Artificial Intelligence (see [vB90]). The classification of *Temporal Constraint Satisfaction Problems* has been laid down in [BK08b] by Bodirsky and Kára.

More generally, many studies focus on the classification of the complexity of CSPs over reducts of a given structure. First, this framework allows to generalize the study of many known problems. Second, the reducts of a given structure have many strong mathematical properties. For instance, the automorphism groups of reducts of a structure Δ form a lattice which is inversely symmetrical to the lattice of classes of first-order interdefinable reducts of Δ when Δ is ω-categorical. Indeed, an order $\prec$ over classes of first-order interdefinable reducts of Δ can be defined as follows: $\mathrm{cl}(\Gamma) \prec \mathrm{cl}(\Gamma')$ if and only if every relation of Γ has a first-order definition in Γ'. Such properties also hold for other kinds of definability than first-order definability but we will describe them later in this thesis.

There is no general method to reach such kind of classifications for ω-categorical structures. But this is not true anymore when we require the ω-categorical structure to possess the so called *Ramsey property*. A structure Δ has the Ramsey property when the class $\mathcal{C}$ of its finite substructures satisfies the following: for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$, there exists $\mathfrak{C} \in \mathcal{C}$ such that for every colouring of the embeddings of $\mathfrak{A}$ into $\mathfrak{C}$ with finitely many colours there exists a monochromatic copy of $\mathfrak{B}$ in $\mathfrak{C}$, that is, an embedding e of $\mathfrak{B}$ into $\mathfrak{C}$ such that all embeddings of $\mathfrak{A}$ into the image of e have the same colour. This property strongly resembles the famous combinatoric Ramsey property for sets. Bodirsky and Pinsker developed a method for classifying reducts of Ramsey structures over a finite relational signature, an illustration of which is the classification of the CSPs for reducts of the Random Graph (see [BP11]). These results use among others a very powerful result deriving from the Ramsey property: the Canonization Lemma. In short, given an ordered Ramsey structure Δ and n constants $c_1, \ldots, c_n$, every operation $f\colon \mathrm{Dom}(\Delta) \to \mathrm{Dom}(\Delta)$ locally generates over Δ a function which agrees with f on $\{c_1, \ldots, c_n\}$ and which has a very strong property on types which we are going to define: canonicity from $(\Delta; c_1, \ldots, c_n)$ to Δ.

Recall that the *(model theoretic) type* of a tuple $(a_1, \ldots, a_n)$ of elements of a structure Δ is the set of all first-order formulas $\phi(x_1, \ldots, x_n)$ such that $\Delta \models \phi(a_1, \ldots, a_n)$. An operation $f\colon \mathrm{Dom}(\Delta) \to \mathrm{Dom}(\Delta')$ is called *canonical from* Δ *to* Δ' if for all $a_1, \ldots, a_n$, the type of $(f(a_1), \ldots, f(a_n))$ only depends on the type of $(a_1, \ldots, a_n)$. Let f, g be two canonical functions on Δ. We define the equivalence relation $\sim$ as follows: $f \sim g$ whenever for all $(a_1, \ldots, a_n)$ the type of $(f(a_1), \ldots, f(a_n))$ equals the type of $(g(a_1), \ldots, g(a_n))$. The equivalence classes of $\sim$ are called *canonical behaviours*.

When Δ is Ramsey over a finite relational signature, there is only a finite number of canonical behaviours. Thus, the Canonization Lemma previously stated provides a simple technique to climb up step by step the lattice of endomorphism monoids of reducts of Δ or the lattice of polymorphism clones of reducts of Δ (the clone is the natural algebraic structure of the set of polymorphisms of a given structure). However, when the signature is not finite, the number of canonical behaviours is potentially infinite. For this reason, there is no known classification of reducts of a homogeneous structure over a functional signature which is not first-order equivalent to any structure homogeneous on a finite relational language. An example of such a structure would be the countably infinite vector space over $\mathbb{F}_2$.

There is an up to isomorphism unique countably infinite vector space $(V;+)$ over the two-element field $\mathbb{F}_2$. As an alternative definition, $(V;+)$ is the Fraïssé limit of the class of all finite vector spaces over $\mathbb{F}_2$. In particular it is *homogeneous* (i.e., any local isomorphism between substructures can be extended to an automorphism) and *universal* in the sense that it contains an isomorphic copy of all finite or countably infinite vector spaces over $\mathbb{F}_2$ as substructures. It frequently serves as an example or a counterexample for many facts and non-facts in model theory. It is ω-categorical and stable. The structure $(V;+)$ has for example the reduct $(V;\mathrm{Ieq}_4)$, where $\mathrm{Ieq}_4 = \{(a,b,c,d) \mid a+b = c+d\}$, known as the countable-dimensional affine space over $\mathbb{F}_2$. This structure can be found under the name *'Shelah's all purpose counterexample'* in the index of [Hod93].

Before even tackling the issue of a classification of CSPs for reducts of $(V;+)$ that might involve the cartography of the lattice of polymorphism clones of reducts, a number of questions are yet unanswered. For instance, is there a finite or infinite number of reduct of $(V;+)$ up to first-order interdefinability (two structures are *first-order interdefinable* when they are reducts of each other)? What are their characteristics? What method can be used to solve this kind of problems?

In 1991, Thomas [Tho91] conjectured that every countable homogeneous structure with a finite relational language has only finitely many reducts up to first-order interdefinability [Tho91]. The conjecture has been confirmed for some fundamental homogeneous structures like the order of the rationals [Cam76], the Random Graph [Tho91], and the Random Partial Order [PPP$^+$], among others.

We prove in this thesis that the countably infinite vector space over $\mathbb{F}_2$ has exactly four reducts up to first-order interdefinabilty: $(V;\{(x,y,z) \in V^3 \mid x+y=z\})$, $(V;\mathrm{Ieq}_4)$, $(V;0)$, and $(V;=)$. As a consequence of Theorem 7.3.1 of [Hod93], for any ω-categorical relational structure Γ, a relation R has a first-order definition in Γ if and only if R is preserved by all automorphisms of Γ. Any two reducts of $(V;+)$ that are first-order interdefinable have the same automorphism group, because $(V;+)$ is ω-categorical. The automorphism groups of reducts of an ω-categorical structure Δ are exactly the closed subgroups of permutations containing $\mathrm{Aut}(\Delta)$. Hence our theorem is equivalent to a classification of the closed subgroups of permutations containing $\mathrm{Aut}(V;+)$. This classification has been established independently by Bodor, Kalina and Szabó [BKS15] in a direct self-contained proof.

In our case, this first classification result for reducts of $(V;+)$ is in fact a corollary of much finer results we obtain in this thesis, since the approach used in this thesis allows a finer analysis beyond the scale of first-order interdefinability and automorphism groups of reducts. Indeed, this connection between automorphism groups of reducts and first-order interdefinability is the exact translation of the connection between self-embeddings of reducts and existential interdefinability (i.e., interdefinability by formulas of the form $\exists \overline{x}.\Psi(\overline{x})$ where Ψ is quantifier-free). In this way, we establish the cartography of the lattice of self-embedding monoids of reducts of $(V;+)$, which correspond to existential interdefinability between reducts. There are exactly seven self-embedding monoids of reducts of $(V;+)$ up to existential interdefinability.

An even finer analysis allows us to map a sizeable region of the lattice of endomorphism monoids of reducts of $(V;+)$, which corresponds to existential positive interdefinability of reducts. Finally, without making a full study of the lattice of polymorphism clones of reducts of $(V;+)$, we give some of its properties in order to state a key theorem which unlocks the door to the classification of CSPs over reducts of $(V;+)$. Out of the six cases defined in this theorem, this thesis establishes a complexity P/NP-complete dichotomy for four and provides an in-depth analysis for the last two. Our study includes a CSP classification for equality-plus-a-constant reducts which generalise the CSP classification for equality reducts published by Bodirsky and Kára in [BK08a]. The CSP classification for reducts of $(V;\mathrm{Ieq}_4)$ is also given.

In practice, our study is not directly based on the polymorphism clones of reducts of $(V;+)$ because the corresponding lattice is infinite. Consequently, factorising the cases becomes necessary. A crucial property of CSPs is their invariance by homomorphic equivalence, i.e., given two structures Γ_1, Γ_2 over a same signature, if there exists a homomorphism from Γ_1 to Γ_2, and another from Γ_2 to Γ_1, then $\mathrm{CSP}(\Gamma_1)$ and $\mathrm{CSP}(\Gamma_2)$ reduce to each other in polynomial time). Thus, rather than studying reducts of $(V;+)$, we study CSPs over their model-complete cores, following a notion fathered by Bodirsky which is related to the notion of model-companion. A core Δ of a structure Γ is a structure homomorphically equivalent to Γ (and thus possessing the same CSP), and such that every endomorphism of Δ is also a self-embedding of Δ. Also recall that an ω-categorical structure is model-complete whenever its self-embeddings are elementary. By theorem, ω-categorical structure has a unique model-complete core, up to isomorphism. In this thesis, we classify the model-complete cores of reducts of $(V;+)$ up to existential-positive interdefinability. This classification provides the fundamentals to divide the field of study into the six cases previously mentioned.

As previously stated, our method unlocks these results, provided the canonical behaviours over $(V;+)$ are identified. These behaviours can potentially be in infinite number. In this thesis, we give a complete list of the canonical behaviours from $(V;+,<)$ to $(V;+)$ up to local closure, and introduce the concept of weakly canonical functions when we add constants to the signature.

Français

Les problèmes de satisfaction de contraintes apparaissent naturellement dans de nombreux champs de l'informatique : la linguistique informatique, la bio-informatique, le scheduling, intelligence artificielle, vérification, etc. Certains des problèmes de décision les plus connus ont d'ailleurs une expression sous forme de CSP, par exemple la question de savoir si un graphe est 3-coloriable, si un graphe dirigé est acyclique, ou encore 3-SAT. Il existe plusieurs façons de formaliser les problèmes de satisfaction de contraintes; la plus adaptée dans le cadre de cette thèse utilisent des notions de théorie de modèles et de logique du premier ordre. Étant donnée une structure Γ de domaine fini ou infini, sur une signature relationnelle finie, le problème $\mathrm{CSP}(\Gamma)$ est de déterminer, sur l'entrée d'un énoncé primitif positif Ψ (i.e., de la forme $\exists \overline{x}. \bigwedge R_i(\overline{x})$), si Γ satisfait Ψ (on note $\Gamma \models \exists\Psi$).

Il faut néanmoins distinguer les CSP sur des domaines finis, des CSP sur domaines infinis. Les premiers témoignent de recherches actives depuis bientôt trente ans. On y fait notamment référence explicite dans l'article de Dechter et Pearl datant de 1987 (cf. [DP87]). Et avant même que les CSP sur domaines finis ne soient définis en tant que tels, certains résultats étaient déjà connus. Par exemple, dés 1978, Schaefer [Sch78] prouve un théorème qui peut être reformulé de la façon suivante: tout CSP sur un domaine à deux éléments est soit NP-complet, soit résoluble en temps polynomial. En 2006, Bulatov [Bul02a] surenchérit en prouvant un résultat similaire pour les CSP sur les domaines à trois éléments. Entre temps, Feder et Vardi [FV99] énoncent en 1999 la célèbre conjecture, toujours ouverte aujourd'hui, que cette dichotomie de complexité P/NP-complet existe pour l'ensemble des CSP finis. On peut également citer le résultat de dichotomie sur les graphes non dirigés (see [HN90]), alors que la conjecture est toujours ouverte pour les graphes dirigés.

Parmi les approches utilisées pour établir ces résultats de dichotomie, l'une des plus puissante est celle par l'algèbre universelle et la méthode dite des polymorphismes. Le polymorphisme est une généralisation de la notion d'homomorphisme à une arité finie quelconque. En 1968, Geiger [Gei68] établit qu'une relation R a une définition primitive positive dans une structure relationnelle Γ si et seulement si R est préservée par tous les polymorphismes de Γ. Ce même résultat est démontré indépendamment par Bodnarcuk, Kaluznin, Kotov et Romov [BKKR69]. Étant donné un ensemble d'opérations $\mathcal{F}$ et un ensemble de relations $\mathcal{R}$ sur un domaine D, on note $\mathrm{Inv}(\mathcal{F})$ l'ensemble des relations préservées par toutes les opérations de $\mathcal{F}$, et $\mathrm{Pol}(\mathcal{R})$ l'ensemble des polymorphismes de $\mathcal{R}$. Avec ces notations, le théorème précédent se réécrit ainsi: $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle_{\mathrm{pp}}$. Par conséquent, comme la complexité de $\mathrm{CSP}(\Gamma)$ est invariante par l'ajout d'une relation définissable par une formule primitive positive de Γ à Γ, les CSP de deux structures ayant les même polymorphismes se réduisent l'un à l'autre en temps polynomial. Ce lien établi, la complexité de $\mathrm{CSP}(\Gamma)$ est entièrement déterminée par $\mathrm{Pol}(\Gamma)$, l'ensemble des polymorphismes de Γ. En somme, certains polymorphismes dans $\mathrm{Pol}(\Gamma)$ peuvent donner des algorithmes polynomiaux pour résoudre $\mathrm{CSP}(\Gamma)$.

Une opération "*weak near unanimity*" est une opération vérifiant:

$$\forall x, y.(f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \cdots = f(x, x, \ldots, x, y))$$

Le CSP d'une structure relationnelle finie ne possédant pas de tels polymorphismes est nécessairement NP-dur([MM08] et [BKJ05]), et Bulatov, Jeavons et Krokhin [BKJ05] conjecturent (leur formulation est différente mais équivalente) qu'à l'inverse, le CSP d'une structure relationnelle finie possédant un tel polymorphisme est dans P. Une opération *near unanimity* est une opération *weak near unanimity* vérifiant l'égalité $f(y, x, \ldots, x) = x$ pour tout x, y. Par exemple la fonction majorité fait partie de cette classe. La conjecture de Jeavons-Korkhin a été confirmée dans le cas de ces opérations *near unanimity* [JCC98], mais aussi dans celui des opérations binaires à la fois commutatives, associatives et idempotentes. Le cas des fonctions dites de Mal'tsev (i.e., des opérations ternaires vérifiant $f(x, y, y) = f(y, y, x) = x$ pour tout x, y) a lui aussi été traité [Bul02b].

Concernant les CSP sur des domaines infinis, leur introduction se justifie par la richesse de leur expressivité, et par le fait que certains problèmes de décision connus ne sont pas modélisables par des CSP sur des domaines finis, mais le sont par des CSP sur des domaines infinis; par exemple l'acyclicité d'un graphe dirigé. En effet, Bodirsky et Grohe (see [BG08] établissent en 2008 que tout problème computationnel est polynomialement équivalent à un CSP sur un domaine infini. Mais dans quelle mesure peut-on adapter les techniques d'algèbre universelle et sa méthode des polymorphismes à des cas de structures infinies? Le cadre ω-catégorique transpose naturellement l'égalité $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle_{\mathrm{pp}}$ à un domaine infini.

Une structure est ω-catégorique si tous les modèles de sa théorie du premier ordre sont isomorphes. Par le théorème de Ryll-Nardzewski, une structure dénombrable Δ est ω-catégorique si pour tout $n \geq 1$, il n'existe qu'un nombre fini de formules avec n variables libres modulo la théorie du premier ordre de Δ. Un certain nombre de structures célèbres présentent cette propriété d'ω-catégoricité. On citera pour exemple $(\mathbb{Q}, <)$, l'espace vectoriel $(\mathbb{F}^n; +)$ avec $\mathbb{F}$ corps fini, l'algèbre de Boole dénombrable sans atomes, et l'espace vectoriel dénombrable sur $\mathbb{F}_2$.

S'il est vrai que le problème $\mathrm{CSP}(\mathbb{Q}; <)$ se résout facilement en temps polynomial, il est en revanche non trivial de donner la complexité de $\mathrm{CSP}(\Gamma)$ pour tous les *réduits (du premier ordre)* de $(\mathbb{Q}, <)$, où *réduit (du premier ordre)* désigne une structure définissable au premier ordre sur la structure de départ et même domaine que celui de la structure de départ. La classe des CSP de réduits de $(\mathbb{Q}; <)$, nommée *Temporal Constraint Satisfaction Problems*, est très riche; elle contient notamment une reformulation de plusieurs problèmes computationnels connus. Par exemple, le Betweeness problem mentionné dans le livre de Garey et Johnson [GJ78] est modélisable par un CSP sur le réduit de $(\mathbb{Q}; <)$ suivant: $(\mathbb{Q}; \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z) \vee (z < y < x)\})$. On peut également citer le *network consistency problem of the Point Algebra* en intelligence artificielle (cf. [vB90]). La classification des *Temporal Constraint Satisfaction Problems* a été réalisée dans [BK08b] par Bodirsky et Kára.

Plus généralement, de nombreuses études s'intéressent à la classification de la complexité des CSP pour des réduits d'une structure donnée, pour des raisons de généralisation d'une part (liée à l'expressivité de la logique du premier ordre), mais aussi pour des propriétés fortes des réduits. Par exemple, les groupes d'automorphismes des réduits d'une structure Δ donnée forment un treillis inversement symmétrique, par une correspondance de Galois présente dans le cadre ω-catégorique, à celui formé par les réduits de cette structure à inter-définissabilité près, dont la relation d'ordre $\prec$ est la définissabilité au premier ordre (i.e., étant donnés deux réduits Γ, Γ' de la structure ω-catégorique Δ, on a que $\Gamma \prec \Gamma'$ si et seulement si toute relation de Γ est définissable au premier ordre dans Γ'). De telles propriétés existent aussi pour des définissabilité autre que du premier ordre que nous détaillerons par la suite.

Il n'y a pas de méthode générale pour obtenir ce genre de classification pour des structures ω-catégoriques. Mais ceci n'est plus vrai dès lors qu'on considère des structures ω-catégoriques qui possèdent en plus la *propriété de Ramsey*. Une structure Δ a la propriété de Ramsey quand la classe $\mathcal{C}$ de ses sous-structures finies vérifie : pour tout

$\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$, il existe $\mathfrak{C} \in \mathcal{C}$ tel que pour tout coloriage des plongements de $\mathfrak{A}$ dans $\mathfrak{C}$ à l'aide d'un nombre fini de couleurs, il existe une copie monochromatique of $\mathfrak{B}$ dans $\mathfrak{C}$, c'est à dire, un plongement e de $\mathfrak{B}$ dans $\mathfrak{C}$ tel que tous les plongements de $\mathfrak{A}$ dans l'image de e ont la même couleur. Cette propriété ressemble fortement à la fameuse propriété de Ramsey pour les ensembles. Bodirsky et Pinsker ont développé une méthode pour classifier les réduits de structures Ramsey à signature relationnelle finie; une illustration de cette méthode est la classification de la complexité des CSP des réduits du Graphe Aléatoire [BP11]. Ces travaux utilisent notamment un résultat très fort dérivé de cette propriété de Ramsey: le lemme de canonisation. En un mot, dans le cadre d'une structure Δ ordonnée homogène et Ramsey sur une signature relationnelle, et étant donnée n constantes $c_1, \ldots, c_n$, toute fonction f engendre sur Δ une fonction égale à f sur les points $\{c_1, \ldots, c_n\}$ et qui possède une propriété forte sur les types que nous définissons dans le paragraphe qui suit: la canonicité de $(\Delta; c_1, \ldots, c_n)$ vers Δ.

On rappelle que le type modèle-théorique d'un tuple $(a_1, \ldots, a_n)$ d'éléments d'un structure Δ est l'ensemble des formules du premier ordre $\phi(\overline{x})$ sur la signature de Δ telles que $\Delta \models \phi(a_1, \ldots, a_n)$. Une opération $f\colon \mathrm{Dom}(\Delta) \to \mathrm{Dom}(\Delta')$ est dite *canonique de* Δ *vers* Δ' si pour tout $a_1, \ldots, a_n$, le type de $(f(a_1), \ldots, f(a_n))$ ne dépend que du type de $(a_1, \ldots, a_n)$. Soient f, g deux fonctions canoniques sur Δ. On définit la relation d'équivalence $\sim$ comme suit: $f \sim g$ si et seulement si pour tout $(a_1, \ldots, a_n)$, le type de $(f(a_1), \ldots, f(a_n))$ est le même que celui de $(g(a_1), \ldots, g(a_n))$. Les classes d'équivalence de $\sim$ sont appelées *comportements canoniques*.

Quand Δ est homogène avec une signature relationnelle finie, il n'y a qu'un nombre fini de comportements canoniques. Ainsi, le théorème de Ramsey précédemment énoncé fournit une technique simple pour "escalader" le treillis des monoïdes d'endomorphismes ou les clones de polymorphismes (le clone est la structure algébrique naturelle qu'on attribue à l'ensemble des polymorphismes d'une structure). Mais si la signature n'est pas relationnelle finie, ce nombre peut être potentiellement infini. Ainsi, il n'existe à ce jour pas de classification de la complexité des CSP sur les réduits d'une structure homogène fonctionnelle n'étant pas inter-définissable au premier ordre avec aucune structure homogène sur un domaine relationnelle finie. Un exemple de telle structure qui vient naturellement à l'esprit est celui de l'espace vectoriel dénombrable sur $\mathbb{F}_2$.

Il existe un unique espace vectoriel dénombrable $(V; +)$ sur le corps à deux éléments. Il s'agit de la limite de Fraïssé de la classe des espaces vectoriels finis sur $\mathbb{F}_2$. En particulier, c'est une structure homogène et universelle dans le sens où elle contient une copie de tous les espaces vectoriels finis ou dénombrables sur $\mathbb{F}_2$. Il est souvent utilisé comme exemple ou d'objet d'étude en théorie des modèles. Il est d'ailleurs stable et ω-catégorique. La structure $(V; +)$ contient également le réduit $(V; \mathrm{Ieq}_4)$, où $\mathrm{Ieq}_4 = \{(a, b, c, d) \mid a+b = c+d\}$, assimilé à l'espace affine dénombrable sur $\mathbb{F}_2$. Cette structure est connue sous le nom de *'Shelah's all purpose counterexample'* dans l'index de [Hod93].

Avant même d'envisager une classification des CSP des réduits de $(V; +)$, qui passerait éventuellement par la cartographie du treillis des ensembles de polymorphismes de réduits de $(V; +)$, un certain nombre d'inconnues d'ordre algébrique restent à lever. Par exemple, y a-t-il un nombre fini ou infini de réduits de $(V; +)$ à inter-définissabilité du premier ordre près (deux structures étant inter-définissable au premier ordre quand elles sont chacune réduit de l'autre), ou à inter-définissabilité existentielle près (les deux réduits s'inter-définissent à l'aide de formules existentielles)?

En 1991, Thomas [Tho91] conjecture qu'une structure dénombrable homogène sur un langage relationnel fini n'a qu'un nombre fini de réduits à inter-définissabilité au premier ordre près (cf. [Tho91]). Cette conjecture a été confirmée pour un certain nombre de structures homogènes fondamentales comme l'ordre sur les rationnels [Cam76], le graphe aléatoire [Tho91] ou encore l'ordre partiel aléatoire [PPP$^+$].

Nous prouvons dans cette thèse que l'espace vectoriel dénombrable sur $\mathbb{F}_2$ a exactement quatre réduits à inter-définissabilité du premier ordre près :

$$(V; \{(x, y, z) \in V^3 \mid x + y = z\}),\ (V; \mathrm{Ieq}_4),\ (V; 0) \text{ et } (V; =)$$

En conséquence du théorème 7.3.1 de [Hod93], pour toute structure relationnelle ω-catégorique Γ, une relation R est définissable au premier ordre dans Γ si et seulement si R est préservée par tous les automorphismes de Γ. Ainsi, deux réduits quelconques de $(V; +)$ qui sont inter-définissables au premier ordre ont le même groupe d'automorphisme, puisque $(V; +)$ est ω-catégorique. En fait, les groupes d'automorphismes de réduits d'une structure ω-catégorique Δ sont exactement les groupes de permutations clos au sens de la convergence simple et contenant $\mathrm{Aut}(\Delta)$. Ainsi, notre théorème équivaut à la classification des groupes de permutations clos contenant $\mathrm{Aut}(V; +)$. Cette classification a d'ailleurs été établie indépendamment par Bodor, Kalina et Szabó [BKS15] dans une preuve directe et auto-suffisante.

En revanche dans cette thèse, ce premier résultat de classification de réduits n'est en fait qu'un corollaire de résultats plus fins que nous obtenons. Aussi l'approche utilisée dans cette thèse permet d'affiner la granularité de l'analyse qui ne se limite donc pas à l'échelle des groupes d'automorphismes et donc à inter-définissabilité du premier ordre près entre les réduits. En effet, cette correspondance entre groupes d'automorphismes de réduits isomorphes et réduits inter-définissables au premier ordre se décalque également dans les cas des monoïdes d'auto-plongements par rapport aux réduits existentiellement inter-définissables (i.e., par des formules de type $\exists \overline{x}.\Psi(\overline{x})$ où Ψ est sans quantificateur). On établit ainsi la cartographie du treillis des monoïdes d'auto-plongements des réduits de $(V; +)$, correspondant à une inter-définissabilité existentielle. On compte en fait sept monoïdes d'auto-plongements de réduits de $(V; +)$ distincts.

En affinant encore davantage, on parvient à cartographier une région importante du treillis des monoïdes d'endomorphismes, qui correspond à une inter-définissabilité existentielle positive entre réduits. Finalement, sans donner le treillis des ensembles de polymorphismes, on ajuste la finesse de l'analyse au cas des inter-définissabilités primitives positives, ce qui permet d'énoncer le théorème clé qui ouvre la classification des CSP. Celle-ci n'est malheureusement pas tout à fait complète mais les deux cas qu'il reste à traiter sont bien identifiés et ne semblent pas présenter de difficultés particulières par rapport à ceux qui sont effectivement traités. On peut notamment mentionner que notre étude inclut la classification des CSP des réduits de l'égalité plus une constante qui vient compléter la classification des CSP des réduits de l'égalité publiée par Manuel Bodirsky et Jan Kára dans [BK08a]. Celle des réduits de $(V; \mathrm{Ieq}_4)$ est également donnée.

En pratique, l'étude que nous menons sur les CSP des réduits $(V;+)$ ne se base pas systématiquement sur ces réduits. En effet, nous prouvons que le treillis des monoïdes d'endomorphisme de réduits de $(V;+)$ est infini. En conséquence, celui des clones de polymorphisme l'est également. Par conséquent, il est nécessaire de "factoriser" les cas pour classifier l'ensemble des CSP de réduits de $(V;+)$. Mais l'une des propriétés cruciales des CSP est l'invariance par équivalence homomorphique, i.e., étant données deux structures Γ_1, Γ_2 sur un même langage, s'il existe un homomorphisme de Γ_1 vers Γ_2, et un autre de Γ_2 vers Γ_1, alors CSP(Γ_1) et CSP(Γ_2) se réduisent l'un à l'autre en temps polynomial). Ainsi, plutôt que d'étudier le CSP des réduits de $(V;+)$, nous étudions ceux des noyaux modèle-complets complets de ces réduits, d'après une notion introduite par Bodirsky et qui n'est pas sans rappeler la notion de modèle-compagne. Un cœur Δ d'une structure Γ est une structure homomorphiquement équivalente à Γ (et qui possède donc le même CSP), et telle que tout endomorphisme de Δ est un auto-plongement de Δ. Rappelons aussi qu'une structure ω-catégorique est modèle-complète si tout ses auto-plongements sont élémentaires. Par théorème, toute structure ω-catégorique possède un unique cœur modèle-complet, à isomorphisme près. Dans cette thèse, et c'est sans doute la contribution algébrique principale, nous classifions les noyaux modèle-complets des réduits de $(V;+)$ à inter-définissabilité existentielle positive près, et nous nous basons sur cette classification pour décomposer l'étude des CSP sur ces mêmes réduits en un nombre fini de cas correspondants aux neufs noyaux modèle-complets des réduits de $(V;+)$ mis en évidence.

Comme nous le disions précédemment, la méthode que nous utilisons permet d'établir ces résultats à condition d'étudier en détail les comportements canoniques sur $(V;+)$. Ceux-ci peuvent être a priori en nombre fini ou infini. Dans cette thèse, nous établissons la liste complète des comportements canoniques de $(V;+,<)$ vers $(V;+)$ à cloture locale près, et introduisons le concept de fonctions faiblement canoniques après avoir ajouté des constantes à la signature.

Plan

We start by recalling in the preliminaries some basic knowledge of algebra, model theory, Ramsey theory, and CSPs. Then in a second chapter we study some basic relations over the countably infinite vector space (first without adding constants to the signature) which will help us to characterize the canonical behaviours. We give a complete list of the canonical behaviours from $(V;+,<)$ to $(V;+)$ up to local closure, and then we extend our study of canonical behaviours after adding n constants to the signature.

Then, we draw the lattice of endomorphism monoids using these canonical behaviours we constructed to locally generate endomorphism monoids. After having dealt with the Ramsey order, we state the theorem of classification for the automorphism group of reducts of $(V;+)$, and the one for self-embeddings monoids.

To establish the link with the CSPs, we also look for the model-complete core of several outstanding reducts of $(V;+)$ in order to simplify the study of the CSPs, and we state the theorem of classification of model-complete core of reducts up to existential positive inter-definability.

In the last chapter, we start solving one by one those cases, trying to obtain the dichotomy P/NP-complete for the complexity of the CSPs for each case. First we state some general properties, then we deal with the case where we only have equality plus a constant. Then come the cases with the affine relation only (and the Ieq_3 relation over $V \setminus \{0\}$ only that we introduce in the thesis). We finally start to tackle the two remaining cases: the case with Ieq_4 (introduced in the thesis) and 0 in the signature, and what we call the full case (where $\mathrm{End}(\Gamma) = \mathrm{End}(V; +, \neq)$).

Chapter 2

Preliminaries on Model Theory and Ramsey Theory

As we emphasized in the introduction, some properties of the countably infinite vector space over $\mathbb{F}_2$ guide us toward an algebraic approach to classify the complexity of the CSPs of reducts of $(V;+)$. Let us start by recalling basic model theory and universal algebra definitions and properties and introduce notations that will be used throughout this thesis.

2.1 Basic Definitions and Preservation Theorems

Among model theory notions we use, ω-categoricity is of prime import. It allows to reduce the number of n-types to a finite number in order to recover the equality $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle_{\mathrm{pp}}$ we presented in the introduction and which hold for finite domains. The fact that $(V;+)$ is ω-categorical makes the study of canonical functions easier.

Definition 2.1.1. In Model Theory, a *structure* is a triple $\Delta = (D, \sigma, \mathcal{I})$ consisting of a domain D, a signature τ which can contain either relation symbols, multi-arity function symbols or constants, and an interpretation function $\mathcal{I}$ that indicates how the signature is to be interpreted on the domain. A structure with signature τ is called a *τ-structure.*

From now, we will omit the interpretation function whenever there is no ambiguity with the interpretation of each function, relation and constant symbols we use.

Definition 2.1.2. Given a signature τ, a theory T is a set of closed first-order τ-formulas (i.e., without free variables). We say that a τ-structure $\mathcal{M}$ *models* T, and we denote it by $\mathcal{M} \models T$, whenever every formula in T is true in $\mathcal{M}$. We say that a theory T is *complete* when T has a model, and for all closed formula ϕ, either ϕ or $\neg\phi$ belongs to T. Given a structure $\mathcal{M}$, we define the first-order theory of $\mathcal{M}$, and denote it by $\mathrm{Th}(\mathcal{M})$, the set of all closed formulas true in $\mathcal{M}$. Note that $\mathrm{Th}(\mathcal{M})$ is a complete theory.

Definition 2.1.3. Let Δ be a structure of signature τ, let c be a constant symbol of τ, let f be a k-ary function symbol of τ, and R be a relation symbol of arity k in τ. Let g be a unary operation on the domain of D of Δ. We say that:

- *g preserves c^Δ* if $g(c^\Delta) = c^\Delta$;
- *g preserves f^Δ* if for all $\overline{x} \in D^k$, $g(f^\Delta(\overline{x})) = f^\Delta(g(x_1), \ldots, g(x_k))$. In the same spirit,
- *g preserves R^Δ* if for all $\overline{x} \in R^\Delta$, $(g(x_1), \ldots, g(x_k)) \in R^\Delta$. Finally,
- *g strongly preserves R^Δ* if the converse is true, i.e., for all $\overline{x} \in D^k$, if we have $(g(x_1), \ldots, g(x_k)) \in R^\Delta$, then $\overline{x} \in R^\Delta$.

If an operation $g\colon D \to D$ preserves every constant, relation and function of a structure Δ, we call g an *endomorphism* of Δ. If g preserves every function, and strongly preserves every constant and relation of Δ, we call g a *self-embedding* of Δ. If g is a bijective self-embedding of Δ, then g is called an *automorphism* of Δ.

From now on, we write c, g, R instead of $c^\Delta, g^\Delta, R^\Delta$ when there is no possible confusion.

Notation 2.1.4. Given a structure Δ, we denote:

- the set of all automorphisms of Δ by $\mathrm{Aut}(\Delta)$;
- the set of all self-embeddings of Δ by $\mathrm{Emb}(\Delta)$;
- the set of all endomorphisms of Δ by $\mathrm{End}(\Delta)$.

Note that for all Δ, we have $\mathrm{Aut}(\Delta) \subseteq \mathrm{Emb}(\Delta) \subseteq \mathrm{End}(\Delta)$.

Remark 2.1.5. The automorphisms of a structure Δ are exactly the bijective endomorphisms of Δ whose inverse is also an endomorphism of Δ.

Proposition 2.1.6. *Let Γ and Γ' be two structures such that $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(\Gamma')$. Then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma')$.*

Proof. Recall that an automorphism is a bijective self-embedding. Hence, every bijective operation in $\mathrm{Emb}(\Gamma)$ is an automorphism of Γ, and since it is contained in $\mathrm{Emb}(\Gamma')$, it is also an automorphism of Γ'. $\square$

Proposition 2.1.7. *Let α be a self-embedding of a structure Δ, and let γ be a function from $\mathrm{Dom}(\Delta) \to \mathrm{Dom}(\Delta)$ such that $\alpha \circ \gamma$ is a self-embedding of Δ. Then $\gamma \in \mathrm{Emb}(\Delta)$.*

Proof. Assume that γ is not a self-embedding of Δ. There exists a relation R in Δ and a tuple $\overline{x}$ in R such that $(\gamma(x_1), \ldots, \gamma(x_n)) \notin R$. Hence, since $\alpha \in \mathrm{Emb}(\Delta)$, we have $(\alpha(\gamma(x_1)), \ldots, \alpha(\gamma(x_n))) \notin R$. Consequently, $\alpha \circ \gamma \notin \mathrm{Emb}(\Delta)$. $\square$

Definition 2.1.8. Given a domain D, and a set $\mathcal{F}$ of unary operations over this domain, we denote by $\mathrm{Inv}(\mathcal{F})$ the set of relations over D which are preserved by every operation of $\mathcal{F}$.

Definition 2.1.9. A *Galois connection* is a pair of functions $\phi\colon U \to V$ and $\psi\colon V \to U$ between two posets U and V, such that $v \leq \phi(u)$ if and only if $u \leq \psi(v)$ for all $(u,v) \in U \times V$.

Lemma 2.1.10. *Let $\mathcal{R}$ be a set of relations over a domain D, Γ be the structure of domain D whose relations are exactly $\mathcal{R}$, and $\mathcal{F}$ be a subset of unary operations of D, we have:*

$$\mathcal{R} \subseteq \mathrm{Inv}(\mathcal{F}) \Leftrightarrow \mathcal{F} \subseteq \mathrm{End}(\Gamma)$$

Corollary 2.1.11. *Hence, given a domain D, the operators* Inv *and* End *form a Galois connexion between the sets of relations of D and the subsets of D^D.*

Proof. The proof is straightforward by Lemma 2.1.10 and by definition of a Galois connexion. □

Given a Galois connection ϕ/ψ, it is natural to consider the two associated closure operators $\phi \circ \psi$ and $\psi \circ \phi$. In the particular case of the connection $\mathrm{Inv} - \mathrm{End}$, these closure operators have very interesting properties. We introduce here the notion of local closure in order to describe these properties.

Definition 2.1.12. Let $\mathcal{F}$ be a set of unary operations on a set D. The *local closure of* $\mathcal{F}$, denoted by $\overline{\mathcal{F}}$, is the smallest set of operations $\mathcal{F}'$ which contains $\mathcal{F}$ and which satisfies the following property: if g is a unary operation on D such that for every finite subset S of D, there exists $f \in \mathcal{F}'$ such that $g{\upharpoonright}S = f{\upharpoonright}S$, then $g \in \mathcal{F}'$.

We say that a set of unary operations is *locally closed* whenever $\mathcal{F} = \overline{\mathcal{F}}$.

Lemma 2.1.13. *Given a structure Γ, the sets* $\mathrm{Aut}(\Gamma)$, $\mathrm{Emb}(\Gamma)$, *and* $\mathrm{End}(\Gamma)$ *are* locally closed.

Proof. We give the proof for $\mathrm{End}(\Gamma)$, the other cases being similar. Since $\mathcal{F} \subseteq \overline{F}$ for every set $\mathcal{F}$ of operations, we only have to prove that $\overline{\mathrm{End}(\Gamma)} \subseteq \mathrm{End}(\Gamma)$. Let g be an element of $\overline{\mathrm{End}(\Gamma)}$, let R be a relation of Γ, and let $(a_1, \ldots, a_n) \in R$. Let $S := \{a_1, \ldots, a_n\}$. By definition of the local closure, there is an endomorphism f of Γ such that $f{\upharpoonright}S = g{\upharpoonright}S$. Since $f \in \mathrm{End}(\Gamma)$, we have $(f(a_1), \ldots, f(a_n)) \in R$. Hence, $(g(a_1), \ldots, g(a_n))$ is also in R, and consequently: $g \in \mathrm{End}(\Gamma)$. □

Notation 2.1.14. For any set A and any $\mathcal{F} \subseteq A^A$, we denote by $\langle \mathcal{F} \rangle_1$ the closure of $\mathcal{F}$ under composition, i.e., $\langle \mathcal{F} \rangle_1$ is the smallest set of unary operations on A which contains $\mathcal{F}$, and such that, if f, g are two unary operations of $\langle \mathcal{F} \rangle_1$, then $f \circ g$ belongs to $\langle \mathcal{F} \rangle_1$.

Definition 2.1.15. Let $\mathcal{F}$ be a set of unary operations over an infinite set D, and g be two unary operations over D. We say that $\mathcal{F}$ *locally generates* g, or g *is locally generated by* $\mathcal{F}$, whenever $g \in \overline{\langle \mathcal{F} \rangle_1}$.

The following lemma gives a characterization of the closure $\overline{\langle \mathcal{F} \rangle_1}$.

Lemma 2.1.16. *Let $\mathcal{F}$ be a set of unary operations over a set D. Then $\overline{\langle \mathcal{F} \rangle_1}$ is exactly the set of operations g such that for all finite subset S of D, there exists $f \in \langle \mathcal{F} \rangle_1$ such that $g{\upharpoonright}S = f{\upharpoonright}S$.*

Proof. Let $\mathcal{G}$ be the set of operations g such that for all finite subset S of D, there exists $f \in \langle\mathcal{F}\rangle_1$ such that $g\upharpoonright S = f\upharpoonright S$. We first prove that $\mathcal{G}$ is locally closed. Let h be an operation such that for all S, there exists $g \in \mathcal{G}$ such that $h\upharpoonright S = g\upharpoonright S$. Let S be a finite subset of D. There exists $g \in \mathcal{G}$ such that $h\upharpoonright S = g\upharpoonright S$. By definition of $\mathcal{G}$, there exists $f \in \langle\mathcal{F}\rangle_1$ such that $g\upharpoonright S = f\upharpoonright S$. Hence, $h\upharpoonright S = f\upharpoonright S$. Consequently, h belongs to $\mathcal{G}$, and $\mathcal{G}$ is locally closed. Since $\overline{\langle\mathcal{F}\rangle_1}$ is the smallest locally closed set containing $\langle\mathcal{F}\rangle_1$, we have: $\overline{\langle\mathcal{F}\rangle_1} \subseteq \mathcal{G}$.

Conversely, since $\overline{\langle\mathcal{F}\rangle_1}$ is locally closed, it satisfies the following property: for all operation g, if for all finite subset S of D there exists $f \in \overline{\langle\mathcal{F}\rangle_1}$ such that $g\upharpoonright S = f\upharpoonright S$, then g belongs to $\overline{\langle\mathcal{F}\rangle_1}$. Let g be an element of $\mathcal{G}$. By definition, for all finite subset S of D, there exists $f \in \langle\mathcal{F}\rangle_1$ such that $g\upharpoonright S = f\upharpoonright S$. Hence, since $\langle\mathcal{F}\rangle_1$ is contained in $\overline{\langle\mathcal{F}\rangle_1}$, g belongs to $\overline{\langle\mathcal{F}\rangle_1}$, and $\mathcal{G} \subseteq \overline{\langle\mathcal{F}\rangle_1}$. □

From now on, this characterization will always be used for $\overline{\langle\mathcal{F}\rangle_1}$.

The following equality does not require any special hypothesis on the set $\mathcal{F}$. It is an important topological property of one of the closure operators of the Galois connection $\mathrm{Inv} - \mathrm{End}$.

Theorem 2.1.17. *Let $\mathcal{F}$ be a set of unary operations on a domain D. We have the following:*

$$\overline{\langle\mathcal{F}\rangle_1} = \mathrm{End}(\mathrm{Inv}(\mathcal{F}))$$

We now describe the classes of first-order formulas that intervene in the description of closure operators of the connection that we have just stated and various other connections which can be deduced from this connection. These are the existential formulas (ex), which only contain existential quantifiers, and existential positive formulas which are existential formulas whose quantifier-free part does not contain any negation. We finally define the notion of primitive positive formula which is intimately linked to CSPs. This notion also appears in the description of the closure operator of a Galois connection that we introduce in the second part of this thesis.

Definition 2.1.18. A formula is *existential* whenever it is of the form $\exists x_1 \dots x_n \phi(\overline{x})$ where ϕ is a quantifier free formula.

A formula is *existential positive* whenever it is of the form $\exists x_1 \dots x_n \bigvee_{i \leq n} \bigwedge_{j \leq k} \psi_{i,j}$ where the $\psi_{i,j}$ are atomic formulas, i.e., of the form $R(t_1, \dots, t_l)$ with t_i terms and R a relation symbol of the language.

A formula is *primitive positive* whenever it is of the form $\exists x_1 \dots x_n \bigwedge_{i \leq k} \psi_i$ where the ψ_i are atomic formulas, i.e., of the form $R(t_1, \dots, t_l)$ with t_i terms and R a relation symbol of the language.

The notion of definability flows from the previous definitions. Given a structure and a first-order formula, potentially existential, existential positive, or primitive positive, with n free variables, we can define a relation R as the set of tuples of elements of the domain of Γ which satisfy the formula.

Definition 2.1.19. Let Γ be a structure of domain D.

A relation $R \subseteq D^k$ is *first-order definable (resp. existential positive definable, resp. primitive positive) over* Γ if there exists a first-order (resp. existential positive, resp. primitive positive) formula $\phi(\overline{x})$ such that for all $a_1, \ldots, a_k \in D$:

$$(a_1, \ldots, a_k) \in R \Leftrightarrow \Gamma \models \phi(a_1, \ldots, a_k)$$

A function $f: D^k \to D$ is *first-order (resp. existential positive, resp. primitive positive) definable over* Γ if there exists a first-order (resp. existential positive, resp. primitive positive) formula $\phi(\overline{x})$ such that for all $a_1, \ldots, a_k, b \in D$:

$$f(a_1, \ldots, a_k) = b \Leftrightarrow \Gamma \models \phi(a_1, \ldots, a_k, b)$$

A constant $c \in D$ is *first-order definable over* Γ if there exists a first-order formula $\phi(x)$ such that for all $a \in D$:

$$a = c \Leftrightarrow \Gamma \models \phi(a)$$

Two relations R_1, R_2 are *first-order (resp. existential positive, resp. primitive positive) interdefinable over a structure* Γ whenever R_1 is first-order (resp. existential positive, resp. primitive positive) definable over (Γ, R_2) and R_2 is first-order (resp. existential positive, resp. primitive positive) definable over (Γ, R_1).

Two structures Γ_1 and Γ_2 are *first-order (resp. existential positive, resp. primitive positive) interdefinable* when all functions, relations and constants of the first are first-order definable over the second, and vice versa.

Notation 2.1.20. From now, we will use the notations fo, ep and pp to denote first-order, existential positive, and primitive positive.

Let Γ be a relational structure. We denote by $\langle \Gamma \rangle_{\mathrm{fo}}$ (resp. $\langle \Gamma \rangle_{\mathrm{ep}}$, resp. $\langle \Gamma \rangle_{\mathrm{pp}}$) the set of fo-definable (resp. ep-definable, resp. pp-definable) relations over Γ.

Before the crucial properties of closure operators derived from the Galois connection $\mathrm{Inv} - \mathrm{Pol}$ we will use all along this thesis, we introduce the notion of ω-categoricity, born from model theory, acting as a bridge between certain infinite structures and finite structures. Indeed, a structure is ω-categorical whenever its automorphism group is oligomorphic, i.e., the natural action of this group over the n-tuples of elements of the domain has a finite number of orbits for each $n \in \mathbb{N}$. We will start by giving the main definition though we will mostly use the characterization we just stated.

Definition 2.1.21. A structure is *ω-categorical* if there exists a unique countably infinite model of its first-order theory up to isomorphism.

Definition 2.1.22. A permutation group $\mathcal{G}$ over a countably infinite set B is oligomorphic if the natural action of $\mathcal{G}$ over B has only finitely many n-orbits for all $n \geq 1$.

Definition 2.1.23. Given a structure Δ of domain D, the *(complete model-theoretic) type* of a tuple $(a_1, \ldots, a_n)$ of elements of D is the set of formulas $\phi(\overline{x})$ such that $\Delta \models \phi(\overline{a})$. We denote this set by $\mathrm{tp}_\Delta(a_1, \ldots, a_n)$.

The following theorem was independently proved by Engeler, Ryll-Nardzewski, and Svenonius.

Theorem 2.1.24. *Let Δ be a countably infinite structure with countable signature. The following are equivalent:*

- *Δ is ω-categorical;*
- *the automorphism group $\mathrm{Aut}(\Delta)$ is oligomorphic and for each n, every n-orbit of the natural action of $\mathrm{Aut}(\Delta)$ on $\mathrm{Dom}(\Delta)$ is definable;*
- *for all $n \geq 1$, there are finitely many inequivalent formulas with n free variables over Δ;*
- *for all n there is a finite number of distinct n-types over Δ.*

The following remark is crucial and will be used extensively in the proofs of the thesis.

Remark 2.1.25. As a direct consequence of Theorem 2.1.24, given an ω-categorical structure Γ and two n-tuples $\overline{a}, \overline{b}$ of elements of Γ with same type over Γ, then there exists $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha.\overline{a} = \overline{b}$, i.e., $\alpha(a_i) = b_i$ for all $i \leq n$. This follows from the fact that n-orbits of the natural action are definable, and hence, two tuples with same types lay necessarily in the same orbit since no formula can distinguish one from the other.

The following is Corollary 7.3.3 from [Hod93].

Proposition 2.1.26. *Let Δ be an ω-categorical structure over a countable language. Then for every positive integer n, and every pair of n-tuples $\overline{a}, \overline{b}$, the tuples $\overline{a}, \overline{b}$ are in the same orbit under $\mathrm{Aut}(\Delta)$ if and only if $\overline{a}$ and $\overline{b}$ have same type over Δ.*

Here follow the link between automorphisms and first-order formulas, self-embeddings and existential formulas, and endomorphisms and existential positive formulas. Note that the ω-categoricity hypothesis is necessary here in order to obtain the equivalences.

The following is from [Bod04] and [BJ11]:

Theorem 2.1.27. *Let Γ be an ω-categorical or finite structure and R be a relation on $\mathrm{Dom}(\Gamma)$. We have the following:*

- *R has a first-order definition in Γ if and only if R is preserved by all automorphisms of Γ.*
- *R has an existential definition in Γ if and only if R is preserved by all self-embeddings of Γ.*
- *R has an existential positive definition in Γ if and only if R is preserved by all endomorphisms of Γ.*

2.2 Homogeneity and Fraisse Theory

As we mentioned in the introduction, the countably infinite vector space over $\mathbb{F}_2$ has a strong property: it is homogeneous. Recall that a structure is homogeneous if any local isomorphism between substructures can be extended to an automorphism of the structure. Several well known structures are homogeneous, among which the Random Graph, the atomless Boolean algebra, any countably infinite vector space over a finite field, etc.

The study of homogeneous structures and the classification of their first-order reducts is indeed a fruitful branch of mathematics. We recall here the main definitions and properties, as well as an alternative definition stated by Fraïssé. Indeed, a homogeneous structure can be seen as a limit of its finite substructures in a way we shall explain.

Definition 2.2.1. A structure Δ is *homogeneous* if any isomorphism between finite substructures of Δ can be extended to an automorphism of Δ.

Remark 2.2.2. Let B be any infinite set, and $c_1, \ldots, c_n$ be n constant symbols. The structures $(B; =)$ and more generally, $(B; c_1, \ldots, c_n)$ are homogeneous. Indeed, $(B; =)$ is highly transitive by Lemma 4.4.23, and high transitivity straightforwardly implies homogeneity. For $(B; c_1, \ldots, c_n)$, we use the partial high transitivity for the set subset $B \setminus \{c_1, \ldots, c_n\}$, combined with the fact that the constants have to be preserved by any local isomorphism, and we obtain the result.

We now recall Roland Fraïssé's terminology in order to shed a different light over the notion of homogeneity.

Definition 2.2.3. The *age* of a countable structure Δ with a signature τ, is the set of all finitely generated sub-structures of Δ, up to isomorphism. We call an *age* any set which is the age of a structure.

We now define three properties that an age must satisfy in order to generate a homogeneous structure.

Definition 2.2.4. Let S be a countable set of structures over a given signature.

- *Hereditary property* (HP): every finitely generated substructure of a structure Γ of S belongs to S, up to isomorphism.
- *Joint embedding property* (JEP): for all A, B in S, there exists $C \in S$ such that A and B are embeddable in C.
- *Amalgamation property* (AP): for all A, B, C in S, if e_1 is an embedding from A to B, and f_1 is an embedding from A to C, then there exist D in S, an embedding e_2 from B to D, and an embedding f_2 from C to D, such that $e_2 \circ e_1 = f_2 \circ f_1$.

Remark 2.2.5. Note that any age satisfies (HP) and (JEP).

Here comes the famous characterization of homogeneous structures by Rolland Fraïssé.

Theorem 2.2.6. *Let σ be a countable signature and $\mathcal{A}$ be a countable set of finitely generated σ-structures which has (HP), (JEP), and (AP). Then there is a countable σ-structure Δ whose age is $\mathcal{A}$, and which is homogeneous. Δ is called the* Fraïssé limit *of $\mathcal{A}$, and is unique up to isomorphism.*

Definition 2.2.7. We say that a structure Δ is *locally finite* when every finitely generated substructure of Δ is finite.

We say that a structure Δ is *uniformly locally finite* if there is a function $f\colon \omega \to \omega$ such that for every substructure S of Δ, if S has a set of at most n generators, $|\operatorname{Dom}(S)|$ is bounded by $f(n)$.

The following is Corollary 7.3.2 from [Hod93].

Proposition 2.2.8. *An ω-categorical structure is locally finite.*

Before stating an important theorem establishing an equivalence between uniformly locally finite homogeneous structures and ω-categorical structures with quantifier elimination, we show a useful property of uniformly locally finite homogeneous structures. In a word, their automorphism group is enough to locally generate their self-embedding monoid.

Proposition 2.2.9. *Let Δ a homogeneous uniformly locally finite structure, then we have the following:*

$$\overline{\langle \operatorname{Aut}(\Delta)\rangle_1} = \operatorname{Emb}(\Delta)$$

Proof. It is straightforward to prove that $\operatorname{Emb}(\Delta)$ is locally closed and $\operatorname{Aut}(\Delta) \subseteq \operatorname{Emb}(\Delta)$, so $\overline{\operatorname{Aut}(\Delta)} \subseteq \operatorname{Emb}(\Delta)$. Conversely, let $f \in \operatorname{Emb}(\Delta)$ and S be a finite subset of $\operatorname{Dom}(\Delta)$. Since Δ is uniformly locally finite, S is contained in a finite substructure S' of Δ. Then f is a partial isomorphism between S' and $f(S')$, which can be extended to an automorphism of Δ by homogeneity (see Proposition 3.2.30). Therefore, $f \in \overline{\operatorname{Aut}(\Delta)}$. □

Definition 2.2.10. A theory T on a signature τ has quantifier elimination when every τ-formulas is equivalent modulo T, to a quantifier free formula.

Theorem 2.2.11. *Let Γ be a countable structure on a finite signature. The following are equivalent:*

- Γ *is homogeneous and uniformly locally finite;*

- $\operatorname{Th}(\Gamma)$ *is ω-categorical and has quantifier elimination.*

Proof. See Corollary 7.4.2 in [Hod93]. □

Remark 2.2.12. Let B be any infinite set, and $c_1, \ldots, c_n$ be n constant symbols. The structures $(B; =)$ and more generally, $(B; c_1, \ldots, c_n)$ are ω-categorical and have quantifier elimination, since they are homogeneous by Remark 2.2.2, and are relational, so trivially locally finite.

We give a slight variant of Theorem 2.2.11 in a setting which does not require a finite signature anymore. It is Proposition 2.2 from [Cam90]:

Theorem 2.2.13. *Let Γ be an ω-categorical structure with a countable signature. The following are equivalent:*

- Γ *is homogeneous;*
- Γ *has quantifier elimination.*

2.3 Model-complete theories and Model Companions

Abraham Robinson introduced in the fifties many fundamental notions of model theory, particularly the notion of model-completeness and companions, in his attempt to understand elementary embeddings (i.e., embeddings which preserves first-order formulas) between models of a same theory. We give here the basic definitions and properties.

Definition 2.3.1. A map g between a τ-structure Γ and a τ-structure Δ is *elementary* if for all τ-formula $\phi(x_1, \dots, x_n)$, and for all $(a_1, \dots, a_n) \in \mathrm{Dom}(\Gamma)^n$, we have:

$$\Gamma \models \phi(a_1, \dots, a_n) \Rightarrow \Delta \models \phi(g(a_1), \dots, g(a_n))$$

Definition 2.3.2. A theory T is *model-complete* if every embedding between models of T is *elementary.* An ω-categorical structure is *model-complete* if its theory is.

The following two propositions can be found in [Hod93]:

Proposition 2.3.3. *Any theory which admits quantifier elimination is model-complete.*

Proof. Just note that any embedding between two models of a theory which admits quantifier elimination is elementary, since every first-order formula is equivalent to a quantifier-free formula, and g preserves every quantifier-free formula. $\square$

Corollary 2.3.4. *Any uniformly locally finite homogeneous structure over a countable signature is model-complete.*

Proof. We know by Theorem 2.2.11 that any uniformly locally finite homogeneous structure has quantifier elimination. $\square$

Definition 2.3.5. Let T be a theory on a signature τ. We say that a theory U on τ is a *model-companion* of T if the three following conditions hold:

1. U is model-complete;
2. every model of T embeds in a model of U;
3. every model of U embeds in a model of T.

A theory T does not necessarily have a model-companion, but it does if it is ω-categorical, as states the following theorem from D. Saracino (see [Sar73]):

Theorem 2.3.6. *Let T be an ω-categorical theory over a countable signature τ. Then T has an ω-categorical model-companion which is unique up to isomorphism.*

Remark 2.3.7. Given an ω-categorical theory T and its ω-categorical model-companion U, we will also call the unique countable model of U the model-companion on the unique model of T.

2.4 Ramsey Theory

We use Ramsey theory as a tool to prove the existence of canonical functions in a monoid which possesses other functions which are harder to use. Indeed, some homogeneous structures among which the vector space we shall study later, exhibit the so-called Ramsey property which is a generalization of the Ramsey property for sets applied to colouring of embeddings between substructures instead of sets.

This property allows us to locally generate from a given function a simpler and more symmetrical function called canonical. A unary function is canonical from a structure Δ to a structure Δ' whenever the images of two tuples of same type over Δ by f have same type over Δ'. It is important to note that the canonical function thus interpolates the starting function on a chosen finite set.

The following definitions and properties are given for structures with a countable signature containing functions, relations and/or constants.

Notation 2.4.1. We denote by $\binom{\mathfrak{B}}{\mathfrak{A}}$ the set of embeddings from $\mathfrak{A}$ to $\mathfrak{B}$.

Notation 2.4.2. When $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are structures with same signature, and $r \in \mathbb{N}$, we write $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ if for all $\chi\colon \binom{\mathfrak{C}}{\mathfrak{A}} \to \{1, \ldots, r\}$ there exists an $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that χ is monochromatic on $\{f \circ g \mid g \in \binom{\mathfrak{B}}{\mathfrak{A}}\}$.

Definition 2.4.3. A class $\mathcal{C}$ of finite structures has the *Ramsey property* if for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and $k \in \mathbb{N}$, there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$.

Definition 2.4.4. A structure is **Ramsey** if it is homogeneous and its age has the Ramsey property.

Proposition 2.4.5. *Let Δ be an totally ordered homogeneous structure. Then the following are equivalent.*

1. *Δ is Ramsey.*
2. *for every finite substructure $\mathfrak{B}$ of Δ and $r \in \mathbb{N}$, there exists a finite substructure $\mathfrak{C}$ of Δ such that for all non-isomorphic substructures $\mathfrak{A}_1, \ldots, \mathfrak{A}_l$ of $\mathfrak{B}$, and all $\chi_i\colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to \{1, \ldots, r\}$ there exists $e \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $|\chi_i(e \circ \binom{\mathfrak{B}}{\mathfrak{A}_i})| = 1$ for all $i \leq l$.*

Proof. The ($\Leftarrow$) is clear by definition of Ramsey. We now prove the ($\Rightarrow$) direction. For all $i \leq l$, let $\chi_i \colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to \{1, \dots, r\}$ be arbitrary. Since $\mathfrak{C}_l \to (\mathfrak{C}_{l-1})^{\mathfrak{A}_l}_r$, there exists an $e_l \in \binom{\mathfrak{C}_l}{\mathfrak{C}_{l-1}}$ with $|\chi_l(e_l \circ \binom{\mathfrak{C}_{l-1}}{\mathfrak{A}})| = 1$. Inductively, suppose that we have already defined $e_i \in \binom{\mathfrak{C}_l}{\mathfrak{C}_{i-1}}$ for an $i \in \{2, \dots, l\}$ such that for all $j \in \{i, \dots, l\}$ we have $|\chi_j(e_i \circ \binom{\mathfrak{C}_{i-1}}{\mathfrak{A}_j})| = 1$. Define $\chi'_{i-1} \colon \binom{\mathfrak{C}_{i-1}}{\mathfrak{A}_{i-1}} \to \{1, \dots, r\}$ by $\chi'_{i-1}(e) := \chi_{i-1}(e_i \circ e)$ for all $e \in \binom{\mathfrak{C}_{i-1}}{\mathfrak{A}_{i-1}}$. Since $\mathfrak{C}_{i-1} \to (C_{i-2})^{\mathfrak{A}_{i-1}}_r$ there exists an $f \in \binom{\mathfrak{C}_{i-1}}{\mathfrak{C}_{i-2}}$ such that $|\chi'_{i-1}(f \circ \binom{\mathfrak{C}_{i-2}}{\mathfrak{A}_{i-1}})| = 1$. It follows that $|\chi_{i-1}(e_i \circ f \circ \binom{\mathfrak{C}_{i-2}}{\mathfrak{A}_{i-1}})| = 1$. The inductive assumption implies that $|\chi_j(e_i \circ f \circ \binom{\mathfrak{C}_{i-2}}{\mathfrak{A}_j})| = 1$ for all $j \in \{i, \dots, l\}$. Hence, $|\chi_j(e_i \circ f \circ \binom{\mathfrak{C}_{i-2}}{\mathfrak{A}_j})| \leq 1$ for all $j \in \{i-1, \dots, l\}$. Define $e_{i-1} := e_i \circ f$. Then the map $e_1 \in \binom{\mathfrak{C}}{\mathfrak{B}}$ has the desired properties from the statement of the proposition. □

Definition 2.4.6. A *type condition* on a product of structures $\Gamma \times \Delta$ is a pair (p, q) where p, q are complete n-types on respectively Γ and Δ. A function $f \colon \mathrm{Dom}(\Gamma) \to \mathrm{Dom}(\Delta)$ *satisfies a type condition* (p, q) if for all n-tuples $(a_1, \dots, a_n)$ of type p, $(f(a_1), \dots, f(a_n))$ has type q.

A *behaviour* $\mathcal{T}$ *from* Γ *to* Δ is a set of type conditions on $\Gamma \times \Delta$ such that for all types p, q_1, q_2 of Γ:

$$(p, q_1) \in \mathcal{T} \wedge (p, q_2) \in \mathcal{T} \Rightarrow q_1 = q_2$$

A function f *has behaviour* $\mathcal{T}$ if f satisfies all the type conditions in $\mathcal{T}$.

Definition 2.4.7. A behaviour $\mathcal{T}$ is *complete* if for all types p of Γ, there exists a type q of Δ such that $(p, q) \in \mathcal{T}$. A function f is *canonical from* Γ *to* Δ if there exists a complete behaviour $\mathcal{T}$ such that f has behaviour $\mathcal{T}$.

Equivalently, given two structures Γ and Δ of domain V, a function $f \colon V \to V$ is *canonical from* Γ *to* Δ if the type of the image of an n-tuple $(a_1, \dots, a_n)$ only depends on the type of $(a_1, \dots, a_n)$, i.e., for all $a_1, \dots, a_n, b_1, \dots, b_n \in \mathrm{Dom}(\Gamma)$:

$$\mathrm{tp}_\Gamma(a_1, \dots, a_n) = \mathrm{tp}_\Gamma(b_1, \dots, b_n) \Rightarrow \mathrm{tp}_\Delta(f(a_1), \dots, f(a_n)) = \mathrm{tp}_\Delta(f(b_1), \dots, f(b_n))$$

Definition 2.4.8. The type of an n-tuple $(a_1, \dots, a_n)$ is an *injective n-type* if $a_i \neq a_j$ for all $1 \leq i \neq j \leq n$.

Remark 2.4.9. To prove that a function is canonical, it is enough to check the property for injective types.

Notation 2.4.10. Let D be a set, $f \colon D \to D$ be a map, n be an integer, and R, S be two n-ary relations on D. We denote by $f(R) = S$ the fact that for every n-tuple $(a_1, \dots, a_n)$ in R, the n-tuple $(f(a_1), \dots, f(a_n))$ is in S.

Lemma 2.4.11. *Let f and g be two canonical operations from a structure Γ to locally finite homogeneous structure Δ. Assume that f and g have the same behaviour. Then together with the automorphisms of Δ, they locally generate each other, i.e:*

$$f \in \overline{\langle \{g\} \cup \mathrm{Aut}(\Delta)\rangle_1} \text{ and } g \in \overline{\langle \{f\} \cup \mathrm{Aut}(\Delta)\rangle_1}$$

Proof. Let $S = \{a_1, \ldots, a_n\}$ be a tuple of elements of $\mathrm{Dom}(\Gamma)$. Since f and g have the same behaviour, we have $\mathrm{tp}_\Delta(f(a_1), \ldots, f(a_n)) = \mathrm{tp}_\Delta(g(a_1), \ldots, g(a_n))$. Let S_1 be the finite substructure of Δ generated by $\{f(a_i) \mid i \leq n\}$, and let S_2 be the finite substructure of Δ generated by $\{g(a_i) \mid i \leq n\}$. We define $h\colon S_1 \to S_2$ such that $h(f(a_i)) = g(a_i)$ for all $i \leq n$, and h is defined by natural extension for the remaining terms in S_1. Note that h is a local isomorphism between two substructures of Δ. Since Δ is homogeneous, there exists $\alpha \in \mathrm{Aut}(\Delta)$ such that $h \subset \alpha$. So $\alpha \circ f{\upharpoonright}S = g{\upharpoonright}S$, and g together with $\mathrm{Aut}(\Delta)$ locally generate f. ☐

Proposition 2.4.12. *Let Δ be an ω-categorical totally ordered Ramsey structure with domain D, let Γ be an ω-categorical structure with domain B, and let $f\colon D \to B$ be an operation. Then for all finite subsets S of D there is an automorphism α of Δ so that the operation $x \mapsto f(\alpha(x))$ is canonical on S as a map from Δ to Γ.*

Proof. Let S be a finite subset of D and let $\mathfrak{S}$ be the substructure of Δ generated by $\mathfrak{S}$. Since Δ is ω-categorical, it is locally finite by Proposition 2.2.8, so $\mathfrak{S}$ is finite. Let c be the cardinal of $\mathrm{Dom}(\mathfrak{S})$ and let r be the number of distinct n-types over Γ (this number is finite since Γ is ω-categorical using Theorem 2.1.24). Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mathfrak{l}$ be all the non-isomorphic substructures of $\mathfrak{S}$. By Proposition 2.4.5, there exists a finite substructure $\mathfrak{C}$ of Δ such that for all $\chi_i\colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to \{1, \ldots, r\}$ there exists $e \in \binom{\mathfrak{C}}{\mathfrak{S}}$ such that $|\chi_i(e \circ \binom{\mathfrak{S}}{\mathfrak{A}_i})| = 1$ for all $i \leq l$. Since Δ is homogeneous, and since e is a local isomorphism between $\mathfrak{S}$ and $e(\mathfrak{S}) \subseteq \mathfrak{C}$, there exists $\alpha \in \mathrm{Aut}(\Delta)$ which extends e. For $i \leq l$, we define $\chi_i\colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to \{1, \ldots, r\}$ as follows: for all $g \in \binom{\mathfrak{C}}{\mathfrak{A}_i}$, $\chi_i(g) = \mathrm{tp}_\Gamma(f \circ g(a_1), \ldots, f \circ g(a_k))$ where $\{a_1, \ldots, a_k\} = \mathrm{Dom}(\mathfrak{A}_i)$, and $a_j < a_{j+1}$ for all $j \leq k - 1$. Note that the fact that Δ is totally ordered makes χ_i well-defined.

Let n be an integer, and $\overline{a}, \overline{a}'$ be two n-tuples of elements of S with same type over Δ. We show that $f \circ \alpha$ is canonical on S as an operation from Δ to Γ. Let $\mathfrak{A}$ be the substructure of $\mathfrak{S}$ generated by $\overline{a}$, and $\mathfrak{A}'$ be the substructure of $\mathfrak{S}$ generated by $\overline{a}'$. Since $\overline{a}, \overline{a}'$ have same type over Δ, $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic, hence, there exists $t \leq l$ such that $\mathfrak{A}_t$ is isomorphic to $\mathfrak{A}$ and $\mathfrak{A}'$. Let h be an isomorphism from $\mathfrak{A}_t$ to $\mathfrak{A}$, and h' be an isomorphism from $\mathfrak{A}_t$ to $\mathfrak{A}'$. Note that $e \circ h$ and $e \circ h'$ are both embeddings from $\mathfrak{A}_t$ to $\mathfrak{C}$. Consequently, since $|\chi_t(e \circ \binom{\mathfrak{S}}{\mathfrak{A}_t})| = 1$ by assumption, we have:

$$\mathrm{tp}_\Gamma(f \circ e \circ h(b_1), \ldots, f \circ e \circ h(b_k)) = \mathrm{tp}_\Gamma(f \circ e \circ h'(b_1), \ldots, f \circ e \circ h'(b_k))$$

where $\{b_1, \ldots, b_k\} = \mathrm{Dom}(\mathfrak{A}_t)$, and $b_j < b_{j+1}$ for all $j \leq k - 1$. And since α extends e, we have:

$$\mathrm{tp}_\Gamma(f \circ \alpha \circ h(b_1), \ldots, f \circ \alpha \circ h(b_k)) = \mathrm{tp}_\Gamma(f \circ \alpha \circ h'(b_1), \ldots, f \circ \alpha \circ h'(b_k))$$

which implies:

$$\mathrm{tp}_\Gamma(f \circ \alpha(a_1), \ldots, f \circ \alpha(a_n)) = \mathrm{tp}_\Gamma(f \circ \alpha(a'_1), \ldots, f \circ \alpha(a'_n))$$

which concludes the proof. ☐

The following is Proposition 10 from [BPT11].

Lemma 2.4.13. *Let Δ be totally ordered Ramsey, and let $c_1, \ldots, c_n \in \Delta$. Then $(\Delta, c_1, \ldots, c_n)$ is Ramsey as well.*

The following is known as König's Lemma, and will be used in various proofs in this thesis. In one word, we will consider (potentially a part of) the finitely many distinct n-types of an ω-categorical structure as the n-th level of a tree, in order to prove that certain operation exists.

Theorem 2.4.14. *If $\mathcal{T}$ is a tree with infinitely many nodes such that every node has finite degree (that is, each node is adjacent to only finitely many other nodes) then $\mathcal{T}$ contains an infinitely long simple path, that is, a path with no repeated nodes.*

Theorem 2.4.15 (Canonization Lemma)**.** *Let Δ be an ω-categorical totally ordered Ramsey structure and let Γ be an ω-categorical structure. Let $f\colon \mathrm{Dom}(\Delta) \to \mathrm{Dom}(\Gamma)$, and $c_1, \ldots, c_n \in \mathrm{Dom}(\Delta)$. Then there is a function in*

$$S = \overline{\{\beta \circ f \circ \alpha \mid \beta \in \mathrm{Aut}(\Gamma), \alpha \in \mathrm{Aut}(\Delta, c_1, \ldots, c_n)\}}$$

which is canonical as a unary function from $(\Delta, c_1, \ldots, c_n)$ to Γ, where $(\Delta, c_1, \ldots, c_n)$ is the structure Δ with added constants $c_1, \ldots, c_n$.

Proof. First note that by Lemma 2.4.13, $\Delta' := (\Delta, c_1, \ldots, c_n)$ is also an ω-categorical Ramsey structure. By Proposition 2.4.12, for all finite subsets S of $D := \mathrm{Dom}(\Delta')$ there is an automorphism α of Δ' so that the operation $x \mapsto f(\alpha(x))$ is canonical on S as a map from Δ' to Γ.

Let $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite substructures of Δ' such that:

$$\bigcup_{n\mathbb{N}} \mathrm{Dom}(\mathfrak{S}_n) = D$$

Such a sequence exists since Δ' is ω-categorical, so by Proposition 2.2.8, it is locally finite, i.e., every finitely generated substructure is finite.

We denote by S_n the domain of $\mathfrak{S}_n$. Consider the following infinite tree $\mathcal{T}$ whose vertices lie on levels $1, 2, \ldots$. The vertices at the n-th level are all the complete types over Γ of tuples of the form $(\delta(f(\beta(s_1^n))), \ldots, \delta(f(\beta(s_k^n))))$ with $S_n =: \{s_1^n, \ldots, s_k^n\}$ and $s_1^n < \cdots < s_k^n$, β being an automorphism of Δ' such that $f \circ \beta$ is canonical on S_n as a map from Δ' to Γ, and $\delta \in \mathrm{Aut}(\Gamma)$. We then say that such a vertex *corresponds* to the map $\delta \circ f \circ \beta$. Note that if a map $\delta \circ f \circ \beta$ *corresponds* to a vertex, then $\delta' \circ f \circ \beta$ also *corresponds* to this vertex for all $\delta' \in \mathrm{Aut}(\Gamma)$, since types are preserved by automorphisms of Γ. We say that a node N (at the n-th level) *corresponding* to a map $\delta \circ f \circ \beta$ is a descendant of a node M *corresponding* to a map $\delta' \circ f \circ \gamma$ (at the m-th level) if:

$$\mathrm{tp}_\Gamma(\delta(f(\beta(s_1^n))), \ldots, \delta(f(\beta(s_k^n)))) \subseteq \mathrm{tp}_\Gamma(\delta'(f(\gamma(s_1^m))), \ldots, \delta'(f(\gamma(s_l^m))))$$

Note that $\mathcal{T}$ has finitely many vertices at each level since Γ is ω-categorical, and so by Theorem 2.1.24, there is a finite number of distinct n-types for all n. We now prove that $\mathcal{T}$ has at least one vertex at each level by induction, but this is straightforward by assumption since for all finite subsets S of $D := \mathrm{Dom}(\Delta')$ there is an automorphism α of Δ' so that the operation $x \mapsto f(\alpha(x))$ is canonical on S as a map from Δ' to Γ. Hence, by König's lemma 2.4.14, $\mathcal{T}$ has an infinite branch $(N_0, \dots, N_n, \dots)$. For $n \geq 0$, we choose a map $\delta_n \circ f \circ \beta_n$ *corresponding* to the node N_n such that $\delta_n f \circ \beta_n \upharpoonright S_n = \delta_{n+1} \circ f \circ \beta_{n+1} \upharpoonright S_n$ for all n. This is possible by Proposition 2.1.26 since Γ is ω-categorical, and by the fact that the set of operations *corresponding* to a node is stable by left composition with automorphisms of Γ.

We now define $g := \cup_{n \geq \mathbb{N}} \delta_n \circ f \circ \beta_n \upharpoonright S_n$. The operation g is well defined and canonical from Δ' to Γ and clearly belongs to $\overline{\{\beta \circ f \circ \alpha \mid \beta \in \mathrm{Aut}(\Gamma), \alpha \in \mathrm{Aut}(\Delta, c_1, \dots, c_n)\}}$ by definition.

$\square$

We state here a corollary that will be of much use in this thesis. Note that a similar statement exists for functions of arbitrary arity.

Corollary 2.4.16. *Let Δ be an ω-categorical structure which has an expansion $(\Delta; <)$ which is ω-categorical totally ordered Ramsey and let f be any operation from $\mathrm{Dom}(\Delta) \to \mathrm{Dom}(\Delta)$, and $c_1, \dots, c_n \in \mathrm{Dom}(\Delta)$ be n constants. Then $\{f\} \cup \mathrm{Aut}(\Delta)$ locally generates a function that*

- *agrees with f on $\{c_1, \dots, c_n\}$, and*
- *is canonical as a k-ary function from $(\Delta; <, c_1, \dots, c_n)$ to Δ.*

Proof. Apply Theorem 2.4.15 to the structures $(\Delta; <, c_1, \dots, c_n)$ and Δ. Note also that $\mathrm{Aut}(\Delta; <, c_1, \dots, c_n) \subseteq \mathrm{Aut}(\Delta)$. $\square$

Chapter 3

The Countable Boolean Vector Space

3.1 General Linear Algebra

A vector space is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. Any vector space V over a field $\mathbb{F}$ must satisfy the following axioms:

- Associativity of addition: for all $u, v, w \in V$, $u + (v + w) = (u + v) + w$;
- Commutativity of addition: for all $u, v, w \in V$, $u + v = v + u$;
- Identity element of addition: $\exists x_0 \in V$ such that for all u, $x_0 + u = u$;
- Identity element of scalar multiplication: for all $v \in V$, $1_{\mathbb{F}}.v = v$;
- Distributivity of scalar multiplication over addition: for all $u, v \in V$, for all $a \in \mathbb{F}$, $a(u + v) = au + av$;
- Inverse elements of addition: for all $v \in V$, $\exists v' \in V$ such that $v + v' = 0$;
- Compatibility of scalar multiplication with field multiplication: for all $a, b \in \mathbb{F}$, and for all $v \in V$, $a(bv) = (ab)v$;
- Distributivity of scalar multiplication with field addition: for all $a, b \in \mathbb{F}$, for all $v \in V$, $(a + b)v = av + bv$.

Example 3.1.1. *The simplest example of a vector space over a field $\mathbb{F}$ is the set of n-tuples of elements of $\mathbb{F}$, equipped with the component-wise addition and multiplication.*

A more elaborate example, as we stated in the introduction, is the infinite dimension vector space over the two-element field $\mathbb{F}_2$. In this case, we only have two scalars: 0 and 1, and the vectors are countable tuples of $\mathbb{F}_2$.

Definition 3.1.2. Let $\mathcal{V}$ be an infinite vector space over a field $\mathbb{F}$, finite or infinite.

Let $(a, b_1, \ldots, b_n)$ be vectors of $\mathcal{V}$. We say that a is *a linear combination of* $b_1, \ldots, b_n$ if there are $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that $a = \sum_{i=1}^{n} \lambda_i b_i$.

A family of vectors of $\mathcal{V}$ is *linearly independent* if none of them can be written as a linear combination of finitely many other vectors in the family.

A family $(b_n)_{n\in\mathbb{N}}$ of vectors of $\mathcal{V}$ *spans* $D \subseteq \mathcal{V}$ if:

$$D = \{x \in V \mid \exists n,\ \exists i_1 < \cdots < i_n,\ \exists \overline{\alpha} \in \mathbb{F}^n,\ x = \sum_{j=1}^{n} \alpha_j b_{i_j}\}$$

Note that if $(b_n)_{n\in\mathbb{N}}$ is linearly independent, the tuple $\overline{\alpha}$ which appears in the decomposition of x with respect to the family $(b_n)_{n\in\mathbb{N}}$ is unique for every $x \neq 0$.

Any linearly independent family of vectors of a vector space $\mathcal{V}$ which spans $\mathcal{V}$ is called a *basis* of $\mathcal{V}$.

We say that $\mathcal{V}$ has *dimension* n if it has a basis of size n. We say that $\mathcal{V}$ has *infinite dimension* if it has a countable basis of vectors.

Vector spaces exhibit the following crucial property: any free family can be completed to a basis of the space, whether the dimension is finite or infinite.

Fact 3.1.3. *Let $\mathcal{V}$ be a countably infinite vector space over a field $\mathbb{F}$. Let I be a set of indices and $(b_i)_{i\in I}$ be a linearly independent family of vectors of $\mathcal{V}$. Then there exists a set J such that $I \cap J = \emptyset$ and a family of vectors $(b_j)_{j\in J}$ such that $(b_i)_{i\in I\cup J}$ is a basis of $\mathcal{V}$. We say that we* complete *the family $(b_i)_{i\in I}$ to a basis of $\mathcal{V}$.*

Following a model theoretic approach, and in order to list the unary canonical functions over $(V;+)$, we start by defining basic relations which will be the ever-present tools for defining every other definable relation over $(V;+)$. Some of these relations isolate types hereafter called “important” types. These “important” types exhibit the property that their image by a canonical function over $(V;+)$ is enough to characterize the behaviour of this function.

From now, when we do not specify otherwise, we will only be considering the countably infinite vector space over $\mathbb{F}_2$.

Notation 3.1.4. In the following, we denote the countably infinite $\mathbb{F}_2$-vector space by $(V;+)$, where $+$ is a function symbol.

3.2 Outstanding Relations and Homogeneity of the Vector Space

Following a model theoretic approach, and in order to understand the lattice of endomorphism monoid of reducts and later list the unary canonical functions over $(V;+)$, we start by defining basic relations which will be the ever-present tools for defining every other definable relation over $(V;+)$. Some of these relations isolate types hereafter called "outstanding" types. These "outstanding" types exhibit the property that their image by a canonical function over $(V;+)$ is enough to characterize the behaviour of this function. In the following, we will sometimes assimilate a type to a formula which isolates it, since in an ω-categorical structure, every type is isolated.

From now, when we do not specify otherwise, we will only be considering the countably infinite vector space over $\mathbb{F}_2$.

3.2.1 The Eq_i relations

We start by describing the most natural class of definable relations over $(V;+)$: the class $(\mathrm{Eq}_k)_{k\in\mathbb{N}}$. A k-tuple of elements of V is in Eq_k if their sum is 0. It is clear that every relation Eq_k is not a type of $(V;+)$ since one element can appear twice in a tuple of Eq_k.

Definition 3.2.1. For $k \geq 1$, let Eq_k denote the relation $\{(x_1,\dots,x_k) \in V^k \mid x_1+\cdots+x_k = 0\}$.

Note that every Eq_k is primitive positive definable by Eq_3.

Lemma 3.2.2. *For all $i \geq 3$, $\mathrm{Eq}_i \in \langle (V;\mathrm{Eq}_3)\rangle_{\mathrm{pp}}$, i.e., for all $i \geq 3$, Eq_i has a primitive positive definition over $(V;\mathrm{Eq}_3)$.*

Proof. We prove that for all $n \geq 3$, if Eq_n has a primitive positive definition over $(V;\mathrm{Eq}_3)$, then Eq_{n+1} also does. Let n be an integer greater than 3. Then for all $\overline{x}$, we have:

$$\mathrm{Eq}_{n+1}(x_1,\dots,x_{n+1}) \Leftrightarrow \exists u.\, \mathrm{Eq}_n(x_1,\dots,x_{n-1},u) \wedge \mathrm{Eq}_3(u,x_n,x_{n+1})$$

□

Less intuitively, every Eq_{2k} is primitive positive definable by Eq_4, for all $k \geq 1$.

Lemma 3.2.3. *For all $i \geq 1$, $\mathrm{Eq}_{2i} \in \langle (V;\mathrm{Eq}_4)\rangle_{\mathrm{pp}}$, i.e., for all $i \geq 1$, Eq_{2i} has a primitive positive definition over $(V;\mathrm{Eq}_4)$.*

Proof. We prove that for all $n \geq 2$, if Eq_{2n} has a primitive positive definition over $(V;\mathrm{Eq}_4)$, then Eq_{2n+2} also does. Let n be an integer greater than 2. Then for all $\overline{x}$, we have:

$$\mathrm{Eq}_{2n+2}(x_1,\dots,x_{2n+2}) \Leftrightarrow \exists u.\, \mathrm{Eq}_{2n}(x_1,\dots,x_{2n-1},u) \wedge \mathrm{Eq}_4(u,x_{2n},x_{2n+1},x_{2n+2})$$

□

3.2.2 The $\mathrm{Eq}_i^{\neq 0}$ relations

We now give a non-zero version of $(\mathrm{Eq}_i)_{i\geq 1}$:

Definition 3.2.4. For $i \geq 1$, we define the relation $\mathrm{Eq}_i^{\neq 0}$ as follows: $\mathrm{Eq}_i^{\neq 0}(x_1, \dots, x_i)$ if and only if $\mathrm{Eq}_i(x_1, \dots, x_i)$ and $x_j \neq 0$ for all $1 \leq j \leq i$.

Remark 3.2.5. Note that $\mathrm{Eq}_i^{\neq 0}$ is still not a type if $i \geq 4$. Indeed, $\mathrm{Eq}_6^{\neq 0}(x_1, \dots, x_6)$ is consistent with either $x_1 + x_2 + x_3 = 0$ or $x_1 + x_2 + x_3 \neq 0$, for instance. Or $\mathrm{Eq}_4^{\neq 0}(x_1, x_2, x_3, x_4)$ with either $x_1 = x_2 = x_3 = x_4$ or $x_1 = x_2 \neq x_3 = x_4$ for instance.

Lemma 3.2.6. *We have the following:* $\mathrm{Eq}_i^{\neq 0} \in \langle (V; \mathrm{Eq}_3^{\neq 0}) \rangle_{\mathrm{pp}}$ *for all* $i \geq 2$*, i.e., for all* $i \geq 2$*,* $\mathrm{Eq}_i^{\neq 0}$ *has a primitive positive definition over* $(V; \mathrm{Eq}_3^{\neq 0})$*.*

Proof. By an inductive argument over i. First note that for all $\overline{x} \in (V \setminus \{0\})^3$:

$$\mathrm{Eq}_3^{\neq 0}(x_1, x_2, x_3) \Leftrightarrow \exists u.\, \mathrm{Eq}_5^{\neq 0}(x_1, x_2, x_3, u, u)$$

Then, if $\mathrm{Eq}_i^{\neq 0}$ has a pp-definition on $(V; \mathrm{Eq}_5^{\neq 0})$ for some $i \geq 3$, we have for all $\overline{x} \in (V \setminus \{0\})^{i+1}$:

$$\mathrm{Eq}_{i+1}^{\neq 0}(x_1, \dots, x_{i+1}) \Leftrightarrow \exists u, v.\, \mathrm{Eq}_i^{\neq 0}(x_1, \dots, x_{i-2}, u, v) \wedge \mathrm{Eq}_5^{\neq 0}(u, v, x_{i-1}, x_i, x_{i+1})$$

Consequently, $\mathrm{Eq}_i^{\neq 0} \in \langle (V; \mathrm{Eq}_5^{\neq 0}) \rangle_{\mathrm{pp}}$ for all $i \geq 2$.

Now we prove that $\mathrm{Eq}_5^{\neq 0} \in \langle (V; \mathrm{Eq}_3^{\neq 0}) \rangle_{\mathrm{pp}}$. To see this, we first define an ad hoc relation R as follows:

$$R(x, y, z, t, u) \Leftrightarrow \exists a, b.\, \mathrm{Eq}_3^{\neq 0}(x, y, a) \wedge \mathrm{Eq}_3^{\neq 0}(a, z, b) \wedge \mathrm{Eq}_3^{\neq 0}(b, t, u)$$

Note that R is distinct from $\mathrm{Eq}_5^{\neq 0}$ since, for instance, $(a, a, a+b, a, b) \notin R$. Now, we only have to check that the following formula does the job:

$$\begin{aligned} \mathrm{Eq}_5^{\neq 0}(x, y, z, u, v) \Leftrightarrow \exists t_1, t_1', t_2, t_2', t_3, t_3', t_4, t_4', t_5, t_5'.\, & \mathrm{Eq}_3^{\neq 0}(x, t_1, t_1') \\ & \wedge \mathrm{Eq}_3^{\neq 0}(y, t_2, t_2') \wedge \mathrm{Eq}_3^{\neq 0}(z, t_3, t_3') \\ & \wedge \mathrm{Eq}_3^{\neq 0}(u, t_4, t_4') \wedge \mathrm{Eq}_3^{\neq 0}(v, t_5, t_5') \\ & \wedge R(t_1, t_2, t_3, t_4, t_5) \wedge R(t_1', t_2', t_3', t_4', t_5') \end{aligned}$$

Note that thanks to the use of the R relation, the tuple $(a, a, a+b, a, b)$ now belongs to the defined relation. □

Lemma 3.2.7. *We have the following:* $\mathrm{Eq}_{2i}^{\neq 0} \in \langle (V; \mathrm{Eq}_4^{\neq 0}) \rangle_{\mathrm{pp}}$ *for all* $i \geq 1$*, i.e., for all* $i \geq 1$*,* $\mathrm{Eq}_{2i}^{\neq 0}$ *has a primitive positive definition over* $(V; \mathrm{Eq}_4^{\neq 0})$*.*

Proof. By an inductive argument over i. First note that for all $\overline{x} \in (V \setminus \{0\})^4$:

$$\mathrm{Eq}_4^{\neq 0}(x_1, x_2, x_3, x_4) \Leftrightarrow \exists u.\, \mathrm{Eq}_6^{\neq 0}(x_1, x_2, x_3, x_4, u, u)$$

Assume now that $\mathrm{Eq}_{2i}^{\neq 0}$ has a pp-definition on $(V; \mathrm{Eq}_6^{\neq 0})$ for some $i \geq 1$. For all $\overline{x} \in (V \setminus \{0\})^{2i+2}$, we have:

$$\mathrm{Eq}_{2i+2}^{\neq 0}(x_1, \ldots, x_{2i+2}) \Leftrightarrow \exists u, v.\, \mathrm{Eq}_{2i}^{\neq 0}(x_1, \ldots, x_{i-2}, u, v) \wedge \mathrm{Eq}_6^{\neq 0}(u, v, x_{2i-1}, x_{2i}, x_{2i+1}, x_{2i+2})$$

Now we prove that $\mathrm{Eq}_6^{\neq 0} \in \langle (V; \mathrm{Eq}_4^{\neq 0}) \rangle_{\mathrm{pp}}$. To see this, we first define an ad hoc relation R as follows:

$$R(x, y, z, t, u, v) \Leftrightarrow \exists a.\, \mathrm{Eq}_4^{\neq 0}(x, y, z, a) \wedge \mathrm{Eq}_4^{\neq 0}(a, t, u, v)$$

Note that R is distinct from $\mathrm{Eq}_6^{\neq 0}$ since, for instance, $(a, a, a, a, a, a) \notin R$. Now, we only have to check that the following formula does the job:

$$\begin{aligned}\mathrm{Eq}_6^{\neq 0}(x, y, z, u, v, w) \Leftrightarrow \exists t_1, t_1', t_2, t_2', t_3, t_3'.\, &\mathrm{Eq}_4^{\neq 0}(x, y, t_1, t_1') \\ &\wedge \mathrm{Eq}_4^{\neq 0}(z, u, t_2, t_2') \wedge \mathrm{Eq}_4^{\neq 0}(v, w, t_3, t_3') \\ &\wedge R(t_1, t_2, t_3, t_1', t_2', t_3')\end{aligned}$$

Note that thanks to the use of the R relation, the tuple (a, a, a, a, a, a) now belongs to the defined relation. □

3.2.3 The $\mathrm{Eq}_i^{\mathrm{inj}}$ relations

We now give an injective version of the relations Eq_i.

Definition 3.2.8. For $n \geq 1$, we define the relation $\mathrm{Eq}_n^{\mathrm{inj}}$ as follows:

$\mathrm{Eq}_n^{\mathrm{inj}}(x_1, \ldots, x_n)$ if and only if $\mathrm{Eq}_n(x_1, \ldots, x_n)$ and $x_i \neq x_j$ for all $1 \leq j < i \leq n$.

Remark 3.2.9. Note that $\mathrm{Eq}_i^{\mathrm{inj}}$ is still not a type if $i \geq 4$. Indeed, $\mathrm{Eq}_4^{\mathrm{inj}}(x_1, \ldots, x_4)$ is consistent with either $x_1 + x_2 + x_3 = 0$ and $x_4 = 0$, or $x_1 + x_2 + x_3 \neq 0$, for instance.

Lemma 3.2.10. *We have the following:* $\mathrm{Eq}_{2i}^{\mathrm{inj}} \in \langle (V; \mathrm{Eq}_4^{\mathrm{inj}}) \rangle_{\mathrm{pp}}$ *for all* $i \geq 1$*, i.e., for all* $i \geq 1$*,* $\mathrm{Eq}_{2i}^{\mathrm{inj}}$ *has a primitive positive definition over* $(V; \mathrm{Eq}_4^{\mathrm{inj}})$*.*

Proof. First note that $\neq$ is pp-definable in $(V; \mathrm{Eq}_4^{\mathrm{inj}})$ since $x \neq y$ if and only if there exist u, v s.t. $\mathrm{Eq}_4^{\mathrm{inj}}(x, y, u, v)$. By an inductive argument over n. Assume that $\mathrm{Eq}_{2n}^{\mathrm{inj}}$ has a pp-definition on $(V; \mathrm{Eq}_4^{\mathrm{inj}})$ for some $n \geq 1$. For all $\overline{x} \in V^{2n+2}$, we have:

$$\begin{aligned}\mathrm{Eq}_{2n+2}^{\mathrm{inj}}(x_1, \ldots, x_{2n+2}) \Leftrightarrow \exists \overline{u}.\, &\mathrm{Eq}_{2n}^{\neq 0}(x_1, \ldots, x_{2n-1}, u_1) \wedge \mathrm{Eq}_4^{\mathrm{inj}}(u_1, x_{2n}, u_2, u_3) \\ &\wedge \mathrm{Eq}_4^{\mathrm{inj}}(u_2, u_3, x_{2n+1}, x_{2n+2}) \wedge \bigwedge_{i<j} x_i \neq x_j\end{aligned}$$

□

Lemma 3.2.11. *We have the following:* $\mathrm{Eq}_{2i} \in \langle (V; \mathrm{Eq}_4^{\mathrm{inj}}) \rangle_{\mathrm{pp}}$ *for all* $i \geq 1$*, i.e.,* Eq_{2i} *has a primitive positive definition over* $(V; \mathrm{Eq}_4^{\mathrm{inj}})$*.*

Proof. First note that for all $x, y, z, t \in V$:

$$\begin{aligned} \mathrm{Eq}_4(x, y, z, t) \Leftrightarrow \exists \bar{t}, \bar{u}, \bar{v}, \bar{w}.\, & \mathrm{Eq}_4^{\mathrm{inj}}(t, t_1, t_2, t_3) \wedge \mathrm{Eq}_4^{\mathrm{inj}}(x, x_1, x_2, x_3) \\ & \wedge \mathrm{Eq}_4^{\mathrm{inj}}(y, y_1, y_2, y_3) \wedge \mathrm{Eq}_4^{\mathrm{inj}}(z, z_1, z_2, z_3) \\ & \wedge \mathrm{Eq}_{12}^{\mathrm{inj}}(t_1, t_2, t_3, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \end{aligned}$$

But since $\mathrm{Eq}_{12}^{\mathrm{inj}} \in \langle (V; \mathrm{Eq}_4^{\mathrm{inj}}) \rangle_{\mathrm{pp}}$ by Lemma 3.2.10, Eq_4 is pp-definable over $(V; \mathrm{Eq}_4^{\mathrm{inj}})$. Hence by Lemma 3.2.3, $\mathrm{Eq}_{2i} \in \langle (V; \mathrm{Eq}_4^{\mathrm{inj}}) \rangle_{\mathrm{pp}}$ for all $i \geq 1$. ☐

3.2.4 The Ieq_i and Ind_i relations

In order to characterize canonical functions over $(V; +)$, we attempt to isolate important types of $(V; +)$. The free family of size k is the simplest k-type of $(V; +)$. We denote it by Ind_k. Contrariwise, as we previously showed, the relation Eq_k is not a type. We need to streamline this relation to a new relation Ieq_k in order to get a type. A k-tuple of elements of V is in Ieq_k if all its elements are non zero, and their sum is 0, and every strict sub-family of elements of the tuple is free.

Definition 3.2.12. We define the following relations:

- Ind_k denotes the k-ary relation which contains exactly the linearly independent k-tuples of elements of V. In other words, $(x_1, \ldots, x_k) \in \mathrm{Ind}_k$ if and only if for all $l \leq k$ and all $1 \leq i_1 < \cdots < i_l \leq k$, we have: $x_{i_1} + \cdots + x_{i_l} \neq 0$. In particular, a tuple in Ind_k does not contain 0.
- Ieq_k denotes the k-ary relation which contains exactly the k-tuples $(x_1, \ldots, x_k)$ such that $x_1 + \cdots + x_k = 0$ and any strict subfamily of $(x_1, \ldots, x_k)$ is linearly independent. In other words, $(x_1, \ldots, x_k) \in \mathrm{Ieq}_k$ if and only if $x_1 + \cdots + x_k = 0$ and for all $l \leq k-1$ and all $1 \leq i_1 < \cdots < i_l \leq k$, we have: $x_{i_1} + \cdots + x_{i_l} \neq 0$. In particular, a tuple in Ieq_k does not contain 0.

Remark 3.2.13. Note that Ieq and Ind clearly isolate a complete type.

Remark 3.2.14. Note that for $n = 3$, Ieq_3 clearly equals $\mathrm{Eq}_3^{\neq 0}$ and $\mathrm{Eq}_3^{\mathrm{inj}}$, but this is not true for bigger n.

Corollary 3.2.15. *We have the following:* $\mathrm{Eq}_i^{\neq 0} \in \langle (V; \mathrm{Ieq}_3) \rangle_{\mathrm{pp}}$ *for all* $i \geq 2$*.*

Proof. Combine Remark 3.2.14 with Lemma 3.2.6. ☐

Note that since $(a, a, a, a) \in \mathrm{Eq}_4^{\neq 0}$, the relation Ieq_4 is strictly contained in $\mathrm{Eq}_4^{\neq 0}$. Nevertheless, we show that:

Lemma 3.2.16. *We have the following:* $\mathrm{Eq}_4^{\neq 0} \in \langle (V; \mathrm{Ieq}_4) \rangle_{\mathrm{pp}}$

Proof. We just have to check that the following formula does the job:

$$\begin{aligned}\mathrm{Eq}_4^{\neq 0}(x,y,z,t) \Leftrightarrow &\exists x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,t_1,t_2,t_3.\, \mathrm{Ieq}_4(x,x_1,x_2,x_3)\\ &\wedge \mathrm{Ieq}_4(y,y_1,y_2,y_3) \wedge \mathrm{Ieq}_4(z,z_1,z_2,z_3) \wedge \mathrm{Ieq}_4(t,t_1,t_2,t_3)\\ &\wedge \mathrm{Ieq}_4(x_1,y_1,z_1,t_1) \wedge \mathrm{Ieq}_4(x_2,y_2,z_2,t_2)\\ &\wedge \mathrm{Ieq}_4(x_3,y_3,z_3,t_3) \wedge \mathrm{Ieq}_4(x_4,y_4,z_4,t_4)\end{aligned}$$

□

Corollary 3.2.17. *We have the following:* $\mathrm{Eq}_{2i}^{\neq 0} \in \langle (V;\mathrm{Ieq}_4)\rangle_{\mathrm{pp}}$ *for all* $i \geq 2$.

Proof. Combine Lemma 3.2.16 and Lemma 3.2.7. □

Lemma 3.2.18. *Let i be an integer greater than 3, and let $j < i$ be a smaller integer. Then for all $\overline{x} \in V^j$, we have:*

$$\mathrm{Ind}_j(x_1,\dots,x_j) \Leftrightarrow \exists x_{j+1}\dots x_i.\, \mathrm{Ieq}_i(x_1,\dots,x_i)$$

Hence, Ind_j *has a primitive positive definition in* $(V;\mathrm{Ieq}_i)$.

Proof. The proof is straightforward by definition of Ind_j and Ieq_i. □

As it was the case for Eq_i, Ieq_{2j+1} pp-defines Ieq_i for all $i \geq 2$. Likewise for Ieq_{2j} which pp-defines Ieq_{2i} for all $i \geq 1$.

Lemma 3.2.19. *Let $i_0 \neq 0$ be a fixed natural number. We have the following:*

- *for all $j \geq 3$, Ieq_j has a primitive positive definition over $(V;\mathrm{Ieq}_{2i_0+1},\mathrm{Ind}_{j-1})$;*
- *for all $j \geq 1$, Ieq_{2j} has a primitive positive definition over $(V;\mathrm{Ieq}_{2i_0+2},\mathrm{Ind}_{2j-1})$.*

Remark 3.2.20. It is important to note that in fact Ieq_3 pp-defines every Ieq_i for all $i \geq 3$, and hence it also pp-defines Ind_n for all n. The proof is not easy to write, and since it is not necessary to get such a strong statement, we prove a slightly weaker one. To give an intuition for a relational proof, assume that we already pp-defined Ieq_4 with Ieq_3. We then have:

$$\begin{aligned}\mathrm{Ieq}_5(x,y,z,t,u) \text{ if and only if } &\exists w\, \mathrm{Ieq}_3(x,y,w) \wedge \mathrm{Ieq}_4(w,z,t,u)\\ &\wedge \text{ every subfamily of } (x,y,z,t,u) \text{ of size 4 is linearly independent}\end{aligned}$$

And x is linearly independent from z,t,y can be written as follows:

$$\begin{aligned}\exists a_1,a_2,b_1,b_2,c_1,c_2.\, &\mathrm{Ieq}_4(a_1,z,t,y) \wedge \mathrm{Ieq}_3(a_1,a_2,x)\\ &\wedge \mathrm{Ieq}_3(b_1,z,t) \wedge \mathrm{Ieq}_3(b_1,b_2,x)\\ &\wedge \mathrm{Ieq}_3(c_1,z,y) \wedge \mathrm{Ieq}_3(c_1,c_2,x)\\ &\wedge \dots\end{aligned}$$

The expression $\exists a_1,a_2.\, \mathrm{Ieq}_4(a_1,z,t,y) \wedge \mathrm{Ieq}_3(a_1,a_2,x)$ implies that $z+t+y \neq x$, and the expression $\exists b_1,b_2.\, \mathrm{Ieq}_3(b_1,z,t) \wedge \mathrm{Ieq}_3(b_1,b_2,x)$ implies that $x+z+t \neq 0$.

Proof. First note that by Lemma 3.2.18, Ieq_{2i_0+1} pp-defines Ind_{2i_0-1}. We now prove a particular case of the first part of the lemma: for every $j \geq 3$, Ieq_j has a primitive positive definition over $(V; \mathrm{Ieq}_3, \mathrm{Ind}_{j-1})$. Our proof is by induction on $j \geq 3$. For $j = 3$ the result is obvious. Suppose that for some $j \geq 3$ the relation Ieq_j is primitive positive definable over $(V; \mathrm{Ieq}_3, \mathrm{Ind}_{j-1})$. We show the proposition for $j + 1$. First note that $\mathrm{Ind}_{j-1}(x_1, \dots, x_{j-1}) \leftrightarrow \exists t \, \mathrm{Ind}_j(x_1, \dots, x_{j-1}, t)$. Then:

$$\begin{aligned}\mathrm{Ieq}_{j+1}(x_1, \dots, x_{j+1}) \leftrightarrow \exists z \, (&\mathrm{Ieq}_j(x_1, \dots, x_{j-1}, z) \wedge \mathrm{Ieq}_3(z, x_j, x_{j+1}) \\ &\wedge \bigwedge_{1 \leq l \leq j+1} \mathrm{Ind}_j(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_{j+1}))\end{aligned}$$

Now let us prove that Ieq_{2i_0} has a primitive positive definition over $(V; \mathrm{Ieq}_{2i_0+1}, \mathrm{Ind}_{2i_0-1})$:

$$\begin{aligned}\mathrm{Ieq}_{2i_0}(x_1, \dots, x_{2i_0}) \leftrightarrow \exists t_1 \dots \exists t_{i_0+1} \, (&\mathrm{Ieq}_{2i_0+1}(x_1, \dots, x_{i_0}, t_1, \dots, t_{i_0+1}) \\ &\wedge \mathrm{Ieq}_{2i_0+1}(t_1, \dots, t_{i_0+1}, x_{i_0+1}, \dots, x_{2i_0+1}) \\ &\wedge \bigwedge_{1 \leq l \leq 2i_0} \mathrm{Ind}_{2i_0-1}(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_{2i_0}))\end{aligned}$$

Finally, Ieq_3 has a primitive positive definition over $(V; \mathrm{Ieq}_{2i_0+1})$. Before giving it, we just note that $\mathrm{Ind}_2(x, y) \Leftrightarrow \exists t_1, \dots, t_{2i_0-1}. \, \mathrm{Ieq}_{2i_0+1}(x, y, t_1, \dots, t_{2i_0-1})$, and that:

$$\mathrm{Ind}_{2i_0-1}(x_1, \dots, x_{2i_0-1}) \Leftrightarrow \exists t_1, t_2. \, \mathrm{Ieq}_{2i_0+1}(x_1, \dots, x_{2i_0-1}, t_1, t_2)$$

Note also that:

$$\begin{aligned}\mathrm{Ieq}_{2i_0}(x_1, \dots, x_{2i_0}) \Leftrightarrow \exists t_1, \dots, t_{i_0+1}. \, &\mathrm{Ieq}_{2i_0+1}(x_1, \dots, x_{i_0}, t_1, \dots, t_{i_0+1}) \\ &\wedge \mathrm{Ieq}_{2i_0+1}(x_{i_0+1}, \dots, x_{2i_0}, t_1, \dots, t_{i_0+1}) \\ &\wedge \bigwedge_{1 \leq l \leq 2i} \mathrm{Ind}_{2i_0-1}(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_{2i_0})\end{aligned}$$

Hence, we have:

$$\begin{aligned}\mathrm{Ieq}_3(x_1, x_2, x_3) \leftrightarrow \exists t_1, \dots, t_{2i_0-1} \, (&\mathrm{Ieq}_{2i_0+1}(x_1, x_2, t_1, \dots, t_{2i_0-1}) \\ &\wedge \mathrm{Ieq}_{2i_0}(t_1, \dots, t_{2i_0-1}, x_3) \wedge \mathrm{Ind}_2(x_1, x_3) \wedge \mathrm{Ind}_2(x_2, x_3))\end{aligned}$$

This concludes the first part of the proof.

The proof of the second part is essentially the same. We first prove by induction that for all $j \geq 1$, Ieq_{2j} has a primitive positive definition in $(V; \mathrm{Ieq}_4, \mathrm{Ind}_{2j-1})$ (the proof is exactly similar to the previous proof). Then we prove that Ieq_4 is pp-definable in $(V; \mathrm{Ieq}_{2i})$ for all $i \geq 1$. Indeed, once we noted that $\mathrm{Ind}_3(x, y, z) \Leftrightarrow \exists t_1, \dots, t_{2i-3}. \, \mathrm{Ieq}_{2i}(x, y, z, t_1, \dots, t_{2i-3})$, we have:

$$\begin{aligned}\mathrm{Ieq}_4(x, y, z, t) \Leftrightarrow \exists t_1, \dots, t_{2i-2}. \, &\mathrm{Ieq}_{2i}(x, y, t_1, \dots, t_{2i-2}) \wedge \mathrm{Ieq}_{2i}(z, t, t_1, \dots, t_{2i-2}) \\ &\wedge \mathrm{Ind}_3(x, y, z) \wedge \mathrm{Ind}_3(x, y, t) \wedge \mathrm{Ind}_3(y, z, t)\end{aligned}$$

Then combining the two successive definitions, we obtain the desired result. $\square$

Notation 3.2.21. From now on, we call a type isolated by Ieq_n or Ind_n an *outstanding* type. Furthermore, we will often assimilate the type with the formula which isolates it.

This rather intuitive lemma establishes that a n-type over $(V;+)$ is unambiguously characterized by a conjunction of "outstanding" types Ieq_i and Ind_i. Let τ be the signature of the relational structure $\mathfrak{V} = (V; \mathrm{Ieq}_1, \mathrm{Ieq}_2, \dots, \mathrm{Ind}_1, \mathrm{Ind}_2, \dots)$.

Lemma 3.2.22. *For every complete n-type p over $(V;+)$, there exists a conjunction of atomic τ-formulas ϕ such that ϕ defines over $\mathfrak{V}$ the same relation as p does over $(V;+)$.*

Proof. Following Remark 3.2.32, we can suppose that a complete type is isolated by a conjunction of equalities (of the form $x_1 + \dots + x_j = 0$) and inequalities (of the form $x_1 + \dots + x_j \neq 0$). Let p be a complete type realized by a tuple $(a_1, \dots, a_n)$ and $\phi_1 \wedge \dots \wedge \phi_k \wedge \psi_1 \wedge \dots \wedge \psi_l$ its description in terms of equalities ϕ_i and inequalities ψ_i. We describe how to modify this conjunction to obtain a conjunction that can be translated easily to a τ-formula.

We say that an equation ϵ' is a *proper sub-equation* of an equality or inequality $\epsilon \in p$ if $\epsilon' \in p$ and the set of variables of ϵ' is strictly contained in the set of variables of ϵ.

We repeat the following as long as possible:

- For $i = 0$ to k, if $e_i :\Leftrightarrow x_1 + \dots + x_s = 0$ has a proper sub-equation $e'_i :\Leftrightarrow x_{i_1} + \dots + x_{i_t} = 0$, it also has its complement $e''_i :\Leftrightarrow \sum_{j \notin \{i_1, \dots, i_t\}} x_j = 0$ as a sub-equation. We then substitute $e'_i \wedge e''_i$ for e_i. Obviously, the new conjunction obtained after these substitutions still isolates p.
- For $i = 0$ to l, if $n_i :\Leftrightarrow x_1 + \dots + x_s \neq 0$ contains a proper sub-equation $e'_i :\Leftrightarrow x_{i_1} + \dots + x_{i_j} = 0$, we have necessarily $n'_i :\Leftrightarrow \sum_{j \notin \{i_1, \dots, i_s\}} x_j \neq 0$. Substitute e'_i, n'_i for n_i. Obviously, the new conjunction obtained after these substitutions still isolates p.

After this procedure, we obtain a new conjunction which is logically equivalent over $(V;+)$ to the previous one. We denote it by:

$$\epsilon_1(x_{11}, \dots, x_{1s_1}) \wedge \dots \wedge \epsilon_r(x_{r1}, \dots, x_{rs_r}) \wedge \theta_1(x_{1'1}, \dots, x_{1's'_1}) \wedge \dots \wedge \theta_m(x_{m1}, \dots, x_{ms'_m})$$

where ϵ_i are minimal equations and θ_i are minimal inequalities (they do not contain any proper sub-equation).

We now define the τ-formula ϕ as follows:

$$\bigwedge_{i=1}^{r} \mathrm{Ieq}_{s_i}(x_{i1}, \dots, x_{is_i}) \wedge \bigwedge_{i=1}^{m} \mathrm{Ind}_{s'_i}(x_{i'1}, \dots, x_{i's'_i})$$

It is straightforward to show that for all $a_1, \dots, a_n \in V$:

$$p = \mathrm{tp}(a_1, \dots, a_n) \Leftrightarrow \mathfrak{V} \models \phi(a_1, \dots, a_n)$$

Indeed, let $(a_1, \dots, a_n) \in V^n$ be such that $p = \mathrm{tp}(a_1, \dots, a_n)$, and let $i \in \{1, \dots, r\}$. We will show that $\mathfrak{V} \models \mathrm{Ieq}_{is_i}(a_{i1}, \dots, a_{is_i})$. We already know that $(V;+) \models \epsilon_i(a_{i1}, \dots, a_{is_i})$

so $a_{i1} + \cdots + a_{is_i} = 0$. Since ϵ_i is minimal, it contains no proper sub-equation, so we have $\mathfrak{V} \models \mathrm{Ieq}_{is_i}(a_{i1}, \ldots, a_{is_i})$. Similarly, we prove that $\mathfrak{V} \models \mathrm{Ind}_{s'_i}(a_{i'1}, \ldots, a_{i's'_i})$ for all $i \in \{1, \ldots, m\}$.

Conversely, let us suppose that $\mathfrak{V} \models \phi(a_1, \ldots, a_n)$. We want to prove that the type p equals $\mathrm{tp}(a_1, \ldots, a_n)$. Let $i \in \{1, \ldots, r\}$, since $\mathfrak{V} \models \mathrm{Ieq}_{is_i}(a_{i1}, \ldots, a_{is_i})$, we have $(V; +) \models \epsilon_i(a_{i1}, \ldots, a_{is_i})$. Let $i \in \{1, \ldots, m\}$. Since $\mathfrak{V} \models \mathrm{Ind}_{s'_i}(a_{i'1}, \ldots, a_{i's'_i})$, we have $(V; +) \models \theta_i(a_{i'1}, \ldots, a_{i's'_i})$. Consequently, $p = \mathrm{tp}(a_1, \ldots, a_n)$. □

Lemma 3.2.23. *We have the following:*

- $(V; +)$ *and* $(V; \mathrm{Eq}_3)$ *are first-order interdefinable;*
- $(V; \mathrm{Eq}_3)$ *and* $(V; \mathrm{Ieq}_3)$ *are first-order interdefinable. Furthermore,* Eq_3 *is existential positive definable over* $(V; \mathrm{Ieq}_3, 0)$*;*
- $(V; \mathrm{Ieq}_3)$ *and* $(V; \mathrm{Ieq}_4)$ *are first-order interdefinable.*

Proof. For the first part, note that Eq_3 is the graph of $+$. For the second part, we have, by definition of , Ieq_3 is first-order definable over $(V; \mathrm{Eq}_3)$. Conversely, first note that $x = 0 \Leftrightarrow \forall y, z \neg \mathrm{Ieq}_3(x, y, z)$. Then:

$$x + y = z \Leftrightarrow \mathrm{Ieq}_3(x, y, z) \vee (x = 0 \wedge y = 0 \wedge z = 0) \vee (x = 0 \wedge y = z) \\ \vee (y = 0 \wedge x = z) \vee (z = 0 \wedge x = y)$$

For the third part, since $(V; \mathrm{Ieq}_3)$ and $(V; \mathrm{Eq}_3)$ are first-order interdefinable, and since Ieq_4 is first-order definable on $(V; \mathrm{Eq}_3)$ by definition, Ieq_4 is first-order definable on $(V; \mathrm{Ieq}_3)$. Conversely, first note that $\mathrm{Ind}_2(x, y) \Leftrightarrow \exists z, t.\, \mathrm{Ieq}_4(x, y, z, t)$. Then:

$$\mathrm{Ieq}_3(x, y, z) \Leftrightarrow \forall t. \neg\, \mathrm{Ieq}_4(x, y, z, t) \wedge \mathrm{Ind}_2(x, y) \wedge \mathrm{Ind}_2(x, z) \wedge \mathrm{Ind}_2(y, z)$$

□

Corollary 3.2.24. *We have:* $\mathrm{Aut}(V; +) = \mathrm{Aut}(V; \mathrm{Eq}_3) = \mathrm{Aut}(V; \mathrm{Ieq}_3) = \mathrm{Aut}(V; \mathrm{Ieq}_4)$.

Proof. Combine Theorem 2.1.27 and Lemma 3.2.23. □

3.2.5 Homogeneity

Now we shall establish one of the strongest properties of $(V; +)$: homogeneity. We will also legitimize the use of a functional signature because, as we will prove, $(V; +)$ is not interdefinable with any homogeneous structure over a finite relational signature.

Since the signature of $(V; +)$ is functional, we can not know for sure that a finitely generated substructure is finite. Even if that were true, is $(V; +)$ uniformly locally finite, i.e., there is a function $f \colon \omega \to \omega$ such that for every substructure S of $(V; +)$, if S has a set of at most n generators, $|\,\mathrm{Dom}(S)|$ is bounded by $f(n)$? The two following lemmas allow us to use the previously stated Theorem 2.2.11 in order to show that $(V; +)$ is ω-categorical and has quantifier elimination.

Lemma 3.2.25. *A substructure of* $(V;+)$ *is finitely generated if and only if it is finite.*

Proof. $(\Leftarrow)$ is straightforward. For $(\Rightarrow)$, let us consider a finitely generated substructure $(B,+)$ of $(V;+)$, and let $\{a_1,\dots,a_n\}$ be a set of generators. Since $a+a=0$ for all $a\in V$, we have $B=\{0\}\cup\{a_{i_1}+\cdots+a_{i_k}\mid k\leq n,\ 1\leq i_1<\cdots<i_k\leq n\}$. So there is a clear injection from B to $\mathcal{P}(\{a_1,\dots,a_n\})$, so $|B|\leq 2^n$. □

Corollary 3.2.26. *The structure* $(V;+)$ *is locally finite.*

Remark 3.2.27. The same proof gives us that $(V;+)$ is uniformly locally finite. Indeed, we define $f(n)=2^n$, and suppose that $(B,+)$ is a finitely generated substructure of $(V;+)$ with less than n generators. Then its cardinality is bounded by $f(n)$.

Remark 3.2.28. It is straightforward to verify that any substructure of $(V;+)$ is a vector space over $\mathbb{F}_2$. Hence by Lemma 3.2.25, the finitely generated substructures of $(V;+)$ and the class of all finite vector spaces over $\mathbb{F}_2$ coincides, up to isomorphism.

Notation 3.2.29. Let $E\subseteq V$. We write $\mathrm{Vect}(E)$ for the substructure of $(V;+)$ generated by E. It is also called the *substructure* of V generated by E.

Proposition 3.2.30. $(V;+)$ *is* homogeneous*.*

Proof. Let us consider two finite substructures A_1 and A_2 of $(V;+)$ and $\sigma\colon A_1\to A_2$ an isomorphism between A_1 and A_2. Let $(b_1,\dots,b_n)$ be a basis of A_1. Since σ is an isomorphism from A_1 to A_2, $(\sigma(b_1),\dots,\sigma(b_n))$ is a basis of A_2. We complete $(b_1,\dots,b_n)$ to a basis $(b_1,\dots,b_n,c_1,\dots,c_i,\dots)$ of $(V;+)$ and $(\sigma(b_1),\dots,\sigma(b_n))$ to a basis $(\sigma(b_1),\dots,\sigma(b_n),d_1,\dots,d_n,\dots)$ of $(V;+)$. Let $\alpha\colon V\to V$ be a mapping such that $\alpha(b_i)=\sigma(b_i)$ for all $i\leq n$, $\alpha(c_j)=d_j$ for all j and α satisfies $\alpha(x+y)=\alpha(x)+\alpha(y)$ for all the other vectors (here we use the fact that there exists only one way to express a non-zero vector as a finite sum of elements of the basis). The map α we have built is clearly an automorphism of $(V;+)$ extending σ. □

Using Fraïssé's terminology defined in Section 2.2, we prove that $(V;+)$ is indeed the Fraïssé limit of the class of finite vector spaces over $\mathbb{F}_2$. Indeed, it is clear that the age of the countably infinite vector space over $\mathbb{F}_2$ is the class of finite vector spaces over $\mathbb{F}_2$. By Remark 2.2.5, the class $\mathcal{K}$ of all finite vector space over $\mathbb{F}_2$ satisfies the (HP) and (JEP) properties. Since we already prove that $(V;+)$ is homogeneous, we only have to prove that $\mathcal{K}$ has the amalgamation property. Then, by unicity of the Fraïssé limit, we will conclude that $(V;+)$ is the Fraïssé limit of the class of all finite vector spaces over $\mathbb{F}_2$. But this is straightforward since we have the following fact: a vector space V_1 over a field $\mathbb{F}$ is embeddable in another $\mathbb{F}$-vector space V_2 if and only if the dimension of V_1 is smaller or equal to the dimension of V_2. Hence, the amalgamation diagram is very easy to close.

After having given this new characterization of the countably infinite vector space over $\mathbb{F}_2$, we state without transition the following two useful propositions:

Proposition 3.2.31. $(V;+)$ *is ω-categorical and has quantifier elimination.*

Proof. A direct consequence of Theorem 2.2.11 since $(V;+)$ is homogeneous by Proposition 3.2.30 and locally finite by Corollary 3.2.26. □

Remark 3.2.32. As a consequence of homogeneity of $(V;+)$, any complete type over $(V;+)$ is isolated by a conjunction of atomic formulas and negations of atomic formulas.

Proposition 3.2.33. $\mathrm{End}(V;+,\neq) = \mathrm{Emb}(V;+) = \overline{\mathrm{Aut}(V;+)}$

Proof. Direct application of Proposition 2.2.9 combined with the fact that when the signature of a structure is purely functional, its self-embeddings are exactly its injective endomorphisms. □

As mentioned in the introduction, if $(V;+)$ were interdefinable with a homogeneous structure with a finite relational signature, the number of canonical behaviours over $(V;+)$ would be assuredly finite, and their study would be simpler.

Proposition 3.2.34. $(V;+)$ *is not first-order interdefinable with any homogeneous structure with a finite relational signature.*

Proof. Suppose that there exists a homogeneous structure Γ with a finite relational signature, and first-order interdefinable with $(V;+)$. We have $\mathrm{Aut}(V;+) = \mathrm{Aut}(\Gamma)$ by Theorem 2.1.27. Let m be the maximal arity of the relations appearing in Γ's signature, and let us consider two $(m+1)$-tuples of vectors $(x_1,\ldots,x_{m+1})$ and $(y_1,\ldots,y_{m+1})$ such that $(y_1,\ldots,y_{m+1})$ is linearly independent, and for all integers $i_1,\ldots,i_m$ such that $1 \leq i_1 < \cdots < i_m \leq m+1$, $(x_{i_1},\ldots,x_{i_m})$ is linearly independent, but $\sum_{i=1}^{m+1} x_i = 0$. Let $k \leq m$ and $1 \leq i_1 < \cdots < i_k \leq m+1$ be any sequence of integers. We have $\mathrm{tp}_{(V;+)}(x_{i_1},\ldots,x_{i_k}) = \mathrm{tp}_{(V;+)}(y_{i_1},\ldots,y_{i_k})$. Since $(V;+)$ is homogeneous, there exists an automorphism $\alpha \in \mathrm{Aut}(V;+) = \mathrm{Aut}(\Gamma)$ such that $\alpha(x_{i_l}) = y_{i_l}$ for all $l \leq k$. So $\mathrm{tp}_\Gamma(x_{i_1},\ldots,x_{i_k}) = \mathrm{tp}_\Gamma(y_{i_1},\ldots,y_{i_k})$ for any $k \leq m$ and $1 \leq i_1 < \cdots < i_k \leq m+1$. Since Γ has relations of arity at most m and admits quantifier elimination by Theorem 2.2.11 (since it is homogeneous and uniformly locally finite because relational), $\mathrm{tp}_\Gamma(x_1,\ldots,x_{m+1}) = \mathrm{tp}_\Gamma(y_1,\ldots,y_{m+1})$. Because Γ is homogeneous, there exists $\beta \in \mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;+)$ such that $\beta(x_i) = y_i$ for all $1 \leq i \leq m+1$, a contradiction to the fact that $\mathrm{tp}_{(V;+)}(x_1,\ldots,x_{m+1}) \neq \mathrm{tp}_{(V;+)}(x_1,\ldots,y_{m+1})$. □

Nevertheless, since $(V;+)$ is ω-categorical, it is first-order interdefinable with an homogeneous structure with infinite relational signature. Here, instead of expanding the language with every definable relation over $(V;+)$, we find a well-chosen structure which does the job: $(V;(\mathrm{Eq}_i)_{i\geq 1})$.

Remark 3.2.35. Every atomic formula on the language of $(V;+)$ can be seen as an atomic formula on the language of $(V;(\mathrm{Eq}_i)_{i\geq 1})$, and conversely. Indeed, $x_1+\cdots+x_n = x_{n+1}+\cdots+x_{n+m}$ if and only if $\mathrm{Eq}_{n+m}(x_1,\ldots,x_{n+m})$. As a consequence, $\mathrm{tp}_{(V;(\mathrm{Eq}_i)_{i\geq 1})}(\overline{a}) = \mathrm{tp}_{(V;(\mathrm{Eq}_i)_{i\geq 1})}(\overline{b})$ if and only if $\mathrm{tp}_{(V;+)}(\overline{a}) = \mathrm{tp}_{(V;+)}(\overline{b})$, for all n-tuples $\overline{a},\overline{b}$ of elements of V.

Proposition 3.2.36. *The structure $(V;+)$ is first-order interdefinable with the homogeneous structure $(V;(\mathrm{Eq}_i)_{i\geq 1})$.*

Proof. First note that $\mathrm{Aut}(V;+) \subseteq \mathrm{Aut}(V;(\mathrm{Eq}_i)_{i\geq 1})$ by Theorem 2.1.27, since every Eq_i has a first-order definition over $(V;+)$. The converse is also true since Eq_3 is exactly the graph of the function $+$. So $\mathrm{Aut}(V;+) = \mathrm{Aut}(V;(\mathrm{Eq}_i)_{i\geq 1})$. We now prove that $(V;(\mathrm{Eq}_i)_{i\geq 1})$ is homogeneous. Let $\overline{a}$ and $\overline{b}$ be two n-tuples of elements of V such that $\mathrm{tp}_{(V;(\mathrm{Eq}_i)_{i\geq 1})}(\overline{a}) = \mathrm{tp}_{(V;(\mathrm{Eq}_i)_{i\geq 1})}(\overline{b})$. By Remark 3.2.35, $\mathrm{tp}_{(V;+)}(\overline{a}) = \mathrm{tp}_{(V;+)}(\overline{b})$. Since $(V;+)$ is homogeneous, there exists $\alpha \in \mathrm{Aut}(V;+) = \mathrm{Aut}(V;(\mathrm{Eq}_i)_{i\geq 1})$ such that $\alpha(\overline{a}) = \overline{b}$. Consequently, $(V;(\mathrm{Eq}_i)_{i\geq 1})$ is homogeneous. ☐

Corollary 3.2.37. *The structure $(V;+)$ is first-order interdefinable with the homogeneous structure $(V;(\mathrm{Ieq}_i)_{i\geq 1})$.*

Proof. First note that for all x, y, z, we have:

$$x + y = z \Leftrightarrow \mathrm{Ieq}_3(x,y,z) \vee (\mathrm{Ieq}_1(x) \wedge y = z) \vee (\mathrm{Ieq}_1(y) \wedge x = z) \vee (\mathrm{Ieq}_1(z) \wedge x = y)$$

Hence, $(V;+)$ is first-order interdefinable with $(V;(\mathrm{Ieq}_i)_{i\geq 1})$. We already know that $(V;(\mathrm{Eq}_i)_{i\geq 1})$ is homogeneous by Proposition 3.2.36. But it is straightforward to see that every Eq_i has an quantifier free positive definition over $(V;(\mathrm{Ieq}_i)_{i\geq 1})$: indeed, it is very natural to see an equation as a disjunction of conjunction of "primitive" equations. For instance:

$$\mathrm{Eq}_4(x,y,z,t) \Leftrightarrow \mathrm{Ieq}_4(x,y,z,t) \vee (\mathrm{Ieq}_3(x,y,z) \wedge \mathrm{Ieq}_1(t)) \vee (\mathrm{Ieq}_2(x,y) \wedge \mathrm{Ieq}_2(z,t)) \vee \cdots$$

Hence, $(V;(\mathrm{Ieq}_i)_{i\geq 1})$ has quantifier elimination. As a reduct of an ω-categorical structure, it is ω-categorical. Hence $(V;(\mathrm{Ieq}_i)_{i\geq 1})$ is homogeneous by Theorem 2.2.13. ☐

Definition 3.2.38. A structure Γ is called *k-transitive* if for any two k-tuples (s,t) of distinct elements from $\mathrm{Dom}(\Gamma)$, there is an $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(s) = t$, where the action of α on tuples is componentwise, i.e., $\alpha(s_1, \ldots, s_k) = (\alpha(s_1), \ldots, \alpha(s_k))$.

We conclude this section with two lemmas which will allow us to apply the properties of 2-transitive structures to various definable structures over $(V;+)$.

Lemma 3.2.39. *The structure $(V \setminus \{0\};(\mathrm{Eq}_i^{\neq 0})_{i\geq 2})$ is 2-transitive.*

Proof. Let $x_1 \neq x_2$ and $y_1 \neq y_2$ be two pairs of distinct elements of $V \setminus \{0\}$. By Fact 3.1.3, we can complete (x_1, x_2) to a basis $(x_1, \ldots, x_n \ldots)$ of $(V;+)$, and similarly, we can complete (y_1, y_2) to a basis $(y_1, \ldots, y_n \ldots)$ of $(V;+)$. So there exists $\alpha \in \mathrm{Aut}(V;+)$ such that $\alpha(x_i) = y_i$ for all $i \leq 1$. But since $\mathrm{Aut}(V;+) = \mathrm{Aut}(V;(\mathrm{Eq}_i)_{i\geq 1})$, and since 0 is first-order definable in $(V;+)$, Ind_1 is preserved by every automorphism of $(V;+)$, so we have: $\alpha \in \mathrm{Aut}(V \setminus \{0\};(\mathrm{Eq}_i)_{i\geq 2}^{\neq 0})$. ☐

Lemma 3.2.40. *The structure $(V \setminus \{0\};(\mathrm{Ieq}_i)_{i\geq 2})$ is 2-transitive.*

Proof. The proof is the same as the previous one. ☐

3.3 Outstanding Functions of the Vector Space

3.3.1 Equivalence Relation for functions

Before defining a series of functions which play a key-role in our understanding of the reducts of the countably infinite boolean vector space, we define an equivalence relation over functions from $V \to V$, in order to manipulate equivalence classes of functions instead of functions in the incoming proofs.

Definition 3.3.1. Given two functions f, g from $V \to V$, we say that $f \backsim_{\text{op}} g$ whenever there exist $\alpha, \beta \in \text{Emb}(V;+)$ such that $\alpha \circ f = \beta \circ g$.

Proposition 3.3.2. *The binary relation $\backsim_{\text{op}}$ is an equivalence relation over V^V.*

Proof. The proof is straightforward. □

Lemma 3.3.3. *Let f, g be two functions from $V \to V$. Then $f \backsim_{\text{op}} g$ if and only if for all finite subset S of V, there exists $\alpha \in \text{Aut}(V;+)$ such that $f = \alpha \circ g$.*

The idea of the following proof is taken from Lemma 74 in [**?**].

Proof. We prove that if for all finite subset S of V, there exists $\alpha \in \text{Aut}(V;+)$ such that $f = \beta \circ g$, then $f \backsim_{\text{op}} g$, the converse being straightforward. Construct a rooted tree as follows. Each vertex of the tree lies on some level $n \in \mathbb{N}$. Let $(v_n)_{n \geq 0}$ be an enumeration of V. Let F_n be the set of partial isomorphisms of $(V;+)$ with domain $V_n := \{v_0, \dots, v_n\}$, and define the equivalence relation $\sim$ on F_n^2 as follows: $(\alpha_1, \alpha_2) \sim (\beta_1, \beta_2)$ if there exists $\delta \in \text{Aut}(V;+)$ such that $\alpha_i = \delta \circ \beta_i$ for $i \in \{1, 2\}$. Note that for each n, the relation $\sim$ has finitely many equivalence classes on F_n^2, by ω-categoricity of $(V;+)$.

Now, the vertices of the tree on level n are precisely the equivalence classes E of $\sim$ on F_n^2 such that for every (equivalently, for some) $(\alpha_1, \alpha_2) \in E$ and $x \in V$ satisfying $\{f(x), g(x)\} \subseteq V_n$ we have $\alpha_1(f(x)) = \alpha_2(g(x))$.

The equivalence class of the partial map with the empty domain V_0 becomes the root of the tree, on level $n = 0$. We define adjacency in the tree by restriction as follows: when E is a vertex on level n, and E' a vertex on level $n+1$, and E contains (α_1, α_2) and E' contains (α_1', α_2') such that $\alpha_j = \alpha_j' \upharpoonright V_n$ for $j \in \{1, 2\}$ then we make E and E' adjacent in the tree. Note that the resulting rooted tree is finitely branching. By assumption, the tree has vertices on all levels. Hence, by Theorem 2.4.14, there exists an infinite path $(E_0, \dots, E_n, \dots)$ in the tree where E_i is from level $i \in \mathbb{N}$.

We define $\gamma_1, \gamma_2 \in \text{Emb}(V;+)$ as follows. Suppose γ_1, γ_2 are already defined on V_n such that $\alpha_j = \gamma_j \upharpoonright V_n$ for $\left\{^{j \in \{1,2\}}\right\}$, and $(\alpha_1, \alpha_2) \in E_n$. We want to define γ_1, γ_2 on v_{n+1} in such a way that $(\gamma_1 \upharpoonright V_{n+1}, \gamma_2 \upharpoonright V_{n+1}) \in E_{n+1}$. Since E_n and E_{n+1} are adjacent in the tree, there exist $(\beta_1, \beta_2) \in E_n$ and $(\beta_1', \beta_2') \in E_{n+1}$ such that $\beta_j = \beta_j' \upharpoonright V_{n+1}$ for $j \in \{1, 2\}$. By definition of $\sim$, there exists $\delta \in \text{Aut}(V;+)$ such that $\alpha_j = \delta \circ \beta_j$ for $j \in \{1, 2\}$. We now define $\alpha_j' := \delta \circ \beta_j'$ for $j \in \{1, 2\}$. We have $(\alpha_1', \alpha_2') \in E_{n+1}$ and $\alpha_j' \upharpoonright V_n = \delta \circ \beta_j' \upharpoonright V_n = \delta \circ \beta_j = \alpha_j$. Hence, α_j' extends α_j for $j \in \{1, 2\}$. We finally define $\gamma_j(v_{n+1}) := \alpha_j'(v_{n+1})$. □

From now on, we will use both the definition and its characterization of the relation $\backsim_{\text{op}}$.

Notation 3.3.4. Given a function f from $V \to V$, we denote by $\text{cl}(f)$ the equivalence class of f with respect to $\backsim_{\text{op}}$.

Proposition 3.3.5. *Let f, g be two functions over V such that $g \in \text{cl}(f)$. Then $f \in \overline{\langle \{g\} \cup \text{Aut}(V;+) \rangle}_1$ and $g \in \overline{\langle \{f\} \cup \text{Aut}(V;+) \rangle}_1$*

Proof. The proof is straightforward by Lemma 3.3.3. □

Corollary 3.3.6. *Let f, g be two functions from $V \to V$ such that $g \in \text{cl}(f)$ and let Γ be a reduct of $(V;+)$. Then $f \in \text{End}(\Gamma)$ if and only if $g \in \text{End}(\Gamma)$.*

Proof. Since Γ is a reduct of $(V;+)$, $\text{Aut}(V;+) \subseteq \text{Aut}(\Gamma) \subseteq \text{End}(\Gamma)$ by Theorem 2.1.27. Assume that f is in $\text{End}(\Gamma)$. Since $\text{End}(\Gamma)$ is locally closed, it contains $\overline{\langle \{f\} \cup \text{Aut}(V;+) \rangle}_1$ by Proposition 3.2.33. Hence, every g in $\text{cl}(f)$ also belongs to $\text{End}(\Gamma)$ by Proposition 3.3.5. □

3.3.2 Quasi-identity functions id, id*, and id$^{\text{ni}}$

We now define various functions derived from the identity function, and distinguished by their image of 0.

Definition 3.3.7. Let $a, c \neq 0$ and $h \in \text{Emb}(V;+)$ be such that $a \notin h(V)$. We define the maps id and id* from $V \to V$ as follows:

- $\text{id}(x) = x$ for all $x \in V$
- $\text{id}^*(0) = a$ and $\text{id}^*(x) = h(x)$ for all $x \in V \setminus \{0\}$
- $\text{id}^{\text{ni}}(0) = h(c)$ and $\text{id}^{\text{ni}}(x) = h(x)$ for all $x \in V \setminus \{0\}$.

Remark 3.3.8. The functions id, id*, and id$^{\text{ni}}$ preserve Ieq_i and Ind_i for all $i \geq 2$ since these relations are defined on $V \setminus \{0\}$ and since the restriction of $\text{id}, \text{id}^*, \text{id}^{\text{ni}}$ to $V \setminus \{0\}$ equals the restriction of a self-embedding of $(V;+)$.

We first prove that these definitions do not depend on the choice of the self-embedding h nor the vectors a, c in the sense that taking another $h' \in \text{Emb}(V;+)$ and $a' \notin h(V)$ define a function in the same equivalence class.

Lemma 3.3.9. *We have:* $\text{cl}(\text{id}) = \text{Emb}(V;+)$.

Proof. Straightforward application of Proposition 2.1.7. □

Lemma 3.3.10. *The class $\text{cl}(\text{id}^*)$ is the set of functions g such that there exist $h' \in \text{Emb}(V;+)$ and $d \notin h(V)$ such that $g(0) = d$ and $g(x) = h'(x)$ for all $x \neq 0$.*

Proof. Let g be such that there exist $h' \in \mathrm{Emb}(V;+)$ and $d \notin h(V)$ such that $g(0) = d$ and $g(x) = h'(x)$ for all $x \neq 0$. By Lemma 3.3.9, h and h' are both in $\mathrm{cl}(\mathrm{id})$. Hence, there exist $\alpha, \beta \in \mathrm{Emb}(V;+)$ such that $\alpha \circ h = \beta \circ h'$. Since $a \notin h(V)$ and $d \notin h'(V)$, we can choose α, β such that $\alpha(a) = \beta(d)$. Consequently, g belongs to $\mathrm{cl}(\mathrm{id}^*)$.

The converse inclusion is straightforward by definition of id^* and $\backsim_{\mathrm{op}}$. $\square$

Lemma 3.3.11. *The equivalence class* $\mathrm{cl}(\mathrm{id}^{\mathrm{ni}})$ *is the set of functions* g *such that there exist* $h' \in \mathrm{Emb}(V;+)$ *and* $d \neq 0$ *such that* $g(0) = h'(d)$ *and* $g(x) = h'(x)$ *for all* $x \neq 0$.

Proof. The proof is similar to the proof of Lemma 3.3.10. $\square$

From now on, we will assimilate id^* and $\mathrm{id}^{\mathrm{ni}}$ to their equivalence classes $\mathrm{cl}(\mathrm{id}^*)$ and $\mathrm{cl}(\mathrm{id}^{\mathrm{ni}})$. In the following, we give the list of the endomorphism monoids locally generated by the quasi-identity functions we just defined.

Lemma 3.3.12. *The function* $\mathrm{id}^{\mathrm{ni}}$ *together with* $\mathrm{Aut}(V;+)$ *locally generates* id^*.

Proof. By definition of $\mathrm{id}^{\mathrm{ni}}$, there exists $c \neq 0$ such that $\mathrm{id}^{\mathrm{ni}}(0) = \mathrm{id}^{\mathrm{ni}}(c)$. Let α be a self-embedding of $(V;+)$ such that $c \notin \alpha(V)$. Then $\mathrm{id}^{\mathrm{ni}} \circ \alpha$ belongs to $\mathrm{cl}(\mathrm{id}^*)$. Hence by Proposition 3.3.5, $\mathrm{id}^{\mathrm{ni}}$ together with $\mathrm{Aut}(V;+)$ locally generates id^*. $\square$

Hence, given a reduct Γ of $(V;+)$ such that $\mathrm{id}^{\mathrm{ni}} \in \mathrm{End}(\Gamma)$, we have that $\mathrm{id}^* \in \mathrm{End}(\Gamma)$.

Notation 3.3.13. We denote by $A \sqcup B$ the disjoint union of two sets A and B.

Proposition 3.3.14. *We have:*

$$\mathrm{End}(V;\mathrm{Ieq}_3) = \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{nij}) \sqcup \mathrm{cl}(\mathrm{id}^*) = \overline{\langle \{\mathrm{id}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}$$

Proof. By Lemma 3.3.12 and Proposition 3.3.5, we have:

$$\mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{nij}) \sqcup \mathrm{cl}(\mathrm{id}^*) \subseteq \overline{\langle \{\mathrm{id}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}$$

By Remark 3.3.8, $\mathrm{id}^{\mathrm{ni}}$ preserve Ieq_3, so using Corollary 3.2.24 we obtain:

$$\overline{\langle \{\mathrm{id}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1} \subseteq \mathrm{End}(V;\mathrm{Ieq}_3)$$

Conversely, consider $f \in \mathrm{End}(V;\mathrm{Ieq}_3)$, we will show that:

$$f \in \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{nij}) \sqcup \mathrm{cl}(\mathrm{id}^*)$$

Since f preserves Ieq_3 and since $\mathrm{Ind}_2 \in \langle (V;\mathrm{Ieq}_3)\rangle_{\mathrm{ep}}$ by Lemma 3.2.18, the relation Ind_2 is preserved by f by Theorem 2.1.27 and hence, f is injective on $V \setminus \{0\}$. We define $g(x) := f(x)$ for all $x \neq 0$, and $g(0) = 0$. We start by proving that g is a self-embedding of $(V;+)$. Let $x \neq y$ be two non-zero elements of V. We have $g(x+y) = f(x+y)$, but since $\mathrm{Ieq}_3(x+y,x,y)$ and f preserves Ieq_3, $f(x+y) = f(x) + f(y)$. Hence, $g(x+y) = f(x) + f(y) = g(x) + g(y)$. If $x = y$, then $g(x) = g(y)$, and $g(x+y) = g(0) = 0$, so $g(x+y) = g(x) + g(y)$. If $x \neq 0$, $g(x+0) = g(x) = g(x) + g(0)$. Consequently, $g \in \mathrm{Emb}(V;+)$.

We distinguish three cases now. Either $f(0) \notin g(V)$, and in this case, f belongs to $\mathrm{cl}(\mathrm{id}^*)$ by Lemma 3.3.12. Or there exists $c \neq 0$ such that $f(0) = f(c)$, and in this case, f belongs to $\mathrm{cl}(\mathrm{id}^{\mathrm{ni}})$ by Lemma 3.3.11. Or $f(0) = 0$, and in this case, $f \in \mathrm{Emb}(V;+)$, which is equal to $\mathrm{cl}(\mathrm{id})$ by Lemma 3.3.9. □

Corollary 3.3.15. *We have:*

$$\mathrm{End}(V;\mathrm{Ieq}_3,\neq) = \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) = \overline{\langle\{\mathrm{id}^*\} \cup \mathrm{Aut}(V;+)\rangle_1}$$

Proof. By Proposition 3.3.14, since id^* is injective, we already have:

$$\mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \subseteq \overline{\langle\{\mathrm{id}^*\} \cup \mathrm{Aut}(V;+)\rangle_1} \subseteq \mathrm{End}(V;\mathrm{Ieq}_3,\neq)$$

Since every endomorphism of $(V;\mathrm{Ieq}_3,\neq)$ preserves $\neq$, it is injective. Further more, we have $\mathrm{End}(V;\mathrm{Ieq}_3,\neq) \subseteq \mathrm{End}(V;\mathrm{Ieq}_3) = \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}})$. Since $\mathrm{cl}(id^{\mathrm{ni}})$ only contains not injective functions, we conclude: $\mathrm{End}(V;\mathrm{Ieq}_3,\neq) \subseteq \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*)$. □

3.3.3 Quasi-affine functions: af, af^*, af', and $\mathrm{af}^{\mathrm{ni}}$

We now define a series of functions derived from vectorial translations. Given that these functions are equal on the $V \setminus \{0\}$ part, their different values on 0 will manifest different important properties.

Definition 3.3.16. Let $a, c, d \neq 0$ be arbitrary elements of V and h be an arbitrary self-embedding of $(V;+)$ such that $\{a, a+d\} \cap h(V) = \emptyset$. We define the functions af, af', af^*, and $\mathrm{af}^{\mathrm{ni}}$ as follows:

- $\mathrm{af}(0) = 0$ and $\mathrm{af}(x) = h(x) + a$ for all $x \in V \setminus \{0\}$
- $\mathrm{af}^*(0) = d$ and $\mathrm{af}^*(x) = h(x) + a$ for all $x \in V \setminus \{0\}$
- $\mathrm{af}'(x) = h(x) + a$ for all $x \in V$
- $\mathrm{af}^{\mathrm{ni}}(0) = h(c) + a$ and $\mathrm{af}^{\mathrm{ni}}(x) = h(x) + a$ for all $x \neq 0$

Remark 3.3.17. The maps $\mathrm{af}, \mathrm{af}^*$, af', and $\mathrm{af}^{\mathrm{ni}}$ clearly violate Ieq_{2i+1} but preserve Ieq_{2i} for all $i \geq 1$.

The function af' is especially remarkable for it is the only one which presents some kind of uniformity in the sense that it is the exact composition of a self-embedding of $(V;+)$ and a translation by a non-zero vector. Thus, af' is the only one which preserves Eq_4.

Notation 3.3.18. For any element $a \neq 0$, we write t_a for the translation of vector a, i.e., $t_a(x) = x + a$ for all $x \in V$.

The following two lemmas are a bit technical but they will allow us to prove that we can think to the functions $\mathrm{af}, \mathrm{af}', \mathrm{af}^*$ and $\mathrm{af}^{\mathrm{ni}}$ as classes of functions in the sense that changing h, a, d, c to define $\mathrm{af}, \mathrm{af}', \mathrm{af}^*, \mathrm{af}^{\mathrm{ni}}$ will result in a function in the same equivalence class.

Lemma 3.3.19. *Let B be a finite substructure of $(V;+)$ and $u \in V \setminus B$. Then* $\mathrm{Vect}(t_u(B)) = B \cup t_u(B)$.

Proof. Let $C = \mathrm{Vect}(t_u(B))$. Since C is generated by $t_u(B)$, we have $t_u(B) \subseteq C$. Note also that every element x of B can be written as the sum of two elements of C: $x = (x+u) + (0+u)$, so $B \subseteq C$. Consequently, $B \cup t_u(B) \subseteq C$. We now show that $C \subseteq t_u(B) \cup B$, which comes straightforward from the fact that the sum of two elements of B is in B (because B is a substructure of V), the sum of two elements of $t_u(B)$ is also in B (because $u + u = 0$) and the sum of one element of B and one element of $t_u(B)$ is in $t_u(B)$. Consequently, $C = B \cup t_u(B)$. □

Lemma 3.3.20. *Let C_1, C_2 be two finite substructures of $(V;+)$ of same dimension, σ an isomorphism between C_1 and C_2, $a \in V \setminus C_1$ and $b \in V \setminus C_2$. Let D_1 be the substructure of $(V;+)$ generated by $t_a(C_1)$ and D_2 be the substructure of $(V;+)$ generated by $t_b(C_2)$. Then we have the following:*

- *$D_1 = C_1 \cup t_a(C_1)$ and $D_2 = C_2 \cup t_b(C_2)$.*
- *the map $\tau\colon D_1 \to D_2$ such that $\tau(x) = \sigma(x)$ if $x \in C_1$ and $\tau(x) = \sigma(x+a) + b$ for all $x \in D_1 \setminus C_1$ is an isomorphism of vector spaces between D_1 and D_2.*
- *Consequently, there exists an automorphism of $(V;+)$ sending D_1 to D_2.*

Proof. In Lemma 3.3.19, we already showed that $D_1 = C_1 \cup t_a(C_1)$ and $D_2 = C_2 \cup t_b(C_2)$.

The map τ is an injection as a composition of injections, let us show that it is surjective. Let $y \in D_2$. Either $y \in C_2$, then we have $\tau(\sigma^{-1}(y)) = y$, or $y \in t_a(C_2)$. In this case, there exists $z \in C_2$ such that $y = z + b$. Let $x = \sigma^{-1}(z) + a$. Note that since $z \in C_2$, $\sigma^{-1}(z) \in C_1$, so $x \in t_a(C_1)$. Then $\tau(x) = \sigma(x+a) + b = \sigma(\sigma^{-1}(z)) + b = y$. Consequently, τ is a bijective map from D_1 to D_2. It remains to show that for all $x, y \in D_1$, $\tau(x+y) = \tau(x) + \tau(y)$, which comes straightforward with a small calculation. For instance, if $x \in C_1$ and $y \in D_1 \setminus C_1$, then $x+y \in t_a(C_1)$, so $\tau(x+y) = \sigma(x+y+a)+b = \sigma(x) + \sigma(y+a) + b = \tau(x) + \tau(y)$. Thus, τ is an isomorphism of vector spaces from D_1 to D_2.

The last part is straightforward by homogeneity of $(V;+)$ since D_1 and D_2 are finite vector spaces. □

Lemma 3.3.21. *The class* $\mathrm{cl}(\mathrm{af})$ *is the set of functions g such that there exist $h' \in$* $\mathrm{Emb}(V;+)$, *$a' \notin h(V)$ such that $g(0) = 0$ and $g(x) = h'(x) + a'$ for all $x \neq 0$.*

Proof. For this proof, we will use the characterization of the equivalence classes of $\backsim_{\mathrm{op}}$ given in Lemma 3.3.3. Let $a' \in V \setminus \{0\}$, let h' be a self-embedding of $(V;+)$ such that $a' \notin h'(V)$, and let g be the function defined as follows: $g(0) = 0$ and $g(x) = h'(x) + a'$

for all $x \neq 0$. Let $S := \{c_1, \dots, c_n\}$ be a finite subset of V. We prove that there exists $\tau \in \mathrm{Aut}(V;+)$ such that $\tau(\mathrm{af}'(x)) = f(x)$ for all $x \in S$. Recall that $\mathrm{af}(0) = 0$ and $\mathrm{af}(x) = h(x) + a$ for all $x \neq 0$ with $h \in \mathrm{Emb}(V;+)$ and $a \notin h(V)$. Since h and h' are both self-embeddings of $(V;+)$, $\mathrm{tp}_{(V;+)}(h(c_1), \dots, h(c_n)) = \mathrm{tp}_{(V;+)}(h'(c_1), \dots, h'(c_n))$. By ω-categoricity of $(V;+)$, there exists an automorphism σ such that $\sigma(h'(c_i)) = h(c_i)$ for all $i \leq n$. Let C_1 be the substructure of $(V;+)$ generated by $h(\{c_1, \dots, c_n\})$ and C_2 be the substructure of $(V;+)$ generated by $h'(\{c_1, \dots, c_n\})$. Note that C_1 and C_2 have same dimension and $\sigma\lceil C_1$ is an isomorphism of vector spaces from C_1 to C_2. Also note that since $a \notin h(V)$, $a \notin C_1$ and since $a' \notin h'(V)$, $a' \notin C_2$. Hence by Lemma 3.3.20, there exists an automorphism τ of $(V;+)$ such that $\tau(\mathrm{af}'(c_i)) = f(c_i)$. □

Lemma 3.3.22. *The class* $\mathrm{cl}(\mathrm{af}^*)$ *is the set of functions* g *such that there exist* $h' \in \mathrm{Emb}(V;+)$, $a', d' \neq 0$ *such that* $\{a', a'+d'\} \cap h(V) = \emptyset$, *and such that* $g(0) = d'$ *and* $g(x) = h'(x) + a'$ *for all* $x \neq 0$.

Proof. Easy adaptation of the proof of Lemma 3.3.21. □

Lemma 3.3.23. *The class* $\mathrm{cl}(\mathrm{af}')$ *is the set of functions* g *such that there exist* $h' \in \mathrm{Emb}(V;+)$, $a' \notin h(V)$ *such that* $g(x) = h'(x) + a'$ *for all* $x \in V$.

Proof. Easy adaptation of the proof of Lemma 3.3.21. □

Lemma 3.3.24. *The class* $\mathrm{cl}(\mathrm{af}^{\mathrm{ni}})$ *is the set of functions* g *such that there exist* $h' \in \mathrm{Emb}(V;+)$, $a' \notin h(V)$ *and* $c' \neq 0$ *such that* $g(0) = h'(c') + a'$ *and* $g(x) = h'(x) + a'$ *for all* $x \neq 0$.

Proof. Easy adaptation of the proof of Lemma 3.3.21. □

The following proposition is very important since it establishes some sort of reverse statement to the fact that the functions $\mathrm{af}, \mathrm{af}', \mathrm{af}^*$, and $\mathrm{af}^{\mathrm{ni}}$ preserve Ieq_4. Indeed, a function which preserves Ieq_4 is equal on $V \setminus \{0\}$ to the composition of a self-embedding of $(V;+)$ and a translation.

Proposition 3.3.25. *Let* f *be a function which preserves* Ieq_4. *Then there exists* $h \in \mathrm{Emb}(V;+)$ *and* $v \notin h(V \setminus \{0\})$ *such that* $f(x) = h(x) + v$ *for all* $x \in V \setminus \{0\}$.

Proof. Let $f \in \mathrm{End}(V; \mathrm{Ieq}_4)$. First note that since f preserves Ieq_4, f must be injective on $V \setminus \{0\}$, as Ind_2 is pp-definable in $(V; \mathrm{Ieq}_4)$. Let us suppose that f preserves Ieq_3. By Proposition 3.3.14, f belongs to $\mathrm{cl}(\mathrm{id})$ or $\mathrm{cl}(\mathrm{id}^*)$, which implies by Lemmas 3.3.9 and 3.3.10 that f has the required form.

If f does not preserve Ieq_3, there exist two non zero distinct vectors b_0, b_1 such that $f(b_0) + f(b_1) \neq f(b_0 + b_1)$. We now show that there exists $h \in \mathrm{Emb}(V;+)$ and $v \notin h(V)$ such that $f(x) = h(x) + v$ for all $x \in V \setminus \{0\}$. Let us define v as follows: $v := f(b_0) + f(b_1) + f(b_0 + b_1)$. Note that $v \neq 0$ by assumption. Let $h(x) = f(x) + v$ for all $x \in V \setminus \{0\}$ and $h(0) = 0$. We prove that h' is a self-embedding of $(V;+)$. We start by proving that h is injective. By injectivity of f on $V \setminus \{0\}$, h is clearly injective on $V \setminus \{0\}$. Now assume that there exists $c \neq 0$ such that $h(c) = f(c) + v = 0$.

In this case, $f(c) = f(b_0) + f(b_1) + f(b_0 + b_1)$. Since f preserves Ieq_4, we also have $f(c) = f(b_0) + f(b_1) + f(b_0 + b_1 + c)$. Hence, $f(b_0 + b_1 + c) = f(b_0 + b_1)$, a contradiction with the injectivity of f on $V \setminus \{0\}$.

Now, let x, y be two vectors such that $\{x, y, x + y\} \cap \mathrm{Vect}(b_0, b_1) = \emptyset$. Since we have $\mathrm{Ieq}_4(x, y, b_0+b_1, x+y+b_0+b_1)$, we have: $f(x+y+b_0+b_1) = f(x)+f(y)+f(b_0+b_1)$. Since $\mathrm{Ieq}_4(x+y, b_0, b_1, b_0+b_1+x+y)$, we have: $f(b_0+b_1+x+y) = f(x+y)+f(b_0)+f(b_1)$. Hence, $f(x) + f(y) + f(b_0 + b_1) = f(x + y) + f(b_0) + f(b_1)$, which is equivalent to: $f(x + y) + v = f(x) + v + f(y) + v$, i.e., $h(x + y) = h(x) + h(y)$ for all x, y such that $\{x, y, x+y\} \notin \mathrm{Vect}(b_0, b_1)$. Then by easy calculations, we solve the cases where x belongs to $\mathrm{Vect}(b_0, b_1)$, which concludes the proof. ☐

Lemma 3.3.26. *We have:* $\mathrm{af}^* \in \langle\{\mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1$.

Proof. By definition, there exists $c \neq 0$ such that $\mathrm{af}^{\mathrm{ni}}(0) = h(c)+d$ and $\mathrm{af}^{\mathrm{ni}}(x) = h(x)+a$ for all $x \neq 0$. Let $\alpha \in \mathrm{Emb}(V;+)$ such that $c \notin \alpha(V)$. Then $\mathrm{af}^{\mathrm{ni}} \circ\alpha$ belongs to $\mathrm{cl}(\mathrm{af}^*)$. We conclude by Proposition 3.3.5. ☐

We now describe the endomorphism monoids locally generated by these freshly defined functions.

Corollary 3.3.27. *We have:*

$$\begin{aligned}(V;\mathrm{Ieq}_4) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}})\\ &= \overline{\langle\{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}, \mathrm{af}', \mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}\end{aligned}$$

Proof. The inclusions $\mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \subseteq \overline{\langle\{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}, \mathrm{af}', \mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}$ and:

$$\overline{\langle\{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}, \mathrm{af}', \mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1} \subseteq \mathrm{End}(V;\mathrm{Ieq}_4)$$

are straightforward by Proposition 3.3.5 and Remark 3.3.17.

Conversely, let f be an endomorphism of $(V;\mathrm{Ieq}_4)$. By Proposition 3.3.25, there exists $h \in \mathrm{Emb}(V;+)$ and $v \notin h(V \setminus \{0\})$ such that $f(x) = h(x) + v$ for all $x \in V \setminus \{0\}$. By Lemmas 3.3.9, 3.3.10, 3.3.11, 3.3.21, 3.3.22, 3.3.23, 3.3.24, depending on the value of $f(0)$ and v, f belongs to $\mathrm{cl}(\mathrm{id})$, $\mathrm{cl}(\mathrm{id}^*)$, $\mathrm{cl}(\mathrm{af})$, $\mathrm{cl}(\mathrm{af}')$, $\mathrm{cl}(\mathrm{af}^{\mathrm{ni}})$, or $\mathrm{cl}(\mathrm{af}^*)$ by exhaustivity of this list. ☐

Corollary 3.3.28. *We have:*

$$\begin{aligned}(V;\mathrm{Ieq}_4, \neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \sqcup \mathrm{cl}(\mathrm{af}')\\ &= \overline{\langle\{\mathrm{id}^*, \mathrm{af}, \mathrm{af}', \mathrm{af}^*\} \cup \mathrm{Aut}(V;+)\rangle_1}\end{aligned}$$

Proof. Apply Corollary 3.3.27 but ignore all references to non injective functions, i.e., $\mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}})$. ☐

Recall that we denote by Eq_4 the following 4-ary relation:

$$\mathrm{Eq}_4 := \{(x, y, z, t) \in V^4 \mid x + y + z + t = 0\}$$

Proposition 3.3.29. *Let f be an injective mapping which preserves Eq_4. Then there exists $h \in \mathrm{Emb}(V; +)$ and $v \in V$ such that $f(x) = h(x) + v$ for all $x \in V$.*

Proof. Let $f \in \mathrm{End}(V; \mathrm{Eq}_4, \neq)$. We now show that there exists $h \in \mathrm{Emb}(V; +)$ and $v \in V$ such that $f(x) = h(x) + v$ for all x. Let $h(x) := f(x) + f(0)$, and let $x, y \in V$. Since $\mathrm{Eq}_4(x, y, x + y, 0)$, we have $\mathrm{Eq}_4(f(x), f(y), f(x + y), f(0))$, i.e., $f(x + y) = f(x) + f(y) + f(0)$. Hence, $h(x + y) = f(x + y) + f(0) = f(x) + f(y) = h(x) + h(y)$. Since f is injective, h is injective, so $h \in \mathrm{Emb}(V; +)$. So $f(x) = h(x) + f(0)$ for all $x \in V$, with $h \in \mathrm{Emb}(V; +)$. □

Recall that we denote by t_b the translation of vector $b \in V$, i.e., $t_b(x) = x + b$ for all $x \in V$.

Lemma 3.3.30. *Let b be a non zero vector. The class $\mathrm{cl}(t_b)$ is the set of functions g such that there exists $h \in \mathrm{Emb}(V; +)$ such that $g(x) = h(x + b)$ for all $x \in V$.*

Proof. Let g be a function such that there exists $h \in \mathrm{Emb}(V; +)$ such that $g(x) = h(x+b)$ for all $x \in V$. Then $g = h \circ t_b$, which straightforwardly implies that g belongs to $\mathrm{cl}(t_b)$. The converse inclusion is obvious by definition of $\backsim_{\mathrm{op}}$. □

Lemma 3.3.31. *Let b be a non zero vector. Then* $\mathrm{af}' \in \overline{\langle \{t_b\} \cup \mathrm{Aut}(V; +)\rangle_1}$.

Proof. Let $h \in \mathrm{Emb}(V; +)$ such that $b \notin h(V)$. Then $\mathrm{af}' = t_b \circ h$. □

Proposition 3.3.32. *We have:*

$$\mathrm{End}(V; \mathrm{Eq}_4, \neq) = \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup (\sqcup_{b \neq 0} \mathrm{cl}(t_b)) = \overline{\langle \{t_b\} \cup \mathrm{Aut}(V; +)\rangle_1}$$

Proof. Note that: $\overline{\langle \{t_b\} \cup \mathrm{Aut}(V; +)\rangle_1} = \overline{\langle \{\mathrm{af}', t_b\} \cup \mathrm{Aut}(V; +)\rangle_1}$ by Lemma 3.3.31. Consequently, by Lemmas 3.3.9, 3.3.23, 3.3.30 we have:

$$\mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup (\sqcup_{b \neq 0} \mathrm{cl}(t_b)) \subseteq \overline{\langle \{t_b\} \cup \mathrm{Aut}(V; +)\rangle_1}$$

It is straightforward to see that $\overline{\langle \{t_b\} \cup \mathrm{Aut}(V; +)\rangle_1} \subseteq \mathrm{End}(V; \mathrm{Eq}_4, \neq)$ by definition of a translation and local closure of $\mathrm{End}(V; \mathrm{Ieq}_4, \neq)$.

Conversely, let $f \in \mathrm{End}(V; \mathrm{Eq}_4, \neq)$. Since f preserves Eq_4, by Proposition 3.3.29, there exists $h \in \mathrm{Emb}(V; +)$ and $v \in V$ such that $f(x) = h(x) + b = t_b(h(x))$. We conclude by a case distinction:

- If $b = 0$, f belongs to $\mathrm{cl}(\mathrm{id})$;
- If $b \notin h(V)$, f belongs to $\mathrm{cl}(\mathrm{af}')$;
- Else, f belongs to $\mathrm{cl}(t_b)$ for some $b \neq 0$.

□

Corollary 3.3.33. *We have the following property:*

$$\mathrm{End}(V;\mathrm{Eq}_4,\mathrm{Ind}_1,\neq)=\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af}')=\overline{\langle\{\mathrm{af}'\}\cup\mathrm{Aut}(V;+)\rangle_1}$$

Proof. By Proposition 3.3.32, we have: $\mathrm{End}(V,\mathrm{Eq}_4,\mathrm{Ind}_1,\neq)\subseteq\mathrm{End}(V;\mathrm{Eq}_4,\neq)=\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af}')\sqcup\mathrm{cl}(t_b)$. Note that if a function f preserves Ind_1, then $f(x)\neq 0$ for all $x\neq 0$. Hence, f can not belong to $\mathrm{cl}(t_b)$ by Lemma 3.3.30. □

Definition 3.3.34. We define the 4-ary relation $Z_1\subseteq V^4$ as follows:

$$Z_1=\{(x,y,z,t)\in V^4\mid x=0\wedge\mathrm{Ieq}_3(y,z,t)\}$$

Proposition 3.3.35. *We have the following:*

$$\mathrm{End}(V;\mathrm{Ieq}_4,Z_1\cup\mathrm{Ind}_4,\neq)=\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af}^*)=\overline{\langle\{\mathrm{af}^*\}\cup\mathrm{Aut}(V;+)\rangle_1}$$

Proof. By Proposition 3.3.5, we have: $\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af}^*)\subseteq\overline{\langle\{\mathrm{af}^*\}\cup\mathrm{Aut}(V;+)\rangle_1}$. Furthermore, since af^* clearly preserves $(V;\mathrm{Ieq}_4,Z_1\cup\mathrm{Ind}_4,\neq)$, and since an endomorphism monoid is locally closed, we have:

$$\overline{\langle\{\mathrm{af}^*\}\cup\mathrm{Aut}(V;+)\rangle_1}\subseteq\mathrm{End}(V;\mathrm{Ieq}_4,Z_1\cup\mathrm{Ind}_4,\neq)$$

We now show that $\mathrm{End}(V;\mathrm{Ieq}_4,Z_1\cup\mathrm{Ind}_4,\neq)\subseteq\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af}^*)$. Let f be an element of $\mathrm{End}(V;\mathrm{Ieq}_4,Z_1\cup\mathrm{Ind}_4,\neq)$. Since f preserves Ieq_4 and $\neq$, by Corollary 3.3.28, f belongs to one of the following classes: $\mathrm{cl}(\mathrm{id})$, $\mathrm{cl}(\mathrm{id}^*)$, $\mathrm{cl}(\mathrm{af})$, $\mathrm{cl}(\mathrm{af}')$, or $\mathrm{cl}(\mathrm{af}^*)$. Since f preserves $Z_1\cup\mathrm{Ind}_4$, it is straightforward to see that f belongs to $\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af}^*)$. □

Definition 3.3.36. We define the 4-ary relation $R_1\subseteq V^4$ as follows:

$$R_1=\{(x,\sigma(y),\sigma(z),\sigma(t))\in V^4\mid x=y\wedge\mathrm{Ind}_3(y,z,t),\ \text{with }\sigma\in\mathrm{Perm}(\{y,z,t\})\}$$

Note that $\mathrm{af}^{\mathrm{ni}}$ preserves $Z_1\cup R_1\cup\mathrm{Ind}_4$.

Proposition 3.3.37. *We have:*

$$\mathrm{End}(V;\mathrm{Ieq}_4,Z_1\vee R_1\vee\mathrm{Ind}_4)=\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af}^{\mathrm{ni}})\sqcup\mathrm{cl}(\mathrm{af}^*)=\overline{\langle\{\mathrm{af}^{\mathrm{ni}}\}\cup\mathrm{Aut}(V;+)\rangle_1}$$

Proof. Since $\mathrm{af}^{\mathrm{ni}}$ together with $\mathrm{Aut}(V;+)$ locally generates af^*, we have by Proposition 3.3.5: $\mathrm{cl}(\mathrm{af}^{\mathrm{ni}})\sqcup\mathrm{cl}(\mathrm{af}^*)\subseteq\overline{\langle\{\mathrm{af}^{\mathrm{ni}}\}\cup\mathrm{Aut}(V;+)\rangle_1}$. By definition of $\mathrm{af}^{\mathrm{ni}}$ and the fact that an endomorphism monoid is locally closed, we have:

$$\overline{\langle\{\mathrm{af}^{\mathrm{ni}}\}\cup\mathrm{Aut}(V;+)\rangle_1}\subseteq\mathrm{End}(V;\mathrm{Ieq}_4,Z_1\vee R_1\vee\mathrm{Ind}_4)$$

Conversely, let f be an endomorphism of $(V;\mathrm{Ieq}_4,Z_1\vee R_1\vee\mathrm{Ind}_4)$. By Corollary 3.3.27, since f preserves Ieq_4, f belongs to one of the following classes: $\mathrm{cl}(\mathrm{id})$, $\mathrm{cl}(\mathrm{id})$, $\mathrm{cl}(\mathrm{id}^{\mathrm{ni}})$, $\mathrm{cl}(\mathrm{id}^*)$, $\mathrm{cl}(\mathrm{af})$, $\mathrm{cl}(\mathrm{af}')$, $\mathrm{cl}(\mathrm{af}^*)$, or $\mathrm{cl}(\mathrm{af}^{\mathrm{ni}})$. Assume that f does not belong to $\mathrm{cl}(\mathrm{id})$. Since f preserves $Z_1\vee R_1\vee\mathrm{Ind}_4$, $f(0)\notin\{0,v\}$ and $v\neq 0$. Consequently, f belongs to either $\mathrm{cl}(\mathrm{af}^*)$ or $\mathrm{cl}(\mathrm{af}^{\mathrm{ni}})$. □

Proposition 3.3.38. *We have the following property:*

$$\mathrm{End}(V;\mathrm{Ieq}_4,0)=\mathrm{End}(V;\mathrm{Ieq}_4,0,\neq)=\mathrm{cl}(\mathrm{id})\sqcup\mathrm{cl}(\mathrm{af})=\overline{\langle\{\mathrm{af}\}\cup\mathrm{Aut}(V;+)\rangle_1}$$

Proof. The proof uses exactly the same method as Proposition 3.3.35. We just have to note that if a function f preserves Ieq_4 and 0, then it is necessarily injective, so it preserves $\neq$. □

3.3.4 Generic functions gen and gen*

As opposed to the functions id and id* which preserve a vast amount of relations, we define two "generic" functions, gen and gen*, which violate many relations in the sense that they send every family of distinct non-zero elements of V to a free family of $(V;+)$.

Definition 3.3.39. Let $(b_n)_{n\geq 0}$ be an arbitrary fixed infinite linearly independent family of vectors of $(V;+)$. Let us consider an enumeration $(a_i)_{i\in\mathbb{N}}$ of V such that $a_0=0$. We define the maps gen and gen* from $V\to V$ as follows:

- $\mathrm{gen}(0)=0$ and $\mathrm{gen}(a_i)=b_i$ for all $i\geq 1$;
- $\mathrm{gen}^*(a_i)=b_i$ for all $i\in\mathbb{N}$.

Remark 3.3.40. The maps gen and gen* sends any tuple in Ieq_n to a tuple in Ind_n but preserve Ind_n for all $n\geq 3$.

In fact, we have more. Let $(d_n)_{n\geq 0}$ be an infinite family of pairwise distinct non zero vectors of V. Then $(\mathrm{gen}(d_n))_{n\geq 0}$ is linearly independent, as is $(\mathrm{gen}^*(d_n))_{n\geq 0}$.

We now prove that we can consider gen and gen* independently from the choice of the basis $(b_n)_{n\geq 0}$ of $(V;+)$ or the enumeration $(a_n)_{n\geq 0}$ we used to defined them.

Lemma 3.3.41. *The class* cl(gen) *(resp.* cl(gen*)*) is the set of functions g which preserve* 0 *and such that for all integer n and for all family $(a'_k)_{k\leq n}$ of distinct elements of $V\setminus\{0\}$ (resp. of distinct elements of V), $(g(a'_k))_{k\leq n}$ is linearly independent.*

Proof. We give the proof for gen, the proof for gen* being similar. We proceed by double inclusions. Let g be a function which preserves 0 and such that for all finite tuple $\overline{x}$ of pairwise non zero distinct elements of V, $(g(x_1),\dots,g(x_n))$ is linearly independent. Let $(a_1,\dots,a_n)$ be a tuple of distinct elements of $V\setminus\{0\}$. It is straightforward to see that $\mathrm{tp}_{(V;+)}(g(a_1),\dots,g(a_n))=\mathrm{tp}_{(V;+)}(\mathrm{gen}(a_1),\dots,\mathrm{gen}(a_n))$ by property of gen and g. Hence there exists $\alpha\in\mathrm{Aut}(V;+)$ such that $\mathrm{gen}(a_i)=\alpha(g(a_i))$ for all $i\leq n$. Consequently, g belongs to cl(gen) by Proposition 3.3.5.

The second inclusion is straightforward by definition of cl(gen). □

The following proposition can be a bit unintuitive but it strengthens the difference between af′ and the three functions id*, af, and af*. Indeed, any of the three latter functions together with a translation of a non-zero vector and $\mathrm{Aut}(V;+)$ locally generates the function gen*. To the contrary, the uniformity of af′ makes it "compatible" with the translations, and thus af' together with $\mathrm{Aut}(V;+)$ and a translation of a non-zero vector locally generates $\mathrm{End}(V;\mathrm{Eq}_4,\neq)$ (see Proposition 3.3.32). Consequently, we can break a lot more relations combining $\mathrm{id}^*, \mathrm{af}$, or af^* with a translation, than combining af' and a translation.

Proposition 3.3.42. *Let Γ be a first-order reduct of $(V;+)$ such that $t_b \in \mathrm{End}(\Gamma)$ with $b \neq 0$. Also assume that at least one of the following functions preserves Γ: $\mathrm{id}^*, \mathrm{af}, \mathrm{af}^*$. Then gen^* belongs to $\mathrm{End}(\Gamma)$.*

Proof. First note that $\{\mathrm{af}, \mathrm{af}^*\} \subseteq \overline{\langle\{t_b, \mathrm{id}^*\} \cup \mathrm{Aut}(V;+)\rangle_1}$, since $t_b \circ \mathrm{id}^*$ has the same belongs to $\mathrm{cl}(\mathrm{af}^*)$, and $t_{\mathrm{id}^*(0)} \circ \mathrm{id}^*$ belongs to $\mathrm{cl}(\mathrm{af})$. Similarly, we have: $\{\mathrm{id}^*, \mathrm{af}^*\} \subseteq \overline{\langle\{t_b, \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1}$, and $\{\mathrm{id}^*, \mathrm{af}\} \subseteq \overline{\langle\{t_b, \mathrm{af}^*\} \cup \mathrm{Aut}(V;+)\rangle_1}$. Consequently, we can always assume that $\{\mathrm{id}^*, \mathrm{af}, \mathrm{af}^*\} \subseteq \mathrm{End}(\Gamma)$. We now prove that gen^* belongs to $\mathrm{End}(\Gamma)$. It is sufficient to prove that for all finite subset K of V, there exists $g_K \in \mathrm{End}(\Gamma)$ such that the family $(g_K(x))_{x\in K}$ is free. We prove it by induction on the size of the subset K.

If $|K| = 2$ with $0 \in K$, then the property is satisfied since id^* is injective and $0 \notin \mathrm{id}^*(K)$. We now suppose that the property is satisfied for all subsets of size n containing 0. Let K' be a subset of V of size $n+1$ containing 0. There exists $(x_0, x_1, \dots, x_n = 0) \in V^{n+1}$ such that $K' = \{x_0\} \cup K$ with $K := \{x_1, \dots, x_n\}$. By induction, there exists $g_K \in \mathrm{End}(\Gamma)$ such that $(g_K(x))_{x\in K}$ is linearly independent. We then define $g_{K'}$ as follows: $g_{K'} := \mathrm{af}^* \circ t_{g_K(x_0)} \circ g_K$, i.e., there exist $h \in \mathrm{Emb}(V;+)$, $d \notin h(V)$ and $c \neq 0$ such that $c + d \notin h(V)$ such that $g_{K'}(x_0) = c$ and $g_{K'}(x) = h(g_K(x) + g_K(x_0)) + d$, for all $x \neq x_0$. Note that $g_{K'}$ belongs to $\mathrm{End}(\Gamma)$ since it is a composition of three endomorphisms of γ.

We now prove that $(g_{K'}(x))_{x\in K'}$ is linearly independent. First note that $g_{K'}(x) \neq 0$ for all $x \in V$, since $0 \notin \mathrm{af}^*(V)$. Suppose now that there exist $\{x_{i_1}, \dots, x_{i_k}\} \subseteq \{x_1, \dots, x_n\}$ such that $\sum_{j\leq k} g_{K'}(x_{i_j}) = 0$. We distinguish two cases:

- either k is odd. In this case, the previous sum simplifies as follows:

  $$d + h(\sum_{j\leq k} g_K(x_{i_j}) + g_K(x_0)) = 0$$

  This contradicts the fact that $d \notin h(V)$.

- or k is even. In this case, the previous sum simplifies as follows:

  $$h(\sum_{j\leq k} g_K(x_{i_j}) + g_K(x_0)) = 0$$

  This implies: $\sum_{j\leq k} g_K(x_{i_j}) = 0$, which contradicts the fact that $(g_K(x))_{x\in K}$ is a free family of V.

Suppose now that there exist $1 \leq i_1 < \cdots < i_k \leq n$ such that $g_{K'}(x_0) + \sum_{j\leq k} g_{K'}(x_{i_j}) = 0$. We distinguish two cases:

- either k is odd. In this case, the previous sum simplifies as follows:

$$c + d + h(\sum_{j \leq k} g_K(x_{i_j}) + g_K(x_0)) = 0$$

  This contradicts the fact that $d \notin \mathrm{Vect}(h(V), c)$.

- or k is even. In this case, the previous sum simplifies as follows:

$$c + h(\sum_{j \leq k} g_K(x_{i_j}) + g_K(x_0)) = 0$$

  This contradicts the fact that $c \notin h(V)$.

Consequently, $(g_{K'}(x))_{x \in K'}$ is a free family of V, and for all finite subset K of V containing 0, there exists an endomorphism g_K of Γ, and an automorphism $\alpha \in \mathrm{Aut}(V; +)$, such that $\alpha(g_K(x)) = \mathrm{gen}^*(x)$ for all $x \in K$. Hence, $\mathrm{gen}^* \in \mathrm{End}(\Gamma)$. $\square$

3.4 Canonical Functions over the Vector Space

3.4.1 Ramsey Order on $(V; +)$

In order to apply Corollary 2.4.16 and locally generate canonical functions, we cite a result from Kechris, Pestov, and Todorcevic that defines an order over $(V; +)$ which makes the structure $(V; +, <)$ Ramsey.

Notation 3.4.1. We denote by $\mathcal{V}\mathbb{F}_2$ the class of finite vector spaces over $\mathbb{F}_2$.

Fact 3.4.2. *The class $\mathcal{V}\mathbb{F}_2$ is a Fraïssé class.*

Proof. We only have to prove that this class has the (JEP), (HP) and (AP), which is straightforward. $\square$

Proposition 3.4.3. *$(V; +)$ has an expansion $(V; +, <)$ by a dense linear order unbounded over $V \setminus \{0\}$ which is Ramsey.*

Proof. If V_0 is a finite dimensional vector space over $\mathbb{F}_2$ of dimension n and $\mathcal{B}$ is a basis for V_0, then every ordering $b_1 < \cdots < b_n$ of $\mathcal{B}$ gives an ordering on V_0 by $\alpha_1 b_1 + \cdots + \alpha_n b_n < \beta_1 b_1 + \cdots + \beta_n b_n \Leftrightarrow (\alpha_n < \beta_n) \vee (\alpha_n = \beta_n \wedge \alpha_{n-1} < \beta_{n-1}) \vee \cdots$, i.e., $<$ is the anti-lexicographical ordering induced by the ordering of $\mathcal{B}$. A natural ordering of V_0 is one induced in this way by an ordering of a basis, with 0 as the minimum.

Let $\mathcal{OV}\mathbb{F}_2$ be the ordered class of all $(V_0, <)$ such that V_0 is a finite-dimensional vector space and $<$ a natural ordering on V_0. Thomas shows in [Tho86] that this is a Fraïssé class, and its Fraïssé limit is an expansion of $(V; +)$ denoted by $(V; +, <)$. Hence $(V; +, <)$ is homogeneous, ω-categorical.

Fact 3.4.4. *The class $\mathcal{OV}\mathbb{F}_2$ has the Ramsey property. Hence, $(V; +, <)$ is Ramsey.*

The proof can be found just before Theorem 6.12 in [KPT05] and follows from the fact that $\mathcal{V}\mathbb{F}_2$ has the Ramsey property (as shown by Graham and Rothschild in [GR71]). Consequently, since $(V;+,<)$ is homogeneous, $(V;+,<)$ is Ramsey.

We now prove that $<$ is dense, and 0 is its minimum. Let x, y be two non-zero distinct elements of V. Let $x \neq 0$ in V. The $\mathbb{F}_2$-vector space $(\{x,0\};+)$ is in the class $\mathcal{OV}\mathbb{F}_2$, and $0 < x$ is forced in this settings by definition of the natural ordering on finite vector spaces. Consequently, for all $x \in V \setminus \{0\}$, $0 < x$. Let x', y' be two distinct non-zero elements of V such that there exists z' in V such that $x' < z' < y'$ (such elements exist because V is infinite and $<$ is linear). Then the mapping σ defined on $\mathrm{Vect}(x', y')$ by : $\sigma(x') = x$, $\sigma(y') = y$, $\sigma(0) = 0$, and $\sigma(x' + y') = x + y$ is a local isomorphism between two substructures of $(V;+,<)$. Since $(V;+,<)$ is homogeneous, there exists an automorphism α of $(V;+,<)$ such that $\sigma \subseteq \alpha$. Consequently, $x < \alpha(z') < y$, and $<$ is a dense linear order with minimum 0. It is straightforward to prove that $<$ has no maximum element. $\square$

Corollary 3.4.5. *The structure $(V;+,<)$ is a totally ordered ω-categorical Ramsey structure which admits quantifier elimination.*

Proof. By Proposition 3.4.3, $(V;+,<)$ is a totally ordered Ramsey structure. Hence $(V;+,<)$ is homogeneous. Since $(V;+,<)$ is clearly uniformly locally finite, it is ω-categorical and admits quantifier elimination by Theorem 2.2.11. $\square$

3.4.2 Classification without constants

In this section, we describe the injective canonical functions from $(V;+,<)$ to $(V;+)$ to eventually give a full classification.

Lemma 3.4.6. *Let f be a canonical function from $(V;+)$ to $(V;+)$ and g be a function from $V \to V$ such that $f \backsim_{\mathrm{op}} g$. Then g is canonical with the same behaviour as f. In other words, the equivalence class of a canonical function f from $(V;+)$ to $(V;+)$ modulo $\backsim_{\mathrm{op}}$ is a canonical function with same behaviour.*

Proof. The proof is straightforward by definition of $f \backsim_{\mathrm{op}} g$ and canonicity, since the self-embeddings of $(V;+)$ preserve every type. $\square$

Corollary 3.4.7. *Let f be a canonical function from $(V;+)$ to $(V;+)$. Then $\mathrm{cl}(f)$ is exactly the set of canonical functions from $(V;+)$ to $(V;+)$ which has the same behaviour as f.*

Proof. Lemma 3.4.6 gives the first direction of the equivalence. We now show that any canonical function g with same behaviour as f belongs to $\mathrm{cl}(f)$. We use the characterization of $\mathrm{cl}(f)$ given in Lemma 3.3.3. Let $S := \{s_1, \ldots, s_n\}$ be a finite subset of V. Since f and g are canonical with same behaviour, we have $\mathrm{tp}_{(V;+)}(f(s_1), \ldots, f(s_n)) = \mathrm{tp}_{(V;+)}(g(s_1), \ldots, g(s_n))$. Hence, by ω-categoricity of $(V;+)$, there exists $\alpha \in \mathrm{Aut}(V;+)$ such that $\alpha(f(s_i)) = g(s_i)$ for all $i \leq n$. Hence, g belongs to $\mathrm{cl}(f)$. $\square$

Observation 3.4.8. A canonical function from $(V;+)$ to $(V;+)$ is also canonical from $(V;+,<)$ to $(V;+)$.

Lemma 3.4.9. *Let f be a canonical injection from $(V;+,<)$ to $(V;+)$. Then f preserves* Ind_n *for all $n \geq 2$.*

Proof. Suppose that f does not preserve Ind_i for all $i \geq 2$, and let n be the smallest integer such that Ind_n is not preserved. There exists $(x_1,\ldots,x_n)$ such that $\mathrm{Ind}_n(x_1,\ldots,x_n)$ and $\neg\,\mathrm{Ind}_n(f(x_1),\ldots,f(x_n))$. Then $f(x_1)+\cdots+f(x_n)=0$, i.e., $f(x_1)=f(x_2)+\cdots+f(x_n)$. Since $(V;+)$ has infinite dimension and $(x_1,\ldots,x_n)$ is linearly independent, and since $<$ is an unbounded dense linear order on $V\setminus\{0\}$, we can find x_1' such that $(x_1',x_2,\ldots,x_n)$ is linearly independent and x_1' is ordered as x_1 with respect to $(x_2,\ldots,x_n)$. Then we have $\mathrm{tp}_{(V;+,<)}(x_1,\ldots,x_n)=\mathrm{tp}_{(V;+,<)}(x_1',x_2,\ldots,x_n)$. Because f is canonical, we have $f(x_1')=f(x_2)+\cdots+f(x_n)=f(x_1)$, a contradiction to the injectivity of f. □

As the canonical injections of $(V;+)$ preserve the relations Ind_n for all n, they have to either preserve Ieq_n, or send them to Ind_n.

Corollary 3.4.10. *Let f be a canonical injection from $(V;+,<)$ to $(V;+)$ and n be a positive integer. Then either f preserves* Ieq_n *(i.e., $f(\mathrm{Ieq}_n)=\mathrm{Ieq}_n$), or $f(\mathrm{Ieq}_n)=\mathrm{Ind}_n$.*

Proof. Suppose that f does not preserve Ieq_n. There exists $(x_1,\ldots,x_n)$ such that $\mathrm{Ieq}_n(x_1,\ldots,x_n)$ and $\neg\,\mathrm{Ieq}_n(f(x_1),\ldots,f(x_n))$. By Lemma 3.4.9, for all $k<n$ and all $1\leq i_1<\cdots<i_k\leq n$ we have $\mathrm{Ind}_k(x_{i_1},\ldots,x_{i_k})$. Since $\neg\,\mathrm{Ieq}_n(f(x_1),\ldots,f(x_n))$, we have $f(x_1)+\cdots+f(x_n)\neq 0$. Consequently, $\mathrm{Ind}_n(f(x_1),\ldots,f(x_n))$. Now let us consider $(y_1,\ldots,y_n)$ such that $\mathrm{Ieq}_n(y_1,\ldots,y_n)$. There exists a permutation α of $\{1,\ldots,n\}$ such that $(y_{\alpha(1)},\ldots,y_{\alpha(n)})$ and $(x_1,\ldots,x_n)$ are ordered in the same way. Consequently, $\mathrm{tp}_{(V;+,<)}(x_1,\ldots,x_n)=\mathrm{tp}_{(V;+,<)}(y_{\alpha(1)},\ldots,y_{\alpha(n)})$, so because f is canonical from $(V;+,<)$ to $(V;+)$, we have:

$$\mathrm{tp}_{(V;+)}(f(x_1),\ldots,f(x_n))=\mathrm{tp}_{(V;+)}(f(y_{\alpha(1)}),\ldots,f(y_{\alpha(n)}))$$

Hence, we have $\mathrm{Ind}_n(f(y_{\alpha(1)}),\ldots,f(y_{\alpha(n)}))$, which implies $\mathrm{Ind}_n(f(y_1),\ldots,f(y_n))$. To conclude: $f(\mathrm{Ieq}_n)=\mathrm{Ind}_n$. □

Proposition 3.4.11. *The function* id^* *is an injective canonical function from $(V;+)$ to $(V;+)$ which preserves* Ieq_n *and* Ind_n *for all $n\geq 2$.*

Proof. By Remark 2.4.9, we only have to verify that for all n-type p, there exist a n-type q such that $\mathrm{id}^*(p)=q$. Let $\overline{a}$ be an n-tuple of pairwise distinct elements of V and let p be the type of $\overline{a}$. If $\overline{a}$ does not contain 0, then $\mathrm{id}^*(p)=p$ by definition of id^* on $V\setminus\{0\}$. If $\overline{a}$ contains 0, we can assume that $a_1=0$ and $a_i\neq 0$ for all $i\geq 2$. Since $(a_2,\ldots,a_n)$ is a tuple of elements of $V\setminus\{0\}$, $\mathrm{id}^*(\mathrm{tp}_{(V;+)}(a_2,\ldots,a_n))=\mathrm{tp}_{(V;+)}(a_2,\ldots,a_n)$ as we just showed in the previous case. Now, since $\mathrm{id}^*(0)\notin h(V)$ by definition of id^*, $\mathrm{id}^*(0)$ is linearly independent of $(\mathrm{id}^*(a_2),\ldots,\mathrm{id}^*(a_n))$. Let q be the type of $(\mathrm{id}^*(a_1),\ldots,\mathrm{id}^*(a_n))$. Then $\mathrm{id}^*(p)=q$. □

Remark 3.4.12. id and id* are canonical functions, but id$^{\mathrm{ni}}$ is not. Indeed, let $d \neq c$ such that $\mathrm{tp}_{(V;+)}(0, c) = \mathrm{tp}_{(V;+)}(0, d)$. We have $\mathrm{id}^{\mathrm{ni}}(0) = \mathrm{id}^{\mathrm{ni}}(c)$ but $\mathrm{id}^{\mathrm{ni}}(0) \neq \mathrm{id}^{\mathrm{ni}}(d)$ since h is injective.

We now focus on generic functions.

Proposition 3.4.13. *The functions* gen *and* gen* *are injective canonical functions from* $(V;+)$ *to* $(V;+)$ *which send* Ieq_n *to* Ind_n *for all* $n \geq 3$.

Proof. We give the proof for gen*, the proof for gen is analogous. By Remark 3.3.40, gen* sends all injective n-types to Ind_n, which isolates a type. Hence by Remark 2.4.9, gen* is canonical. □

Note that every canonical injection which sends Ieq_n to Ind_n for all $n \geq 3$ locally generates together with $\mathrm{Aut}(V;+)$ either gen or gen*. This proposition emphasizes the role of the relations Ieq_n and Ind_n in our article. More formally:

Proposition 3.4.14. *Let f be a canonical injection of $(V;+)$ such that $f(\mathrm{Ieq}_n) = \mathrm{Ind}_n$ for all $n \geq 3$. Then f together with $\mathrm{Aut}(V;+)$ locally generates either* gen, *or* gen*, *depending whether f preserves 0 or not.*

Proof. We show that for every finite subset D of $V \setminus \{0\}$, there exists a function $g_D \in \overline{\langle \{f\} \cup \mathrm{Aut}(V;+) \rangle_1}$ such that g_D is injective over $(V;+)$, and $g_D(D)$ is a linearly independent family of vectors. The inductive property is clearly satisfied for every subset D of cardinality one. Suppose now that it is true for every subset D of cardinality n, and let $x \in V \setminus (D \cup \{0\})$. We now consider $g_{D\cup\{x\}} := f \circ g_D$. Since f is a canonical injection of $(V;+)$ such that $f(\mathrm{Ieq}_i) = \mathrm{Ind}_i$ for all i, and since $g_D(D)$ is linearly independent by inductive assumption, it is straightforward to see that $g_{D\cup\{x\}}(D)$ is linearly independent. Suppose now that $g_{D\cup\{x\}}(D \cup \{x\})$ is not linearly independent. Then, there exists $x_1, \ldots, x_j \in D$ such that $\mathrm{Ieq}_{j+1}(f(g_D(x)), f(g_D(x_1)), \ldots, f(g_D(x_j)))$. Since $g_D(D)$ is linearly independent, there are two possibilities. Either

- $\mathrm{Ind}_{j+1}(g_D(x), g_D(x_1), \ldots, g_D(x_j))$, in which case, we would have the following: $\mathrm{Ind}_{j+1}(f(g_D(x)), f(g_D(x_1)), \ldots, f(g_D(x_j)))$, since f preserves Ind_i for all i by Observation 3.4.8 and Lemma 3.4.9. Or,

- there exists $1 \leq i_1 < \cdots < i_k \leq j$ such that $\mathrm{Ieq}_{k+1}(g_D(x), g_D(x_{i_1}), \ldots, g_D(x_{i_k}))$, in which case there is again a contradiction since $g_D(\mathrm{Ieq}_{j+1}) = \mathrm{Ind}_{j+1}$. If $k = j$, we obtain an immediate contradiction since $f(\mathrm{Ieq}_{j+1}) = \mathrm{Ind}_{j+1}$. If $k < j$, there exists $i_0 \leq j$ such that $i_0 \notin \{i_1, \ldots, i_k\}$. By assumption, $g_D(x_{i_0})$ does not belong to $\mathrm{Vect}(g_D(x), g_D(x_1), \ldots, g_D(x_{i_0-1}), g_D(x_{i_0+1}), \ldots, g_D(x_j))$. Let us consider $y_0 \notin \mathrm{Vect}(g_D(x), g_D(x_1), \ldots, g_D(x_n))$. We have the following equality of types:

$$\begin{aligned}&\mathrm{tp}_{(V;+)}(g_D(x), g_D(x_1), \ldots, g_D(x_{i_0-1}), g_D(x_{i_0}), g_D(x_{i_0+1}), \ldots, g_D(x_j))\\ &\qquad = \mathrm{tp}_{(V;+)}(g_D(x), g_D(x_1), \ldots, g_D(x_{i_0-1}), y_0, g_D(x_{i_0+1}), \ldots, g_D(x_j))\end{aligned}$$

  Consequently, by canonicity of f, we have $f(g_D(x_{i_0})) = f(y_{i_0})$, which contradicts the injectivity of f.

Hence, $g_{D\cup\{x\}}(D \cup \{x\})$ is a set of linearly independent vectors, and the property is satisfied for every subset of cardinality $n + 1$. We can now conclude that either gen, or gen* belong to the local closure of $\langle \{f\} \cup \mathrm{Aut}(V;+)\rangle_1$. □

We now focus on quasi-affine functions.

Remark 3.4.15. Note that $\mathrm{af}^{\mathrm{ni}}$ is not canonical, even if, as we show in the next Proposition, $\mathrm{af}, \mathrm{af}^*$, and af' are. Indeed, by definition of $\mathrm{af}^{\mathrm{ni}}$, there exists $c \neq 0$ such that $\mathrm{af}^{\mathrm{ni}}(c) = \mathrm{af}^{\mathrm{ni}}(0)$. Let $c' \neq c$ be an element of $V \setminus \{0\}$. We have $\mathrm{tp}_{(V;+)}(0, c) = \mathrm{tp}_{(V;+)}(0, c')$, but: $\mathrm{af}^{\mathrm{ni}}(0) = \mathrm{af}^{\mathrm{ni}}(c)$ and $\mathrm{af}^{\mathrm{ni}}(0) \neq \mathrm{af}^{\mathrm{ni}}(c')$.

Proposition 3.4.16. *The functions* af*,* af**, and* af$'$ *are injective canonical functions from* $(V;+)$ *to* $(V;+)$*.*

Proof. Let p be a n-type and $(c_1, \ldots, c_n), (d_1, \ldots, d_n)$ be two tuples of V of type p. We will show that $\mathrm{tp}(\mathrm{af}'(c_1, \ldots, c_n)) = \mathrm{tp}(\mathrm{af}'(d_1, \ldots, d_n))$. Since h is a self-embedding of $(V;+)$, we have $\mathrm{tp}_{(V;+)}(h(c_1, \ldots, c_n)) = \mathrm{tp}_{(V;+)}(h(d_1, \ldots, d_n)$. Since $(V;+)$ is ω-categorical, there exists an automorphism σ of $(V;+)$ such that $h(d_i) = \sigma(h(c_i))$ for all $i \leq n$. Recall that t_a stands for the translation of vector a. Let C_1 be the substructure of $(V;+)$ generated by $h(\{c_1, \ldots, c_n\})$, C_2 be the substructure of $(V;+)$ generated by $h(\{d_1, \ldots, d_n\})$, D_1 be the substructure of $(V;+)$ generated by $t_a(C_1)$ and D_2 be the substructure of $(V;+)$ generated by $t_a(C_2)$. Note that $\sigma{\restriction}C_1$ is an isomorphism of vector spaces sending C_1 to C_2. By Lemma 3.3.20, since $a \notin C_1 \cup C_2$, we have: $D_1 = C_1 \cup t_a(C_1)$, $D_2 = C_2 \cup t_a(C_2)$, and the mapping $\tau\colon D_1 \to D_2$ such that $\tau(x) = \sigma(x)$ if $x \in C_1$, and $\tau(x) = \sigma(x + a) + a$ if $x \in D_1 \setminus C_1$, is an isomorphism of vector spaces between D_1 and D_2. Since $a \notin h(V)$, we have $c_i + a \in D_1 \setminus C_1$, so:

$$\tau(\mathrm{af}'(c_i)) = \tau(h(c_i) + a) = \sigma(h(c_i) + a + a) + a = \sigma(h(c_i)) + a = h(d_i) + a = \mathrm{af}'(d_i)$$

for all $i \leq n$, we conclude that $\mathrm{tp}(\mathrm{af}'(c_1, \ldots, c_n)) = \mathrm{tp}(\mathrm{af}'(d_1, \ldots, d_n))$.

For af, similarly, if $c_i = 0$, we have $d_i = 0$, then $\tau(\mathrm{af}(c_i)) = \tau(0) = 0 = \mathrm{af}(d_i)$. If $c_i \neq 0$, $\tau(\mathrm{af}(c_i)) = \tau(h(c_i)+a) = \sigma(h(c_i)+a+a)+a = \sigma(h(c_i))+a = h(d_i)+a = \mathrm{af}(d_i)$, so we conclude that $\mathrm{tp}(\mathrm{af}(c_1, \ldots, c_n)) = \mathrm{tp}(\mathrm{af}(d_1, \ldots, d_n))$.

The proof for af* is analogous since 0 is sent to a non-zero vector $d \notin h(V \setminus \{0\})$. □

We are now able to classify the injective canonical behaviours from $(V;+,<)$ to $(V;+)$. Despite the functional signature, there is a finite number of them up to local closure.

Theorem 3.4.17. *Let f be a canonical injection from $(V;+,<)$ to $(V;+)$. Then exactly one of the following cases holds:*

- *f belongs to* cl(id) *or* cl(id*)*;*
- *f belongs to* cl(af), cl(af$'$)*, or* cl(af*)*;*
- *f together with* $\mathrm{Aut}(V;+)$ *locally generates either* gen *or* gen*.

Proof. First note that since f is injective, it preserves $\neq$. By Lemma 3.4.9, f preserves Ind_n for all $n \geq 2$. Let us suppose that f does not preserve Ieq_4. Due to Lemma 3.2.19 and Corollary 3.4.10, we have that $f(\mathrm{Ieq}_n) = \mathrm{Ind}_n$ for all $n \geq 3$. Hence, by Proposition 3.4.14, we can conclude that depending on whether f preserves 0 or not, f together with $\mathrm{Aut}(V;+)$ locally generates either gen or gen*.

If f preserves Ieq_4, then by Corollary 3.3.28, f belongs to one of the following classes: cl(id), cl(id*), cl(af), cl(af*), or cl(af$'$), since the other classes listed in the corollary contain non injective functions. $\square$

Corollary 3.4.18. *Let f be a canonical injection from $(V;+)$ to $(V;+)$. Then exactly one of the following cases holds:*

- *f has the same behaviour on $(V;+)$ as* id *or* id**;*
- *f has the same behaviour on $(V;+)$ as* af, af$'$*, or* af**;*
- *f together with* $\mathrm{Aut}(V;+)$ *locally generates* gen *or* gen**.*

Proof. The proof is a direct application of Theorem 3.4.17 combined with Observation 3.4.8 and Corollary 3.4.7. $\square$

The following proposition explains why we only focus on studying injective canonical functions. Indeed, non injective canonical functions are constant over at least one 1-orbit with strictly more than one elements. There is two distinct 1-orbit on $(V;+)$: an element can either be 0 or distinct from 0. Hence, we have the following:

Proposition 3.4.19. *Let f be a canonical non injective function from $(V;+,<)$ to $(V;+)$. Then f is constant on $V \setminus \{0\}$.*

Proof. Let $c \neq d$ be two non-zero vectors such that $f(c) = f(d)$, and assume that $c < d$. Let a be a vector distinct from c, d, and 0, such that $c < a$. Such an a exists since $<$ is linearly and unbounded on $V \setminus \{0\}$. Since $\mathrm{tp}_{(V;+,<)}(c,d) = \mathrm{tp}_{(V;+,<)}(c,a)$, and since f is canonical, we have $f(a) = f(c) = f(d)$ and this is true for all $a \in V \setminus \{0, c, d\}$ such that $c < a$. Let $a' \in V \setminus \{0, c, d\}$ be such that $a' < c$. Then $\mathrm{tp}_{(V;+,<)}(a', c) = \mathrm{tp}_{(V;+,<)}(d, c)$, and hence, $f(a') = f(c) = f(d)$. To conclude, f is constant on $V \setminus \{0\}$.

If $c = 0$, we have $\mathrm{tp}_{(V;+)}(c, d) = \mathrm{tp}_{(V;+)}(c, a)$ for all $a \neq 0$. Because f is canonical, we have that: $\mathrm{tp}_{(V;+,<)}(f(c), f(d)) = \mathrm{tp}_{(V;+,<)}(f(c), f(a))$, so $f(a) = f(0)$. Therefore, f is constant on V. $\square$

3.4.3 Outstanding Types with Constants

As we did for the structure $(V;+)$, we now isolate outstanding types on the same structure, but adding n constants to the signature.

Notation 3.4.20. Let $c_1, \ldots, c_n$ be n fixed vectors of V. In the following, we denote by C the finite vector space generated by $c_1, \ldots, c_n$.

Definition 3.4.21. We define the following relations:

- Gen_k^u denotes the k-ary relation that contains all tuples $(x_1, \dots, x_k) \in V$ such that for all $c \in C$, for every $l < k$ and for all $1 \leq i_1 < \dots < i_l \leq k$, we have:

$$x_{i_1} + \dots + x_{i_l} \neq c$$

- Ind_k^u denotes $\{(x_1, \dots, x_k) \in V^k \mid \forall c \in C,\, x_1 + \dots + x_k \neq c \wedge \mathrm{Gen}_k^u(x_1, \dots, x_k)\}$.

- for $c \in C$, Ieq_k^c denotes $\{(x_1, \dots, x_k) \in V^k \mid x_1 + \dots + x_k = c \wedge \mathrm{Gen}_k^u(x_1, \dots, x_k)\}$.

- More generally, we define the relation Ieq_k^d for all $d \in V$ as follows:

$$\mathrm{Ieq}_k^d = \{(x_1, \dots, x_k) \in V^k \mid x_1 + \dots + x_k = d \wedge \mathrm{Gen}_k^u(x_1, \dots, x_k)\}$$

Remark 3.4.22. Note that Ind_k^u isolates the type over $(V; +, c_1, \dots, c_n)$ of k-tuples $(a_1, \dots, a_k)$ such that $\mathrm{Ind}_n(a_1, \dots, a_k)$ and $\mathrm{Vect}(a_1, \dots, a_k) \cap C = \{0\}$. This notion is a generalisation of the notion of linear independence, since $\overline{a}$ is now linearly independent with respect to the finite vector space C, which strictly contains the vector space $\{0\}$. Note also that if $d \in C$, the relation Ieq_k^d isolates a k-type of $(V; +, c_1, \dots, c_n)$. But if $d \notin C$, this relation is strictly contained in Ind_k^u and is not a type anymore.

Let $\mathfrak{V}$ be the structure of domain V with the following relations: Ieq_i^c and Ind_i^u for every $c \in C$ and every $i \geq 1$. Let τ be its signature.

Lemma 3.4.23. *For every complete n-type p over $(V; +, c_1, \dots, c_n)$, there exists a conjunction of atomic τ-formulas ϕ such that ϕ defines over $\mathfrak{V}$ the same relation as p does over $(V; +, c_1, \dots, c_n)$.*

Proof. Following Remark 3.2.32, we can suppose that a complete type is isolated by a conjunction of equalities (of the form $x_1 + \dots + x_j = c$) and inequalities (of the form $x_1 + \dots + x_j \neq c$). Let p be a complete type realized by a tuple $(a_1, \dots, a_n)$ and $\phi_1 \wedge \dots \wedge \phi_k \wedge \psi_1 \wedge \dots \wedge \psi_l$ its description in terms of equalities ϕ_i and inequalities ψ_i. We describe how to modify this conjunction to obtain a conjunction that can be translated easily to a τ-formula.

We say that an equation ϵ' is a *proper sub-equation* of an equation or inequality $\epsilon \in p$ if $\epsilon' \in p$ and the set of variables of ϵ' is strictly contained in the set of variables of ϵ.

We repeat the following as long as possible:

- For $i = 0$ to k, if $e_i :\Leftrightarrow x_1 + \dots + x_s = c$ has a proper sub-equation $e_i' :\Leftrightarrow x_{i_1} + \dots + x_{i_t} = c'$, it also has its complement $e_i'' :\Leftrightarrow \sum_{j \notin \{i_1, \dots, i_t\}} x_j = c + c'$ as a sub-equation. We then substitute $e_i' \wedge e_i''$ for e_i. Obviously, the new conjunction obtained after these substitutions still isolates p.

- For $i = 0$ to l, if $n_i :\Leftrightarrow x_1 + \dots + x_s \neq c$ contains a proper sub-equation $e_i' :\Leftrightarrow x_{i_1} + \dots + x_{i_j} = c'$, we have necessarily that $n_i' :\Leftrightarrow \sum_{j \notin \{i_1, \dots, i_s\}} x_j \neq c + c'$. Substitute e_i', n_i' for n_i. Obviously, by semantic equivalence, the new conjunction obtained after these substitutions still isolates p.

After this procedure, we obtain a new conjunction logically equivalent over the structure $(V;+,c_1,\dots,c_n)$ to the previous one. We denote it by:

$$\epsilon_{1,c_{l_1}}(x_{11},\dots,x_{1s_1})\wedge\cdots\wedge\epsilon_{r,c_{l_r}}(x_{r1},\dots,x_{rs_r})\wedge\theta_1(x_{1'1},\dots,x_{1's'_1})$$
$$\wedge\cdots\wedge\theta_m(x_{m1},\dots,x_{ms'_m})$$

where $\epsilon_{i,c_{l_i}}$ are minimal equations of value c_{l_i}, and θ_i are minimal inequalities (they do not contain any proper sub-equations).

We now define the τ-formula $\phi = \bigwedge_{i=1}^{r} \mathrm{Ieq}_{s_i}^{c_{s_i}}(x_{i1},\dots,x_{is_i}) \wedge \bigwedge_{i=1}^{m} \mathrm{Ind}_{s'_i}^{u}(x_{i'1},\dots,x_{i's'_i})$. It is straightforward to show that for all $a_1,\dots,a_n\in V$, $p=\mathrm{tp}(a_1,\dots,a_n)\Leftrightarrow\mathfrak{V}\models\phi(a_1,\dots,a_n)$. Indeed, let $(a_1,\dots,a_n)\in V^n$ be such that $p=\mathrm{tp}(a_1,\dots,a_n)$, and let $i\in\{1,\dots,r\}$. We will show that $\mathfrak{V}\models\mathrm{Ieq}_{s_i}^{c_{s_i}}(a_{i1},\dots,a_{is_i})$. We already know that $(V;+)\models\epsilon_{i,c_{l_i}}(a_{i1},\dots,a_{is_i})$ so $a_{i1}+\cdots+a_{is_i}=c_{l_i}$. Since $\epsilon_{i,c_{l_i}}$ is minimal, it contains no proper sub-equation, so we have $\mathfrak{V}\models\mathrm{Ieq}_{s_i}^{c_{l_i}}(a_{i1},\dots,a_{is_i})$. Similarly, we prove that $\mathfrak{V}\models\mathrm{Ind}_{s'_i}^{u}(a_{i'1},\dots,a_{i's'_i})$ for all $i\in\{1,\dots,m\}$.

Conversely, let us suppose that $\mathfrak{V}\models\phi(a_1,\dots,a_n)$ and we prove that the type p equals $\mathrm{tp}(a_1,\dots,a_n)$. Let $i\in\{1,\dots,r\}$, since $\mathfrak{V}\models\mathrm{Ieq}_{s_i}^{c_{l_i}}(a_{i1},\dots,a_{is_i})$, we have $(V;+)\models\epsilon_{i,c_{l_i}}(a_{i1},\dots,a_{is_i})$. Let $i\in\{1,\dots,m\}$. Since $\mathfrak{V}\models\mathrm{Ind}_{s'_i}^{u}(a_{i'1},\dots,a_{i's'_i})$, we have $(V;+)\models\theta_i(a_{i'1},\dots,a_{i's'_i})$. Consequently, $p=\mathrm{tp}(a_1,\dots,a_n)$. □

3.4.4 Tackling the Problem, Order non-Included

In the following, we define a concept of weakly canonical functions, which are functions which are canonical with respect to the very primitive types we call "outstanding". We present the list of the injective weakly canonical functions from $(V;+,c_1,\dots,c_k)$ to $(V;+)$ and use them to more or less explicitly locally generate canonical functions from $(V;+)$ to $(V;+)$.

Notation 3.4.24. In the following, we denote by C the finite vector space generated by $\bar{c}$.

Remember that we called an n-type p of $(V;+)$ *outstanding* if p is isolated by either Ieq_n or Ind_n for some n. The same kind of definition holds for the structure $(V;+,\bar{c})$:

Definition 3.4.25. A n-type p of $(V;+,\bar{c})$ is *outstanding* whenever p is isolated by either Ieq_n^c for some $c\in\bar{c}$ and some n, or by Ind_n^u.

Definition 3.4.26. We call f a weakly canonical function from $(V;+,c_1,\dots,c_k)$ to $(V;+)$ if f sends any outstanding type of $(V;+,c_1,\dots,c_k)$ to an outstanding type of $(V;+)$.

Similarly to Lemma 3.4.9 for canonical functions of $(V;+)$ without constants, weakly canonical injections send linearly independent tuples to linearly independent tuples.

Lemma 3.4.27. *Let f be a weakly canonical injection from $(V;+,c_1,\dots,c_k)$ to $(V;+)$. Then $f(\mathrm{Ind}^u_n)=\mathrm{Ind}_n$ for all $n\geq 2$.*

Proof. Suppose that it is not the case, and let n be the smallest integer such that Ind^u_n is not sent to Ind_n. There exists $(x_1,\dots,x_n)$ such that $\mathrm{Ind}^u_n(x_1,\dots,x_n)$ and $\neg\,\mathrm{Ind}_n(f(x_1),\dots,f(x_n))$. Then $f(x_1)+\dots+f(x_n)=0$, i.e., $f(x_1)=f(x_2)+\dots+f(x_n)$. Since $(V;+)$ has infinite dimension and $(x_1,\dots,x_n)$ are linearly independent, we can take x_1' such that $\mathrm{Ind}^u_{n+1}(x_1',x_2,\dots,x_n)$. Then we have:

$$\mathrm{tp}_{(V;+,\bar{c})}(x_1,\dots,x_n)=\mathrm{tp}_{(V;+,\bar{c})}(x_1',x_2,\dots,x_n)$$

Because f is weakly canonical, we have $f(x_1')=f(x_2)+\dots+f(x_n)=f(x_1)$, which contradicts the injectivity of f. □

Consequently, as we prove in the next corollary, the image of an equation by a weakly canonical injection must be either an equation, or a family of independent vectors.

Corollary 3.4.28. *Let f be a weakly canonical injection from $(V;+,c_1,\dots,c_k)$ to $(V;+)$, $c\in C$, and n be a positive integer. Then either $f(\mathrm{Ieq}^c_n)=\mathrm{Ieq}_n$, or $f(\mathrm{Ieq}^c_n)=\mathrm{Ind}_n$.*

Proof. Suppose that $f(\mathrm{Ieq}^c_n)\neq\mathrm{Ieq}_n$. There exists $(x_1,\dots,x_n)$ such that we have: $\mathrm{Ieq}^c_n(x_1,\dots,x_n)$ and $\neg\,\mathrm{Ieq}_n(f(x_1),\dots,f(x_n))$. By Lemma 3.4.27, for all $k<n$ and all $1\leq i_1<\dots<i_k\leq n$ we have $\mathrm{Ind}^u_k(x_{i_1},\dots,x_{i_k})$. Hence: $\mathrm{Ind}_n(f(x_1),\dots,f(x_n))$. Consequently, by weakly canonicity of f, we have $f(\mathrm{Ieq}^c_n)=\mathrm{Ind}_n$. □

The following lemma is crucial in order to understand a certain class of weakly canonical functions.

Lemma 3.4.29. *Let $f\colon V\to V$ such that $f(\mathrm{Ieq}^c_n)=\mathrm{Ieq}_n$ for some $n\geq 3$ and $c\in C$, and which sends Ind^u_m to Ind_m for all $m\in\mathbb{N}$. Then $f(\mathrm{Ieq}^0_4)=\mathrm{Ieq}_4$.*

Proof. Let $(u_1,u_2,u_3,u_4)\in\mathrm{Ieq}^0_4$. Since $f(\mathrm{Ind}^u_m)=\mathrm{Ind}_m$ for all $m\in\mathbb{N}$, we have either $\mathrm{Ieq}_4(f(u_1),f(u_2),f(u_3),f(u_4))$, or $\mathrm{Ind}_4(f(u_1),f(u_2),f(u_3),f(u_4))$. Since we have $\mathrm{Ieq}^0_4(u_1,u_2,u_3,u_4)$, there exists $x_1,\dots,x_{n-2}$ such that $\mathrm{Ind}^u_n(u_i,x_1,\dots,x_{n-2})$ for all $i\in\{1,2,3,4\}$, and $u_1+u_2=c+\sum_{i=1}^{n-2}x_i=u_3+u_4$. Since $f(\mathrm{Ieq}^c_n)=\mathrm{Ieq}_n$, we have:

$$\mathrm{Ieq}_n(f(u_1),f(u_2),f(x_1),\dots,f(x_{n-2}))\text{ and }\mathrm{Ieq}_n(f(u_3),f(u_4),f(x_1),\dots,f(x_{n-2}))$$

which implies $\mathrm{Ieq}_4(f(u_1),f(u_2),f(u_3),f(u_4))$. □

Lemma 3.4.30. *Consider the structure $(V;+,c_1,\dots,c_n)$, and let f be a map which sends Ieq^0_4 to Ieq_4. Then there exists $h\in\mathrm{Emb}(V;+)$ and $a\in V\setminus h(V\setminus C)$ such that $f(x)=h(x)+a$ for all $x\in V\setminus C$.*

Proof. Let W be such that $W\oplus C=V$, and let $(b_i)_{i\in\mathbb{N}}$ be a basis of W. We define the mapping $h(x)=f(x)+f(b_0)+f(b_1)+f(b_0+b_1)$ for every $x\in W$ and we show that $h(x+y)=h(x)+h(y)$ for every $x,y\in W$, by induction on $(b_i)_{i\in\mathbb{N}}$. First note that this equality is true for every $x,y\in\mathrm{Vect}(b_0,b_1)$ (the proof is straightforward). Let us suppose that

$h(x+y) = h(x) + h(y)$ for all $x, y \in \mathrm{Vect}(b_0, \ldots, b_n)$. We first prove that $h(x' + b_{n+1}) = h(x') + h(b_{n+1})$ for all $x' \in \mathrm{Vect}(b_0, \ldots, b_n)$. Let $a_1, a_2 \neq 0$ be such that $\mathrm{Ieq}_3^0(a_1, a_2, x')$. Since $f(\mathrm{Ieq}_4^0) = \mathrm{Ieq}_4$, we have that $f(a_1 + a_2 + b_{n+1}) = f(a_1) + f(a_2) + f(b_{n+1})$. Hence, $h(x' + b_{n+1}) = h(a_1 + a_2 + b_{n+1}) = h(a_1) + h(a_2) + h(b_{n+1}) = h(x') + h(b_{n+1})$ since $x' \in \mathrm{Vect}(b_0, \ldots, b_n)$ (recursive assumption). Now let $x, y \in \mathrm{Vect}(b_0, \ldots, b_{n+1})$. We do a case distinction whether b_{n+1} appears in the decomposition of x or y. Assume for instance that $x := x' + b_{n+1}$ with $x' \in \mathrm{Vect}(b_0, \ldots, b_n)$ and $y \in \mathrm{Vect}(b_0, \ldots, b_n)$. Then by what we just proved and by the recursive assumption, we have: $h(x+y) = h(x' + b_{n+1} + y) = h(x' + y) + h(b_{n+1}) = h(x') + h(b_{n+1}) + h(y) = h(x' + b_{n+1}) + h(y) = h(x) + h(y)$. The other cases are similar. To conclude, we have $h(x+y) = h(x) + h(y)$ for all $x, y \in W$.

We now prove that h can be extended to a self-embedding of $(V; +)$. For $x \in W \setminus \{0\}$, we define $h(x + c) := f(x + c) + f(b_0) + f(b_1) + f(b_0 + b_1)$. We then prove that $h(x_1 + x_2 + c) = h(x_1) + h(x_2 + c)$ for all $x_1, x_2 \in W$ such that $x_1 + x_2 \notin C$. Let a, a' be elements of W such that $\mathrm{Ieq}_3^0(a, a', x_1)$. Since $f(\mathrm{Ieq}_4^0) = \mathrm{Ieq}_4$, we have that $f(x_1 + x_2 + c) = f(a) + f(a') + f(x_2 + c)$. Hence, $h(x_1 + x_2 + c) = h(a) + h(a') + h(x_2 + c) = h(x_1) + h(x_2) + c$. Now we prove that for all $x, y \in V \setminus C$ and such that $x + y \notin C$, we have: $h(x+y) = h(x) + h(y)$. Note that the only case we have not dealt with yet is the case where $x := x' + c_1$ and $y := y' + c_2$ with $x', y' \in W$. Let $x_1, x_2 \in W$ be such that $\mathrm{Ieq}_3^0(x_1, x_2, x')$. Since $f(\mathrm{Ieq}_4^0) = \mathrm{Ieq}_4$, we have: $f(x+y) = f(x_1 + x_2 + c_1 + y' + c_2) = f(x_1) + f(x_2 + c_1) + f(y' + c_2)$. Hence, $h(x+y) = h(x_1 + x_2 + c_1 + y' + c_2) = h(x_1) + h(x_2 + c_1) + h(y' + c_2) = h(x_1 + x_2 + c_1) + h(y' + c_2) = h(x) + h(y)$ (since we proved previously that $h(x_1) + h(x_2 + c) = h(x_1 + x_2 + c)$ for all $x_1, x_2 \in W$ such that $x_1 + x2 \notin V$). Finally, we define $h(c) := h(b_0 + c) + h(b_0)$. It remains to prove that h is uniquely defined on V and that $h(x+y) = h(x) + h(y)$ for all $x, y \in V$, which is an easy calculation. □

The following lemma is some sort of reverse statement to the two previous lemmas. Indeed, any function "derived" from a translation is also a weakly canonical function. This statement is a generalization of Proposition 3.4.16 to the case of $(V; +)$ with constants.

Lemma 3.4.31. *Let $h \in \mathrm{Emb}(V; +)$, $a \notin h(V \setminus C)$, and f an injective function such that $f(x) = h(x) + a$ for all $x \in V \setminus C$. Then f is a weakly canonical injection from $(V; +, c_1, \ldots, c_n)$ to $(V; +, c_1, \ldots, c_n)$.*

Proof. The proof follows exactly the same pattern as Proposition 3.4.16, but this time, 0 is not the only constant. □

We are now able to classify the injective weakly canonical functions. We divide them into three classes: the id-functions, the af-functions, and the gen-functions.

Lemma - Definition 3.4.32. *Let f be a weakly canonical injection from the structure $(V; +, c_1, \ldots, c_n)$ to $(V; +)$. Then exactly one of the following cases holds:*

- *Either $f(\mathrm{Ieq}_4^0) = \mathrm{Ieq}_4$. In this case, there exists $h \in \mathrm{Emb}(V; +)$ and $a \notin h(V \setminus C)$ such that $f(x) = h(x) + a$ for all $x \in V \setminus C$.*

– *If* $a = 0$*,* f *is called an* id-function*.*

– *If* $a \neq 0$*,* f *is called an* af-function*.*

- *Or* $f(\mathrm{Ieq}_4^0) \neq \mathrm{Ieq}_4$*. In this case,* $f(\mathrm{Ieq}_i^c) = \mathrm{Ind}_i$ *for all* $i \geq 3$*, all* $c \in C$*, and* f *is called a* gen-function*.*

Proof. First assume that $f(\mathrm{Ieq}_4^0) = \mathrm{Ieq}_4$. Then by Lemma 3.4.30, there exists $h \in \mathrm{Emb}(V;+)$ and $a \in V \setminus h(V \setminus C)$ such that $f(x) = h(x) + a$ for all $x \in V \setminus C$. So f is either an id-function, or an af-function. Now assume that $f(\mathrm{Ieq}_4^0) \neq \mathrm{Ieq}_4$. In this case, $f(\mathrm{Ieq}_i^c) = \mathrm{Ind}_i$ for all $i \geq 3$, all $c \in C$ by Lemma 3.4.29 and Corollary 3.4.28. And f is a gen-function. □

As we expect, a gen-function locally generates together with $\mathrm{Aut}(V;+)$ either gen or gen*.

Lemma 3.4.33. *Let* f *be a* gen*-function. Then* f *together with* $\mathrm{Aut}(V;+)$ *locally generates either* gen *or* gen**.*

Proof. Let $\alpha \in \mathrm{Emb}(V;+)$ such that $\alpha(V) \oplus C = V$ and consider $g := f \circ \alpha$. Then g is a canonical injection from $(V;+)$ to $(V;+)$ such that $g(\mathrm{Ieq}_n) = \mathrm{Ind}_n$ for all $n \geq 3$. Depending whether $g(0) = 0$ or $g(0) \neq 0$, g together with $\mathrm{Aut}(V;+)$ locally generates gen or gen* by Proposition 3.4.14. □

Proposition 3.4.34. *Let* f *be an* id*-function or an* af*-function and* $n \geq 1$*. By definition, there exists* $h \in \mathrm{Emb}(V;+)$ *and* $a \notin h(V \setminus C)$ *such that* $f(x) = h(x) + a$ *for all* $x \in V \setminus C$*. Then* $f(\mathrm{Ieq}_{2n+1}^c) = \mathrm{Ieq}_{2n+1}^{h(c)+a}$ *and* $f(\mathrm{Ieq}_{2n}^c) = \mathrm{Ieq}_{2n+1}^{h(c)}$*.*

Proof. Let us take $\overline{a}$ such that $\mathrm{Ieq}_{2n+1}^c(\overline{a})$. Then $a_1 + \cdots + a_{2n+1} = c$, so $f(a_1) + \cdots + f(a_{2n+1}) = h(a_1) + a + \cdots + h(a_{2n+1}) + a = h(a_1 + \cdots + a_{2n+1}) + a = h(c) + a$. And since $a \notin h(V \setminus C)$, we can conclude that $\mathrm{Ieq}_{2n+1}^{h(c)+a}(f(a_1), \ldots, f(a_{2n+1}))$. The proof for Ieq_{2n} is exactly the same. □

We prove in the following a series of technical lemmas which allows us to locally generate together with $\mathrm{Aut}(V;+)$ gen-functions from id-functions or af-functions with special values on the constants of C. In one word, whenever an id-function or an af-function breaks an equation of vectors of C of size 4, this can be propagated, thanks to local closure and composition, to every equation in order to generate a gen-function.

Proposition 3.4.35. *Let* (c_1, c_2, c_3, c_4) *be a tuple of* Ieq_4 *and let* f *be an* id*-function such that* $\sum_{i=1}^4 f(c_i) \neq 0$*. Then* f *together with* $\mathrm{Aut}(V;+)$ *locally generates a* gen*-function.*

Proof. By definition, there exists $h \in \mathrm{Emb}(V;+)$ such that $f(x) = h(x)$ for all $x \in V \setminus C$. First suppose that there exists $i \in \{1, 2, 3, 4\}$ such that $f(c_i) = 0$, for instance $f(c_1) = 0$. Then consider $\gamma \in \mathrm{Emb}(V;+)$ such that $\gamma(V) \cap C = \{0\}$, and $\{h^{-1}(f(0))\} \cap \gamma(V) = \emptyset$. We define $g = f \circ \gamma \circ f$. It is straightforward to check that g is an id-function such that

$g(x) = h(\gamma(h(x)))$ for all $x \in V \setminus C$, and $g(c_i) = h(\gamma(f(c_i))) \neq 0$ for all $i \in \{2,3,4\}$ because $f(c_i) \neq 0$ for all $i \in \{2,3,4\}$. We also have $g(c_1) = f(0) \neq 0$ by injectivity of f. Finally, $\sum_{i=1}^4 g(c_i) = f(0)+h(\gamma(f(c_2)+f(c_3)+f(c_4)) \neq 0$ because $f(c_2)+f(c_3)+f(c_4) \neq 0$ and $h^{-1}(f(0)) \notin \gamma(V) \setminus \{0\}$.

Consequently, we can suppose in the following that $f(c_i) \neq 0$ for all $i \leq 4$. We start by supposing that $f(c_i) = h(c_i)$ for all $i \leq 4$. We have:

$$\sum_{i=1}^{4} f(c_i) = h(c_1 + c_2 + c_3 + c_4) = h(0) = 0$$

a contradiction with the assumption that $\sum_{i=1}^4 f(c_i) \neq 0$. Hence, we can assume that $f(c_1) \neq h(c_1)$.

To show that f together with $\mathrm{Aut}(V;+)$ locally generates a gen-function, we will show that for every finite subset S of V, for every $n \geq 3$, for every $(x_1, \ldots, x_n) \in S$ such that $\forall i,\ x_i \neq 0$ and $x_1 + \cdots + x_n = 0$, there exists $g \in \overline{\langle f \cup \mathrm{Aut}(V;+)\rangle_1}$ such that $g(x_1) + \cdots + g(x_n) \neq 0$ and g does not create new equations, i.e., $n \geq 3$, for every $(x_1, \ldots, x_n) \in S$ such that $\forall i,\ x_i \neq 0$ and $x_1 + \cdots + x_n \neq 0$, we have $g(x_1) + \cdots + g(x_n) \neq 0$. Then, since there is only a finite number of equations, we will be able to break them all by composing with every g we found, knowing that this composition will be finite.

Let S be a finite substructure of V and $(a_1, \ldots, a_n) \in S^n$ such that $a_1 + \cdots + a_n = 0$ for a certain $n \geq 3$. Since $f(c_1) \neq 0$, there exists $\alpha \in \mathrm{Emb}(V;+)$ such that $\alpha(a_1) = c_1$, $h^{-1}(f(c_1)) \notin \alpha(V \setminus \{c_1\})$, and $C \setminus \{0, c_1\} \cap \alpha(V) = \emptyset$. We have $\mathrm{Ieq}_{2k-1}^{c_1}(\alpha(a_2), \ldots, \alpha(a_n))$, which implies $\mathrm{Ieq}_{2k-1}^{h(c_1)}(f(\alpha(a_2)), \ldots, f(\alpha(a_n)))$ by Proposition 3.4.34. We thus have $f(\alpha(a_2)) + \cdots + f(\alpha(a_n)) = h(c_1)$, but since $f(\alpha(a_1)) = f(c_1) \neq h(c_1)$, we conclude $f(\alpha(a_1)) + f(\alpha(a_2)) + \cdots + f(\alpha(a_n)) \neq 0$.

We now prove that no new equations were created in $f(\alpha(S))$. Suppose that $x_1 + \cdots + x_m \neq 0$ and $f(\alpha(x_1)) + \cdots + f(\alpha(x_m)) = 0$. We then have $\alpha(x_1) + \cdots + \alpha(x_m) \neq 0$. Suppose that $\alpha(x_i) \neq c_1$ for all $i \leq m$. Then, $\sum_{i=1}^m f(\alpha(x_i)) = h(\alpha(\sum_{i=1}^m x_i)) \neq 0$ because $x_1 + \cdots + x_m \neq 0$. Suppose now that $\alpha(x_1) = c_1$. Then if $\sum_{i=2}^m x_i \neq 0$, $\sum_{i=1}^m f(\alpha(x_i)) = f(c_1) + h(\alpha(\sum_{i=2}^m x_i)) = f(c_1) + f(\alpha(\sum_{i=2}^m x_i)) \neq 0$ because f is injective and $\alpha(x_2) + \cdots + \alpha(x_m) \neq c_1$. If $\sum_{i=2}^m x_i = 0$, $\sum_{i=1}^m f(\alpha(x_i)) = f(c_1) + h(\alpha(\sum_{i=2}^m x_i)) = f(c_1) \neq 0$.

□

Proposition 3.4.36. *Let (c_1, c_2, c_3, c_4) be a tuple of Ieq_4 and let f be an* id*-function such that $\sum_{i=1}^4 f(c_i) = 0$ but $\neg\,\mathrm{Ieq}_4(f(c_1), f(c_2), f(c_3), f(c_4))$. Then f together with $\mathrm{Aut}(V;+)$ locally generates a* gen*-function.*

Proof. By definition, there exists $h \in \mathrm{Emb}(V;+)$ such that $f(x) = h(x)$ for all $x \in V \setminus C$. Recall that $\mathrm{Ieq}_4(c_1, c_2, c_3, c_4)$ forces $c_i \neq 0$ for all $i \leq 1$. Note also that since $\sum_{i=1}^4 f(c_i) = 0$ but $\neg\,\mathrm{Ieq}_4(f(c_1), f(c_2), f(c_3), f(c_4))$, we can assume that $f(c_1) = 0$, and $\sum_{i=2}^4 f(c_i) = 0$, else we are back in Proposition 3.4.35. Consequently, since $c_1 \neq 0$, $f(0) \neq 0$, otherwise f would not be injective. Consider $\gamma \in \mathrm{Emb}(V;+)$ such that $\gamma(V) \cap C = \{0\}$, and

$\{h^{-1}(f(0))\} \cap \gamma(V) = \emptyset$. We define $g = f \circ \gamma \circ f$. It is straightforward to check that g is an id-function such that $g(x) = h(\gamma(h(x)))$ for all $x \in V \setminus C$, and $g(c_i) = h(\gamma(f(c_i))) \neq 0$ for all $i \in \{2,3,4\}$ because $f(c_i) \neq 0$ for all $i \in \{2,3,4\}$. We also have $g(c_1) = f(0) \neq 0$ by injectivity of f. Finally, $\sum_{i=1}^{4} g(c_i) = f(0) + h(\gamma(f(c_2) + f(c_3) + f(c_4)) = f(0) \neq 0$. Then we are back in the settings of Proposition 3.4.35, so f together with $\mathrm{Aut}(V;+)$ locally generates a gen-function. $\square$

Proposition 3.4.37. *Let (c_1, c_2, c_3, c_4) be a tuple of* Ieq_4 *and let f be an* af*-function such that $\sum_{i=1}^{4} f(c_i) \neq 0$. Then f together with* $\mathrm{Aut}(V;+)$ *locally generates a* gen*-function.*

Proof. First suppose that there exists $i \in \{1,2,3,4\}$ such that $f(c_i) = 0$, for instance $f(c_1) = 0$. Then consider $\gamma \in \mathrm{Emb}(V;+)$ such that $\gamma(V) \cap C = \{0\}$, and $\{h^{-1}(f(0)), h^{-1}(a), h^{-1}(f(0)+a)\} \cap \gamma(V) = \emptyset$. We define $g = f \circ \gamma \circ f$. It is straightforward to check that g is an af-function such that $g(x) = h(\gamma(h(x))) + h(\gamma(a)) + a$ for all $x \in V \setminus C$, and $g(c_i) = h(\gamma(f(c_i))) + a \neq 0$ for all $i \in \{2,3,4\}$ because $h^{-1}(a) \notin \gamma(V)$ for all $i \in \{2,3,4\}$ by injectivity of f. We also have $g(c_1) = f(0) \neq 0$ by injectivity of f. Finally, $\sum_{i=1}^{4} g(c_i) = f(0) + h(\gamma(f(c_2) + f(c_3) + f(c_4)) + a \neq 0$ because $f(c_2) + f(c_3) + f(c_4) \neq 0$ and $h^{-1}(f(0) + a) \notin \gamma(V) \setminus \{0\}$.

Consequently, we can suppose in the following that $f(c_i) \neq 0$ for all $i \leq 4$. We start by supposing that $f(c_i) = h(c_i) + a$ for all $i \leq 4$. We have $\sum_{i=1}^{4} f(c_i) = h(c_1 + c_2 + c_3 + c_4) = h(0) = 0$, a contradiction. So we can suppose that $f(c_1) \neq h(c_1) + a$. We first assume that $f(c_1) \neq a$. To show that f together with $\mathrm{Aut}(V;+)$ locally generates a gen-function, we will show that for every finite substructure S of V, for every $n \geq 3$, for every $(x_1, \ldots, x_n) \in S$ such that $\forall i,\ x_i \neq 0$ and $x_1 + \cdots + x_n = 0$, there exists $g \in \overline{\langle f \cup \mathrm{Aut}(V;+)\rangle_1}$ such that $g(x_1) + \cdots + g(x_n) \neq 0$ and g does not create new equations, i.e., $n \geq 3$, for every $(x_1, \ldots, x_n) \in S$ such that $\forall i,\ x_i \neq 0$ and $x_1 + \cdots + x_n \neq 0$, we have $g(x_1) + \cdots + g(x_n) \neq 0$. Then, since there is only a finite number of equations, we will be able to break them all by composing with every g we found, knowing that this composition will be finite.

Let S be a finite substructure of V and $(a_1, \ldots, a_n) \in S^n$ such that $a_1 + \cdots + a_n = 0$ for a certain $n \geq 3$. We first suppose that $n = 2k$. Since $f(c_1) \neq 0$, there exists $\alpha \in \mathrm{Emb}(V;+)$ such that $\alpha(a_1) = c_1$, $h^{-1}(f(c_1)) \notin \alpha(V \setminus \{c_1\})$, and $C \setminus \{0, c_1\} \cap \alpha(V) = \emptyset$. We have $\mathrm{Ieq}_{2k-1}^{c_1}(\alpha(a_2), \ldots, \alpha(a_n))$, which implies $\mathrm{Ieq}_{2k-1}^{h(c_1)+a}(f(\alpha(a_2)), \ldots, f(\alpha(a_n)))$ by Proposition 3.4.34. We thus have $f(\alpha(a_2)) + \cdots + f(\alpha(a_n)) = h(c_1) + a$, but since $f(\alpha(a_1)) = f(c_1) \neq h(c_1) + a$, we conclude $f(\alpha(a_1)) + f(\alpha(a_2)) + \cdots + f(\alpha(a_n)) \neq 0$. Then we suppose that $n = 2k+1$. Let $\beta \in \mathrm{Emb}(V;+)$ such that $C \setminus \{0\} \cap \beta(V) = \emptyset$. By definition of f, we have $f(\beta(a_1)) + \cdots + f(\beta(a_n)) = h(\beta(a_1 + \cdots + a_{2k+1})) + a \neq 0$ since $a \notin h(V \setminus C)$.

We now prove that no new equations were created in $f(\alpha(S))$ nor $f(\beta(S))$. We first prove this assumption for α, and then for β. Suppose that $x_1 + \cdots + x_m \neq 0$ and $f(\alpha(x_1)) + \cdots + f(\alpha(x_m)) = 0$. We then have $\alpha(x_1) + \cdots + \alpha(x_m) \neq 0$. Suppose that $\alpha(x_i) \neq c_1$ for all $i \leq m$. Then, if m is even, $\sum_{i=1}^{m} f(\alpha(x_i)) = h(\alpha(\sum_{i=1}^{m} x_i)) \neq 0$ because $x_1 + \cdots + x_m \neq 0$. If m is odd, then $\sum_{i=1}^{m} f(\alpha(x_i)) = h(\alpha(\sum_{i=1}^{m} x_i)) + a \neq 0$

since $a \notin h(V \setminus C)$. Suppose now that $\alpha(x_1) = c_1$. Then if m is even and $\sum_{i=2}^m x_i \neq 0$, $\sum_{i=1}^m f(\alpha(x_i)) = f(c_1) + h(\alpha(\sum_{i=2}^m x_i)) + a = f(c_1) + f(\alpha(\sum_{i=2}^m x_i)) \neq 0$ because f is injective and $\alpha(x_2) + \cdots + \alpha(x_m) \neq c_1$. If m is even and $\sum_{i=2}^m x_i = 0$, $\sum_{i=1}^m f(\alpha(x_i)) = f(c_1) + h(\alpha(\sum_{i=2}^m x_i)) + a = f(c_1) + a \neq 0$ because $f(c_1) \neq a$. If m is odd, then $\sum_{i=1}^m f(\alpha(x_i)) = f(c_1) + h(\alpha(\sum_{i=2}^m x_i)) \neq 0$ since $\alpha(x_2) + \cdots + \alpha(x_m) \neq c_1$ and $f(c_1) \notin h(\alpha(V \setminus \{c_1\}))$. We now do the proof for β. Suppose that $x_1 + \cdots + x_m \neq 0$ and $f(\beta(x_1)) + \cdots + f(\beta(x_m)) = 0$. We then have $\beta(x_1) + \cdots + \beta(x_m) \neq 0$. If m is even, $\sum_{i=1}^m f(\beta(x_i)) = h(\beta(\sum_{i=1}^m x_i)) \neq 0$ because $x_1 + \cdots + x_m \neq 0$. If m is odd, then $\sum_{i=1}^m f(\beta(x_i)) = h(\beta(\sum_{i=1}^m x_i)) + a \neq 0$ since $a \notin h(V \setminus C)$.

Now suppose that $f(c_1) = a$, and there exists $i \in \{2, 3, 4\}$ such that $f(c_i) \neq h(c_i) + a$. Note that $f(c_i)$ is necessarily distinct from a by injectivity of f, and by assumption, $f(c_i) \neq 0$. Then we apply exactly the same proof as before using this c_i in place of c_1.

Now suppose that $f(c_1) = a$ and $f(c_i) = h(c_i) + a$ for all $i \in \{2, 3, 4\}$. Let then consider $\gamma \in \mathrm{Emb}(V; +)$ such that $\gamma(V) \cap C = \{0\}$, and $g = f \circ \gamma \circ f$. It is straightforward to check that g is an af-function such that $g(x) = h(\gamma(h(x))) + h(\gamma(a)) + a$ for all $x \in V \setminus C$. We also note that $g(c_1) = h(\gamma(a)) \notin \{0, h\gamma(a)) + a\}$, $g(c_i) = h(\gamma(h(c_i))) + h(\gamma(a)) + a \neq 0$ for all $i \in \{2, 3, 4\}$, and $\sum_{i=1}^4 g(c_i) = h(\gamma(a)) + a + \sum_{i=2}^4 h(\gamma(h(c_i))) + h(\gamma(a)) + a = h(\gamma(a)) + a + h(\gamma(h(c_1))) + h(\gamma(a)) + a = h(\gamma(c_1)) \neq 0$. Consequently, we are back in the first case. □

Corollary 3.4.38. *Let f be a weakly canonical injection from $(V; +, c_1, c_2, c_3, c_4)$ to $(V; +)$ such that $\mathrm{Ieq}_4(c_1, c_2, c_3, c_4)$ and $\sum_{i=1}^4 f(c_i) \neq 0$. Then f together with $\mathrm{Aut}(V; +)$ locally generates* gen *or* gen**.*

Proof. By Lemma-Definition 3.4.32, f is either an id-function, an af-function, or a gen-function. If f is an id-function, then f together with $\mathrm{Aut}(V; +)$ locally generates a gen-function by Proposition 3.4.35. So by Lemma 3.4.33, f together with $\mathrm{Aut}(V; +)$ locally generates gen or gen*. If f is a gen-function, then again by Lemma 3.4.33, f together with $\mathrm{Aut}(V; +)$ locally generates gen or gen*. Finally, if f is an af-function, then f together with $\mathrm{Aut}(V; +)$ locally generates a gen-function by Proposition 3.4.37. Hence, by Lemma 3.4.33, f together with $\mathrm{Aut}(V; +)$ locally generates gen or gen*. □

Proposition 3.4.39. *Let (c_1, c_2, c_3, c_4) be a tuple of Ieq_4 and let f be an* af*-function such that $\sum_{i=1}^4 f(c_i) = 0$ but $\neg\, \mathrm{Ieq}_4(f(c_1), f(c_2), f(c_3), f(c_4))$. Then f together with $\mathrm{Aut}(V; +)$ locally generates either a* gen*-function, or* af$'$.

Proof. First note that since $\sum_{i=1}^4 f(c_i) = 0$ but $\neg\, \mathrm{Ieq}_4(f(c_1), f(c_2), f(c_3), f(c_4))$, we can suppose that $f(c_1) = 0$, and $\sum_{i=2}^4 f(c_i) = 0$. Consider $\gamma \in \mathrm{Emb}(V; +)$ such that $\gamma(V) \cap C = \{0\}$, and $\{h^{-1}(f(0)), h^{-1}(a), h^{-1}(f(0)+a)\} \cap \gamma(V) = \emptyset$. We define $g = f \circ \gamma \circ f$. It is straightforward to check that g is an af-function such that $g(x) = h(\gamma(h(x))) + h(\gamma(a)) + a$ for all $x \in V \setminus C$, and $g(c_i) = h(\gamma(f(c_i))) + a \neq 0$ for all $i \in \{2, 3, 4\}$ because $h^{-1}(a) \notin \gamma(V)$ for all $i \in \{2, 3, 4\}$ by injectivity of f. We also have $g(c_1) = f(0) \neq 0$ by injectivity of f. Finally, $\sum_{i=1}^4 g(c_i) = f(0) + h(\gamma(f(c_2) + f(c_3) + f(c_4)) + a = f(0) + a$.

Then we have two distinct cases: either $f(0) + a \neq 0$ and we are back in the settings of Proposition 3.4.37, and f together with $\mathrm{Aut}(V;+)$ locally generates a gen-function, or $f(0) = a$.

We now deal with the case $f(0) = a$. Let $\gamma' \in \mathrm{Emb}(V;+)$ such that $\gamma'(V) \cap C = \{0\}$. We then define $g' = f \circ \gamma'$. Note that $g'(0) = f(0) = a$ and $g'(x) = f(\gamma'(x)) = h(\gamma'(x)) + a$ for all $x \in V$ (because $\gamma'(V) \cap C = \{0\}$). Consequently, since $h \circ \gamma' \in \mathrm{Emb}(V;+)$, and $a \notin h(\gamma'(V))$ (because $a \neq 0$, $a \notin h(V \setminus C)$, and $\gamma'(V) \cap C = \{0\}$), we conclude that g' belongs to $\mathrm{cl}(\mathrm{af}')$ by Lemma 3.3.23, and hence locally generates af' by Proposition 3.3.5. $\square$

Using the previous lemmas, it is now possible to state the following theorem: any weakly canonical injection which breaks Ieq_4 on $c_1, c_2, c_3, c_4 := c_1 + c_2 + c_3$ locally generates together with $\mathrm{Aut}(V;+)$ one of the following functions: gen, gen*, or af$'$.

Theorem 3.4.40. *Let f be a weakly canonical injection from $(V;+,c_1,c_2,c_3,c_4)$ to $(V;+)$, such that $\mathrm{Ieq}_4(c_1,c_2,c_3,c_4)$ but $\neg\,\mathrm{Ieq}_4(f(c_1),f(c_2),f(c_3),f(c_4))$. Then f together with $\mathrm{Aut}(V;+)$ locally generates either gen, gen*, or af$'$.*

Proof. By Lemma-Definition 3.4.32, f is either an id-function, an af-function, or a gen-function. If we are in the two first cases, we apply straightforwardly Propositions 3.4.35, 3.4.36, 3.4.37, and 3.4.39 to obtain the result. If f is a gen-function, then by Lemma 3.4.33, f together with $\mathrm{Aut}(V;+)$ locally generates gen or gen*. $\square$

3.4.5 Tackling the Problem, Order Included

Recall that the structure $(V;+)$ has an homogeneous expansion $(V;+,<)$ by a dense linear order with minimal element 0, unbounded on $V \setminus \{0\}$, such that $\mathrm{Aut}(V;+,<)$ is extremely amenable.

Lemma 3.4.41. *Consider the structure $(V;+,<,c_1,c_2,c_3,c_4)$ with $\mathrm{Ieq}_4(c_1,\ldots,c_4)$, and suppose that $(\mathrm{Vect}(c_1,c_2,c_3),<,\bar{c})$ is ordered anti-lexicographically with respect to the basis (c_1,c_2,c_3). Then there exists a self-embedding δ of $(V;+,c_1,c_2,c_3,c_4)$ such that $\delta(x) > c$ for all $c \in \mathrm{Vect}(C)$, and all $x \in V \setminus \mathrm{Vect}(c_1,c_2,c_3,c_4)$.*

Proof. We build δ by induction. We denote $\mathrm{Vect}(\{c_1,c_2,c_3\})$ by C. Suppose that there exists an embedding δ_S from $(V;+,c_1,c_2,c_3)$ to $(V;+)$ such that $\delta(x) > c$ for all $x \in \mathrm{Vect}(S \cup C) \setminus C$, and such that $\delta_S(\mathrm{Vect}(C \cup S))$ is a vector space anti-lexicographically ordered with respect to a basis starting by $c_1 < c_2 < c_3$. Let us show that this property can be extended to $\mathrm{Vect}(C \cup S \cup \{a\})$ for any a.

The substructure $\delta_S(\mathrm{Vect}(C \cup S \cup \{a\}))$ is a finitely generated vector space which belongs to the age of $(V;+,<)$, so it is anti-lexicographically ordered with respect to one of its basis $b_1 < b_2 < \cdots$. Note that we supposed that $\delta_S(\mathrm{Vect}(C \cup S))$ is anti-lexicographically ordered with respect to a basis starting by $c_1 < c_2 < c_3$. Let k be its dimension. Hence, $(\mathrm{Vect}(b_1,\ldots,b_k);+,<)$ is isomorphic to $(\delta_S(\mathrm{Vect}(C \cup S);+,<)$ via an isomorphism σ such that $\sigma(b_i) = c_i$ for all $i \leq 3$. By homogeneity of $(V;+)$, there exists an automorphism α of $(V;+)$ such that $\sigma \subseteq \alpha$. Note that $\alpha(x) > c$ for all $x \in \delta_S(\mathrm{Vect}(C \cup S \cup \{a\}) \setminus \mathrm{Vect}(C \cup S))$.

Let τ be a local isomorphism from $(\alpha(\delta_S(\mathrm{Vect}(C \cup S))), +, <)$ to $(\delta_S(\mathrm{Vect}(C \cup S))), +, <)$ such that $\tau(\alpha(c_i)) = c_i$, and more generally, $\tau(\alpha(\delta_S(x))) = \delta_S(x)$ for all $x \in \alpha(\delta_S(\mathrm{Vect}(C \cup S)))$. Since $(V; +, <)$ is homogeneous, there exists an automorphism β of $(V; +, <)$ such that $\tau \subseteq \beta$. Note that for all $x \in \alpha(\delta_S(\mathrm{Vect}(C \cup S \cup \{a\}) \setminus \mathrm{Vect}(C \cup S)))$, $\beta(x) > c$ for all $c \in C$. Hence, for all $x \in \mathrm{Vect}(C \cup S \cup \{a\}) \setminus C$, we have $\beta \circ \alpha \circ \delta_S(x) > c$ for all $c \in C$.

Let $\delta_{S \cup \{a\}} := \beta \circ \alpha \circ \delta_S$. Clearly, for all $c \in C$, $\delta_{S \cup \{a\}}(c) = c$, and by the remark we just did, for all $x \in \mathrm{Vect}(C \cup S \cup \{a\}) \setminus C$, we have $\delta_{S \cup \{a\}}(x) > c$ for all $c \in C$.

Let us consider a chain of finite sets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \ldots V$ such that $\bigcup_{i \geq 1} A_i = V$, and let us define $\delta := \bigcup_{i \geq 1} \delta'_{A_i}$ with δ'_{A_i} being the restriction of δ_{A_i} to $\mathrm{Vect}(A_i \cup C)$. It is straightforward to see that the mapping δ satisfies the required properties. □

Lemma 3.4.42. *Let f be a canonical injection from $(V; +, <, c_1, \ldots, c_k)$ to $(V; +)$. Then $f(\mathrm{Ind}_n^u) = \mathrm{Ind}_n$ for all $n \geq 2$.*

Proof. Suppose that it is not the case, and let n be the smallest integer such that Ind_n^u is not sent to Ind_n. There exists $(x_1, \ldots, x_n)$ such that $\mathrm{Ind}_n^u(x_1, \ldots, x_n)$ and $\neg\,\mathrm{Ind}_n(f(x_1), \ldots, f(x_n))$. Then $f(x_1) + \cdots + f(x_n) = 0$, i.e., $f(x_1) = f(x_2) + \cdots + f(x_n)$. Since $(V; +)$ has infinite dimension and $(x_1, \ldots, x_n)$ are linearly independent, and since $<$ is an unbounded dense linear order on $V \setminus \{0\}$, we can take x'_1 such that $\mathrm{Ind}_{n+1}^u(x_1, \ldots, x_n, x'_1)$ and x'_1 is placed in the same position as x_1 with respect to the order $<$ and the constants of C. Then we have:

$$\mathrm{tp}_{(V;+,<,\bar{c})}(x_1, \ldots, x_n) = \mathrm{tp}_{(V;+,<,\bar{c})}(x'_1, x_2, \ldots, x_n)$$

Because f is canonical, we have $f(x'_1) = f(x_2) + \cdots + f(x_n) = f(x_1)$, which contradicts the injectivity of f. □

The following theorem build a bridge between canonical functions with the order and constants, and weakly canonical functions. If we send the maximum of points in an orbital called "high orbital" which is above every vectors of C, and if we use the property that the relations Ieq_n are symmetrical in the sense that $\mathrm{Ieq}_4(x_1, x_2, x_3, x_4) \Leftrightarrow \mathrm{Ieq}_4(x_2, x_3, x_4, x_1)$, we can locally generate together with $\mathrm{Aut}(V; +)$ weakly canonical injections over $(V; +, c_1, c_2, c_3, c_4)$ without the order.

Proposition 3.4.43. *Let (c_1, c_2, c_3, c_4) be a tuple of Ieq_4 and let f be a canonical injection from $(V; +, <, c_1, c_2, c_3, c_4)$ to $(V; +)$ such that $\neg\,\mathrm{Ieq}_4(f(c_1), \ldots, f(c_4))$. Then f together with $\mathrm{Aut}(V; +)$ locally generates a weakly canonical injection from $(V; +, c_1, c_2, c_3, c_4)$ to $(V; +)$, which agrees with f on c_1, c_2, c_3, c_4.*

Proof. We denote by C the vector space generated by c_1, c_2, c_3. By Lemma 3.4.41, there exists a self-embedding δ of $(V; +, c_1, c_2, c_3, c_4)$ such that for all $x \in V \setminus \mathrm{Vect}(c_1, c_2, c_3, c_4)$, $\delta(x) > c$, for all $c \in C$. Hence, $f \circ \delta$ is an injection locally generated by f together with $\mathrm{Aut}(V; +)$. Let $\alpha \in \mathrm{Aut}(V; +)$ such that $\alpha \circ f \circ \delta(c_i) = f(c_i)$ for all $i \leq 4$. We now prove that $g := \alpha \circ f \circ \delta$ is a weakly canonical injection from $(V; +, c_1, \ldots, c_4)$ to $(V; +)$.

Recall that g is a weakly canonical injection if it sends any outstanding n-type of $(V;+,c_1,c_2,c_3,c_4)$ to an outstanding n-type of $(V;+)$. Let p be an outstanding n-type of $(V;+,c_1,c_2,c_3,c_4)$. Suppose first that $p = \mathrm{Ind}_n^u$, we prove that $g(p) = \mathrm{Ind}_n$. Let $(a_1,\ldots,a_n) \in \mathrm{Ind}_n^u$. For all $i \leq n$, we have $\delta(a_i) > c$ for all $c \in \mathrm{Vect}(c_1,c_2,c_3,c_4)$, by definition of δ, and $\mathrm{Ind}_n(\delta(a_1),\ldots,\delta(a_n))$, since δ is an embedding from $(V;+,\overline{c})$ to $(V;+)$. Hence, we have $\mathrm{Ind}_n^u(\delta(a_1),\ldots,\delta(a_n))$, by an easy induction on $n \geq 3$. Since f is a canonical injection from $(V;+,<,\overline{c})$ to $(V;+)$, Lemma 3.4.42 gives us:

$$\mathrm{Ind}_n(f(\delta(a_1)),\ldots,f(\delta(a_n)))$$

Consequently, $g(\mathrm{Ind}_n^u) = \mathrm{Ind}_n$.

We now suppose that $p = \mathrm{Ieq}_n^c$ for a $c \in \mathrm{Vect}(c_1,\ldots,c_4)$. Since $g(\mathrm{Ind}_n^u) = \mathrm{Ind}_n$, we already know that either $g(Eq_n^c) = \mathrm{Ieq}_n^d$ for a $d \in \mathrm{Vect}(c_1,c_2,c_3,c_4)$, or $g(Eq_n^c) = \mathrm{Ind}_n$, by definition of Ieq_n^c. In both cases, p is sent to an outstanding n-type of $(V;+)$. Consequently, g is a weakly canonical function from $(V;+,c_1,\ldots,c_n)$ to $(V;+)$. □

We conclude by the following theorem which complete our toolbox. We will now be able to locally generate either $\mathrm{gen}, \mathrm{gen}^*$, or af' in every situation we need them.

Theorem 3.4.44. *Let (c_1,c_2,c_3,c_4) be a tuple of Ieq_4 and let f be a canonical injection from $(V;+,<,c_1,c_2,c_3,c_4)$ to $(V;+)$ such that $\neg\,\mathrm{Ieq}_4(f(c_1),\ldots,f(c_4))$. Then f together with $\mathrm{Aut}(V;+)$ locally generates either af', gen, or gen^*.*

Proof. By Proposition 3.4.43, f together with $\mathrm{Aut}(V;+)$ locally generates a weakly canonical injection g from $(V;+,c_1,c_2,c_3,c_4)$ to $(V;+)$, which agrees with f on c_1,c_2,c_3,c_4. Now we prove that g together with $\mathrm{Aut}(V;+)$ locally generates either af', gen, or gen^*. But this comes straightforwardly from Theorem 3.4.40, since we have just proved that g is a weakly canonical injection. Finally, since f together with $\mathrm{Aut}(V;+)$ locally generates g, we conclude with: g together with $\mathrm{Aut}(V;+)$ locally generates either gen, gen^*, or af'. □

3.5 Reducts Classification up to First-order, Existential and Existential Positive Interdefinability

This section state the main algebraic classification results we obtain for automorphism groups of reducts, self-embedding monoids of reducts, and endomorphism monoids of reducts of $(V;+)$.

In order to do so, we study the link between certain endomorphisms and self-embeddings of reducts and the associated automorphism group or self-embedding monoid. We show for instance that if gen is a self-embedding of a reduct Γ of $(V;+)$, then Γ is in fact a reduct of $(V;0)$, and thus, $\mathrm{Aut}(\Gamma)$ contains the set of bijections of V which preserve 0.

3.5.1 The $(V;0)$ Case

Recall that a structure Γ is called *k-transitive* if for any two k-tuples (s,t) of distinct elements from $\mathrm{Dom}(\Gamma)$, there is an $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(s) = t$, where the action of α on tuples is componentwise, i.e., $\alpha(s_1,\dots,s_k) = (\alpha(s_1),\dots,\alpha(s_k))$. Furthermore, Γ is called *highly transitive* if it is k-transitive for all $k \geq 1$.

If Γ is 2-transitive, $\mathrm{Aut}(\Gamma)$ has exactly two 2-orbits, namely $\{(x,y) \mid x = y\}$ and $\{(x,y) \mid x \neq y\}$. Note that a structure Γ with domain D is highly transitive if and only if it is a reduct of $(D;=)$.

We now state a crucial property of the reducts Γ such that gen or gen* are endomorphisms or self-embeddings of Γ.

Proposition 3.5.1. *Let Γ be a reduct of $(V;+)$. We have the following:*

- *if* $\mathrm{gen}^* \in \mathrm{End}(\Gamma)$*, then Γ is homomorphically equivalent to a reduct of* $(V;=)$.
- *if* $\mathrm{gen}^* \in \mathrm{Emb}(\Gamma)$*, then Γ is a reduct of* $(V;=)$.
- *if* $\mathrm{gen} \in \mathrm{End}(\Gamma)$*, then Γ is homomorphically equivalent to a reduct of* $(V;0)$.
- *if* $\mathrm{gen} \in \mathrm{Emb}(\Gamma)$*, then Γ is a reduct of* $(V;0)$.

Proof. Suppose that $\mathrm{gen}^* \in \mathrm{End}(\Gamma)$ (resp. $\mathrm{gen}^* \in \mathrm{Emb}(\Gamma)$), and let Γ' be the substructure of Γ induced by $B = \mathrm{gen}^*(V)$. We show that $\mathrm{Aut}(B,=) \subseteq \mathrm{Aut}(\Gamma')$. Indeed, let $(b_1,\dots,b_n,\dots)$ be an enumeration of B and $\beta \in \mathrm{Aut}(B,=)$. Since $(b_1,\dots,b_n,\dots)$ is linearly independent (by definition of gen*), we can complete it into a basis of $(V;+)$: $(b_1,\dots,b_n,\dots) \cup (s_1,\dots,s_k,\dots)$. Let α be defined as follows: $\alpha(x) = \beta(x)$ for all $x \in B$ and $\alpha(s_i) = s_i$ for all $i \in \mathbb{N}$. The mapping α, sending a basis of $(V;+)$ to another basis of $(V;+)$, induces an automorphism of $(V;+)$ that will also be denoted α. Since Γ is a reduct of $(V;+)$, α is also an automorphism of Γ. We now prove that β is an automorphism of Γ': we already know that β is a bijection of $\mathrm{gen}^*(V)$. Now let R be a relational symbol of the signature of Γ', and let us take $(u_1,\dots,u_k) \in R^{\Gamma'}$. Since Γ' is induced from Γ, we have $(u_1,\dots,u_k) \in R^{\Gamma}$, so $(\alpha(u_1),\dots,\alpha(u_k)) \in R^{\Gamma}$ since α is an automorphism of Γ. Since $\alpha \restriction B = \beta$, $(\beta(u_1),\dots,\beta(u_k)) \in R^{\Gamma}$ and finally $(\beta(u_1),\dots,\beta(u_k)) \in R^{\Gamma'}$ since Γ' is induced from Γ and $(u_1,\dots,u_k) \in B^k$. Conversely, suppose that $(u_1,\dots,u_k) \notin R^{\Gamma'}$. Exactly the same proof gives $(\beta(u_1),\dots,\beta(u_k)) \notin R^{\Gamma'}$. Therefore $\mathrm{Aut}(B,=) \subseteq \mathrm{Aut}(\Gamma')$, which means, by ω-categoricty of Γ', that it is a reduct of $(B,=)$, trivially isomorphic to a reduct Δ of $(V;=)$. So gen* is a homomorphism from Γ to Γ' which is isomorphic to a first-order Δ of $(V;=)$. The identity map is a homomorphism from Γ' to Γ, so since Δ and Γ' are isomorphic, there exists also a homomorphism from Δ to Γ, and Γ is homomorphically equivalent to a reduct of $(V;=)$. Respectively, gen* is an isomorphism from Γ to Γ' which is isomorphic to a reduct Δ of $(V;=)$, so Γ is a reduct of $(V;=)$).

Suppose now that $\mathrm{gen} \in \mathrm{End}(\Gamma)$ (resp. $\mathrm{gen} \in \mathrm{Emb}(\Gamma)$), and let Γ' be the substructure of Γ induced by $B = \mathrm{gen}(V)$. First note that $\mathrm{Aut}(B,0) \subseteq \mathrm{Aut}(\Gamma')$. Indeed, let $\beta \in \mathrm{Aut}(B,0)$; we can show in a same way of the first case, that there exists an automorphism α of $(V;+)$ containing β, so $\beta \in \mathrm{Aut}(\Gamma')$ (the same proof stands). To conclude, it is sufficient to note that gen is a homomorphism from Γ to Γ', and id is a homomorphism from Γ' to Γ, so Γ is homomorphically equivalent to a reduct of $(V;0)$. Respectively, note that gen is an isomorphism from Γ to Γ', so Γ is a reduct of $(V;0)$. $\square$

Lemma 3.5.2. *Let Γ be a reduct of $(V;0)$ such that there exists an endomorphism g of Γ which is not injective. Then one of the following cases holds:*

- *Γ has an endomorphism f such that $f(x) = 0$ for all $x \in V \setminus \{0\}$;*
- *there exists $y_0 \neq 0$ such that Γ has an endomorphism f such that $f(x) = y_0$ for all $x \in V \setminus \{0\}$;*
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ind}_2)$.

Proof. Since Γ is a reduct of $(V;0)$, we have $\mathrm{Aut}(V;0) \subseteq \mathrm{Aut}(\Gamma)$, and hence $\mathrm{End}(V;0,\neq) = \mathrm{Emb}(V;0) = \overline{\mathrm{Aut}(V;0)} \subseteq \mathrm{End}(\Gamma)$. First assume that there exist two distinct elements x_0, x_1 of $V \setminus \{0\}$ such that $g(x_0) = g(x_1) = 0$. Then by a straightforward recursive argument ended by a local closure limit, g locally generates together with $\mathrm{Aut}(V;0)$ a function f such that $f(x) = 0$ for all $x \in V \setminus \{0\}$.

Now assume that there exist two distinct elements x_0, x_1 of $V \setminus \{0\}$ such that $g(x_0) = g(x_1) = y_0 \neq 0$. Then by a straightforward recursive argument ended by a local closure limit, g locally generates together with $\mathrm{Aut}(V;0)$ a function f such that $f(x) = y_0$ for all $x \in V \setminus \{0\}$.

Now assume that there exists $x_0 \neq 0$ such that $g(x_0) = g(0) = 0$. Then by a straightforward recursive argument ended by a local closure limit, g locally generates together with $\mathrm{Aut}(V;0)$ a function f such that $f(x) = 0$ for all $x \in V$.

Finally, assume that we are not in one of the previous cases. Then every endomorphism of Γ is injective over $V \setminus \{0\}$. Hence, there exists $x_0, y_0 \neq 0$ such that $g(x_0) = g(0) = y_0 \neq 0$. Assume that there exists an endomorphism g' of Γ and an element $x_1 \neq 0$ such that $g'(x_1) = 0$, and let $\alpha \in \mathrm{Aut}(V;0)$ be such that $\alpha(y_0) = x_1$. Then $g' \circ \alpha \circ g$ is an endomorphism of Γ such that $g'(\alpha(g(0))) = g'(\alpha(g(x_0))) = 0$, and we are back in case three, which contradicts the assumption that we are not in one of the previous cases. Hence, every endomorphism of Γ preserves Ind_1 (which is equal to the unary relation $V \setminus \{0\}$). In this case, we show that $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ind}_2)$. Indeed, if f is an endomorphism of Γ, it is injective over $V \setminus \{0\}$, and $f(V \setminus \{0\} \subseteq V \setminus \{0\}$. Consequently, f preserves Ind_2. Conversely, if f preserves Ind_2, then either f is injective, and belongs to $\mathrm{End}(V;0,\neq) \subseteq \mathrm{End}(\Gamma)$, or f is not injective. In this case, since f is injective over $V \setminus \{0\}$ (indeed, f preserves Ind_2), and since f preserves Ind_1 (because Ind_1 is existential positive definable over $(V; \mathrm{Ind}_2)$ as follows: $\mathrm{Ind}_1(x) \Leftrightarrow \exists y \, \mathrm{Ind}_2(x,y)$), there exists $x_0' \neq 0$ such that $f(x_0') = f(0) \neq 0$. But it straightforward to see that any f of this form locally generates together with $\mathrm{Aut}(V;0)$ every other function f' with the same form. Hence, since $\mathrm{End}(\Gamma)$ also contains such a function, we have: $\mathrm{End}(V; \mathrm{Ind}_2) \subseteq \mathrm{End}(\Gamma)$, and finally: $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ind}_2)$. $\square$

Proposition 3.5.3. *Let Γ be a reduct of $(V;0)$. Then exactly one of the following cases holds:*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; 0, \neq)$*;*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \neq)$*;*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ind}_2)$*;*

- Γ *has a non injective endomorphism, and* Γ *is homomorphically equivalent to a structure with at most two elements.*

Proof. Let $R_1, \ldots, R_k$ be k relations such that $\Gamma := (V; R_1, \ldots, R_k)$. We first assume that Γ has an endomorphism which violates 0, and that every endomorphism of Γ is injective. Consider the structure $f(\Gamma)$ defined as follows: $f(\Gamma) := (f(V); f(R_1), \ldots, f(R_k))$ where $f(R) := \{(f(x_1), \ldots, f(x_r)) \mid \overline{x} \in R\}$. Even if it means composing f with a well-chosen automorphism of $(V; 0)$, we can assume that $f(V) \subseteq V \setminus \{0\}$. Indeed, if there exists $x_0 \in V$ such that $f(x_0) = 0$, then x_0 is distinct from 0 by the fact that f does not preserve 0. Hence, taking $\alpha \in \mathrm{Emb}(V; 0)$ such that $x_0 \notin \alpha(V)$, we have: $f \circ \alpha \in \mathrm{End}(\Gamma)$, and $f \circ \alpha(V) \subseteq V \setminus \{0\}$ since f is injective.

We show that $f(\Gamma)$ is isomorphic to Γ and highly transitive. Indeed, let $\overline{b} := (b_1, \ldots, b_n) \in f(V)^n$ be such that $b_i \neq b_j$ whenever $i \neq j$. Let $b' := (b'_1, \ldots, b'_n) \in f(V)^n$ be such that $b'_i \neq b'_j$. Since $f(V) \subseteq V \setminus \{0\}$, there exists $\beta \in \mathrm{Aut}(V; 0)$ such that $\beta.b = b'$. Since $\mathrm{Aut}(V; 0) \subseteq \mathrm{Aut}(\Gamma)$, this implies that $f(\Gamma)$ is highly transitive. Because f is an injective endomorphism of Γ, $f(\Gamma)$ is isomorphic to Γ, so Γ is highly transitive too. Since $\mathrm{End}(\Gamma)$ is locally closed, it must contain all the injections of V^V. Indeed, since $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V; =) \subseteq \mathrm{End}(\Gamma)$, we have $\mathrm{End}(V; \neq) = \mathrm{Emb}(V; =) \subseteq \mathrm{End}(\Gamma)$, and consequently: $\mathrm{End}(\Gamma) = \mathrm{End}(V; \neq)$.

Assume now that every endomorphism of Γ is injective and preserves 0. Since $(V; 0)$ is homogeneous and uniformly locally finite, $\mathrm{Emb}(V; 0) = \overline{\mathrm{Aut}(V; 0)}$ by Proposition 2.2.9. Hence, since $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V; 0)$, we have $\mathrm{Emb}(V; 0) = \mathrm{End}(V; 0, \neq) \subseteq \mathrm{End}(\Gamma)$. And since every endomorphism of Γ is injective, we have $\mathrm{End}(\Gamma) = \mathrm{End}(V; 0, \neq)$.

Finally, assume that Γ has a non injective endomorphism. By Lemma 3.5.2, one of the following cases holds:

- Γ has an endomorphism f such that $f(x) = 0$ for all $x \in V \setminus \{0\}$. In this case, Γ is homomorphically equivalent to a structure with at most two elements;

- there exists $y_0 \neq 0$ such that Γ has an endomorphism f such that $f(x) = y_0$ for all $x \in V \setminus \{0\}$. In this case, Γ is homomorphically equivalent to a structure with at most two elements;

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ind}_2)$.

□

Corollary 3.5.4. *Let* Γ *be a reduct of* $(V; 0)$*. Then either* $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; 0)$*, or* $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; =)$*.*

Proof. Assume that $\Gamma := (V; R_1, \ldots, R_k)$, and consider the reduct Γ' defined as follows:

$$\Gamma' := (V; R_1, \neg R_1, \ldots, R_k, \neg R_k, \neq)$$

We have that Γ' is a reduct of $(V; 0)$ such that $\mathrm{Emb}(\Gamma) = \mathrm{End}(\Gamma')$. Hence, by Proposition 3.5.3, exactly one of the following cases holds:

- $\mathrm{End}(\Gamma') = \mathrm{End}(V; 0, \neq) = \mathrm{Emb}(V; 0)$;
- $\mathrm{End}(\Gamma') = \mathrm{End}(V; \neq) = \mathrm{Emb}(V; =)$;
- $\mathrm{End}(\Gamma') = \mathrm{End}(V; \mathrm{Ind}_2)$;
- Γ' has a non injective endomorphism, and Γ' is homomorphically equivalent to a structure with at most two elements.

But in our settings, the two last cases can not occur since every endomorphism of Γ' is injective (because $\neq$ is preserved by every endomorphism of Γ'). ☐

Corollary 3.5.5. *Let* Γ *be a reduct of* $(V; 0)$*. Then either* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V; 0)$*, or* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V; =)$.

Proof. Since Γ is a reduct of $(V; 0)$, we know that $\mathrm{Aut}(V; 0) \subseteq \mathrm{Aut}(\Gamma) \subseteq \mathrm{Aut}(V; =)$. Let us distinguish two cases:

- either all automorphisms of Γ preserve 0. Then trivially: $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V; 0)$.
- or there exists an automorphism α of Γ which does not preserve 0. In this case, we show in the second case of the proof of Corollary 3.5.4 that $\alpha(\Gamma)$ was highly transitive, and because α is an automorphism , we conclude that Γ is highly transitive, so it is a reduct of $(V; =)$ and $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V; =) \simeq S_\omega$.

☐

3.5.2 The Affine Case

We benefit from the fact that we have introduced the relations Eq_i by defining one of the most important reducts of $(V; +)$ which we already mentioned in the introduction: the affine reduct, denoted by $(V; \mathrm{Eq}_4)$. Unlike $(V; +)$, 0 is not first-order definable over $(V; \mathrm{Eq}_4)$. One of the consequences following the link between automorphisms and first-order formulas (see Theorem 2.1.27) is that its automorphism group is distinct from $\mathrm{Aut}(V; +)$. In fact, it strictly contains it.

Notation 3.5.6. Recall that we defined the relation $\mathrm{Eq}_4 \subseteq V^4$ as follows. For all $x, y, u, v \in V^4$:

$$\mathrm{Eq}_4(x, y, u, v) \Leftrightarrow x + y + u + v = 0$$

Notation 3.5.7. We denote by id the identity map. Recall that for any element a of V, we write t_a for the translation of vector a, i.e., $t_a(x) = x + a$ for all $x \in V$.

Theorem 3.5.8. *Let* $a \in V \setminus \{0\}$. *Then* $\langle \mathrm{Aut}(V;+) \cup \{t_a\}\rangle_1 = \mathrm{Aut}(V;\mathrm{Eq}_4)$.

Proof. We first check that t_a is an automorphism of $(V;\mathrm{Eq}_4)$. Because $t_a \circ t_a = \mathrm{id}$, t_a is bijective. Let $(x_1,x_2,x_3,x_4) \in \mathrm{Eq}_4$, then: $x_1+x_2+x_3+x_4 = 0$, so $x_1+a+x_2+a+x_3+a+x_4+a = 0$, which means $(t_a(x_1), t_a(x_2), t_a(x_3), t_a(x_4)) \in \mathrm{Eq}_4$.

Let $f \in \mathrm{Aut}(V;\mathrm{Eq}_4)$ be arbitrary. We distinguish two cases: first suppose that f preserves 0. We show that $f \in \mathrm{Aut}(V;+)$. Indeed, let x,y,z be such that $\mathrm{Eq}_4(x,y,z,0)$. Since f preserves Eq_4, we have: $\mathrm{Eq}_4(f(x),f(y),f(z),f(0))$, so: $f(x)+f(y) = f(z)+0 = f(z)$, which means that f preserves $+$. Since f is bijective, it is an automorphism of $(V;+)$.

Let us suppose now that f does not preserve 0. We show that $f \in \langle \mathrm{Aut}(V;+) \cup \{t_a\}\rangle_1$. Indeed: let $\beta \in \mathrm{Aut}(V;+)$ be such that $\beta(f(0)) = a$ (such a β exists because $\mathrm{tp}_{(V;+)}(f(0)) = \mathrm{tp}_{(V;+)}(a)$ and $(V;+)$ is ω-categorical). Let us now consider $\gamma := t_a \circ \beta \circ f$. It is clearly an automorphism of $(V;\mathrm{Eq}_4)$ preserving 0, so by what we have just proved, γ is an automorphism of $(V;+)$. But since $f = \beta^{-1} \circ t_a \circ \gamma$, f belongs to $\langle \mathrm{Aut}(V;+) \cup \{t_a\}\rangle_1$. □

In the affine case, we show a very strong property of the reducts of $(V;+)$ of which af' is a self-embedding. Indeed, a reduct Γ of $(V;+)$ such that $\mathrm{af}' \in \mathrm{Emb}(\Gamma)$ is in fact a reduct of $(V;\mathrm{Eq}_4)$. Note that this property is not true anymore when af^* or af are self-embeddings of Γ, because of their special value on 0 and their lack of "uniformity".

Proposition 3.5.9. *Let* Γ *be a reduct of* $(V;+)$ *such that* $\mathrm{af}' \in \mathrm{Emb}(\Gamma)$. *Then* Γ *is a reduct of* $(V;\mathrm{Eq}_4)$.

Proof. We will prove that $\mathrm{Aut}(V;\mathrm{Eq}_4) \subseteq \mathrm{Aut}(\Gamma)$. Recall that we showed in Theorem 3.5.8 that $\mathrm{Aut}(V;\mathrm{Eq}_4) = \langle \mathrm{Aut}(V;+) \cup \{t_d\}\rangle_1$ for any $d \neq 0$, where $t_d(x) = x+d$ for all $x \in V$. Also recall that $\mathrm{af}'(x) = \alpha(x)+a$ for all $x \in V$ where $a \neq 0$ and $\alpha \in \mathrm{Emb}(V;+)$ is such that $a \notin \alpha(V)$. It follows from Lemma 3.3.23 and Proposition 3.3.5 that af' together with $\mathrm{Aut}(V;+)$ locally generates the map $h(x) = \alpha(x)+\alpha(a)+a$ since $\alpha(a)+a \notin \alpha(V)$ (indeed, if $\alpha(a)+a \in \alpha(V)$ then there exists b such that $\alpha(b) = \alpha(a)+a$, so $a = \alpha(b+a)$, which contradicts the fact that $a \notin \alpha(V)$). Therefore, since $\mathrm{Emb}(\Gamma)$ is locally closed, $h \in \mathrm{Emb}(\Gamma)$.

We will prove that $t_a \in \mathrm{Aut}(\Gamma)$. Let R be a n-ary relation of Γ, and let $(a_1,\dots,a_n) \in V^n$ be such that $R(a_1,\dots,a_n)$. Since $h \in \mathrm{Emb}(\Gamma)$, we have $R(\alpha(a_1)+\alpha(a)+a, \dots, \alpha(a_n)+\alpha(a)+a)$. So $R(\alpha(a_1+a)+a,\dots,\alpha(a_n+a)+a)$, i.e., $R(\mathrm{af}'(a_1+a),\dots,\mathrm{af}'(a_n+a))$. Therefore, since $\mathrm{af}' \in \mathrm{Emb}(\Gamma)$, we have $R(a_1+a,\dots,a_n+a)$, i.e., $R(t_a(a_1),\dots,t_a(a_n))$. Since $t_a \circ t_a = \mathrm{id}$, we conclude: $R(a_1,\dots,a_n) \Leftrightarrow R(t_a(a_1),\dots,t_a(a_n))$, and $t_a \in \mathrm{Aut}(\Gamma)$. Because Γ is a reduct of $(V;+)$, we have $\mathrm{Aut}(V;+) \subseteq \mathrm{Aut}(\Gamma)$. Consequently: $\mathrm{Aut}(V;\mathrm{Eq}_4) = \langle \mathrm{Aut}(V;+) \cup \{t_a\}\rangle_1 \subseteq \mathrm{Aut}(\Gamma)$. □

Theorem 3.5.10. *Let* Γ *be a reduct of* $(V;\mathrm{Eq}_4)$. *Then either* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Eq}_4)$, *or* $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;=)$.

Proof. We suppose that $\mathrm{Aut}(\Gamma) \neq \mathrm{Aut}(V;\mathrm{Eq}_4)$. Then there exists $\gamma \in \mathrm{Aut}(\Gamma)$ which violates Eq_4, i.e., there exist $c_1,c_2,c_3,c_4 \in V$ such that:

$$\mathrm{Eq}_4(c_1,c_2,c_3,c_4) \wedge \neg\,\mathrm{Eq}_4(\gamma(c_1),\gamma(c_2),\gamma(c_3),\gamma(c_4))$$

Claim: we can assume that $\mathrm{Ieq}_4(c_1,c_2,c_3,c_4)$. Indeed, if it is not the case, let $a \in V \setminus \{c_1,c_2,c_3,c_4\}$ and $\gamma' = \gamma \circ t_a$. Then γ' is locally generated by γ together with $\mathrm{Aut}(V;\mathrm{Eq}_4)$ since $t_a \in \mathrm{Aut}(V;\mathrm{Eq}_4) \subseteq \mathrm{Aut}(\Gamma)$, and γ' has the good property since $t_a(c_i) \neq 0$ for all $i \in \{1,2,3,4\}$.

Since $(V;+,<,c_1,c_2,c_3,c_4)$ is an ω-categorical totally ordered Ramsey structure by Corollary 3.4.5 combined with Lemma 2.4.13, we can apply Corollary 2.4.16. Hence γ together with $\mathrm{Aut}(V;+)$ locally generates a canonical function f from $(V;+,<,c_1,c_2,c_3,c_4)$ to $(V;+)$, which agrees with γ on c_1,c_2,c_3,c_4. Hence, $f(0)=0$, $\mathrm{Ieq}_4(c_1,c_2,c_3,c_4)$, and $\mathrm{Ind}_3(f(c_1),f(c_2),f(c_3)))$. Since γ is injective, f is injective. Therefore, since $f(0)=0$, and f is canonical from $(V;+,<,c_1,c_2,c_3,c_4)$ to $(V;+)$, by Theorem 3.4.44 we know that either:

- f together with $\mathrm{Aut}(V;+)$ locally generates af$'$ but by checking Propositions 3.4.35, 3.4.36, 3.4.37, 3.4.39, and 3.4.33, this implies that the function f is an af-function such that $\mathrm{Ieq}_4(c_1,c_2,c_3,c_4)$ and $\sum_{i=1}^{4}(f(c_i))=0$, which contradicts the fact that $\sum_{i=1}^{4}(\gamma(c_i)) \neq 0$ and f agrees with γ on c_1,c_2,c_3,c_4.

- f together with $\mathrm{Aut}(V;+)$ locally generates gen*. Hence gen$^* \in \mathrm{Emb}(\Gamma)$. In this case, we show in Proposition 3.5.1 that Γ is a reduct of $(V;0)$. We then apply Corollary 3.5.5 to obtain that either $\mathrm{Aut}(\Gamma)=\mathrm{Aut}(V;0)$, or $\mathrm{Aut}(\Gamma)=\mathrm{Aut}(V;=)$. But if $\mathrm{Aut}(\Gamma)=\mathrm{Aut}(V;0)$, 0 would be first-order definable over Γ, so it would also be first-order definable over $(V;\mathrm{Eq}_4)$ since Γ is one of its reduct, which contradicts the fact that $\mathrm{Aut}(V;\mathrm{Eq}_4,0)=\mathrm{Aut}(V;+) \neq \mathrm{Aut}(V;\mathrm{Eq}_4)$.

- or f together with $\mathrm{Aut}(V;+)$ locally generates gen. Thus gen $\in \mathrm{Emb}(\Gamma)$,and by Proposition 3.5.1, Γ is a reduct of $(V;=)$. Consequently, $\mathrm{Aut}(\Gamma)=\mathrm{Aut}(V;=)$ by Corollary 3.5.5.

□

3.5.3 A Slice of the Lattice of Endomorphism Monoids

Time has come to sketch the lattice of the endomorphism monoids of reducts of $(V;+)$. Even though, as we show, this lattice is infinite, the various functions we defined in the previous subsections allow us to sketch an interesting part of it. In the following section, we keep listing endomorphism monoids locally generated by functions derived from the identity map and translations.

Definition 3.5.11. We define the 4-ary relations $Z_2,S,T_1,T_2,T_3,T_4,T_5,T_6,T_7 \subseteq V^4$ as follows:

- Recall that $R_1 = \{(x,\sigma(y),\sigma(z),\sigma(t)) \in V^4 \mid x=y \wedge \mathrm{Ind}_3(y,z,t),\ \text{with } \sigma \in \mathrm{Perm}(\{y,z,t\})\}$
- $R_2 = \{(x,\sigma(y),\sigma(z),\sigma(t)) \in V^4 \mid x=y \wedge \mathrm{Ieq}_3(y,z,t),\ \text{with } \sigma \in \mathrm{Perm}(\{y,z,t\})\}$

- Recall that $Z_1 = \{(x,y,z,t) \in V^4 \mid x = 0 \wedge \mathrm{Ieq}_3(y,z,t)\}$
- $Z_2 = \{(x,y,z,t) \in V^4 \mid x \notin \{0,y,z,t\} \wedge \mathrm{Ieq}_3(y,z,t)\}$
- $Z_3 = \{(x,y,z,t) \in V^4 \mid x = 0 \wedge \mathrm{Ind}_3(y,z,t)\}$
- $T_1 = \{(x,y,z,t) \in V^4 \mid Z_1(x,y,z,t) \vee Z_2(x,y,z,t) \vee \mathrm{Ieq}_4(x,y,z,t) \vee \mathrm{Ind}_4(x,y,z,t)\}$
- $T_2 = \{(x,y,z,t) \in V^4 \mid Z_1(x,y,z,t) \vee Z_2(x,y,z,t) \vee \mathrm{Ind}_4(x,y,z,t)\}$
- $T_3 = \{(x,y,z,t) \in V^4 \mid Z_1(x,y,z,t) \vee Z_2(x,y,z,t) \vee Z_3(x,y,z,t) \vee \mathrm{Ind}_4(x,y,z,t)\}$
- $T_4 = \{(x,y,z,t) \in V^4 \mid Z_1(x,y,z,t) \vee \mathrm{Ieq}_4(x,y,z,t) \vee Z_3(x,y,z,t) \vee \mathrm{Ind}_4(x,y,z,t)\}$
- $T_5 = \{(x,y,z,t) \in V^4 \mid Z_1(x,y,z,t) \vee Z_3(x,y,z,t) \vee \mathrm{Ind}_4(x,y,z,t)\}$
- $T_6 = \{(x,y,z,t) \in V^4 \mid Z_1(x,y,z,t) \vee \mathrm{Ieq}_4(x,y,z,t) \vee \mathrm{Ind}_4(x,y,z,t)\}$
- $T_7 = \{(x,y,z,t) \in V^4 \mid T_3(x,y,z,t) \vee \mathrm{Ieq}_4(x,y,z,t)\}$

Lemma 3.5.12. *We have the following:*

- $\mathrm{af}^* \in \overline{\langle \{\mathrm{id}^*, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1}$
- $\mathrm{af}^* \in \overline{\langle \{\mathrm{id}^*, \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1}$
- $\mathrm{af}^* \in \overline{\langle \{\mathrm{af}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1}$

Proof. Easy verification. □

Proposition 3.5.13. *We have the following properties:*

$$\begin{aligned}
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4,T_1,\neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^*, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4,T_2,\neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^*, \mathrm{af}^*\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4,T_3,\neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^*, \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4,T_4,\neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{af}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4,T_5,\neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{af}, \mathrm{af}^*\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4,T_6,\neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{af}', \mathrm{af}^*\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4,\neq) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^*, \mathrm{af}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1}
\end{aligned}$$

Proof. The proofs of these properties start with Corollary 3.3.28 and Lemma 3.5.12, and use exactly the same method as used in the proof of Proposition 3.3.35. $\square$

Lemma 3.5.14. *We have the following properties:*

- $\mathrm{af}^* \in \overline{\langle\{\mathrm{id}^*, \mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}$
- $\mathrm{af}^* \in \overline{\langle\{\mathrm{af}, \mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}$
- $\mathrm{af}^* \in \overline{\langle\{\mathrm{af}', \mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}$

Proof. There exists $x_0 \neq 0$ such that $\mathrm{af}^{\mathrm{ni}}(0) = \mathrm{af}^{\mathrm{ni}}(x_0)$. Let us take representatives $\mathrm{id}^*, \mathrm{af}, \mathrm{af}'$ such that $x_0 \notin \mathrm{id}^*(V) \cup \mathrm{af}(V) \cup \mathrm{af}'(V)$. Then it is straightforward to see that for all $x \in V$:

$$\mathrm{af}^*(x) = \mathrm{af}^{\mathrm{ni}} \circ \mathrm{id}^*(x) = \mathrm{af}^{\mathrm{ni}} \circ \mathrm{af}(x) = \mathrm{af}^{\mathrm{ni}} \circ \mathrm{af}'(x)$$

$\square$

Proposition 3.5.15. *We have the following properties:*

$$\begin{aligned}
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, Z_1 \cup R_1 \cup \mathrm{Ind}_4) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_2 \cup R_1) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{id}^*, \mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_5 \cup R_1) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{af}^{\mathrm{ni}}, \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_6 \cup R_1) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{af}^{\mathrm{ni}}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_1 \cup R_1) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{id}^*, \mathrm{af}^{\mathrm{ni}}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_3 \cup R_1) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{id}^*, \mathrm{af}^{\mathrm{ni}}, \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_4 \cup R_1) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{af}^{\mathrm{ni}}, \mathrm{af}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_7 \cup R_1) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle\{\mathrm{id}^*, \mathrm{af}^{\mathrm{ni}}, \mathrm{af}', \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1}
\end{aligned}$$

Proof. The proofs of these properties start with Corollary 3.3.27, and use exactly the same method as used in the proof of Proposition 3.3.35. $\square$

Lemma 3.5.16. *We have the following properties:*

- $\mathrm{id}^* \in \overline{\langle \{\mathrm{id}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1}$
- $\mathrm{af}^{\mathrm{ni}} \in \overline{\langle \{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}^*\} \cup \mathrm{Aut}(V;+)\rangle_1}$
- $\{\mathrm{af}^{\mathrm{ni}}, \mathrm{af}^*\} \subseteq \overline{\langle \{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1}$
- $\{\mathrm{af}^{\mathrm{ni}}, \mathrm{af}^*\} \subseteq \overline{\langle \{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1}$

Proof. First note that it is straightforward to see that for all $x \in V$:

$$\mathrm{af}^{\mathrm{ni}}(x) = \mathrm{af}^* \circ \mathrm{id}^{\mathrm{ni}}(x) = \mathrm{af} \circ \mathrm{id}^{\mathrm{ni}}(x) = \mathrm{af}' \circ \mathrm{id}^{\mathrm{ni}}(x)$$

Let us take representatives $\mathrm{af}, \mathrm{af}^*, \mathrm{af}'$ such that $x_1 \notin \mathrm{af}(V) \cup \mathrm{af}^*(V) \cup \mathrm{af}'(V)$. Then it is straightforward to see that for all $x \in V$:

$$\mathrm{af}^*(x) = \mathrm{id}^{\mathrm{ni}} \circ \mathrm{af}(x) = \mathrm{id}^{\mathrm{ni}} \circ \mathrm{af}'(x)$$

□

Proposition 3.5.17. *We have the following properties:*

$$\begin{aligned}
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_3) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{id}^*) \\
&= \overline{\langle \{\mathrm{id}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_2 \cup R_1 \cup R_2) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}^*\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_3 \cup R_1 \cup R_2) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4, T_1 \cup R_1 \cup R_2) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1} \\
\bullet\, \mathrm{End}(V;\mathrm{Ieq}_4) &= \mathrm{cl}(\mathrm{id}) \sqcup \mathrm{cl}(\mathrm{id}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{id}^*) \sqcup \mathrm{cl}(\mathrm{af}^{\mathrm{ni}}) \sqcup \mathrm{cl}(\mathrm{af}) \sqcup \mathrm{cl}(\mathrm{af}') \sqcup \mathrm{cl}(\mathrm{af}^*) \\
&= \overline{\langle \{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}, \mathrm{af}'\} \cup \mathrm{Aut}(V;+)\rangle_1}
\end{aligned}$$

Proof. The proofs of these properties start with Corollary 3.3.27, and use exactly the same method as used in the proof of Proposition 3.3.35. □

Lemma 3.5.18. *Let* Γ *be a reduct of* $(V;+)$ *such that* $\mathrm{End}(\Gamma) \subseteq \mathrm{End}(V;\mathrm{Ieq}_4)$. *Assume that there exists* $f \in \mathrm{End}(\Gamma)$ *and* $c \neq 0$ *such that* $f(0) = f(c) \neq 0$. *Then exactly one of the following holds:*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V;\mathrm{Ieq}_3)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V;\mathrm{Ieq}_4, Z_1 \cup R_1 \cup \mathrm{Ind}_4)$

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_1 \cup R_1)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_1 \cup R_1 \cup R_2)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_2 \cup R_1)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_2 \cup R_1 \cup R_2)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_3 \cup R_1)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_3 \cup R_1 \cup R_2)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_4 \cup R_1)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_5 \cup R_1)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_6 \cup R_1)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_7 \cup R_1)$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4)$

Proof. By Proposition 3.3.25, the only non injective functions f which belong to the monoid $\mathrm{End}(V; \mathrm{Ieq}_4)$ are of the form $f(0) = h(c) + v$ and $f(x) = h(x) + v$ with $v \notin h(V \setminus \{0\})$ and $c \neq 0$. Such an f together with $\mathrm{Aut}(V; +)$ clearly locally generates and is locally generated by either $\{\mathrm{id}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V; +)$ or $\{\mathrm{af}^{\mathrm{ni}}\} \cup \mathrm{Aut}(V; +)$. Hence, $\{\mathrm{id}^{\mathrm{ni}}, \mathrm{af}^{\mathrm{ni}}\} \cap \mathrm{End}(\Gamma) \neq \emptyset$. Since the list given in Propositions 3.5.15 and 3.5.17, combined with the equalities from Propositions 3.3.14 and 3.3.37, corresponds to an exhaustive combination of functions from $\{\mathrm{id}^*, \mathrm{af}, \mathrm{af}', \mathrm{af}^*\}$ with $\mathrm{id}^{\mathrm{ni}}$ and $\mathrm{af}^{\mathrm{ni}}$, we have: $\mathrm{End}(\Gamma)$ equals one of monoids listed in the statement of the lemma. □

Lemma 3.5.19. *Let* Γ *be a reduct of* $(V; +)$ *such that* $\mathrm{End}(\Gamma) \nsubseteq \mathrm{End}(V; \mathrm{Ieq}_4)$ *but* Ind_2 *is preserved by every endomorphism of* Γ*, and such that there exists* $f \in \mathrm{End}(\Gamma)$ *and* $x_0 \neq 0$ *satisfying:* $f(0) = f(x_0) \neq 0$*. Then* $\mathrm{End}(\Gamma) \cap \{\mathrm{gen}, \mathrm{gen}^*\} \neq \emptyset$*.*

Proof. Suppose that there exists $(c_1, c_2, c_3, c_4) \in \mathrm{Ieq}_4$ such that $x_0 \notin \{c_1, c_2, c_3, c_4\}$, and $\neg\, \mathrm{Ieq}_4(f(c_1), f(c_2), f(c_3), f(c_4))$. Since $0 \notin f(V)$, we have $\sum_{i \leq 4} f(c_i) \neq 0$. Let $h \in \mathrm{Emb}(V; +)$ such that $\{c_1, c_2, c_3, c_4\} \subseteq h(V)$ but $x_0 \notin h(V)$. For $i \leq 4$, we define d_i such that $h(d_i) = c_i$. The function $f \circ h$ is an injection such that $\mathrm{Ieq}_4(d_1, d_2, d_3, d_4)$ and $\sum_{i \leq 4} f(h(d_i)) \neq 0$. Since $(V; +, <, c_1, c_2, c_3, c_4)$ is an ω-categorical totally ordered Ramsey structure by Corollary 3.4.5 combined with Lemma 2.4.13, we can apply Corollary 2.4.16. Hence, $f \circ h$ together with $\mathrm{Aut}(V; +)$ locally generates a canonical function f_1 from $(V; +, <, c_1, c_2, c_3, c_4)$ to $(V; +)$ which agrees with $f \circ h$ on d_1, d_2, d_3, d_4 (which implies $\sum_{i \leq 4} f_1(d_i) \neq 0$). Since $f \circ h$ is injective, f_1 is injective. By Proposition 3.4.43, f_1 together with $\mathrm{Aut}(V; +)$ locally generates a weakly canonical injection f_1' from $(V; +, d_1, d_2, d_3, d_4)$ to $(V; +)$ such that: $\sum_{i \leq 4} f_1'(d_i) \neq 0$. So, by Corollary 3.4.38, f_1' together with $\mathrm{Aut}(V; +)$ locally generates either gen or gen^*, and $\{\mathrm{gen}, \mathrm{gen}^*\} \cap \mathrm{End}(\Gamma) \neq \emptyset$.

We now suppose that for all $(x_1, x_2, x_3, x_4) \in \mathrm{Ieq}_4$ such that $x_0 \notin \{x_1, x_2, x_3, x_4\}$, we have: $(f(x_1), f(x_2), f(x_3), f(x_4)) \in \mathrm{Ieq}_4$. Let $h \in \mathrm{Emb}(V;+)$ such that $x_0 \notin h(V)$. Then $f \circ h$ is an injective function which preserves Ieq_4. Let $g \in \mathrm{End}(\Gamma)$ such that g violates Ieq_4. There exists $(d_1, d_2, d_3, d_4) \in \mathrm{Ieq}_4$ such that:

$$(g(d_1), g(d_2), g(d_3), g(d_4)) \notin \mathrm{Ieq}_4$$

Since g preserves Ind_2, g is injective on $V \setminus \{0\}$ and since Ind_1 is pp-definable by Ind_2 (indeed, for all $x, y \neq 0$, $\mathrm{Ind}_1(x) \Leftrightarrow \exists y \, \mathrm{Ind}_2(x, y)$), then $g(V \setminus \{0\}) \subseteq V \setminus \{0\}$. So $\sum_{i \leq 4} g(d_i) \neq 0$. Let $(c_1, c_2, c_3, c_4) \in \mathrm{Ieq}_4$ such that $x_0 \notin \{c_1, c_2, c_3, c_4\}$. Then we have $(f(h(c_1)), f(h(c_2)), f(h(c_3)), f(h(c_4))) \in \mathrm{Ieq}_4$. Let $\alpha \in \mathrm{Aut}(V;+)$ such that $\alpha(c_i) = d_i$ for all $i \leq 4$ (such an α exists since $\mathrm{tp}_{(V;+)}(c_1, c_2, c_3, c_4) = \mathrm{tp}_{(V;+)}(d_1, d_2, d_3, d_4)$). Then consider the function $g' := g \circ \alpha \circ f \circ h$. We have g' injective since $\alpha \circ f \circ h$ is injective on V, and since $\alpha(f(h(V))) \subseteq V \setminus \{0\}$ with g injective on $V \setminus \{0\}$. Consequently, we have an injective function g' which sends $(c_1, c_2, c_3, c_4) \in \mathrm{Ieq}_4$ to a tuple satisfying $\sum_{i \leq 4} g'(c_i) \neq 0$. We are now back in the settings of the first part of the proof. Hence, $\{\mathrm{gen}, \mathrm{gen}^*\} \cap \mathrm{End}(\Gamma) \neq \emptyset$. □

Proposition 3.5.20. *Let Γ be a reduct of $(V;+)$ such that Γ has a non-injective endomorphism. Then either Γ is homomorphically equivalent to a structure with at most two elements, or $\{\mathrm{gen}, \mathrm{gen}^*\} \cap \mathrm{End}(\Gamma) \neq \emptyset$, or $\mathrm{End}(\Gamma)$ belongs to the list of monoids given in Lemma 3.5.18.*

Proof. Exactly one of the following cases holds:

- there exist $x_0, x_1 \neq 0$ such that $x_0 \neq x_1$, and $f(x_0) = f(x_1) = y_0 \neq 0$. In this case, we prove that Γ has an endomorphism which is constant on $V \setminus \{0\}$. Indeed, assume that for every subset S of $V \setminus \{0\}$ of size smaller or equal to n, there exists $g_S \in \mathrm{End}(\Gamma)$ such that $g_S(x) = y_0$ for all $x \in S$. Let $x \in V \setminus (S \cup \{0\})$. If $g_S(x) = y_0$, then we define $g_{S \cup \{x\}} := g_S$ and we are done. Otherwise, let $\alpha \in \mathrm{Aut}(\Gamma)$ be such that $\alpha(g_S(x)) = x_0$ and $\alpha(y_0) = x_1$. Then we define $g_{S \cup \{x\}} := f \circ \alpha \circ g_S$. The function $g_{S \cup \{x\}}$ is an endomorphism of Γ which is constant equal to y_0 on $S \cup \{x, 0\}$. Consequently, by local closure of $\mathrm{End}(\Gamma)$, Γ has a constant endomorphism g on $V \setminus \{0\}$, so Γ is homomorphically equivalent to the following 2-element structure: $(\{g(0), y_0\}; R'_1, \ldots, R'_n)$ where R'_i is the restriction of R_i on $\{g(0), y_0\}$, with $(R_i)_{i \in I}$ the relations of Γ.

- there exists x_0 such that $f(0) = f(x_0) = 0$. In this case, we prove that Γ has a constant endomorphism. Indeed, suppose that for every subset S of $V \setminus \{0\}$ of size smaller or equal to n, there exists $g_S \in \mathrm{End}(\Gamma)$ such that $g_S(x) = 0$ for all $x \in S \cup \{0\}$. Let $x \in V \setminus (S \cup \{0\})$. If $g_S(x) = 0$, then we define $g_{S \cup \{x\}} := g_S$ and we are done. Otherwise, let $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(g_S(x)) = x_0$. Then we define $g_{S \cup \{x\}} := f \circ \alpha \circ g_S$. The function $g_{S \cup \{x\}}$ is an endomorphism of Γ which is constant equal to 0 on $S \cup \{x, 0\}$. Consequently, Γ is homomorphically equivalent to a 1-element structure, namely $(\{0\}; R'_1, \ldots, R'_n)$ where R'_i is the restriction of R_i on $\{0\}$, with $(R_i)_{i \in I}$ the relations of Γ.

- there exist $x_0, x_1 \neq 0$ such that $x_0 \neq x_1$, and $f(x_0) = f(x_1) = 0$. In this case, let us consider $f \circ f$. If $f \circ f(0) \neq 0$, then we are back in the first case of this proof. If $f(0) = 0$, then $f \circ f(x_0) = f \circ f(0) = 0$. And we are back in the second case of this proof.

- Ieq_4 is preserved by every endomorphism of Γ and there exists x_0 such that $f(0) = f(x_0) \neq 0$. In this case, $\mathrm{End}(\Gamma)$ belongs to the list of monoids given in Lemma 3.5.18.

- Ieq_4 and Ind_2 are both violated by an endomorphism of Γ, and there exists x_0 such that $f(0) = f(x_0) \neq 0$. In this case, since Ind_2 is violated, there are two possibilities: either there exists $g \in \mathrm{End}(\Gamma)$ and $x'_0, x'_1 \neq 0$ such that $g(x'_0) = g(x'_1)$ and $x'_0 \neq x'_1$, and we are back to case 1 or 3 of this proof. Or, there exists $g \in \mathrm{End}(\Gamma)$ and $x'_0 \neq 0$ such that $g(x'_0) = 0$. In this case, let $\alpha \in \mathrm{Aut}(V;+)$ be such that $\alpha(f(0)) = x'_0$ (such an α exists since $f(0)$ and x'_0 are both distinct from 0), and let us consider the following function: $g' := g \circ \alpha \circ f$. Then $g'(0) = g'(x_0) = 0$, and we are back in case 2.

- Ieq_4 is violated by an endomorphism of Γ but Ind_2 is preserved by every endomorphism of Γ, and there exists x_0 such that $f(0) = f(x_0) \neq 0$. Then by Lemma 3.5.19 we have $\mathrm{End}(\Gamma) \cap \{\mathrm{gen}, \mathrm{gen}^*\} \neq \emptyset$.

□

Theorem 3.5.21. *Let Γ be a reduct of $(V;+)$. Then at least one of the following cases holds:*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_3,\neq), \mathrm{End}(V;\mathrm{Ieq}_3)\}$
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4,\neq), \mathrm{End}(V;\mathrm{Ieq}_4)\}$
- $\mathrm{End}(\Gamma) = \mathrm{End}(V;\mathrm{Ieq}_4, 0)$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Eq}_4,\mathrm{Ind}_1,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, Z_1 \cup R_1 \cup \mathrm{Ieq}_4)\}$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4, Z_1 \cup \mathrm{Ind}_4,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, Z_1 \cup R_1 \cup \mathrm{Ind}_4)\}$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4, T_1,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, T_1 \cup R_1), \mathrm{End}(V;\mathrm{Ieq}_4, T_1 \cup R_1 \cup R_2)\}$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4, T_2,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, T_2 \cup R_1), \mathrm{End}(V;\mathrm{Ieq}_4, T_2 \cup R_1 \cup R_2)\}$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4, T_3,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, T_3 \cup R_1), \mathrm{End}(V;\mathrm{Ieq}_4, T_3 \cup R_1 \cup R_2)\}$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4, T_4,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, T_4 \cup R_1)\}$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4, T_5,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, T_5 \cup R_1)\}$;
- $\mathrm{End}(\Gamma) \in \{\mathrm{End}(V;\mathrm{Ieq}_4, T_6,\neq), \mathrm{End}(V;\mathrm{Ieq}_4, T_6 \cup R_1)\}$;

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_7 \cup R_1)$*;*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Eq}_4, \neq)$*;*

- $\mathrm{End}(\Gamma) \cap \{\mathrm{gen}, \mathrm{gen}^*\} \neq \emptyset$*;*

- Γ *is homomorphically equivalent to a structure with at most two elements.*

See figure 3.5.3.

Proof. Let Γ be a reduct of $(V; +)$. If Γ has a non-injective endomorphism, then we are in the various cases described in Proposition 3.5.20. We now assume that all the endomorphisms of Γ are injective, i.e., preserve $\neq$. We first suppose that $\mathrm{End}(\Gamma) \subseteq \mathrm{End}(V; \mathrm{Ieq}_4, \neq)$. By Proposition 3.3.25, for all $f \in \mathrm{End}(\Gamma)$ there exists $h \in \mathrm{Emb}(V; +)$, $v \notin h(V)$, and $c \in V$ such that $f(0) = c$ and $f(x) = h(x) + v$ for all $x \neq 0$. Depending on the values of v and c, f belongs to one of the following classes: $\mathrm{cl}(\mathrm{id})$, $\mathrm{cl}(\mathrm{id}^*)$, $\mathrm{cl}(\mathrm{af})$, $\mathrm{cl}(\mathrm{af}')$, or $\mathrm{cl}(\mathrm{af}^*)$ by Corollary 3.3.28. Since the list given in Proposition 3.5.13, combined with the equalities from Propositions 3.3.15, 3.3.35, 3.3.38, and Corollary 3.3.33, corresponds to an exhaustive union of these classes, we have $\mathrm{End}(\Gamma)$ equals one of the monoids listed in the theorem.

We now assume that $\mathrm{End}(\Gamma) \not\subseteq \mathrm{Ieq}_4$. There exists $f \in \mathrm{End}(\Gamma)$ which does not preserve Ieq_4. We first assume that $\mathrm{End}(\Gamma) \subseteq \mathrm{End}(V; \mathrm{Eq}_4, \neq)$. By Proposition 3.3.29, there exists $h \in \mathrm{Emb}(V; +)$ and $v \in V$ such that $f(x) = h(x) + v$ for all $x \in V$. Since f does not preserve Ieq_4, we have: $v \in h(V)$. Consequently, f together with $\mathrm{Aut}(V; +)$ locally generates every translation. Indeed, let v' be a non-zero element of V, and $S := \{0, x_1, \ldots, x_n\}$ be a finite subset of V. Then:

$$\begin{aligned}
\mathrm{tp}_{(V;+)}(f(x))_{x \in S} &= \mathrm{tp}_{(V;+)}(h(x) + v)_{x \in S} \\
&= \mathrm{tp}_{(V;+)}(\alpha(h(x)) + v')_{x \in S} \\
&= \mathrm{tp}_{(V;+)}(t_{v'}(\alpha(h(x))))_{x \in S}
\end{aligned}$$

with $\alpha \in \mathrm{Aut}(V; +)$ such that $\alpha(v) = v'$. So $t_{v'}$ belongs to $\overline{\langle \{f\} \cup \mathrm{Aut}(V; +) \rangle_1}$ for every $v' \in V$. By Proposition 3.3.32, $\mathrm{End}(V; \mathrm{Eq}_4, \neq) = \overline{\langle \{t_v\} \cup \mathrm{Aut}(V; +) \rangle_1}$, hence:

$$\mathrm{End}(V; \mathrm{Eq}_4, \neq) \subseteq \mathrm{End}(\Gamma) \subseteq \mathrm{End}(V; \mathrm{Eq}_4, \neq)$$

Consequently, $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Eq}_4, \neq)$.

We can now assume that there exists $f, g \in \mathrm{End}(\Gamma)$ such that f_1 does not preserve Ieq_4, and f_2 does not preserve Eq_4. There exist $(c_1, c_2, c_3, c_4) \in V$ such that $\mathrm{Ieq}_4(c_1, c_2, c_3, c_4)$ and $\neg\, \mathrm{Ieq}_4(f_1(c_1), f_1(c_2), f_1(c_3), f_1(c_4))$. There also exists $(d_1, d_2, d_3, d_4) \in V^4$ such that:

$$\mathrm{Eq}_4(d_1, d_2, d_3, d_4) \text{ and } \neg\, \mathrm{Ieq}_4(f_2(d_1), f_2(d_2), f_2(d_3), f_2(d_4))$$

First suppose that $\mathrm{Ieq}_4(d_1, d_2, d_3, d_4)$. Then f_2 together with $\mathrm{Aut}(V; +)$ locally generates a canonical function g_2 from $(V; +, <, d_1, d_2, d_3, d_4)$ to $(V; +)$ which agrees with f_2 on

d_1, d_2, d_3, d_4 (which implies $\neg\, \mathrm{Eq}_4(g_2(d_1), g_2(d_2), g_2(d_3), g_2(d_4))$). Since f_2 is injective, g_2 is injective. By Proposition 3.4.43, g_2 together with $\mathrm{Aut}(V;+)$ locally generates a weakly canonical injection g'_2 over $(V;+,d_1,d_2,d_3,d_4)$ such that:

$$\neg\, \mathrm{Eq}_4(g'_2(d_1), g'_2(d_2), g'_2(d_3), g'_2(d_4))$$

Consequently, by Corollary 3.4.38, g'_2 together with $\mathrm{Aut}(V;+)$ locally generates either gen or gen*. So $\{\mathrm{gen}, \mathrm{gen}^*\} \cap \mathrm{End}(\Gamma) \neq \emptyset$.

We can now suppose that $\mathrm{Eq}_4(d_1, d_2, d_3, d_4)$ but $\neg\, \mathrm{Ieq}_4(d_1, d_2, d_3, d_4)$. Hence, we can assume that $d_1 = 0$, and $d_2 + d_3 + d_4 = 0$ with $d_i \neq d_j$ for all $i \neq j$. Since the structure $(V;+,<,c_1,c_2,c_3,c_4)$ is an ω-categorical totally ordered Ramsey structure by Corollary 3.4.5 combined with Lemma 2.4.13, we can apply Corollary 2.4.16. Hence, f_1 together with $\mathrm{Aut}(V;+)$ locally generates a canonical function g_1 from the structure $(V;+,<,c_1,c_2,c_3,c_4)$ to $(V;+)$ which agrees with f_1 on c_1, c_2, c_3, c_4 (which implies that the tuple $(g_1(c_1), g_1(c_2), g_1(c_3), g_1(c_4))$ does not belong to Ieq_4). Since f_1 is injective, g_1 is injective. By Proposition 3.4.43, g_1 together with $\mathrm{Aut}(V;+)$ locally generates a weakly canonical injection g'_1 over $(V;+,d_1,d_2,d_3,d_4)$ such that:

$$\neg\, \mathrm{Ieq}_4(g'_1(c_1), g'_1(c_2), g'_1(c_3), g'_1(c_4))$$

We distinguish two cases here:

- either $\neg\, \mathrm{Eq}_4(g'_1(c_1), g'_1(c_2), g'_1(c_3), g'_1(c_4))$. In this case, by Corollary 3.4.38, g'_1 together with $\mathrm{Aut}(V;+)$ locally generates either gen or gen*. So $\{\mathrm{gen}, \mathrm{gen}^*\} \cap \mathrm{End}(\Gamma) \neq \emptyset$.

- or $\mathrm{Eq}_4(g'_1(c_1), g'_1(c_2), g'_1(c_3), g'_1(c_4))$. Since $\neg\, \mathrm{Ieq}_4(g'_1(c_1), g'_1(c_2), g'_1(c_3), g'_1(c_4))$, we can assume that $\mathrm{tp}_{(V;+)}(d_1, d_2, d_3, d_4) = \mathrm{tp}_{(V;+)}(g'_1(c_1), g'_1(c_2), g'_1(c_3), g'_1(c_4))$, so there exists $\alpha \in \mathrm{Aut}(V;+)$ such that $\alpha(g'_1(c_i)) = d_i$ for all $i \leq 4$. So the function $f_3 := g'_2 \circ \alpha \circ g'_1$ satisfies:

  $$\mathrm{Ieq}_4(c_1, c_2, c_3, c_4) \text{ and } \neg\, \mathrm{Eq}_4(f_3(c_1), f_3(c_2), f_3(c_3), f_3(c_4))$$

  Then f_3 together with $\mathrm{Aut}(V;+)$ locally generates a canonical function g_3 from $(V;+,<,d_1,d_2,d_3,d_4)$ to $(V;+)$ which agrees with f_3 on c_1, c_2, c_3, c_4 (which implies that $\neg\, \mathrm{Eq}_4(g_3(c_1), g_3(c_2), g_3(c_3), g_3(c_4))$). Since f_3 is injective, g_3 is injective. By Proposition 3.4.43, g_3 together with $\mathrm{Aut}(V;+)$ locally generates a weakly canonical injection g'_3 over $(V;+,c_1,c_2,c_3,c_4)$ such that:

  $$\neg\, \mathrm{Eq}_4(g'_3(c_1), g'_3(c_2), g'_3(c_3), g'_3(c_4))$$

  Consequently, by Corollary 3.4.38, g'_3 together with $\mathrm{Aut}(V;+)$ locally generates either gen or gen*. So $\{\mathrm{gen}, \mathrm{gen}^*\} \cap \mathrm{End}(\Gamma) \neq \emptyset$.

$\square$

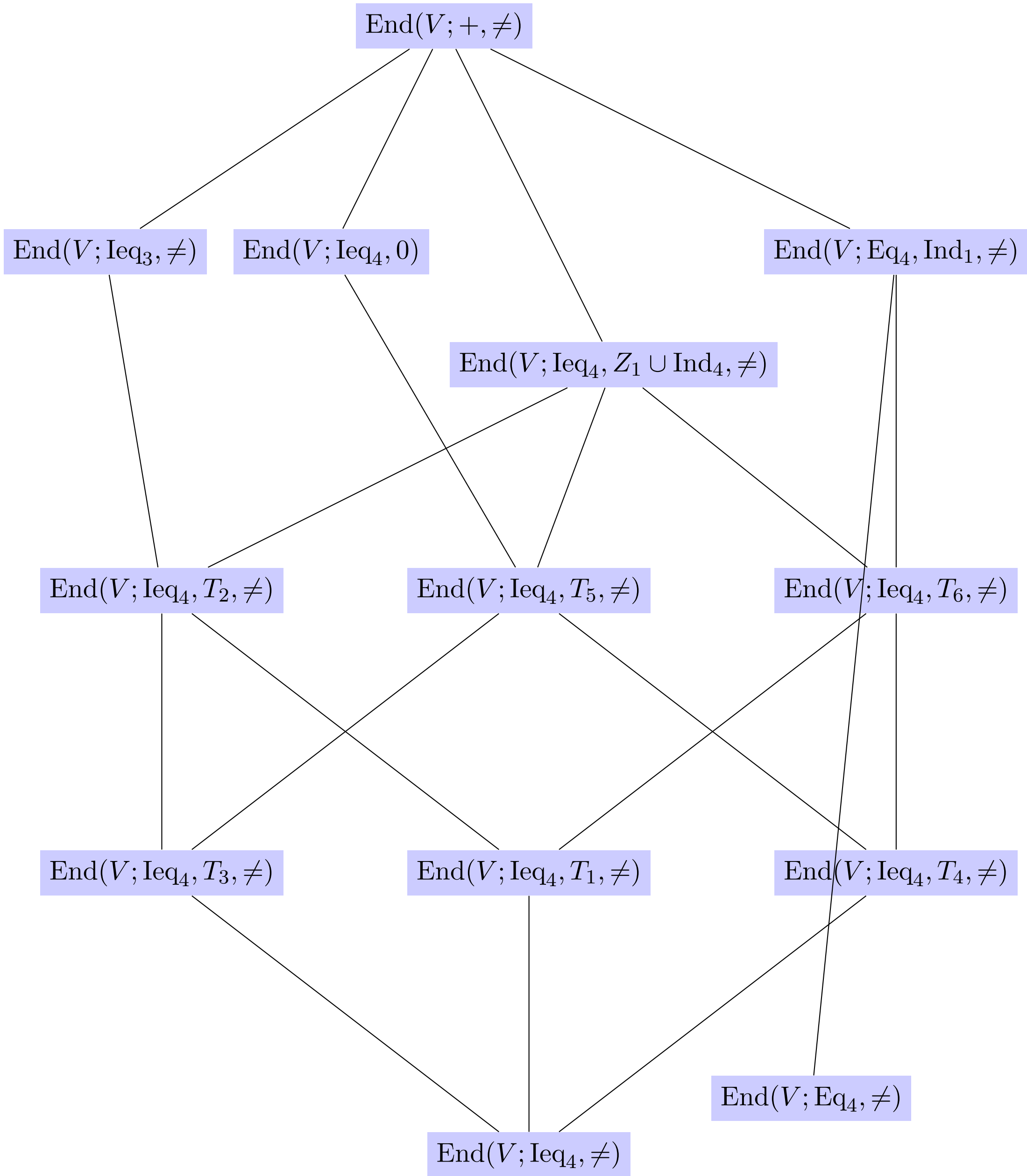

Figure 3.1: The injective endomorphism monoids of reducts of $(V; +)$ which do not contain gen nor gen*.

Proposition 3.5.22. *There exists an infinite chain of endomorphism monoids of reducts of* $(V;+)$*. More precisely:*

$$\cdots \subsetneq \mathrm{End}(V;\mathrm{Ind}_{n+1}) \subsetneq \mathrm{End}(V;\mathrm{Ind}_n) \subsetneq \cdots \subsetneq \mathrm{End}(V;\mathrm{Ind}_2)$$

Proof. First note that Ind_k is existential positive definable in $(V;\mathrm{Ind}_n)$ for all $k \leq n$, since $\mathrm{Ind}_k(x_1,\dots,x_k)$ if and only if $\exists y_{k+1},\dots,y_n.\mathrm{Ind}_n(x_1,\dots,x_k,y_{k+1},\dots,y_n)$. Hence, $\mathrm{End}(V;\mathrm{Ind}_n) \subseteq \mathrm{End}(V;\mathrm{Ind}_k)$ for all $k \leq n$, by Theorem 2.1.27. For $n \geq 3$, let $(a_1,\dots,a_n)$ be elements of V such that and $\mathrm{Ieq}_n(a_1,\dots,a_n)$, and let $(b_k)_{k\geq n+1}$ be an infinite family of linearly independent vectors from $V \setminus \mathrm{Vect}(a_1,\dots,a_n)$. Let $(v_k)_{k\in\mathbb{N}}$ be an enumeration of V such that $v_0 = 0$, and $\mathrm{Ind}_n(v_1,\dots,v_n)$. We define the function agen_n as follows: $\mathrm{agen}_n(0) = 0$, $\mathrm{agen}(v_i) = a_i$ for all $i \leq n$, and $\mathrm{agen}(v_i) = b_i$ for all $i \geq n+1$. Then we clearly have: $\mathrm{agen}_n \in \mathrm{End}(V;\mathrm{Ind}_{n-1}) \setminus \mathrm{End}(V;\mathrm{Ind}_n)$ for all $n \geq 3$. □

3.5.4 The Lattice of the Self-embedding Monoids

Lemma 3.5.23. *Let* Γ *be a first-order reduct of* $(V;+)$*, and assume that* $\mathrm{af}^* \in \mathrm{Emb}(\Gamma)$*. Then* id^* *and* af *also belong to* $\mathrm{Emb}(\Gamma)$*.*

Proof. We first give the proof for af. Let R be a relation of Γ, and let us consider $(0,\dots,0,x_1,\dots,x_n) \in R$ (we suppose for the sake of the notations that all the zeros of the tuple are in the beginning of the tuple, and that $x_1,\dots,x_n \neq 0$). We want to prove that the tuple $(\mathrm{af}(0),\dots,\mathrm{af}(0),\mathrm{af}(x_1),\dots,\mathrm{af}(x_n))$ is also in R. Recall that $\mathrm{af}^*(0) = d$ and $\mathrm{af}^*(x) = h(x) + v$ with $v \notin h(V)$ and $d \notin \mathrm{Vect}(h(V) \cup \{v\})$. Let $h' \in \mathrm{Emb}(V;+)$ and v' such that $v' \notin g(V)$, and $d \notin \mathrm{Vect}(h'(h(V) \cup \{h'(v) + v'\}$. We denote by g the function such that $g(0) = d$ and $g(x) = h'(h(x)) + h'(v) + v'$ (such a self-embedding always exists since $(V;+)$ has infinite dimension. By Lemma 3.3.22 and Proposition 3.3.5, since $h'(v) + v' \notin h' \circ h(V)$, and since $d \notin \mathrm{Vect}(h'(h(V) \cup \{h'(v) + v'\})$, we have af^* together with $\mathrm{Aut}(V;+)$ locally generates g, so $g \in \mathrm{Emb}(\Gamma)$. Hence, $(d,\dots,d,h'(h(x_1)) + h'(v) + v',\dots,h'(h(x_n)) + h'(v) + v') \in R$. For the same reasons, the function g' defined as follows: $g'(0) = d$ and $g'(x) = h'(x) + v'$ belongs to $\mathrm{Emb}(\Gamma)$. Consequently $(0,\dots,0,h(x_1) + v,\dots,h(x_n) + v) \in R$, and $\mathrm{af} \in \mathrm{End}(\Gamma)$.

We now show that if $(0,\dots,0,h(x_1)+v,\dots,h(x_n)+v) \in R$, then $(0,\dots,0,x_1,\dots,x_n)$ is also in R. Using the same techniques, we get:

$$(d,\dots,d,g(h(x_1) + v) + v',\dots,g(h(x_n) + v) + v') \in R$$

So $(d,\dots,d,g(h(x_1)) + g(v) + v',\dots,g(h(x_n)) + g(v) + v') \in R$, and finally, since $\mathrm{af}^* \in \mathrm{Emb}(\Gamma)$, we have $(0,\dots,0,x_1,\dots,x_n) \in R$. To conclude: $\mathrm{af} \in \mathrm{Emb}(\Gamma)$.

The same proof works if v is replaced by 0. So $\mathrm{id}^* \in \mathrm{Emb}(\Gamma)$. □

Theorem 3.5.24. *Let* Γ *be a reduct of* $(V;+)$*. Then exactly one of the following cases holds:*

- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;+)$*;*

- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; \mathrm{Ieq}_3)$;
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; \mathrm{Ieq}_4, 0)$;
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; \mathrm{Ieq}_4)$;
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; \mathrm{Eq}_4)$;
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; 0)$;
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; =)$.

See figure 3.5.4.

Proof. First note that every self-embedding monoid of a reduct Γ of a structure Δ is also an endomorphism monoid of a reduct of Δ. Indeed, if $\Gamma = (D; R_1, \ldots, R_n)$, we define the reduct $\Gamma' := (D; R_1, \neg R_1, \ldots, R_n, \neg R_n, \neq)$ of Δ, and then $\mathrm{Emb}(\Gamma) = \mathrm{End}(\Gamma')$. Furthermore, this monoid must contain only injective functions. By Theorem 3.5.21, one of the following cases holds:

- gen or gen* belongs to $\mathrm{Emb}(\Gamma)$, and in this case, by Proposition 3.5.1, Γ is a reduct of $(V; 0)$, which implies by Corollary 3.5.4 that either $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; 0)$, or $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V; =)$.
- $\mathrm{Emb}(\Gamma)$ does not contain neither gen nor gen*. In this case, since $\mathrm{Emb}(\Gamma) = \mathrm{End}(\Gamma')$, we have: $\mathrm{End}(\Gamma') \cap \{\mathrm{gen}, \mathrm{gen}^*\} = \emptyset$. So by Theorem 3.5.21, $\mathrm{Emb}(\Gamma) = \mathrm{End}(\Gamma')$ equals one of the monoids listed in the theorem. Note that we have a functional description of every monoid of injective functions listed in Theorem 3.5.21. Such descriptions are available in Propositions 3.3.15, 3.3.33, 3.3.35, 3.3.38, and 3.5.13. Looking at these functional descriptions, we proceed with a case distinction:
  - if $\mathrm{af}' \in \mathrm{Emb}(\Gamma)$, then by Proposition 3.5.9, Γ is a reduct of $(V; \mathrm{Eq}_4)$ and contains all the translations. The only possibility in our list is $\mathrm{Emb}(\Gamma) = \mathrm{End}(V; \mathrm{Eq}_4, \neq)$;
  - if $\mathrm{af}' \notin \mathrm{Emb}(\Gamma)$ but $\mathrm{af}^* \in \mathrm{Emb}(\Gamma)$, then by Lemma 3.5.23, id^* and af also belong to $\mathrm{Emb}(\Gamma)$. Hence, the only candidate left is $\mathrm{Emb}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, T_3, \neq)$. Finally, a fast calculation gives $\mathrm{End}(V; \mathrm{Ieq}_4, T_3, \neq) = \mathrm{Emb}(V; \mathrm{Ieq}_4)$;
  - if $\mathrm{af}^*, \mathrm{af}' \notin \mathrm{Emb}(\Gamma)$ but $\mathrm{id}^* \in \mathrm{Emb}(\Gamma)$, then $\mathrm{af} \notin \mathrm{Emb}(\Gamma)$ (because $\{\mathrm{id}^*, \mathrm{af}\}$ together with $\mathrm{Aut}(V; +)$ locally generates af^*, since $\mathrm{af} \circ \mathrm{id}^*$ has the same behaviour as af^*). So the only the case $\mathrm{Emb}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_3, \neq) = \mathrm{Emb}(V; \mathrm{Ieq}_3)$ remains;
  - if $\mathrm{af}^*, \mathrm{af}' \notin \mathrm{Emb}(\Gamma)$ but $\mathrm{af} \in \mathrm{Emb}(\Gamma)$, then $\mathrm{id}^* \notin \mathrm{Emb}(\Gamma)$ (because $\{\mathrm{id}^*, \mathrm{af}\}$ together with $\mathrm{Aut}(V; +)$ locally generates af^*). So the only possibility is $\mathrm{Emb}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, 0) = \mathrm{Emb}(V; \mathrm{Ieq}_4, 0)$;
  - finally, if $\mathrm{id}^*, \mathrm{af}, \mathrm{af}', \mathrm{af}^* \notin \mathrm{Emb}(\Gamma)$, then the only endomorphism monoid left in our list is $\mathrm{End}(V; +, \neq)$. So $\mathrm{Emb}(\Gamma) = \mathrm{End}(V; +, \neq) = \mathrm{Emb}(V; +)$.

□

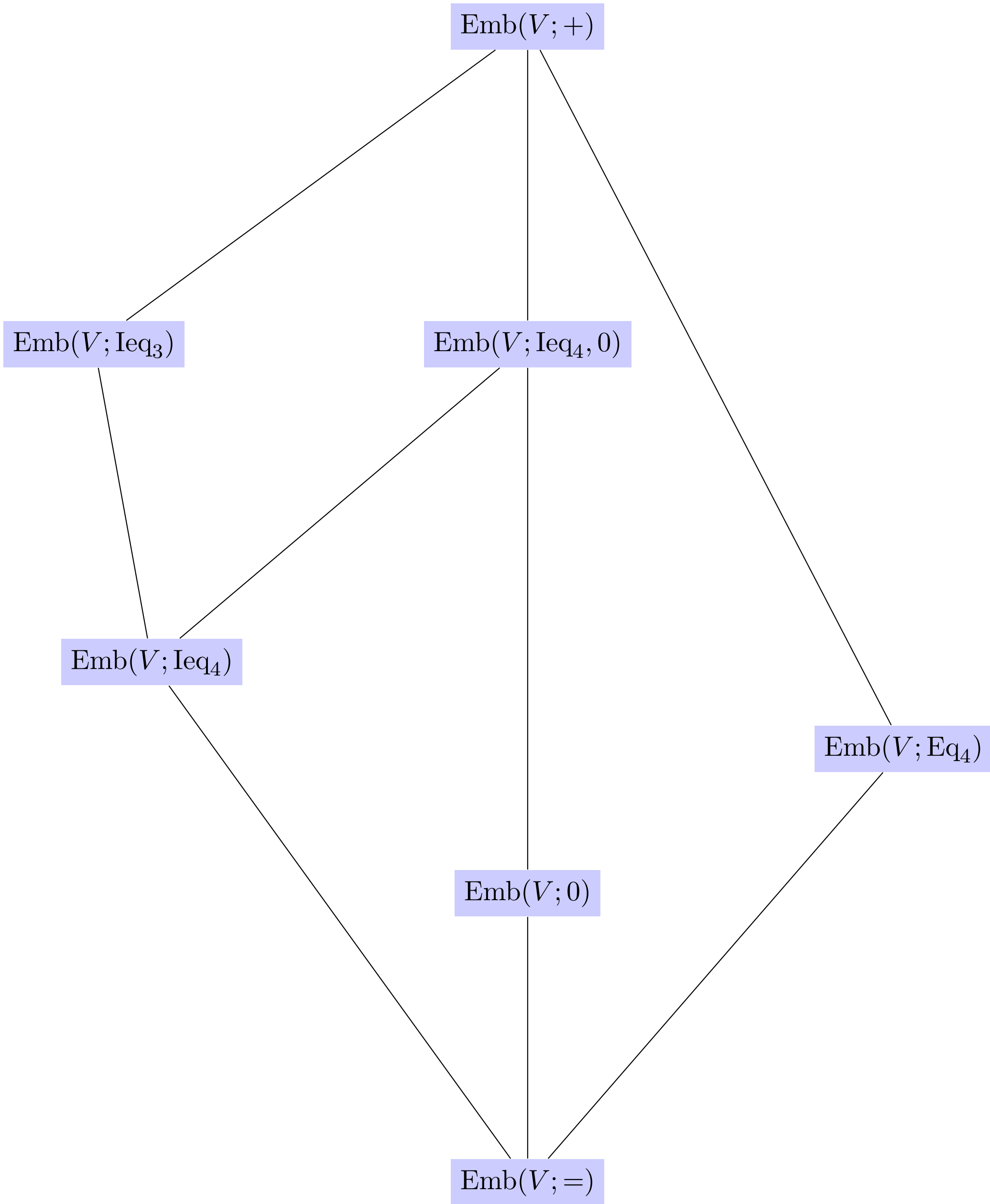

Figure 3.2: The lattice of self-embedding monoids of reducts of $(V;+)$

3.5.5 The Group Classification

Now we have all the tools to build the classification of the automorphism groups of reducts of $(V;+)$: we listed all the canonical functions and proved the theorem which allows us to locally generate them from any function. We also described the lattice of self-embedding monoids of reducts of $(V;+)$. So let us give the sketch of what is coming: we will give two distinct proves for the group classification. The first, and the more natural in our chronology, is a corollary of our self-embedding monoids classification 3.5.24. The second only uses the Canonization Lemma 2.4.16 and the classification of canonical functions.

Notation 3.5.25. Let G, H be two permutation groups acting on a set M, N respectively. We denote by $G \cong_p H$ the fact that G and H as isomorphic as permutation groups, i.e., there exists an isomorphism of abstract group ϕ from G to H, and a bijection α from M to N such that for all $g \in G$, and for all $x \in M$, we have:

$$\phi(g).\alpha(x) = \alpha(g.x)$$

Remark 3.5.26. The following theorem has been proved independently by Kalina, Bodor, and Szabó [BKS15] in a direct self-contained proof. Their idea is to make a case distinction on the stabilizer of 0 of a closed group G of permutations of V strictly containing $\mathrm{Aut}(V;+)$ but not fixing 0, and depending whether this stabilizer is $\mathrm{Aut}(V;+)$ or $\mathrm{Aut}(V;0)$, conclude that G is either $\mathrm{Aut}(V;\mathrm{Eq}_4)$ or $\mathrm{Aut}(V;=)$. In our case, the proof we give is quite distinct. Nonetheless, both proves strongly use the notion of high transitivity in order to establish that a given closed permutation group contains $\mathrm{Aut}(V;0)$.

Theorem 3.5.27. *Let Γ be a reduct of $(V;+)$. Then exactly one of the following cases holds:*

- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;+) \cong_p GL(\omega, \mathbb{F}_2)$
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Eq}_4) \cong_p AGL(\omega, \mathbb{F}_2)$
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;0) = \mathrm{Sym}(V)_0$*, the group of all permutations of V preserving* 0.
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;=) = \mathrm{Sym}(V)$*, the group of all permutations of V.*

See figure 3.5.5

Proof. The first proof we give is a simple consequence of Theorem 3.5.24. We make a case distinction following the list given in the theorem:

- if $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;+)$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;+)$ by Proposition 2.1.6.
- if $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Ieq}_3)$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Ieq}_3)$ by Proposition 2.1.6. By Lemma 3.2.23, $(V;\mathrm{Eq}_3)$ and $(V;\mathrm{Ieq}_3)$ are first-order interdefinable. Hence:

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Ieq}_3) = \mathrm{Aut}(V;\mathrm{Eq}_3) = \mathrm{Aut}(V;+)$$

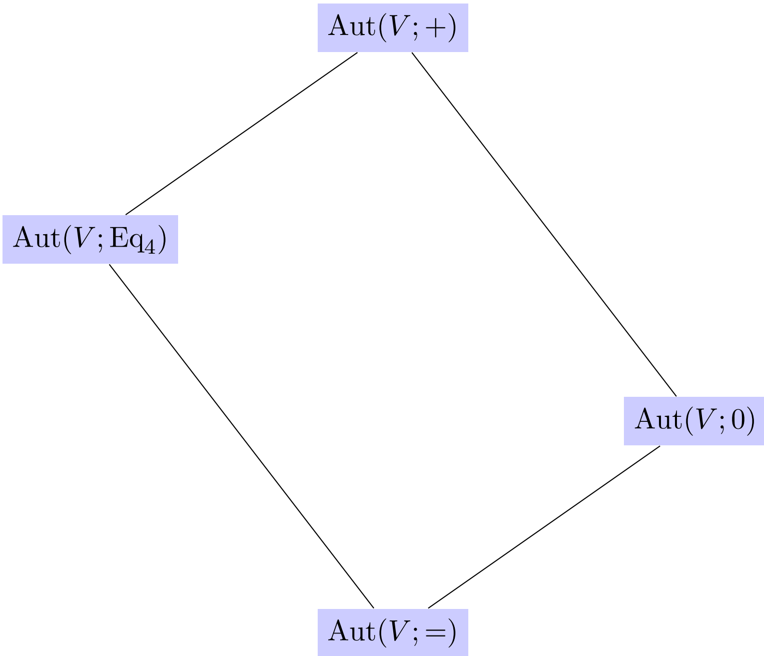

Figure 3.3: The lattice of the automorphism groups of reducts of $(V;+)$

- if $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Ieq}_4, 0)$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Ieq}_4, 0)$ by Proposition 2.1.6. By Lemma 3.2.23, $(V;\mathrm{Eq}_3)$ and $(V;\mathrm{Ieq}_4)$ are first-order interdefinable, so $(V;\mathrm{Eq}_3)$ and $(V;\mathrm{Ieq}_4, 0)$ are also first-order definable since 0 is first-order definable in $(V;\mathrm{Eq}_3)$. Hence:

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Ieq}_4, 0) = \mathrm{Aut}(V;\mathrm{Eq}_3) = \mathrm{Aut}(V;+)$$

- if $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Ieq}_4)$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Ieq}_4)$ by Proposition 2.1.6. By Lemma 3.2.23, $(V;\mathrm{Eq}_3)$ and $(V;\mathrm{Ieq}_4)$ are first-order interdefinable. Hence:

$$\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Ieq}_4) = \mathrm{Aut}(V;\mathrm{Eq}_3) = \mathrm{Aut}(V;+)$$

- if $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Eq}_4)$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Eq}_4)$ by Proposition 2.1.6.
- if $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;0)$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;0)$ by Proposition 2.1.6.
- if $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;=)$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;=)$ by Proposition 2.1.6.

$\square$

The first proof we give is easier to understand and fit better in the chronological development of the article, because it follows the description of the lattice of self-embedding monoids. Nevertheless, it is interesting to give a second proof, which does not use this lattice, and which illustrate better the method of canonical functions.

Proof. We suppose that $\mathrm{Aut}(\Gamma) \neq \mathrm{Aut}(V;+)$, and let $\gamma \in \mathrm{Aut}(\Gamma) \setminus \mathrm{Aut}(V;+)$. Since $(V;\mathrm{Eq}_3)$ and $(V;\mathrm{Ieq}_4)$ are first-order interdefinable by Lemma 3.2.23, Theorem 2.1.27 gives that $\mathrm{Aut}(V;+) = \mathrm{Aut}(V;\mathrm{Ieq}_4)$. Therefore γ violates Ieq_4, i.e., there exist four elements c_1, c_2, c_3, c_4 of V such that:

$$\mathrm{Ieq}_4(c_1, c_2, c_3, c_4) \text{ and } \neg\,\mathrm{Ieq}_4(\gamma(c_1), \gamma(c_2), \gamma(c_3), \gamma(c_4))$$

Since $(V;+,<,c_1,c_2,c_3,c_4)$ is an ω-categorical totally ordered Ramsey structure by Corollary 3.4.5 combined with Lemma 2.4.13, we can apply Corollary 2.4.16. Hence, γ together with $\mathrm{Aut}(V;+)$ locally generates a canonical function f from $(V;+,<,c_1,c_2,c_3,c_4)$ to $(V;+)$ which agrees with γ on c_1, c_2, c_3, c_4 (which implies that $(f(c_1), f(c_2), f(c_3), f(c_4))$ does not belong to Ieq_4). Since γ is injective, f is injective. By Theorem 3.4.44, one of the following case occurs:

- f together with $\mathrm{Aut}(V;+)$ locally generates af', so $\mathrm{af}' \in \mathrm{Emb}(\Gamma)$. In this case, we show in Proposition 3.5.9 that Γ is a reduct of $(V;\mathrm{Eq}_4)$, hence by Theorem 3.5.10, either $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Eq}_4)$ or $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;=)$.
- f together with $\mathrm{Aut}(V;+)$ locally generates gen. Thus $\mathrm{gen} \in \mathrm{Emb}(\Gamma)$. By Proposition 3.5.1, Γ is a reduct of $(V;0)$ and by Corollary 3.5.5, either $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;=)$ or $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;0)$.
- f together with $\mathrm{Aut}(V;+)$ locally generates gen^*. Thus $\mathrm{gen}^* \in \mathrm{Emb}(\Gamma)$, and by Proposition 3.5.1, Γ is a reduct of $(V;=)$. Consequently, $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;=)$ by Corollary 3.5.5.

□

3.6 Model-complete Cores of Reducts

3.6.1 Definitions and General Properties

Recall that two structures Γ and Δ are *homomorphically equivalent* whenever there exists a homomorphism from Γ to Δ, and a homomorphism from Δ to Γ. When two structures are homomorphically equivalent, they have the same CSPs. Consequently, it is useful to introduce the notion of *core* of a structure.

Definition 3.6.1. A *core* is a structure Δ such that all endomorphisms of Δ are self-embeddings, and a structure Δ is a core of Γ if Δ is a core and homomorphically equivalent to Γ.

Proposition 3.6.2. *Let Γ be a relational structure, and Δ be a core of Γ. Then:*

$$|\,\mathrm{Dom}(\Gamma)| \geq |\,\mathrm{Dom}(\Delta)|$$

Proof. Since Δ is a core of Γ, Γ and Δ are homomorphically equivalent. Hence there exist an homomorphism g from Γ to Δ, and a homomorphism h from Δ to Γ. Then $g \circ h$ is an endomorphism of Δ, and since Δ is a core, $g \circ h$ is injective. Consequently, h is an injection from $\mathrm{Dom}(\Delta)$ to $\mathrm{Dom}(\Gamma)$. Hence, $|\,\mathrm{Dom}(\Gamma)| \geq |\,\mathrm{Dom}(\Delta)|$. □

The core of a finite structure is unique. This core is often simpler than the original structure, and by homomorphical equivalence, it has the same CSP. Consequently, in order to study the complexity of the CSP of a structure with a finite domain, it can be easier to study the CSP of its core. There still exist a notion of "core" for a structure with infinite domains, but it is not unique up to isomorphism. That is why Bodirsky introduced the notion of *model-complete core*, which has the following property. Every ω-categorical structure Γ has a unique model-complete core Δ up to isomorphism. As we would guess, $\mathrm{End}(\Delta) = \mathrm{Emb}(\Delta)$, and Δ is model-complete. We start by several definitions to introduce the concept of model-complete core.

Recall that we introduced model-completeness in Definition 2.3.2. We now define the notion of model-complete core, which only applies to ω-categorical structures.

Definition 3.6.3. A structure Δ is called a *model-complete core* whenever Δ is an ω-categorical core whose first-order theory is model-complete.

Given a structure Γ, we call a *model-complete core of* Γ any model-complete core homomorphically equivalent to Γ.

The following is from [Bod07]:

Theorem 3.6.4. *Let Δ be ω-categorical. Then Δ is a core if and only if every existential formula is equivalent to an existential positive formula over Δ. Moreover, the following are equivalent:*

1. *Δ is a model-complete core;*
2. *Every first-order formula is equivalent to an existential positive one over Δ;*
3. *The automorphisms of Δ locally generate the endomorphisms of Δ.*

Corollary 3.6.5. *Let Γ and Δ be two ω-categorical relational structures such that* $\mathrm{End}(\Delta) = \mathrm{End}(\Gamma)$*, and such that Δ is a model-complete core. Then Γ is also a model-complete core.*

Proof. By Theorem 3.6.4, we only have to prove that $\overline{\mathrm{Aut}(\Gamma)} = \mathrm{End}(\Gamma)$. Since Δ is a model-complete core, we have $\mathrm{End}(\Delta) = \overline{\mathrm{Aut}(\Delta)}$. Since $\mathrm{End}(\Delta) = \mathrm{End}(\Gamma)$, it is enough to prove that $\mathrm{Aut}(\Delta) = \mathrm{Aut}(\Gamma)$. But this is straightforward since Γ and Δ are existential positive (and hence first-order) reducts of each-other by Theorem 2.1.27. $\square$

Example 3.6.6. *The structure $(\mathbb{Q};<)$ is a model-complete core. Since $(\mathbb{Q};<)$ has quantifier elimination, it is model-complete, and since every endomorphism of $(\mathbb{Q};<)$ must be injective and strongly preserves $<$, it is a core.*

The following is from [Bod07]:

Theorem 3.6.7. *Every ω-categorical relational structure Γ is homomorphically equivalent to a model-complete core Δ. All model-complete cores of Γ are isomorphic to Δ.*

The following proposition build a bridge between the notion of model-companion presented in section 2.3, and model-complete cores.

Proposition 3.6.8. *Let* $\Gamma := (D; R_1, \ldots, R_k)$ *be an* ω*-categorical structure with finite relational signature, and let* $\Delta' := (D'; S_1, S_1', \ldots, S_k, S_k', B)$ *be the model-complete core of* $\Gamma' := (D; R_1, \neg R_1, \ldots, R_k, \neg R_k, \neq)$*. Then:*

$$\Delta := (D'; S_1, \ldots, S_k) \text{ is the model-companion of } \Gamma.$$

Consequently, $\mathrm{Emb}(\Delta) = \mathrm{End}(\Delta')$.

Proof. Since Δ' is the model-complete core of Γ', there exist two homomorphism $h_1 \colon \Gamma' \to \Delta'$ and $h_2 \colon \Delta' \to \Gamma'$ such that $h_1 \circ h_2$ is an elementary self-embedding of Δ'. We now prove that h_1 is an embedding from Γ to Δ, and that h_2 is an embedding from Δ to Γ.

Since $h_1 \circ h_2$ is injective, h_2 is injective. Let $\overline{a}$ be a tuple of elements of D such that $S_i(h_1(a_1), \ldots, h_1(a_n))$ and assume that $\neg R_i(a_1, \ldots, a_n)$. Then we have:

$$S_i'(h_1(a_1), \ldots, h_1(a_n))$$

Consequently, since h_2 is a homomorphism from Δ' to Γ', we have:

$R_i(h_2(h_1(a_1)), \ldots, h_2(h_1(a_n)))$ and $\neg R_i(h_2(h_1(a_1)), \ldots, h_2(h_1(a_n)))$, a contradiction.

Similarly, $B(h_1(x), h_1(x))$ leads to $h_2(h_1(x)) \neq h_2(h_1(x))$, a contradiction. Consequently, h_1 is an embedding from Γ' to Δ', and a fortiori, h_1 is an embedding from Γ to Δ.

Since $h_1 \circ h_2$ is a self-embedding of Δ', it is straightforward to prove that h_2 is an embedding from Δ' to Γ'.

We finally prove that Δ is model-complete. First note that $S_i' = \neg S_i$ for all $i \leq k$. Indeed, if $\overline{a} \in S_i' \cup S_i$, then $R_i(h_2(a_1), \ldots, h(a_n))$ and $\neg R_i(h_2(a_1), \ldots, h(a_n))$, a contradiction, so we have $S_i' \subseteq \neg S_i$. Now, if $\overline{a} \in \neg S_i \setminus S_i'$, then we have either $h_2(\overline{a}) \in R_i$, or $h_2(\overline{a}) \in \neg R_i$. So $h_1(h_2(\overline{a})) \in S_i$, or $h_1(h_2(\overline{a})) \in S_i'$. But since $h_1 \circ h_2$ is a self-embedding of Δ', and since $\overline{a} \notin S_i'$, we have: $\overline{a} \in S_i$, which contradicts the fact that $\overline{a} \in \neg S_i$. Consequently, $S_i' = \neg S_i$.

As a consequence, every self-embedding of Δ is also a self-embedding of Δ', and hence, it is elementary. So Δ is model-complete. Finally, since Δ' is a model-complete core, $\mathrm{End}(\Delta') = \mathrm{Emb}(\Delta')$. Since Δ is existential interdefinable with Δ', we have $\mathrm{Emb}(\Delta) = \mathrm{Emb}(\Delta')$ by Theorem 2.1.27. Consequently, $\mathrm{Emb}(\Delta) = \mathrm{End}(\Delta')$. $\square$

Proposition 3.6.9. *Let* $\Gamma = (D; R_1, \ldots, R_n)$ *be an existential positive reduct of a relational structure* Δ*. By definition of* Γ*, there exist* n *existential positive formulas* $(\Phi_j)_{j \leq n}$ *on the signature of* Δ *which define the* $(R_j)_{j \leq n}$ *over* Δ*. Let* Δ' *be a structure of domain* D' *which is homomorphically equivalent to* Δ*. Then there exists an existential positive reduct* $\Gamma' := (D'; R_1' \ldots, R_n')$ *of* Δ'*, whose relations are defined as follows:* $\overline{a} \in R_i'$ *if and only if* $\Delta' \models \Phi_i(\overline{a})$*, and such that* h *is a homomorphism from* Γ' *to* Γ*, and* g *is a homomorphism from* Γ *to* Γ'*.*

See Figure 3.6.1

$$
\begin{tikzcd}
\Delta \arrow[r, shift left, "{g,h}"] & \Delta' \arrow[l, shift left] \\
\Gamma \arrow[u, hook, "\mathrm{id}"] \arrow[r, shift left, "{g,h}"] & \Gamma' \arrow[l, shift left] \arrow[u, hook, "\mathrm{id}"']
\end{tikzcd}
$$

Figure 3.4: Commutative Diagram for homomorphically equivalent structures

Proof. By the definition of existential positive definitions, we have $R \in \langle\Delta\rangle_{\text{ep}}$ for every relation R of Γ. We build a structure Γ' which will be an existential positive reduct of Δ' homomorphically equivalent to Γ. Since Δ' and Δ are homomorphically equivalent, there exist two mappings g, h such that g is a homomorphism from Δ to Δ', and h is a homomorphism from Δ' to Δ. And for every relation symbol C in the signature of Δ, we have $g(C^{\Delta}) \subseteq C^{\Delta'}$, and $h(C^{\Delta'}) \subseteq C^{\Delta}$. Assume that $\Gamma = (D; R_1, \ldots, R_n)$, and let $k \leq n$. Since R_k is existential positive definable over Δ, R_k has a definition Φ of the form $\exists t_1 \cdots t_m \bigvee \bigwedge \psi_i(\overline{x}, \overline{t})$ where the ψ_i are relations symbols which belong to the signature of Δ. Let S_k be the relation defined over Δ' by Φ_k , and $\Gamma' = (\mathrm{Dom}(\Delta'), S_1, \ldots, S_n)$. By definition, Γ' is an existential positive reduct of Δ. It is straightforward to prove that g is a homomorphism from Γ to Γ', and h is a homomorphism from Γ' to Γ. $\square$

Corollary 3.6.10. *Let* $\Gamma := (D; R_1, \ldots, R_n)$ *be a reduct of an* ω*-categorical relational structure* $\Delta := (D; S_1, \ldots, S_k)$ *such that* $\mathrm{End}(\Gamma) = \mathrm{End}(\Delta)$*, and let* $\Delta' := (D'; S'_1, \ldots, S'_k)$ *be the model-complete core of* Δ*. Then there exists a reduct* $\Gamma' := (D'; R'_1, \ldots, R'_n)$ *of* Δ' *which is homomorphically equivalent to* Γ*, and such that* $\mathrm{End}(\Gamma') = \mathrm{End}(\Delta')$*. Furthermore, this structure* Γ' *is isomorphic to the model-complete core of* Γ*.*
See Figure 3.6.1

Proof. We build Γ' as specified in Proposition 3.6.9. We use the following notations:

- $\Gamma := (D; R_1, \ldots, R_n)$ and $\Gamma' := (D'; R'_1, \ldots, R'_n)$
- $\Delta := (D; S_1, \ldots, S_k)$ and $\Delta' := (D'; S'_1, \ldots, S'_k)$

Since $\mathrm{End}(\Gamma) = \mathrm{End}(\Delta)$, Γ and Δ are existential positive interdefinable by Theorem 2.1.27. Let g, h be two mappings such that g is a homomorphism from Δ to Δ' and h is a homomorphism from Δ' to Δ. By Proposition 3.6.9, we know that g is also a homomorphism from Γ to Γ', and h is a homomorphism from Γ' to Γ. Since Δ is existential positive definable over Γ, we denote by $\Psi_1, \ldots, \Psi_k$ the existential positive formulas on the signature of Γ which define $S_1, \ldots, S_k$. We show that Δ' is existentially definable in Γ'. More specifically, we prove that $\overline{a} \in S'_i$ if and only if $\Gamma' \models \Psi_i(\overline{a})$. So let $\overline{a} \in S'_i$. Then $h(\overline{a}) \in S_i$, so $\Gamma \models \Psi_i(h(\overline{a})$. Hence, $\Gamma' \models \Psi_i(g(h(\overline{a}))$. Since Δ is a core,

$\mathrm{End}(\Delta) = \mathrm{Emb}(\Delta)$, so $g \circ h$ is a self-embedding of Δ. But since Δ' is model-complete, $\mathrm{Aut}(\Delta')$ locally generates $\mathrm{Emb}(\Delta')$. Consequently, there exists $\alpha \in \mathrm{Aut}(\Delta') \subseteq \mathrm{End}(\Gamma')$ such that $g(h(\overline{a})) = \alpha(\overline{a})$, i.e., $\alpha^{-1}(g(h(\overline{a}))) = \overline{a}$. Hence, $\Gamma' \models \Psi_i(\overline{a})$.

Conversely, we prove that for all $\overline{a}$ and for all $i \leq k$, if $\Gamma' \models \Psi_i(\overline{a})$, then $\overline{a} \in S'_i$. So let us consider $i \leq k$ and $\overline{a}$ such that $\Gamma' \models \Psi_i(\overline{a})$. Then $\Gamma \models \Psi'_i(h(\overline{a}))$, i.e., $h(\overline{a}) \in S_i$, since Ψ_i is an existential positive definition of S_i over Γ'. Consequently, $g(h(\overline{a})) \in S'_i$. Since Δ' is a core, $g \circ h \in \mathrm{End}(\Delta') = \mathrm{Emb}(\Delta')$, and since Δ' is model-complete, $\mathrm{Aut}(\Delta')$ locally generates $\mathrm{Emb}(\Delta')$. Hence, there exists $\alpha \in \mathrm{Aut}(\Delta')$ such that $g(h(\overline{a})) = \alpha(\overline{a})$, i.e., $\alpha^{-1}(g(h(\overline{a}))) = \overline{a}$. So $\overline{a} \in S'_i$. Consequently, for all $i \leq k$, S'_i is defined over Γ' by the existential positive formula Ψ_i. Finally, since every relations of Γ' have an existential positive definition in Δ', we have Γ' and Δ' are existential positive interdefinable, and $\mathrm{End}(\Delta') = \mathrm{End}(\Gamma')$.

Finally, since Δ' is a model-complete core, $\mathrm{Aut}(\Delta')$ locally generates $\mathrm{End}(\Delta')$. But since Δ' and Γ' are existential positive interdefinable, we have $\mathrm{Aut}(\Delta') = \mathrm{Aut}(\Gamma')$, and $\mathrm{End}(\Delta') = \mathrm{End}(\Gamma')$. Consequently, $\mathrm{Aut}(\Gamma')$ locally generates $\mathrm{End}(\Gamma')$, and Γ' is a model-complete core. Γ' being homomorphically equivalent to Γ, we conclude that Γ' is the model-complete core of Γ. $\square$

3.6.2 Special Cases

Proposition 3.6.11. *The structures* $(V; 0, \neq)$ *and* $(V; \neq)$ *are model-complete cores.*

Proof. Since $(V; 0, \neq)$ and $(V; \neq)$ are trivially homogeneous and uniformly locally finite, they admit quantifier elimination by Theorem 2.2.11. Consequently, they both are model-complete cores by Theorem 3.6.4. $\square$

Proposition 3.6.12. *The structure* $(V; +, \neq)$ *is a model-complete core.*

Proof. Combine Theorem 3.6.4 with Proposition 3.2.33. $\square$

Proposition 3.6.13. *The structure* $(V \setminus \{0\}; \mathrm{Ieq}_3)$ *is a model-complete core.*

Proof. First note that $(V \setminus \{0\}; \mathrm{Ieq}_3)$ is ω-categorical. Indeed, since $(V; +)$ is ω-categorical, it has a finite number of n-types for all $n \geq 1$, by Theorem 2.1.24. But if two tuples of non-zero elements of V have the same complete type over $(V; +)$, then they have the same type over $(V \setminus \{0\}; \mathrm{Ieq}_3)$ (this is clear since Ieq_3 fo-defines the graph of $+$). Hence, $(V \setminus \{0\}; \mathrm{Ieq}_3)$ has a finite number of n-types for all $n \geq 1$, and thus, by Theorem 2.1.24, $(V \setminus \{0\}; \mathrm{Ieq}_3)$ is ω-categorical.

Consequently, by Theorem 3.6.4, we only have to prove that the automorphisms locally generate the endomorphisms of $(V \setminus \{0\}; \mathrm{Ieq}_3)$. Let $f \in \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$. Since f preserves Ieq_3, we have $f(x+y) = f(x)+f(y)$ for every non-zero distinct elements x and y of V. Since f preserves $\neq$ (because $\neq$ has the following primitive positive definition over $(V \setminus \{0\}; \mathrm{Ieq}_3)$: $x \neq y$ if and only if $\exists z. \mathrm{Ieq}_3(x, y, z)$), f is injective. Hence, we can complete f to an injective linear function $\tilde{f}$ of $(V; +)$. Conversely, every injective linear function of $(V; +)$ is an endomorphism of $(V \setminus \{0\}; \mathrm{Ieq}_3, \neq)$ when we consider its restriction to $V \setminus \{0\}$. Similarly, an automorphism of $(V \setminus \{0\}; \mathrm{Ieq}_3, \neq)$ can be seen as the restriction of an automorphism of $(V;+)$ on $V\setminus\{0\}$. Consequently, every endomorphism of $(V\setminus\{0\};\mathrm{Ieq}_3,\neq)$ is locally generated by $\mathrm{Aut}(V\setminus\{0\};\mathrm{Ieq}_3,\neq)$. $\square$

Note that $(V\setminus\{0\};\mathrm{Ieq}_3)$ is not homogeneous, since it has no quantifier elimination (for instance $\neq$ is definable in $(V\setminus\{0\};\mathrm{Ieq}_3)$ but needs an $\exists$ quantifier). Nevertheless, we have:

Proposition 3.6.14. *The structure* $(V\setminus\{0\};(\mathrm{Eq}_i^{\neq 0})_{i\geq 2})$ *is homogeneous, and we have the following:* $\mathrm{End}(V\setminus\{0\};(\mathrm{Eq}_i^{\neq 0})_{i\geq 2})=\mathrm{End}(V\setminus\{0\};\mathrm{Ieq}_3)$.

Proof. It is easy to see that $\mathrm{End}(V\setminus\{0\};\mathrm{Ieq}_3)=\mathrm{End}(V\setminus\{0\};(\mathrm{Eq}_i^{\neq 0})_{i\geq 2})$. Indeed, since $\mathrm{Eq}_3^{\neq 0}=\mathrm{Ieq}_3$, we have the $\supseteq$ inclusion. Conversely, since the endomorphisms of Ieq_3 are linear functions (see Proposition 3.3.15), and since all linear functions clearly preserve Eq_i for all i, we obtain the second inclusion. We now prove that $(V\setminus\{0\};(\mathrm{Eq}_i^{\neq 0})_{i\geq 2})$ is homogeneous. Recall that $(V;(\mathrm{Eq}_i)_{i\geq 1})$ is homogeneous by Proposition 3.2.36. It is also ω-categorical as a reduct of an ω-categorical structure. Hence, $(V;(\mathrm{Eq}_i)_{i\geq 1})$ has quantifier elimination by Theorem 2.2.13, and every relation R first-order definable over $(V;(\mathrm{Eq}_i)_{i\geq 1})$ has a quantifier-free definition. Assume that $R\subseteq(V\setminus\{0\})^n$. Then every Eq_i in its definition can be replaced by an $\mathrm{Eq}_i^{\neq 0}$ since its values are all distinct from 0. The last thing we have to see is that every first-order definable relations over $(V\setminus\{0\};(\mathrm{Eq}_i^{\neq 0})_{i\geq 2})$ is first-order definable over $(V;(\mathrm{Eq}_i)_{i\geq 1})$, which is clear. So $(V\setminus\{0\};(\mathrm{Eq}_i^{\neq 0})_{i\geq 2})$ has quantifier elimination, since it is ω-categorical as a reduct of an ω-categorical structure, it is homogeneous by Theorem 2.2.13. $\square$

Proposition 3.6.15. *The structure* $(V;\mathrm{Eq}_4,\neq)$ *is a model-complete core.*

Proof. First note that $(V;\mathrm{Eq}_4)$ is ω-categorical, as a first-order reduct of $(V;+)$ which is ω-categorical. By Theorem 3.6.4, we have to prove that the automorphisms of $(V;\mathrm{Eq}_4,\neq)$ locally generate the endomorphisms of $(V;\mathrm{Eq}_4,\neq)$. But it is exactly what Proposition 3.3.32 states. Consequently, $(V;\mathrm{Eq}_4,\neq)$ is a model-complete core. $\square$

Notation 3.6.16. We define in the following four new relations, and give another notation for the already defined relation $\mathrm{Eq}_4^{\mathrm{inj}}$, in order to describe some of the monoids of endomorphism of reducts. For all $(x,y,z,t)\in V^4$:

- $\mathrm{Inj}_{4,4}(x,y,z,t)$ if and only if x,y,z,t are pairwise distinct.
- $\mathrm{Inj}_{4,3}(x,y,z,t)$ if and only if three of the values are pairwise distinct, and the fourth value equals one of the previous.

  Example: $(0,a,b,0)$ and (a,b,a,d) are both in $\mathrm{Inj}_{4,3}$, for a,b,d pairwise distinct.
- $\mathrm{Eq}_4^{\mathrm{inj}}(x,y,z,t)\Leftrightarrow\mathrm{Eq}_4^{\mathrm{inj}}(x,y,z,t)\Leftrightarrow x+y+z+t=0$ and $\mathrm{Inj}_{4,4}(x,y,z,t)$.
- $\mathrm{NAff}_4^{\mathrm{inj}}(x,y,z,t)$ if and only if $x+y+z+t\neq 0$ and $\mathrm{Inj}_{4,4}(x,y,z,t)$.
- $\mathrm{NAff}_{4,3}^{\mathrm{inj}}(x,y,z,t)$ if and only if $x+y+z+t\neq 0$ and $\mathrm{Inj}_{4,3}(x,y,z,t)$.

Corollary 3.6.17. *The following relational structures are model-complete cores:*

- $(V; \mathrm{Eq}_4^{\mathrm{inj}})$ *and* $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \neq)$
- $(V; \mathrm{Eq}_4, V, \neq)$
- $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Eq}_4^{\mathrm{inj}} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}})$
- $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{NAff}_4^{\mathrm{inj}}, \neq)$ *and* $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{NAff}_4^{\mathrm{inj}} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}})$
- $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Inj}_{4,4}, \neq)$ *and* $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Inj}_{4,4} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}})$

Proof. By Proposition 3.6.15, $(V; \mathrm{Eq}_4, \neq)$ is a model-complete core, so the automorphism group $\mathrm{Aut}(V; \mathrm{Eq}_4, \neq)$ locally generates $\mathrm{End}(V; \mathrm{Eq}_4, \neq)$. To prove that the structures above are model-complete cores, we prove that their endomorphism monoid equals $\mathrm{End}(V; \mathrm{Eq}_4, \neq)$, and their automorphism group equals $\mathrm{Aut}(V; \mathrm{Eq}_4, \neq)$.

In each of the listed cases, the verification is straightforward since each of the listed structures clearly existential positive defines $(V; \mathrm{Eq}_4, \neq)$, and is existential positive definable over $(V; \mathrm{Eq}_4, \neq)$. Hence Theorem 2.1.27 give us that they have the same endomorphism monoid and automorphism group as $(V; \mathrm{Eq}_4, \neq)$. □

Note that $(V; \mathrm{Ieq}_4, \neq)$ is not homogeneous, since it has no quantifier elimination (for instance Eq_6 is definable in $(V; \mathrm{Ieq}_4, \neq)$ but needs an $\exists$ quantifier). Nevertheless, we have:

Proposition 3.6.18. *The structure* $(V; (\mathrm{Eq}_{2i})_{i \geq 1}, \neq)$ *is homogeneous, and we have the following:* $\mathrm{End}(V; (\mathrm{Eq}_{2i})_{i \geq 1}, \neq) = \mathrm{End}(V; \mathrm{Ieq}_4, \neq) = \mathrm{End}(V; \mathrm{Ieq}_4{}^{\mathrm{inj}})$.

Proof. It is straightforward to see that $\mathrm{End}(V; (\mathrm{Eq}_{2i})_{i \geq 1}, \neq) \subseteq \mathrm{End}(V; \mathrm{Ieq}_4, \neq)$, since Ieq_4 equals Eq_4. The other inclusion comes from Lemma 3.2.3, which states that every Eq_{2i} has a primitive positive definition over $(V; \mathrm{Eq}_4)$. This means that any polymorphism of a reduct of $(V; +)$ which preserves Eq_4 (and in particular any endomorphism of this reduct) also preserves Eq_{2i}. Finally, by definition, $\mathrm{Ieq}_4{}^{\mathrm{inj}}$ is pp-definable over $(V; \mathrm{Ieq}_4, \neq)$, and Ieq_4 is ep-definable over $(V; \mathrm{Ieq}_4{}^{\mathrm{inj}})$.

We now prove that $(V; (\mathrm{Eq}_{2i})_{i \geq 1}, \neq)$ is homogeneous. But this is straightforward since we already prove in Proposition 3.6.18 that $(V; (\mathrm{Eq}_{2i})_{i \geq 1})$ is homogeneous, and $\neq$ has first-order definition in $(V; (\mathrm{Eq}_{2i})_{i \geq 1})$ (indeed, $x \neq y$ of and only if $\neg \mathrm{Eq}_2(x, y)$). □

Proposition 3.6.19. *The structure* $(V \cup \{\infty\}, \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$ *is a model-complete core.*

Proof. We proved in Corollary 3.6.17 that $(V; \mathrm{Eq}_4^{\mathrm{inj}})$ is a model-complete core. But the structure $(V \cup \{\infty\}, \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$ is nearly the same as $(V; \mathrm{Eq}_4^{\mathrm{inj}})$ since we only add an ∞ point in the domain, which has to be preserved by every endomorphism. Consequently, the endomorphisms of $(V \cup \{\infty\}, \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$ are exactly the endomorphisms of $(V; \mathrm{Eq}_4^{\mathrm{inj}})$ extended to the ∞ point, i.e., $f \in \mathrm{End}(V \cup \{\infty\}, \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$ if and only if $f \restriction V \in \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$, and $f(\infty) = \infty$. Hence, $\mathrm{Aut}(V \cup \{\infty\}, \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$ locally generates $\mathrm{End}(V \cup \{\infty\}, \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$, and $(V \cup \{\infty\}, \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$ is a model-complete core. □

3.6.3 Classification and Pathway to CSPs

Theorem 3.6.20. *Let* Γ *be a reduct of* $(V;+)$*. Then exactly one of the following cases holds:*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$*;*
- *up to isomorphism, the model-complete core* Γ' *of* Γ *satisfies one of the following property:*
  - $\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$*;*
  - $\mathrm{End}(\Gamma') = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$*;*
  - $\mathrm{End}(\Gamma') = \mathrm{End}(V \cup \{\infty\}; \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$*;*
- Γ *is homomorphically equivalent to a reduct of* $(V;0)$*;*
- Γ *is homomorphically equivalent to a structure with at most two elements.*

Proof. By Theorem 3.5.21, we know that $\mathrm{End}(\Gamma)$ belongs to one of the elements of the list given in the statement of the theorem. Three cases are already done: the case where $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$, the case where Γ is homomorphically equivalent to a reduct of $(V;0)$ (by Proposition 3.5.1, it is the case where $\mathrm{End}(\Gamma) \cap \{\mathrm{gen}, \mathrm{gen}^*\} \neq \emptyset$), or the case where Γ is homomorphically equivalent to a structure with at most two elements.

We now deal with the other cases. In each one of them, we will find the model-complete core of the structures listed, and use Corollary 3.6.10 to obtain the result. Here is how it works: we know by Theorem 3.5.21 that $\mathrm{End}(\Gamma) = \mathrm{End}(\Delta)$ for a given Δ. Then we find the model-complete core Δ' of Δ, and exhibit the two homomorphisms which allow the homomorphic equivalence between Δ and Δ'. Then by Corollary 3.6.10, Γ is homomorphically equivalent to a reduct Γ' of Δ', and whose endomorphism monoid equals $\mathrm{End}(\Delta')$. Finally we prove that for each case, the following is true:

$$\mathrm{End}(\Delta') \in \{\mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3), \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}}), \mathrm{End}(V \cup \{\infty\}; \mathrm{Eq}_4^{\mathrm{inj}}, \infty)\}$$

Since this there are many cases to verify, we will only give, for each case, the model-complete core of the listed structure, and the homomorphisms which allow the homomorphic equivalence: the first one from Δ to Δ', and the second one from Δ' to Δ. Note that some of the following structures are model-complete cores by Proposition 3.6.15, Corollary 3.6.17, and Proposition 3.6.19.

- $(V; \mathrm{Ieq}_3, \neq)$'s model-complete core is $(V \setminus \{0\}; \mathrm{Ieq}_3, \neq)$, with the homomorphisms id^* and id. Furthermore:

$$\mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3, \neq) = \{f\restriction(V \setminus \{0\}) \mid f \in \mathrm{End}(V; \mathrm{Ieq}_3)\}$$

- $(V; \mathrm{Ieq}_3)$'s model-complete core is $(V \setminus \{0\}; \mathrm{Ieq}_3)$, with the homomorphisms id^* and id.

- $(V;\mathrm{Ieq}_4,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\neq)$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\neq)=\mathrm{End}(V;\mathrm{Eq}_4,\neq)$$

- $(V;\mathrm{Ieq}_4)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}})$, with the homomorphisms id and af$'$.

- $(V;\mathrm{Ieq}_4,0)$'s model-complete core is $(V\cup\{\infty\};\mathrm{Eq}_4^{\mathrm{inj}},\infty)$, with the homomorphism g such that $g(0)=\infty$, and $g(x)=x$ for all $x\neq 0$, and the homomorphism h such that $h(\infty)=0$, and $h(x)=\mathrm{af}'(x)$ for all $x\in V$.

- $(V;\mathrm{Eq}_4,\mathrm{Ind}_1,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4,V,\neq)$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4,V,\neq)=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,Z_1\cup R_1\cup\mathrm{Ieq}_4)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Eq}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Eq}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,Z_1\cup\mathrm{Ind}_4,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)$, with the homomorphisms af* and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,Z_1\cup R_1\cup\mathrm{Ind}_4)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Eq}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Eq}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_1,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4},\neq)$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4},\neq)=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_1\cup R_1)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_1\cup R_1\cup R_2)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_2,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)$, with the homomorphisms af^* and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_2\cup R_1)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_2\cup R_1\cup R_2)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_3,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)$, with the homomorphisms af^* and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_3\cup R_1)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_3\cup R_1\cup R_2)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_4,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4},\neq)$, with the homomorphisms id and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4},\neq)=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_4\cup R_1)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{Inj}_{4,4}\cup\mathrm{NAff}_{4,3}^{\mathrm{inj}})=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V;\mathrm{Ieq}_4,T_5,\neq)$'s model-complete core is $(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)$, with the homomorphisms af^* and af'. Furthermore:

$$\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}},\mathrm{NAff}_4^{\mathrm{inj}},\neq)=\mathrm{End}(V;\mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V; \mathrm{Ieq}_4, T_5 \cup R_1)$'s model-complete core is $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{NAff}_4^{\mathrm{inj}} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{NAff}_4^{\mathrm{inj}} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}}) = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V; \mathrm{Ieq}_4, T_6, \neq)$'s model-complete core is $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Inj}_{4,4}, \neq)$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Inj}_{4,4}, \neq) = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V; \mathrm{Ieq}_4, T_6 \cup R_1)$'s model-complete core is $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Inj}_{4,4} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Inj}_{4,4} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}}) = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V; \mathrm{Ieq}_4, T_7 \cup R_1)$'s model-complete core is $(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Eq}_4^{\mathrm{inj}} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}})$, with the homomorphisms id and af$'$. Furthermore:

$$\mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}}, \mathrm{Eq}_4^{\mathrm{inj}} \cup \mathrm{NAff}_{4,3}^{\mathrm{inj}}) = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$$

- $(V; \mathrm{Eq}_4, \neq)$'s model-complete core is $(V; \mathrm{Eq}_4, \neq)$. Furthermore:

$$\mathrm{End}(V; \mathrm{Eq}_4, \neq) = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$$

□

In the next lemma, we denote by id the identity function, by c the function from $\{0,1\} \to \{0,1\}$ such that $c(x) = 1 - x$ for all x, $\overline{0}$ the constant function equal to 0, and $\overline{1}$ the constant function equal to 1.

Lemma 3.6.21. *Let Γ be a structure with at most two elements. Then exactly one of the following cases holds:*

- $\mathrm{End}(\Gamma) = \mathrm{End}(\{0,1\}; =) = \{id, \overline{0}, \overline{1}, c\}$*;*
- $\mathrm{End}(\Gamma) = \mathrm{End}(\{0,1\}; 0) = \{id, \overline{0}\}$*;*
- $\mathrm{End}(\Gamma) = \mathrm{End}(\{0,1\}; 1) = \{id, \overline{1}\}$*;*
- $\mathrm{End}(\Gamma) = \mathrm{End}(\{0,1\}; \Rightarrow) = \{id, \overline{0}, \overline{1}\}$ *where* $\Rightarrow$ *equals* $\{(0,0),(0,1),(1,1)\}$*;*
- $\mathrm{End}(\Gamma) = \mathrm{End}(\{0,1\}; \neq) = \{id, c\}$*;*
- $\mathrm{End}(\Gamma) = \mathrm{End}(\{0,1\}; 0, 1) = \{id\}$*;*
- $\mathrm{End}(\Gamma) = \mathrm{End}(\{0\}; =) = \{id\}$.

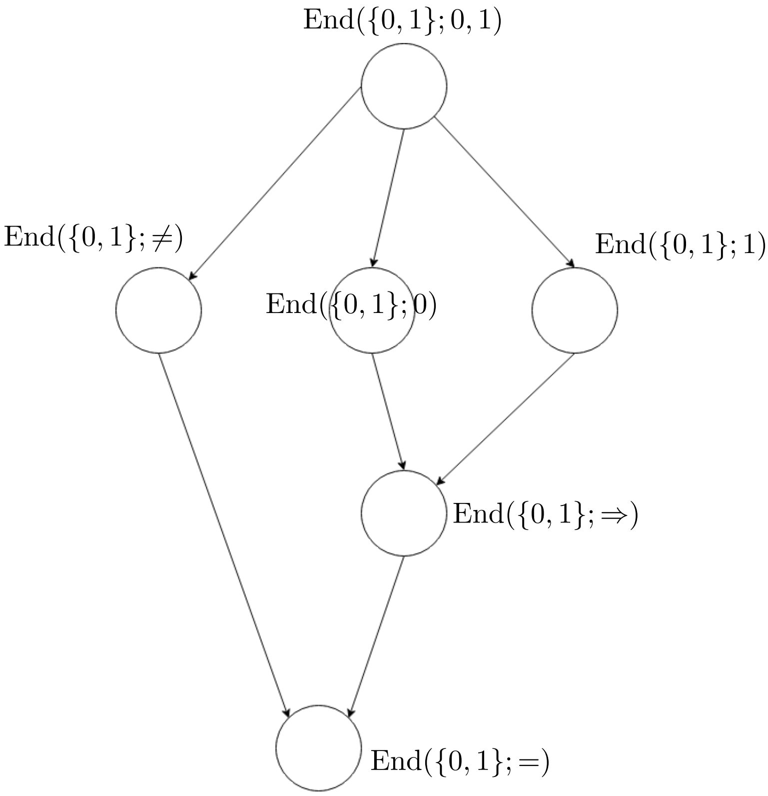

Figure 3.5: The lattice of endomorphism monoids of structures over $\{0,1\}$

See figure 3.6.3.

Proof. Note that $\mathrm{End}(\{0,1\};=) = \{\mathrm{id}, \overline{0}, \overline{1}, c\}$ where $\overline{0}$ and $\overline{1}$ are constant functions, and $c(x) = \neg x$. Now, we just have to combine these functions to obtain the lattice of endomorphism monoids. The last thing to note is that $c \circ \overline{0} = \overline{1}$ and $c \circ \overline{1} = 0$, so we will get only $2^3 - 2$ monoids for the 2-element structure case. □

Corollary 3.6.22. *Let Γ be a reduct of $(V;+)$. Then the model-complete core of Γ is isomorphic to a structure Γ' such that exactly one of the following cases holds:*

1. $\mathrm{End}(\Gamma') = \mathrm{End}(V;+,\neq)$;
2. $\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$;
3. $\mathrm{End}(\Gamma') = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$;
4. $\mathrm{End}(\Gamma') = \mathrm{End}(V \cup \{\infty\}; \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$;
5. $\mathrm{End}(\Gamma') = \mathrm{End}(V; 0, \neq)$;
6. $\mathrm{End}(\Gamma') = \mathrm{End}(V; \neq)$;
7. $\mathrm{End}(\Gamma') = \mathrm{End}(\{0,1\}; \neq)$;
8. $\mathrm{End}(\Gamma') = \mathrm{End}(\{0,1\}; 0, 1)$;
9. $\mathrm{End}(\Gamma') = \mathrm{End}(\{0\}; =)$.

Proof. By Theorem 3.6.20, we can distinguish various cases:

- $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$. Hence Γ is already a model-complete core by Proposition 3.2.33 and Corollary 3.6.5.
- the model-complete core of Γ is isomorphic to a structure Γ' such that $\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$, and we are done;
- the model-complete core of Γ is isomorphic to a structure Γ' such that $\mathrm{End}(\Gamma') = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$, and we are done;
- the model-complete core of Γ is isomorphic to a structure Γ' such that $\mathrm{End}(\Gamma') = \mathrm{End}(V \cup \{\infty\}; \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$, we are done;
- Γ is homomorphically equivalent to a reduct Γ' of $(V;0)$. Hence by Proposition 3.5.3, one of the following sub-case holds:
  - $\mathrm{End}(\Gamma') = \mathrm{End}(V; 0, \neq)$, in which case Γ' is a model-complete core by Proposition 3.6.11 and Corollary 3.6.5. Since Γ' is homomorphically equivalent to Γ, it is isomorphic to the model-complete core of Γ by unicity of the model-complete core of Γ;

– $\mathrm{End}(\Gamma') = \mathrm{End}(V; \neq)$, in which case Γ' is a model-complete core by Proposition 3.6.11 and Corollary 3.6.5. Since Γ' is homomorphically equivalent to Γ, it is isomorphic to the model-complete core of Γ by unicity of the model-complete core of Γ;
– $\mathrm{End}(\Gamma') = \mathrm{End}(V; \mathrm{Ind}_2)$. It is straightforward to see that $(V; \mathrm{Ind}_2)$ is homomorphically equivalent to $(V; \neq)$. Indeed, id is an homomorphism from $(V; \mathrm{Ind}_2)$ to $(V; \neq)$, and gen* is an homomorphism from $(V; \neq)$ to $(V; \mathrm{Ind}_2)$. Furthermore, by Proposition 3.6.11, we have that $(V; \neq)$ is a model-complete core. Hence, by Corollary 3.6.10, there exists a reduct Γ'' of $(V; \neq)$ which is homomorphically equivalent to Γ', and such that $\mathrm{End}(\Gamma'') = \mathrm{End}(V; \neq)$. Furthermore, this structure Γ'' is isomorphic to the model-complete core of Γ'. Since Γ' is homomorphically equivalent to Γ, Γ'' is also isomorphic to the model-complete core of Γ by unicity of the model-complete core up to isomorphism, and we are done.

- Γ is homomorphically equivalent to a structure with at most two elements which has the same model-complete core as Γ by unicity of the model-complete core. Hence the model-complete core of Γ has at most two elements by Proposition 3.6.2, and then by Lemma 3.6.21, we are in one of the remaining cases listed in the Theorem. Note that most of them can not occur since Γ' is a core, and thus, every endomorphism of Γ' is injective.

□

Corollary 3.6.23. *Let Γ be a reduct of $(V; +)$. Then the model-companion of Γ is isomorphic to a structure Γ' such that at least one of the following cases holds:*

1. $\mathrm{Emb}(\Gamma') = \mathrm{End}(V; +, \neq)$;
2. $\mathrm{Emb}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$;
3. $\mathrm{Emb}(\Gamma') = \mathrm{End}(V; \mathrm{Eq}_4^{\mathrm{inj}})$;
4. $\mathrm{Emb}(\Gamma') = \mathrm{End}(V \cup \{\infty\}; \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$;
5. $\mathrm{Emb}(\Gamma') = \mathrm{End}(V; 0, \neq)$;
6. $\mathrm{Emb}(\Gamma') = \mathrm{End}(V; \neq)$;
7. $\mathrm{Emb}(\Gamma') = \mathrm{End}(\{0,1\}; \neq)$;
8. $\mathrm{Emb}(\Gamma') = \mathrm{End}(\{0,1\}; 0, 1)$;
9. $\mathrm{Emb}(\Gamma') = \mathrm{End}(\{0\}; =)$.

Proof. By Proposition 3.6.8, given a reduct Γ of $(V; +)$, there exists a reduct Γ' which is existential interdefinable with Γ whose model-complete core Δ' satisfies: $\mathrm{Emb}(\Delta) = \mathrm{End}(\Delta')$. Since we classified the endomorphism monoids of all reducts of $(V; +)$ in Corollary 3.6.22, we immediately obtain the required classification. □

Corollary 3.6.24. *Let* Γ *be a reduct of* $(V;+)$*. Then the model-companion (resp. the model-complete core) of* Γ *is isomorphic to a structure which is existential (resp. existential positive) interdefinable with exactly one of the following structures:*

1. $(V;+,\neq)$*;*
2. $(V \setminus \{0\}; \mathrm{Ieq}_3)$*;*
3. $(V; \mathrm{Eq}_4^{\mathrm{inj}})$*;*
4. $(V \cup \{\infty\}; \mathrm{Eq}_4^{\mathrm{inj}}, \infty)$*;*
5. $(V; 0, \neq)$*;*
6. $(V; \neq)$*;*
7. $(\{0,1\}; \neq)$*;*
8. $(\{0,1\}; 0, 1)$*;*
9. $(\{0\}; =)$.

Proof. Direct application of Theorem 2.1.27 to Corollaries 3.6.23 and 3.6.22 combined with the fact that all the listed structures in the corollary are cores. □

Chapter 4

Preliminaries on Constraint Satisfaction Problems

4.1 Model Theory and Universal Algebra

We add in this section algebraic material to the preliminaries in order to fully introduce the notion of polymorphism, which is crucial in the study of CSPs. As we will see, a polymorphism is a generalization of the notion of endomorphism to arbitrary arity. As endomorphisms or self-embeddings, polymorphisms appear in a Galois connection which we will also describe.

We start by giving definitions for preservation in a more general setting.

Definition 4.1.1. Let Δ be a structure of signature τ, let c be a constant symbol of τ, let f be a k-ary function symbol of τ, and R be a relation symbol of arity k of τ. Let g be an operation from $D^n \to D$. We say that:

- g *preserves* c^Δ if $g(c^\Delta, \ldots, c^\Delta) = c^\Delta$.
- g *preserves* f^Δ if for all list of k-tuples of size n $(\overline{x}^1, \ldots, \overline{x}^n) \in (D^k)^n$, we have: $g(f^\Delta(\overline{x}^1), \ldots, f^\Delta(\overline{x}^k)) = f^\Delta(g(x_1^1, \ldots, x_1^n), \cdots, g(x_k^1, \ldots, x_k^n))$. In the same spirit,
- g *preserves* R^Δ if for all list of k-tuples of size n $(\overline{x}^1, \ldots, \overline{x}^n) \in (R^\Delta)^n$, we have: $(g(x_1^1, \ldots, x_1^n), \ldots, g(x_k^1, \ldots, x_k^n)) \in R^\Delta$. Finally,
- g *strongly preserves* R^Δ if the converse is true, i.e., for all list of k-tuples of size k $\overline{x}^1, \ldots, \overline{x}^n \in D^k$, if $(g(x_1^1, \ldots, x_1^n), \ldots, g(x_k^1, \ldots, x_k^n)) \in R^\Delta$, then for all $i \leq n$, we have $\overline{x}^i \in R^\Delta$.

Given an operation $g\colon D^n \to D$ over the domain D of a structure Δ, g is called *polymorphism* of Δ if g preserves every function, constants and relations of Δ.

From now on, we write c, g, R instead of $c^\Delta, g^\Delta, R^\Delta$ when there is no possible confusion.

Notation 4.1.2. Given a structure Δ, we denote the set of all polymorphisms of Δ by $\mathrm{Pol}(\Delta)$. Note that for all Δ, we have:

$$\mathrm{Aut}(\Delta) \subseteq \mathrm{Emb}(\Delta) \subseteq \mathrm{End}(\Delta) \subseteq \mathrm{Pol}(\Delta)$$

Lemma 4.1.3. *Let $\mathcal{R}$ be a set of relations over a domain D, let Γ be the structure of domain D and whose relations are exactly $\mathcal{R}$, and let $\mathcal{F}$ be a set of operations with arbitrary finite arities. We have the following:*

$$\mathcal{R} \subseteq \mathrm{Inv}(\mathcal{F}) \Leftrightarrow \mathcal{F} \subseteq \mathrm{Pol}(\Gamma)$$

Proof. The proof is straightforward by definition of Inv and Pol. $\square$

The following is a straightforward application of the previous lemma:

Corollary 4.1.4. *The operators* $\mathrm{Inv} - \mathrm{Pol}$ *form a Galois connection between the sets of relations over a domain D and the subsets of $\bigcup_{n\geq 1} D^{D^n}$.*

In order to describe an interesting property of the closure operator $\mathrm{Pol} \circ \mathrm{Inv}$, we define the notion of local closure for sets of arbitrary arity operations

Definition 4.1.5. Let $\mathcal{F}$ be a set of operations on a set D. The *local closure of $\mathcal{F}$*, denoted by $\overline{\mathcal{F}}$, is the smallest set $\mathcal{F}'$ of operations which satisfies the following property: if g is an operation on D such that for every finite subset S of D, there exists $f \in \mathcal{F}'$ such that $g \upharpoonright S = f \upharpoonright S$, then $g \in \mathcal{F}'$.

A set $\mathcal{F}$ of operations over a domain D is *locally closed* when $\mathcal{F} = \overline{\mathcal{F}}$.

Lemma 4.1.6. *Given a structure Γ, the set* $\mathrm{Pol}(\Gamma)$ *is locally closed.*

Proof. The proof is similar to the proof of Lemma 2.1.13. $\square$

Notation 4.1.7. Given a domain D, we denote by π_n^i the projection over D of arity n over the i-th coordinate.

Notation 4.1.8. For any set A and any $\mathcal{F} \subseteq \bigcup_{n\geq 1} A^{A^n}$, we denote by $\langle \mathcal{F} \rangle$ the closure of $\mathcal{F} \cup \{\pi_n^i \mid i \leq n \in \mathbb{N}\}$ under generalised composition, i.e., $\langle \mathcal{F} \rangle$ is the smallest set of operations on A which contains $\mathcal{F}$ and the projections over A, and such that, if g is an n-ary operation of $\langle \mathcal{F} \rangle$, and $f_1, \ldots, f_n$ are k-ary operations of $\langle \mathcal{F} \rangle$, then the k-ary operation $g(f_1, \ldots, f_n)$ belongs to $\langle \mathcal{F} \rangle$.

Definition 4.1.9. Let f, g two operations with arbitrary arities on an infinite domain D. We say that f *locally generates* g, or g *is locally generated by* f, whenever $g \in \overline{\langle \{f\} \rangle}$.

Recall that given a domain D, and a set $\mathcal{F}$ of operations over this domain, we denote by $\mathrm{Inv}(\mathcal{F})$ the set of relations over D which are preserved by every operation of $\mathcal{F}$.

We state here an important property of one of the closure operator of the Galois connection $\mathrm{Inv} - \mathrm{Pol}$. Note that this equality does not require any hypothesis on the set of operations $\mathcal{F}$.

Theorem 4.1.10. *Let $\mathcal{F}$ be a set of operations on D with arbitrary arities. Then we have the following:*

$$\overline{\langle \mathcal{F} \rangle} = \mathrm{Pol}(\mathrm{Inv}(\mathcal{F}))$$

Recall that a formula is *primitive positive* whenever it is of the form $\exists x_1 \dots x_n \bigwedge_{i \leq k} \psi_i$ where the ψ_i are atomic formulas, i.e., of the form $R(t_1, \dots, t_l)$ with t_i terms and R a relation symbol of the language. We use pp to denote primitive positive, and denote by $\langle \Gamma \rangle_{\mathrm{pp}}$ the set of pp-definable relations over a structure Γ.

We recall here properties of closure operators derived from the Galois connection $\mathrm{Inv} - \mathrm{End}$. We also introduce a crucial property of the closure operator $\mathrm{Inv} \circ \mathrm{Pol}$ from the recently defined Galois connection $\mathrm{Inv} - \mathrm{Pol}$. Note that the ω-categoricity hypothesis is needed here.

The following comes from [Bod04] and [BJ11]:

Theorem 4.1.11. *Let Γ be an ω-categorical or finite structure and R be a relation on* $\mathrm{Dom}(\Gamma)$*. We have the following:*

- *R has a first-order definition in Γ if and only if R is preserved by all automorphisms of Γ.*
- *R has an existential definition in Γ if and only if R is preserved by all self-embeddings of Γ.*
- *R has an existential positive definition in Γ if and only if R is preserved by all endomorphisms of Γ.*
- *R has a primitive positive definition in Γ if and only if R is preserved by all polymorphisms of Γ.*

We give here two lemmas which will be very useful in order to reduce case studies. Indeed, when there exists a polymorphism of a structure Γ which violates a k-ary relation R, we can assume that this polymorphism has arity at most m where m is the number of k-orbits intersecting R.

The following is from [BK08b]:

Lemma 4.1.12. *Let Γ be a relational structure, and let R be a k-ary relation that intersects exactly m orbits of k-tuples of* $\mathrm{Aut}(\Gamma)$*. If Γ has a polymorphism f that violates R, then Γ also has an at most m-ary polymorphism that violates R.*

Proof. Let f' be a polymorphism of Γ of smallest arity l that violates R. Then there are k-tuples $t_1, \dots, t_l \in R$ such that $f'.(t_1, \dots, t_l) \notin R$. For $l > m$ there are two tuples t_i, t_j that lie in the same orbit of k-tuples, and therefore Γ has an automorphism α such that $\alpha.t_j = t_i$. By permuting the arguments of f', we can assume that $i = 1$ and $j = 2$. Then the $(l-1)$-ary operation g defined as $g(x_2, \dots, x_l) := f'(\alpha(x_2), x_2, \dots, x_l)$ is also a polymorphism of Γ, and also violates R, a contradiction. Hence, $l \leq m$. $\square$

As a consequence of this lemma, we have the following corollary:

Corollary 4.1.13. *Let Γ be an ω-categorical relational structure, and let R be a k-ary relation which is an orbit of k-tuples of* $\mathrm{Aut}(\Gamma)$*. If R is preserved by every endomorphism of Γ, then R has a primitive positive definition in Γ.*

Proof. If R is not preserved by every polymorphism of Γ, then there exists by Lemma 4.1.12 an endomorphism f of Γ which would violate R, which contradicts the fact that every endomorphism of Γ preserves R. Thus, $R \in \mathrm{Inv}(\mathrm{Pol}(\Gamma))$. But since Γ is ω-categorical with finite relational signature, we have $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle_{\mathrm{pp}}$. $\square$

Definition 4.1.14. A quantifier-free first-order formula Φ in CNF (conjunctive normal form) over a structure Δ is called *reduced* if every formula obtained from Φ by removing a literal is not equivalent to Φ over Δ.

Remark 4.1.15. Every quantifier-free formula on the signature of a given structure Δ is equivalent over Δ to a reduced formula.

Definition 4.1.16. Let Γ be a relational structure. A formula over the signature of Γ is called *quantifier-free Horn* (or just *Horn*) if it is a conjunction of disjunctive clauses with at most one positive literal.

The following statement is from [BCKvO09] and [BJvO11]:

Theorem 4.1.17. *Let Γ be a structure with an embedding e from Γ^2 into Γ. Then a relation R with a quantifier-free definition in Γ has a quantifier-free Horn definition in Γ if and only if R is preserved by e.*

4.2 Ramsey Theory

We come back to Ramsey Theory in order to state a more general theorem than the Canonization Lemma 2.4.15 stated previously. Indeed, we need such a lemma for arbitrary finite arities. The trick is to use the theorem in a unary setting with a product of structures. Though, we introduce the notion of *full product* of relational structures.

Definition 4.2.1. Let $\Gamma_1, \ldots, \Gamma_k$ be k relational structures. Assume that the signatures $\sigma_1, \ldots, \sigma_k$ of $\Gamma_1, \ldots, \Gamma_k$ are pairwise disjoint (if it is not the case, we rename relation symbols as needed in order to satisfy this property). The *full product* of $\Gamma_1, \ldots, \Gamma_k$ denoted by $\Gamma_1 \boxtimes \cdots \boxtimes \Gamma_k$ is defined as follows:

- $\mathrm{Dom}(\Gamma_1 \boxtimes \cdots \boxtimes \Gamma_k) = \Pi_{i \leq k} \mathrm{Dom}(\Gamma_i)$
- for every $i \leq k$, and every m-ary relation $R \in \sigma_i \cup \{=\}$ an m-ary relation defined by:

$$\{((x_1^1, \ldots, x_1^k), \ldots, (x_m^1, \ldots, x_m^k)) \in (\Pi_{i \leq k} \mathrm{Dom}(\Gamma_i))^m \mid (x_1^i, \ldots, x_m^i) \in R^{\Gamma_i}\}$$

Notation 4.2.2. Given a relational structure Γ of signature σ, we denote by $\Gamma^{[k]}$ the full product $\Gamma \boxtimes \Gamma \boxtimes \cdots \boxtimes \Gamma$ of size k. Note that in order to define the full product, we have to create, for each relation symbol R in σ, k new relation symbols $R_1, \ldots, R_k$ so the signature of each component structure are pairwise disjoint.

It is well known that the Ramsey property is not lost when going to full products (see Lemma 19 from [MBT13]):

Theorem 4.2.3. *Let $\Gamma_1, \ldots, \Gamma_k$ be k ω-categorical ordered Ramsey relational structures. Then $\Gamma_1 \boxtimes \cdots \boxtimes \Gamma_k$ is also ω-categorical ordered Ramsey.*

The following definition is a generalization to arbitrary arity of the notion of unary canonical operation we introduced in Definition 2.4.7.

Definition 4.2.4. Let Δ be a structure with domain D, and Γ a structure with domain C. When $f\colon D^k \to C$ is a function, and S is a subset of D^k we say that f is *canonical on S* if for all n and all n-tuples $t^1, \ldots, t^k$ where $(t_i^1, \ldots, t_i^k) \in S$ for all $i \leq n$, the n-type of $f(t^1, \ldots, t^k)$ in Γ only depends on the n-types of $t^1, \ldots, t^k$ in Δ.

We say that f is *canonical* (as a k-ary map) from Δ to Γ if f is canonical on D^k.

Definition 4.2.5. A topological group is a group G together with a topology on the elements of G such that $(x, y) \mapsto xy^{-1}$ is continuous from G^2 to G.

Definition 4.2.6. A topological group is extremely amenable if and only if any continuous action of the group on a compact Hausdorff space has a fixed point.

The following statement is Theorem 6.1 from [KPT05].

Theorem 4.2.7. *Let σ be a signature containing the order symbol $<$ and let $\mathcal{K}$ be a Fraïssé ordered class. Let $\mathbb{F}$ be the Fraïssé limit of $\mathcal{K}$. Then the following are equivalent:*

(i) $\mathrm{Aut}(\mathbb{F})$ *is extremely amenable*

(ii) $\mathcal{K}$ *has the Ramsey property*

Corollary 4.2.8. *The group* $\mathrm{Aut}(V; +, <)$ *is extremely amenable.*

Proof. Combine Proposition 3.4.3 with Theorem 4.2.7. □

Theorem 4.2.9 (Canonization Lemma)**.** *Let Δ be an ω-categorical totally ordered Ramsey structure and let Γ be an ω-categorical structure. Let $c_1, \ldots, c_m \in \mathrm{Dom}(\Delta)$, and let $f\colon \mathrm{Dom}(\Delta)^k \to \mathrm{Dom}(\Gamma)$ be any function. Then there is a function in*

$$S = \overline{\{\beta \circ f(\alpha_1, \ldots, \alpha_k) \mid \beta \in \mathrm{Aut}(\Gamma), \alpha_i \in \mathrm{Aut}(\Delta, c_1, \ldots, c_n) \text{ for all } i \leq k\}}$$

which is canonical as a k-ary function from $(\Delta, c_1, \ldots, c_n)$ to Γ, where $(\Delta, c_1, \ldots, c_n)$ is the structure Δ with added constants $c_1, \ldots, c_n$.

Proof. By Lemma 2.4.13, the structure $(\Delta; c_1, \ldots, c_n)$ is also totally ordered Ramsey. Let Δ' be the relational structure whose relations are exactly all the first-order definable relations over $(\Delta; c_1, \ldots, c_n)$. By Theorem 2.1.27, $\mathrm{Aut}(\Delta') = \mathrm{Aut}(\Delta; c_1, \ldots, c_n)$, and hence by Theorem 4.2.7, Δ' is an ω-categorical totally ordered Ramsey structure since $\mathrm{Aut}(\Delta') = \mathrm{Aut}(\Delta)$ is extremely amenable. Consequently, by Theorem 4.2.3, $\Delta'^{[k]}$ is also an ω-categorical totally ordered Ramsey structure. Hence by Theorem 2.4.15, there is a function in

$$S = \overline{\{\beta \circ f \circ \alpha \mid \beta \in \mathrm{Aut}(\Gamma), \alpha \in \mathrm{Aut}(\Delta'^{[k]})\}}$$

which is canonical as a unary function from $\Delta'^{[k]}$ to Γ. To conclude the proof, we only have to note that a canonical function from $\Delta'^{[k]}$ to Γ is exactly a k-ary canonical function from $(\Delta; c_1, \ldots, c_k)$ to Γ, which is straightforward by definition of the full product and canonical functions. $\square$

Corollary 4.2.10. *Let $(\Delta; <)$ be an ω-categorical totally ordered Ramsey structure with domain D. Let $c_1, \ldots, c_m \in D$, and let $f\colon D^k \to D$ be any function. Then f together with* $\mathrm{Aut}(\Delta)$ *locally generates a function that*

- *agrees with f on $\{c_1, \ldots, c_n\}$, and*
- *is canonical as a k-ary function from $(\Delta; <, c_1, \ldots, c_n)$ to Δ.*

Proof. Apply Theorem 4.2.9 with the structures $(\Delta; <, c_1, \ldots, c_n)$ and Δ, and note that $\mathrm{Aut}(\Delta; <, c_1, \ldots, c_n) \subseteq \mathrm{Aut}(\Delta)$. $\square$

4.3 Pseudo-varieties and Primitive Positive Interpretations

The notion of primitive positive interpretation will be very useful in the end of this thesis. Indeed, it allows us to code well-known NP-complete problems inside various reducts Γ of $(V; +)$, in order to prove that CSP(Γ) is NP-hard. In fact, we "simulate" a given structure inside a richer structure, defining its domain as a primitive positive definable subset of the original domain, and its relations as primitive positive definable relations over this subset.

Definition 4.3.1. A relational σ-structure Γ has a primitive positive interpretation I in a τ-structure Δ if there exists a natural number d, called the dimension of I, and:

- a τ-formula $\delta_I(x_1, \ldots, x_d)$ (called domain formula),
- for each atomic σ-formula $\phi(y_1, \ldots, y_k)$ a τ-formula $\phi_I(\overline{x_1}, \ldots, \overline{x_k})$ where the x_i denote disjoint d-tuples of distinct variables (called the defining formulas),
- a surjective map $h\colon \delta_I(\mathrm{Dom}(\Delta)^d) \to \mathrm{Dom}(\Gamma)$ (called coordinate map), such that for all atomic σ-formulas ϕ and all tuples $\overline{a_i} \in \delta_I(\mathrm{Dom}(\Delta)^d)$:

$$\Gamma \models \phi(h(\overline{a_1}), \ldots, h(\overline{a_n})) \Leftrightarrow \Delta \models \phi_I(\overline{a_1}, \ldots, \overline{a_n})$$

Definition 4.3.2. An *algebra* is a structure with a purely operational signature. When $\mathcal{K}$ is a class of algebras of the same signature, then:

- $P(\mathcal{K})$ denotes the class of all products of algebras from $\mathcal{K}$.
- $P^{\text{fin}}(\mathcal{K})$ denotes the class of all finite products of algebras from $\mathcal{K}$.
- $S(\mathcal{K})$ denotes the class of all sub-algebras of algebras from $\mathcal{K}$.
- $H(\mathcal{K})$ denotes the class of all homomorphic images of algebras from $\mathcal{K}$.

A class $\mathcal{V}$ of algebras with the same signature is called a *pseudo-variety* if $\mathcal{V}$ contains all homomorphic images, sub-algebras, and direct products of algebras in $\mathcal{V}$, i.e., $H(\mathcal{V}) = S(\mathcal{V}) = P^{\text{fin}}(\mathcal{V})$. The smallest pseudo-variety that contains an algebra A is called *the pseudo-variety generated by A*.

The following is from [Bod08]. It establishes a bridge between primitive positive interpretation and the notion of generated pseudo-variety. This theorem echoes the properties of the closure operators of the $\text{Inv} - \text{Pol}$ Galois connection presented earlier in this thesis: indeed, it strengthens the link between formulas and operations in order to give functional proofs for model theoretic results.

Theorem 4.3.3. *Let Δ be a finite or ω-categorical relational structure, and let P be a polymorphism algebra of Δ. Then a structure Γ has a primitive positive interpretation in Δ if and only if there is an algebra Q in the pseudo-variety generated by P such that all operations of Q are polymorphisms of Γ.*

4.4 Constraint Satisfaction Problems

CSPs naturally show up in various domains of Computer Science: artificial intelligence, scheduling, computational linguistics, computational biology, verification, and algebraic computation. Because of this wide spectrum of domains, there is more than one definition for CSPs. In particular, CSPs can be defined under four distinct perspectives: a homomorphism perspective which is the one we will use in this thesis, a first-order logic perspective where the input is a primitive positive formula, a second order perspective, and a purely syntactic one where the template and the input are given in a syntactic form.

4.4.1 Definitions and Key Properties

Definition 4.4.1. Let Γ be a structure with a finite relational signature τ. Then $\text{CSP}(\Gamma)$ is the computational problem to decide whether a given finite τ-structure Δ homomorphically maps to Γ.

Definition 4.4.2. We say that two structures Γ and Δ are *homomorphically equivalent* whenever there exists a homomorphism from Γ to Δ, and a homomorphism from Δ to Γ.

Proposition 4.4.3. *If two structures* Γ *and* Γ' *are homomorphically equivalent, then* $\mathrm{CSP}(\Gamma) = \mathrm{CSP}(\Gamma')$*.*

Proof. Follows immediately from the definition. $\square$

The following is from [Bod08], although it has been proven before for finite domain constraint satisfaction [BKJ05]. It legitimates the use of primitive positive interpretation introduced in Definition 4.3.1. Without surprise, if a structure contains a primitive positive interpretation of a structure whose CSP is NP-hard, then its CSP is also NP-hard.

Theorem 4.4.4. *Let* Γ *and* Δ *be structures with finite relational signatures. If there is a primitive positive interpretation of* Γ *in* Δ*, then there is a polynomial-time reduction from* $\mathrm{CSP}(\Gamma)$ *to* $\mathrm{CSP}(\Delta)$*.*

Because of the perspective we use to define CSPs, the following proposition, whose proof is straightforward, can be extremely useful in the study of the complexity of CSPs. We introduce shortly the notion of "core" of a structure, which can be conceived as a kind of closure of the original structure by homomorphic equivalence. The core of a structure has the remarkable property that its CSP is polynomial time equivalent to the CSP of the original structure. It is often easier to study the CSP over the core of a structure instead of studying directly the CSP of this structure.

Definition 4.4.5. Given a relational structure Γ with finite signature, $\mathrm{CSP}(\Gamma)$ is the computational problem to decide whether Γ realises a given primitive positive formula.

We here call upon the equality $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle_{\mathrm{pp}}$ combined with the definition of CSPs under the first-order logic perspective in order to establish a reverse link between polymorphisms and complexity. In one word, the more polymorphisms, the easier to solve.

Proposition 4.4.6. *Let* Γ *and* Δ *be two first-order reduct of a same* ω*-categorical structure, and such that* $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta)$*. Then* $\mathrm{CSP}(\Delta)$ *reduces to* $\mathrm{CSP}(\Gamma)$ *in polynomial time.*

Proof. Since $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta)$, we have $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) \supseteq \mathrm{Inv}(\mathrm{Pol}(\Delta))$. Since Γ and Δ are ω-categorical, we have: $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle_{\mathrm{pp}}$ and $\mathrm{Inv}(\mathrm{Pol}(\Delta)) = \langle\Delta\rangle_{\mathrm{pp}}$ by Theorem 4.1.11. Hence, $\langle\Delta\rangle_{\mathrm{pp}} \subseteq \langle\Gamma\rangle_{\mathrm{pp}}$. So $\mathrm{CSP}(\Delta)$ reduces to $\mathrm{CSP}(\Gamma)$ in linear time. Indeed, we only have to convert the input primitive formula on the language of Δ to its translate as a primitive formula on the language of Γ. Note that for every relation R in Δ, its translate as a primitive positive formula in Γ is pre-calculated. $\square$

4.4.2 CSPs over a Boolean Domain

We start by recalling two famous NP-completes problems of propositional logic.
Positive 1-in-3-3SAT
INSTANCE: a propositional 3SAT formula with only positive literals

QUESTION: is there a Boolean assignment for the variables such that in each clause exactly one literal is true?

Positive Not-All-Equal-3SAT
INSTANCE: a propositional 3SAT formula with only positive literals
QUESTION: is there a Boolean assignment for the variables such that in each clause neither all three literals are true nor all three are false?

These two problems will be very useful in order to prove that some particular CSPs over infinite domains are NP-hard. We will establish a reduction between one of these two problems and the CSP over the reduct we are interested in, using Proposition 4.4.6 and Theorem 4.4.4. Before this, we recall Schaefer's famous theorems for two-element sets.

Definition 4.4.7. We define the following relations:

- $\mathrm{NAE} := \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$
- $1\mathrm{IN}3 := \{(0,0,1),(0,1,0),(1,0,0)\}$

It is not hard to see that NAE is pp-definable on $(\{0,1\}, 1\mathrm{IN}3)$.

The previous theorem acts as a basis for the Feder-Vardi conjecture we cited in the introduction. Indeed, it states that we have a P/NP-complete dichotomy for the complexity of CSPs over a boolean domain, while the conjecture states that this dichotomy generalizes to CSPs over all finite domains. Long after Schaefer, Bulatov established in [Bul06] that this dichotomy hold for CSPs over sets with three elements.

Definition 4.4.8. We define the *majority* function f over the boolean domain as follows: for all $x, y, z \in \{0,1\}^3$, $f(x,y,z) = t \in \{0,1\}$ if and only if at least two of the arguments are set to t. We can see this $f(x,y,z)$ as a majority vote over the tuple (x,y,z).

We define the *minority* function g over the boolean domain as follows: for all $x, y, z \in \{0,1\}^3$, $g(x,y,z) = t \in \{0,1\}$ if and only if either exactly one argument is set to t, or the three arguments are set to t. We can this $f(x,y,z)$ as a minority vote over the tuple (x,y,z).

The following is Lemma 5.4.1 from [Bod12].

Lemma 4.4.9. *Let f be an idempotent function on the domain $\{0,1\}$ that violates 1IN3.*

- *If f is binary, then f must be $(x,y) \mapsto min(x,y)$ or $(x,y) \mapsto max(x,y)$*
- *If f is ternary, then f locally generates* $\min, \max$, *the majority or the minority operation.*

Proof. There are four distinct idempotent binary operations, and eight ternary idempotent operations. The proof is therefore a straightforward case distinction. □

The following statement is called Schaefer's dichotomy Theorem, from [Sch78].

Theorem 4.4.10 (Schaefer's dichotomy Theorem)**.** *Let* Γ *be a structure over a two-element universe, let us say* $\{0,1\}$*. Then either* NAE *has a primitive positive definition in* Γ*, and* $\mathrm{CSP}(\Gamma)$ *is NP-complete, or at least one of the following cases holds:*

1. Γ *is preserved by a constant operation.*
2. Γ *is preserved by min. In this case, every relation of* Γ *has a definition by a propositional Horn formula, that is, by a propositional formula in CNF where every clause contains at most one positive literal.*
3. Γ *is preserved by max. In this case, every relation of* Γ *has a definition by a dual-Horn formula, that is, by a propositional formula in CNF where every clause contains at most one negative literal.*
4. Γ *is preserved by the majority operation. In this case, every relation of* Γ *is bi-junctive.*
5. Γ *is preserved by the minority operation. In this case, every relation of* Γ *can be defined by a conjunction of linear equations modulo 2.*

In case (1) to case (5), $\mathrm{CSP}(\Gamma)$ *can be solved in polynomial time.*

Corollary 4.4.11. $\mathrm{CSP}(\{0,1\}; 1\mathrm{IN}3)$ *is NP-complete.*

Proof. Just note that 1IN3 isn't preserved by any constant operation, nor neither of the following operations: min, max, the majority, and the minority function. □

Corollary 4.4.12. *Let* Γ *be a structure over a finite or infinite domain with a finite relational signature. Assume that* Γ *is homomorphically equivalent to structure* Γ' *whose domain has at most two elements. Then* $\mathrm{CSP}(\Gamma)$ *is either in P or NP-complete.*

Proof. By Proposition 4.4.3, $\mathrm{CSP}(\Gamma)$ equals $\mathrm{CSP}(\Gamma')$. If Γ' has one element, then $\mathrm{CSP}(\Gamma)$ is trivially in P. If Γ' has two elements, then Schaefer's dichotomy Theorem (see 4.4.10) gives us that $\mathrm{CSP}(\Gamma')$ is either in P or NP-complete. □

4.4.3 Essentially Unary and Essential Operations

We recall here the notion of essentially unary polymorphism and its enemy: the essential polymorphism. These notions allows to distinguish between operations which depend on all their arguments, and operations which depend on only one argument. When all the polymorphisms of a structure Γ depend on only one argument, it can be shown, modulo the hypothesis that $\neq$ has a primitive positive definition in Γ, that $\mathrm{CSP}(\Gamma)$ is NP-hard for any infinite Γ.

Definition 4.4.13. We say that a k-ary operation f *depends on the argument* $i \in \{1, \ldots, k\}$ if there is no $(k-1)$-ary operation f' such that for all $x_1, \ldots, x_k \in B$:

$$f(x_1, \ldots, x_k) = f'(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_k)$$

We can equivalently characterize k-ary operations that depend on the i-th argument by requiring that there are $(x_1, \ldots, x_k) \in B$ and $x'_i \in B$ such that:

$$f(x_1, \ldots, x_k) \neq f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_k)$$

We say that an operation f is *essentially unary* if there is an $i \in \{1, \ldots, k\}$ and a unary operation f_0 such that $f(x_1, \ldots, x_k) = f_0(x_i)$ for all $\overline{x}$. Operations that are not essentially unary are called *essential*.

Essentially unary operations can be characterized by the fact that they preserve the relation P_3^B defined as follows:

Definition 4.4.14. For any set B, we define the relation P_3^B as follows.

For all $(a, b, c) \in B^3$: $(a, b, c) \in P_3^B$ if and only if $a = b$ or $b = c$

The following is Lemma 5.3.2 from [Bod12]:

Lemma 4.4.15. *Let f be an operation on a set B. Then the following are equivalent:*

- *f is essentially unary.*
- *f preserves P_3^B*

The following is Proposition 5.3.3 from [Bod12]:

Proposition 4.4.16. *Let Γ be an ω-categorical structure of domain B, and let $\mathcal{C}$ be its polymorphism clone. Then the following are equivalent:*

- *Every relation with an existential positive definition in Γ has a primitive positive definition in Γ.*
- *All operations in $\mathcal{C}$ are essentially unary.*
- *The relation P_3^B is contained in $\mathrm{Inv}(\mathcal{C})$.*

Definition 4.4.17. For any set B, we write E_6^B for the relation defined as follows. For all $(x_1, x_2, y_1, y_2, z_1, z_2) \in B^6$, $(x_1, x_2, y_1, y_2, z_1, z_2) \in E_6^B$ if and only if:

$$(x_1 = x_2 \wedge y_1 \neq y_2 \wedge z_1 \neq z_2) \vee (x_1 \neq x_2 \wedge y_1 = y_2 \wedge z_1 \neq z_2) \vee (x_1 \neq x_2 \wedge y_1 \neq y_2 \wedge z_1 = z_2)$$

The following statement is Proposition 5.5.9 from [Bod12]:

Proposition 4.4.18. *For any set B with $|B| \geq 2$, the structure $(\{0, 1\}; \mathrm{1IN3})$ has a primitive positive interpretation in $(B; E_6^B)$, and $\mathrm{CSP}(B; E_6^B)$ is NP-complete.*

The following is from [Bod07]:

Corollary 4.4.19. *Let Γ be an ω-categorical structure. The following are equivalent:*

- *Every relation with a first-order definition also has a primitive positive definition in Γ.*
- *Γ is a model-complete core, and P_3^B is primitive positive definable in Γ.*

Corollary 4.4.20. *For any infinite set B, the complexity of* $\mathrm{CSP}(B; P_3^B, \neq)$ *is NP-complete.*

Proof. First note that $(B; P_3^B, \neq)$ is a reduct of $(B; =)$, hence $\mathrm{Pol}(B; P_3^B, \neq)$ contains every bijection from B to B. Since P_3^B is preserved by all polymorphisms of $(B; P_3^B, \neq)$, all polymorphisms of $(B; P_3^B, \neq)$ are essentially unary by Proposition 4.4.16. Since $\neq$ is preserved by every $f \in \mathrm{Pol}(B; P_3^B, \neq)$, all endomorphisms of $(B; P_3^B, \neq)$ are injective. Hence, they are locally generated by the automorphisms of $(B; P_3^B, \neq)$ since every bijection of V^V is contained in $\mathrm{End}(B; P_3^B, \neq)$. Consequently, $(B; P_3^B, \neq)$ is a model-complete core and pp-defines P_3^B. By Corollary 4.4.19 and Theorem 4.1.11, every first-order definable relation on $(B; =)$ has a primitive positive definition in $(B; P_3^B, \neq)$, and in particular E_6. So by Proposition 4.4.18, we conclude that $\mathrm{CSP}(B; P_3^B, \neq)$ is NP-complete. $\square$

Corollary 4.4.21. *Let Γ be an ω-categorical structure such that every polymorphism are essentially unary, and preserve $\neq$. Then* $\mathrm{CSP}(\Gamma)$ *is NP-hard.*

Proof. Let B be the domain of Γ. By Lemma 4.4.15, every polymorphism of Γ preserve P_3^B. Hence, P_3^B has a primitive positive definition in Γ by Theorem 4.1.11. Since $\neq$ is preserved by every polymorphism of Γ, $\neq$ has also a primitive positive definition in Γ. Consequently, $\mathrm{CSP}(B; P_3^B, \neq)$ trivially reduces to $\mathrm{CSP}(\Gamma)$ (in fact it is contained in it), and by Corollary 4.4.20, $\mathrm{CSP}(\Gamma)$ is NP-hard. $\square$

4.4.4 CSPs over k-transitive Structures

The notion of k-transitivity comes from group theory but can be stated in a more general setting. It is a very strong property shared by some structures where no constants are definable. In one word, a structure is k-transitive when any pair of k distinct elements of the domain can not be distinguished by any first-order formula. Note that a reduct Δ of $(D; =)$ where D is infinite is k-transitive for all $k \geq 1$. We then say that Δ is highly transitive.

Definition 4.4.22. A permutation group G acting on a set B is *k-transitive* if for any two k-tuples s, t of distinct elements of B, there exists $\sigma \in G$ such that $\sigma.s = t$, where the action of σ on tuples is componentwise, i.e., $\sigma.(s_1, \ldots, s_k) = (\sigma(s_1), \ldots, \sigma(s_k))$.

We say that a structure is *k-transitive* whenever its automorphism group is.

We say that a structure is *highly transitive* if it is k-transitive for all $k \geq 1$.

Let us now give three examples of k-transitive automorphism groups which appear in Theorem 3.6.20.

Lemma 4.4.23. *A structure* Γ *with domain* D *is highly transitive if and only if it is a reduct of* $(D; =)$.

Proof. If Γ is a reduct of $(D; =)$, it is clearly highly transitive. Conversely, if Γ is highly transitive, we can show that any bijection from D to D is an automorphism of Γ by a back and forth argument. So $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(D; =)$, and every relation of Γ has a first order definition over $(D; =)$ by Theorem 4.1.11, since $(D; =)$ is ω-categorical. $\square$

CSPs over highly transitive structures have already been classified by Bodirsky and Kára (see [BK08a]). Their complexity presents a P/NP-complete dichotomy.

Theorem 4.4.24. *Let* Γ *be a first-order reduct of* $(D; =)$. *Then* $\mathrm{CSP}(\Gamma)$ *is polynomial-time tractable if* Γ *has a constant unary or an injective binary polymorphism. Otherwise it is NP-complete.*

CSPs over 2-transitive structures have not been classified yet but have been studied in details. Before stating the results we need, we start by giving few lemmas and propositions in a more general setting.

Definition 4.4.25. Let Γ be a structure with relational signature τ. Then we say that $\neq$ is *independent* from Γ if for all primitive positive τ-formula ϕ, if both $\phi \wedge x \neq y$ and $\phi \wedge u \neq v$ are satisfiable over Γ, then $\phi \wedge x \neq y \wedge u \neq v$ is satisfiable over Γ as well.

Definition 4.4.26. A relation $R \subseteq B^k$ is called *intersection-closed* if for all k-tuples $\overline{u}, \overline{v}$ in R, there is a tuple $(w_1, \ldots, w_k) \in R$ such that for all $i, j \leq k$, we have $w_i \neq w_j$ whenever $u_i \neq u_j$ or $v_i \neq v_j$.

The following statement is from [BK08a]:

Lemma 4.4.27. *Let* Γ *be an* ω*-categorical structure where* $\neq$ *is primitive positive definable. Then the following are equivalent:*

- *Inequality is independent from* Γ*;*
- *All primitive positive definable relations in* Γ *are intersection-closed;*
- Γ *has a binary injective polymorphism.*

Definition 4.4.28. Given a set B, we define the relation S_B as follows. For all $a, b, c \in B$:

$$(a, b, c) \in S_B \Leftrightarrow b \neq c \wedge (a = b \vee a = c)$$

The following statement is Corollary 2.3 from [BJvO11]:

Corollary 4.4.29. *Let* B *be an infinite set. Every first-order definable relation in* $(B; =)$ *has a primitive positive definition in* $(B; S_B)$.

The following statement is Lemma 5.3 from [BJvO11]:

Lemma 4.4.30. *Let Γ be a relational structure over an infinite domain B such that B^2, $=$, $\neq$, $\emptyset$ are the only primitive positive definable binary relations. Suppose that Γ contains a relation ϕ such that there are pairwise distinct $1 \leq i, j, k, l \leq n$ for which the following conditions hold:*

- $\phi(x_1, \ldots, x_n) \wedge x_i \neq x_j$ *is satisfiable;*
- $\phi(x_1, \ldots, x_n) \wedge x_k \neq x_l$ *is satisfiable;*
- $\phi(x_1, \ldots, x_n) \wedge x_i \neq x_j \wedge x_k \neq x_l$ *is unsatisfiable.*

Then S_B has a primitive positive definition in Γ.

Remark 4.4.31. It is straightforward to see that if Γ is 2-transitive, the only primitive positive definable binary relations are B^2, $=$, $\neq$, and $\emptyset$.

The following statement is Corollary 5.3.8 from [Bod12]:

Corollary 4.4.32. *Let $\mathcal{F}$ be a locally clone that contains a 2-transitive permutation group G . If there is an $f \in \mathcal{F}$ that violates $\neq$, then $\mathcal{F}$ contains a constant operation.*

The following statement is Corollary 5.3.11 from [Bod12]:

Corollary 4.4.33. *Let Γ be a 2-transitive ω-categorical structure with an essential polymorphism. Then Γ also has a binary essential polymorphism.*

Lemma 4.4.34. *Let Γ be a 2-transitive ω-categorical structure with an essential binary polymorphism f, and suppose that $\neq$ has a primitive positive definition in $\mathfrak{B}$. Then either Γ has a binary injective polymorphism, or S_B has a primitive positive definition over Γ.*

Proof. Assume that Γ does not have a binary injective polymorphism. Since $\neq$ has a primitive positive definition in Γ, Inequality is not independent from Γ by Lemma 4.4.27, i.e., there exists a primitive positive formula ϕ such that both $\phi \wedge x \neq y$ and $\phi \wedge u \neq v$ are satisfiable over Γ, but $\phi \wedge x \neq y \wedge u \neq v$ is unsatisfiable over Γ. Then by Lemma 4.4.30 and Remark 4.4.31, S_B has a primitive positive definition over Γ. $\square$

The following theorem focuses on 2-transitive structures and its shape will be recurrent in this thesis. It is a case distinction which can lead to a CSP classification. Indeed, if a structure has a constant polymorphism, we only need to check whether the corresponding constant tuple satisfies the instance of the CSP. This can be done in polynomial time. If every polymorphism of the 2-transitive structure is essentially unary and $\neq$ is primitive positive definable in this structure, then the CSP is NP-hard, as it is stated in Proposition 4.4.21. Finally, if there exists an injective binary polymorphism, we usually manage to find polynomial time algorithms to solve the CSP. Such algorithms can derive from the Horn resolution.

Theorem 4.4.35. *Let Γ be an ω-categorical 2-transitive structure. Then at least one of the following statements holds:*

- Γ *has a constant polymorphism;*
- *Every polymorphism of* Γ *is essentially unary, and* $\neq$ *has a primitive positive definition in* Γ*;*
- Γ *has a binary injective polymorphism.*

Proof. First suppose that $\neq$ is not primitive positive definable on Γ. Since Γ is 2-transitive, $\mathrm{Pol}(\Gamma)$ contains a 2-transitive permutation group $\mathrm{Aut}(\Gamma)$. And since $\neq$ does not belong to $\langle\Gamma\rangle_{\mathrm{pp}}$, there exists $f \in \mathrm{Pol}(\Gamma)$ such that f violates $\neq$. Then by Corollary 4.4.32, Γ has a constant polymorphism.

In the following, let $B := \mathrm{Dom}(\Gamma)$. Now assume that $\neq$ has a primitive positive definition in Γ. Either every polymorphism of Γ is essentially unary, or Γ has an essential polymorphism. Since Γ is 2-transitive, Γ has a binary essential polymorphism by Corollary 4.4.33. So by Lemma 4.4.34, either Γ has an injective binary polymorphism, or S_B has a primitive positive definition in Γ. In the second case, since P_3^B has a primitive positive definition over $(B; S_B)$ by Corollary 4.4.29, P_3^B has also a primitive positive definition over Γ. So by Proposition 4.4.16, all polymorphisms of Γ are essentially unary, and $\neq \in \langle\Gamma\rangle_{\mathrm{pp}}$ by assumption. $\square$

Chapter 5

Bit Vector CSPs

5.1 Introduction

An example of Bit Vector CSPs can be the following natural algebraic constraint satisfaction problem:

Vector Space CSP
INSTANCE: A set of equalities of the form $x + y = z$ and disequalities of the form $x + y \neq z$ over a set of variables Var.
QUESTION: Can we assign d-dimensional boolean vectors to the variables such that all the equalities and disequalities are satisfied, for some d?

If there is a solution to an instance of this problem, it is straightforward to see that this solution embeds in a vector space of dimension $d = |\operatorname{Var}|$ over $\mathbb{F}_2$. But the more the size of Var is growing, the bigger d can be. Hence, this CSP can not be formulated with a finite template.

The Vector Space CSP can be naturally formulated as a CSP for a reduct of the ω-categorical template $(V; +)$, where $(V; +)$ is the countably infinite vector space over $\mathbb{F}_2$, where that the $+$ is a function and not a relation. In fact, as we showed in Subsection 3.2.5, $(V; +)$ is a uniformly locally finite homogeneous structure which is not first-order equivalent with any homogeneous structure over a finite relational language.

It has been shown in [BCKvO09] that the particular case of the Vector Space CSP is in P (the algorithm uses Gauss Pivot as a subroutine), and many other interesting constraint satisfaction problems can be defined as CSPs over reducts of $(V; +)$, some of them being NP-hard since $(V; +)$ can encode every hard case from the dichotomy Schaefer's Theorem for boolean CSPs (see 4.4.10).

Definition 5.1.1. A CSP over a reduct of $(V; +)$ is called a *Bit Vector CSP*.

By the following proposition, each Bit Vector CSP is in NP.

Proposition 5.1.2. *Let* Γ *be a reduct of* $(V; +)$*. Then* $\operatorname{CSP}(\Gamma)$ *is in NP.*

Proof. Let $\sigma := \{R_1, \ldots, R_n\}$ be the signature of Γ. Since R_i has a first-order definition on $(V;+)$ for all $i \leq n$, and since $(V;+)$ admits quantifier elimination by Proposition 3.2.31, there exists a quantifier free formula ϕ_i in disjunctive normal form which is equivalent to $R_i(\overline{x})$.

Let ϕ be a primitive formula over Γ. Replace in ϕ every constraint $R_i(\overline{x})$ by $\phi_i(\overline{x})$. We get a new formula over $\{+\}$ that we call ϕ'. To prove that CSP(Γ) is in NP, assume that we are given as an oracle the literals which have to be satisfied, one for each clause in ϕ'. Then, forgetting every other literal, we obtain a conjunction of equations and inequations over the set of variables of ϕ. Hence, by a classic Gauss Pivot, we can determine whether the system of equations is satisfiable in polynomial time in the number of equations and then check if this solution is compatible with the sets of inequations, which takes linear time. □

But do we have a similar P/NP-complete dichotomy for Bit Vector CSPs as in Schaefer's Theorem? And how can this problem be tackled? This is where our algebraic study of reducts of $(V;+)$ becomes useful. Indeed the algebraic Theorem 3.6.20 states that given a reduct Γ of $(V;+)$, exactly one of the following cases holds:

- $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$;
- up to isomorphism, the model-complete core Γ' of Γ satisfies one of the following property:
    - $\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$;
    - $\mathrm{End}(\Gamma') = \mathrm{End}(V; {\mathrm{Ieq}_4}^{\mathrm{inj}})$;
    - $\mathrm{End}(\Gamma') = \mathrm{End}(V \cup \{\infty\}; {\mathrm{Ieq}_4}^{\mathrm{inj}}, \infty)$;
- Γ is homomorphically equivalent to a reduct of $(V;0)$;
- Γ is homomorphically equivalent to a structure with at most two elements.

But recall that, by Proposition 4.4.3, the CSPs of two homomorphically equivalent structures reduce to each other, and since the model-complete core of Γ is homomorphically equivalent to Γ, the study of the complexity of Bit Vector CSPs is reduced to a finite number of cases.

In the following sections, we first establish some important results on binary injective polymorphism of reducts of $(V;+)$, and then tackle one by one the cases described in Theorem 3.6.20, starting by reducts of $(V;0)$ which has an interest by itself, as it is the first time CSPs over reducts of an infinite structure containing equality plus a constant are classified. We will also classify CSPs over reducts of $(V \setminus \{0\}; \mathrm{Ieq}_3)$, and CSPs over reducts of $(V; \mathrm{Ieq}_4)$. Two cases still miss to our classification: classifying CSP(Γ) with Γ being a reduct of $(V;+)$ satisfying $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$, or $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, 0)$.

5.2 Generating Binary Injections

This section gathers some technical lemmas which will simplify the CSPs classification later in the thesis. The goal is to locally generate binary injective polymorphisms over $(V \setminus \{0\})^2$ for such operations often allow to adapt an Horn resolution to solve the CSP.

Definition 5.2.1. We start by defining three classes of binary operations:

- $\mathcal{I}(\text{inj}, \text{inj})$ is the set of all binary operations g over V such that $g(0,0) = 0$, $g(x,y) \neq 0$ for all $x, y \neq 0$, and g injective over V^2.
- $\mathcal{I}(0, \text{inj})$ is the set of all binary operations g over V such that $g(x,0) = 0$ for all $x \in V$, $g(x,y) \neq 0$ for all $x, y \neq 0$, and g injective over $(V \setminus \{0\}) \times V$.
- $\mathcal{I}(\text{inj}, 0)$ is the set of all binary operations g over V such that $g(0,y) = 0$ for all $y \in V$, $g(x,y) \neq 0$ for all $x, y \neq 0$, and g injective over $(V \setminus \{0\}) \times V$.
- $\mathcal{I}(0,0)$ is the set of all binary operations g over V such that $g(x,0) = g(0,y) = 0$ for all $x, y \in V$, $g(x,y) \neq 0$ for all $x, y \neq 0$, and g injective over $(V \setminus \{0\})^2$.

Lemma 5.2.2. *Let f be a binary operation such that $x \mapsto f(x,0)$ is injective, $f(0,y) = 0$ for all y, $f(x,y) \neq 0$ for all $x, y \neq 0$, and f is injective on $(V \setminus \{0\})^2$. Then $\overline{\langle \{f\} \cup \mathrm{Aut}(V;+) \rangle} \cap \mathcal{I}(\text{inj}, 0) \neq \emptyset$.*

Proof. We can always assume that $f(x,y) \neq 0$ for all $(x,y) \in (V \setminus \{0\}) \times V$. Indeed if it is not the case, there exist at most two distinct elements (x_1, y_1) and (x_2, y_2) of $(V \setminus \{0\}) \times V$ such that $f(x_1, y_1) = f(x_2, y_2) = 0$ (by the partial injectivity of f). Now, if we consider a self-embedding α of $(V;+)$ such that $x_1, x_2 \notin \alpha(V)$, $(x,y) \mapsto f(\alpha(x), y)$ has the required property. We now define g as follows: for $x, y \in V$, $g(x,y) := f(x, f(x,y))$. It is straightforward to check that g belongs to $\mathcal{I}(\text{inj}, 0)$. □

Lemma 5.2.3. *Let f be a binary operation such that $f(0,0) = 0$, $x \mapsto f(x,0)$ is injective, $y \mapsto f(0,y)$ is injective, $f(x,y) \neq 0$ for all $x, y \neq 0$, and f is injective over $(V \setminus \{0\})^2$. Then $\overline{\langle \{f\} \cup \mathrm{Aut}(V;+) \rangle} \cap \mathcal{I}(\text{inj}, \text{inj}) \neq \emptyset$.*

Proof. It is straightforward to see that the assumptions force f to strongly preserve 0, i.e., $f(x,y) = 0$ if and only if $x = y = 0$. By Corollary 4.2.10, since $(V;+,<)$ is an ω-categorical totally ordered Ramsey structure by Corollary 3.4.5, f together with $\mathrm{Aut}(V;+)$ locally generates a canonical function g from $(V;+,<)$ to $(V;+)$ such that $g(0,0) = 0$. Since $g \in \overline{\{\beta f(\alpha_1, \alpha_2) \mid \beta, \alpha_1, \alpha_2 \in \mathrm{Aut}(V;+,<)\}}$, g is injective over $(V \setminus \{0\})^2$, over $\{0\} \times V$, and over $V \times \{0\}$. Suppose now that $g \notin \mathcal{I}(\text{inj}, \text{inj})$. Then without loss of generality there exist $y_1, x_2, y_2 \in V \setminus \{0\}$ such that $g(0, y_1) = g(x_2, y_2)$. If $x_2 = y_2$, then for all $x, y \neq 0$, $g(0,y) = g(x,x)$ by canonicity of g, which contradicts the fact that g is injective over $(V \setminus \{0\})^2$. If $x_2 < y_2$, then for all $y' \neq 0$ and for all $0 < x < y$, $g(0, y') = g(x,y)$, which contradicts the injectivity of g over $(V \setminus \{0\})^2$. □

Proposition 5.2.4. *Let g be a binary operation over V such that $g(0,0) = 0$, $g(x,y) \neq 0$ for all $x, y \neq 0$, and g injective on $(V \setminus \{0\})^2$. Then g together with* $\mathrm{Aut}(V;+)$ *locally generates an operation h which belongs to one of the following classes:* $\mathcal{I}(\mathrm{inj}, \mathrm{inj})$, $\mathcal{I}(\mathrm{inj}, 0)$, *or* $\mathcal{I}(0, 0)$.

Proof. Let g be as in the assumptions. We make a case distinction:

- First assume that there exists $x_0 \neq 0$ such that $g(x_0, 0) = 0$. We will locally generate with g and $\mathrm{Aut}(V;+)$, an operation g' such that $g'(x,0) = 0$ for all $x \in V$. Let $v_0 = 0, v_1 = x_0, \ldots, v_n, \ldots$ be an enumeration of V, and let V_n be the subset of V defined as follows: $V_n := \{v_0, \ldots, v_n\}$. Consider the following infinite tree $\mathcal{T}$ whose vertices lie on levels $1, 2, \ldots$. The vertices at the n-th level are all the complete types over $(V;+)$ of tuples of the form $(g'(v_i, v_j))_{0 \leq i,j \leq n}$, with g' being an operation of V which satisfies $g'(0,0) = 0$, $g'(x,0) = 0$ for all $x \in V_n$, and g' injective on $(V \setminus \{0\})^2$. We say that a node N corresponding to a map g_N is a descendant of a node M corresponding to a map g_M if:

$$\mathrm{tp}_{(V;+)}(g_M(v_i, v_j))_{0 \leq i,j \leq n} \subseteq \mathrm{tp}_{(V;+)}(g_N(v_i, v_j))_{0 \leq i,j \leq n}$$

  Note that $\mathcal{T}$ has finitely many vertices at each level since $(V;+)$ is ω-categorical, and so by Theorem 2.1.24, there is a finite number of distinct $(n+1)$-types for all n. We now prove that $\mathcal{T}$ has at least one vertex at each level by induction. The initialization of the induction is trivially true since $g(v_1, 0) = 0$. Suppose now that there exists a node at the n-th level, and let g_S be an operation corresponding to this node. If $g_S(v_{n+1}, 0) = 0$, then we define $g_{S'}(x,y) := g_S(x,y)$ for all $x, y \in V$. It is straightforward to check that $g_{S'}$ corresponds to a node in the $(n+1)$-th level of $\mathcal{T}$. If $g_S(v_{n+1}, 0) \neq 0$, let $\alpha \in \mathrm{Aut}(V;+)$ such that $\alpha(g_S(v_{n+1}, 0)) = v_1$. We define $g_{S'}(x,y) := g(\alpha(g_S(x,y)), y)$ for all $x, y \in V$. Note that $g_{S'}(x, 0) = 0$ for all $x \in S'$. Also note that $g_{S'}$ is injective on $(V \setminus \{0\})^2$, and $g_{S'}(x,y) \neq 0$ for all $x, y \neq 0$. So $g_{S'}$ corresponds to a node in the $(n+1)$-th level, whose father corresponds to $g_{S'} \upharpoonright V_n^2$. We now conclude by König's lemma that $\mathcal{T}$ has an infinite branch $(N_0, \ldots, N_n, \ldots)$. For $n \geq 0$, we choose a g_n corresponding to the node N_n such that $g_n \upharpoonright V_n^2 = g_{n+1} \upharpoonright V_n^2$. This choice is made possible by Proposition 2.1.26 since $(V;+)$ is ω-categorical. Such a sequence $(g_n)_{n \geq 1}$ exists since $(V;+)$ is ω-categorical, hence two tuples of same type can be sent one to the other by Proposition 2.1.26. We then define $g' := \cup_{n \geq \mathbb{N}} g_n \upharpoonright V_n^2$. The operation g' is well defined and has the required properties.

  - Now suppose that there exists $y \neq 0$ such that $g'(0,y) = 0$. A similar construction gives us a binary operation h such that $h(x,0) = h(0,y) = 0$ for all $x, y \in V$, $h(x,y) \neq 0$ for all $x, y \neq 0$, and h injective on $(V \setminus \{0\}) \times (V \setminus \{0\})$. So g together with $\mathrm{Aut}(V;+)$ locally generates an operation $h \in \mathcal{I}(0,0)$.
  - Conversely, suppose that for all $y \neq 0$, $g'(0,y) \neq 0$. Then we define an operation h on V as follows: $h(x,y) := g'(y, g'(x,y))$ for all $x, y \in V$. It is straightforward to verify that modulo Lemma 5.2.2, h belongs to $\mathcal{I}(\mathrm{inj}, 0)$,

since $h(x,0) = 0$ for all $x \in V$, $y \mapsto h(0,y)$ is injective, and h is injective on $(V \setminus \{0\})^2$.

- Symmetrically, if there exists y such that $g(0,y) = 0$, we prove that g together with $\mathrm{Aut}(V;+)$ locally generates an operation h which belongs to either $\mathcal{I}(\mathrm{inj},0)$, or $\mathcal{I}(0,0)$.

- Finally, suppose that $g(x,y) \neq 0$ for all $(x,y) \neq (0,0)$. We will prove that g together with $\mathrm{Aut}(V;+)$ locally generates an operation h in $\mathcal{I}(\mathrm{inj},\mathrm{inj})$. First note that the mapping $g'(x,y) := g(x, g(x,y))$ is injective on $(V \setminus \{0\})^2$, and $x \mapsto g'(x,0)$ is injective with respect to x. Then the mapping $h(x,y) := g'(g'(x,y),y)$ is injective on $(V \setminus \{0\})^2$, $x \mapsto h(x,0)$ is injective, $y \mapsto h(0,y)$ is injective, $h(0,0) = 0$, and $h(x,y) \neq 0$ for all $x, y \neq 0$. Note that h is not necessarily injective on V^2 since we can have $x_0, x_1, y \neq 0$ such that $h(x_0, 0) = h'(x_1, y)$ for example. But by Lemma 5.2.3, h together with $\mathrm{Aut}(V;+)$ locally generates an operation which belongs to $\mathcal{I}(\mathrm{inj},\mathrm{inj})$.

□

In fact, the proof of Proposition 5.2.4 gives us a bit more:

Corollary 5.2.5. *Let g be a binary operation over V such that $g(0,0) = 0$, $g(x,y) \neq 0$ for all $x, y \neq 0$, and g injective on $(V \setminus \{0\}) \times (V \setminus \{0\})$. Suppose also that $g(x,0) \neq 0$ and $g(0,x) \neq 0$ for all $x \neq 0$. Then g together with $\mathrm{Aut}(V;+)$ locally generates an operation h which belongs to $\mathcal{I}(\mathrm{inj},\mathrm{inj})$.*

Proof. Only the last case of the proof of Proposition 5.2.4 applies. □

Corollary 5.2.6. *Let g be a binary operation over V such that $g(0,0) = 0$, $g(x,y) \neq 0$ for all $x, y \neq 0$, and g injective on $(V \setminus \{0\}) \times (V \setminus \{0\})$. Suppose also that there exists $x_0, y_0 \neq 0$ such that $g(x_0, 0) = g(0, y_0) = 0$. Then g together with $\mathrm{Aut}(V;+)$ locally generates an operation h which belongs to $\mathcal{I}(0,0)$.*

Proof. Only the first case with first sub-case of the proof of Proposition 5.2.4 applies. □

Lemma 5.2.7. *Let g be a binary operation of V such that $g(0,0) = 0$, g injective on $(V \setminus \{0\})^2$, and $g(x,y) \neq 0$ for all $(x,y) \in V \times (V \setminus \{0\})$. Suppose that there exists $x_0 \neq 0$ such that $g(x_0, 0) \neq 0$. Then g together with $\mathrm{Aut}(V;+)$ locally generates an operation $h \in \mathcal{I}(\mathrm{inj},\mathrm{inj})$.*

Proof. We first prove that there exists an operation g' such that $g'(0,0) = 0$, g' injective over $(V \setminus \{0\})^2$, and $g'(x,y) \neq 0$ for all $(x,y) \neq (0,0)$. Let $v_0 = 0, v_1 = x_0, \ldots, v_n, \ldots$ be an enumeration of V, and let V_n be the subset of V defined as follows: $V_n := \{v_0, \ldots, v_n\}$. Consider the following infinite tree $\mathcal{T}$ whose vertices lie on levels $1, 2, \ldots$. The vertices at the n-th level are all the complete types over $(V;+)$ of tuples of the form $(f(v_i, v_j))_{0 \leq i,j \leq n}$, with f being a binary operation of V which satisfies $f(0,0) = 0$,

$f(x,y) \neq 0$ for all $x,y \in V_n^2$, and f injective on $(V \setminus \{0\})^2$. We say that a node N corresponding to a map f_N is a descendant of a node M corresponding to a map f_M if:

$$\mathrm{tp}_{(V;+)}(f_M(v_i,v_j))_{0\leq i,j\leq n} \subseteq \mathrm{tp}_{(V;+)}(f_N(v_i,v_j))_{0\leq i,j\leq n}$$

Note that $\mathcal{T}$ has finitely many vertices at each level since $(V;+)$ is ω-categorical, and so by Theorem 2.1.24, there is a finite number of distinct $(n+1)$-types for all n. We have to prove that at each level n, there exists at least one node S, with a corresponding operation g_S from V^2 to V locally generated by g together with $\mathrm{Aut}(V;+)$, which is injective on $(V \setminus \{0\})^2$, and such that $g_S(x,y) \neq 0$ for all $(x,y) \in V_n^2$. The initialization of the induction is trivially true since $g(x_0,0) \neq 0$ by assumption. Let g_S be an operation of V corresponding to a node S in the n-th level of the tree. Let $\alpha \in \mathrm{Aut}(V;+)$ such that $\alpha(v_{n+1}) = x_0$. We define $g_{S'}(x,y) := g(\alpha(x), g_S(x,y))$ for all $x,y \in V$. It is straightforward to verify that $g_{S'}$ has the required properties, and so corresponds to a node at the $(n+1)-th$ level. Hence the tree $\mathcal{T}$ has an infinite branch $(g_n)_{n\geq 1}$ by Theorem 2.4.14. Note that we can assume that the g_i are not types but operations which satisfy the following: $g_n{\restriction}V_n = g_{n+1}{\restriction}V_n$ for all $n \geq 1$. Indeed, let g be an operation whose type over V_n belongs to a node S_n, and let h be an operation whose type over V_{n+1} belongs to a child node of S_n, i.e., we have:

$$\mathrm{tp}_{(V;+)}(g(v_i,v_j))_{0\leq i,j\leq n} \subseteq \mathrm{tp}_{(V;+)}(h(v_i,v_j))_{0\leq i,j\leq n+1}$$

By homogeneity of $(V;+)$, there exists an automorphism α such that $\alpha \circ g(x,y) = h(x,y)$ for all $x,y \in V_n$. Hence we can define g' as follows: $g'(x,y) := \cup_{n\geq 1} g_n{\restriction}V_n$. Note that g' is locally generated by g. It is now straightforward to see that g' has the required properties. Then by Corollary 5.2.5, we conclude that g' together with $\mathrm{Aut}(V;+)$ locally generates an operation h which belongs to $\mathcal{I}(\mathrm{inj},\mathrm{inj})$. $\square$

5.3 Polymorphisms of Reducts

This section builds bridges between reducts of $(V;+)$ and their polymorphisms. In particular, we derive from Schaefer's Theorem a list of seven properties one of which (at least) must be satisfied by any reduct of $(V;+)$ which pp-defines $0, \mathrm{Ind}_1, \mathrm{Ind}_2$ and the equivalence relation E we define in the next paragraph, and which allows to naturally quotient the vector space in order to obtain the two element domain needed for Schaefer's Theorem.

Definition 5.3.1. We give here the definition of three relations, two of which have already been defined:

- Ind_1 is the unary relation such that: $x \in \mathrm{Ind}_1$ if and only if $x \neq 0$.
- Ind_2 is the binary relation such that: $(x,y) \in \mathrm{Ind}_2$ if and only if $0 \neq x \neq y \neq 0$.
- N is the binary relation such that: $(x,y) \in N$ if and only if $x = 0 \Leftrightarrow y \neq 0$.

- E is the binary relation such that: $(x, y) \in E$ if and only if $x = 0 \Leftrightarrow y = 0$.

Note that both Ind_1 and Ind_2 consist of only one orbit, and N consists of two orbits. Also note that E is clearly pp-definable in $(V; N)$, the converse being false.

Proposition 5.3.2. *Let Γ be a first-order reduct of $(V; +)$ such that $0, \mathrm{Ind}_1, \mathrm{Ind}_2$ are* pp-*definable over Γ but N is not. Then at least one of the following cases holds:*

- *Γ has no injective binary polymorphism on $(V \setminus \{0\})^2$;*
- *Γ has a polymorphism which belongs to $\mathcal{I}(\mathrm{inj}, \mathrm{inj})$;*
- *Γ has a polymorphism which belongs to $\mathcal{I}(0, 0)$.*

Proof. Suppose that Γ has a binary polymorphism g, injective on $(V \setminus \{0\})^2$. We have $g(0) = 0$, $g(x, y) \neq 0$ for all $x, y \neq 0$. So by Proposition 5.2.4, g together with $\mathrm{Aut}(V; +)$ locally generates a binary operation h which belongs either to $\mathcal{I}(\mathrm{inj}, \mathrm{inj})$, $\mathcal{I}(0, \mathrm{inj})$, $\mathcal{I}(\mathrm{inj}, 0)$, or $\mathcal{I}(0, 0)$.

So let us suppose that there exists a binary polymorphism h of Γ which belongs to $\mathcal{I}(\mathrm{inj}, 0)$ (the case $h \in \mathcal{I}(0, \mathrm{inj})$ is symmetrical). Since N is not in $\langle\Gamma\rangle_{\mathrm{pp}}$, there exists a polymorphism of Γ which violates N. Because N is the union of two 2-orbits, we can suppose that this polymorphism f has arity 2 by Lemma 4.1.12. Since f violates N but preserves 0 and Ind_1, there exists $x, y \neq 0$ such that either $f(x, 0) = f(0, y) = 0$, or: $f(x, 0) \neq 0$ and $f(0, y) \neq 0$. We make a case distinction.

- First suppose that there exists $x_0, y_0 \neq 0$ such that $f(x_0, 0) = f(0, y_0) = 0$. Let δ_1, δ_2 two self-embeddings of $(V; +)$ such that $\delta_1(h(x_0, 0)) = x_0$, and $\delta_2(h(y_0, 0)) = y_0$. Let $f'(x, y) := f(\delta_1(h(x, y)), \delta_2(h(y, x)))$ for all $x, y \in V$. The map f' is clearly injective over $(V \setminus \{0\})^2$, and $f'(x_0, 0) = f'(0, y_0) = 0$. So by Corollary 5.2.6, f' together with $\mathrm{Aut}(V; +)$ locally generates a binary map h which belongs to $\mathcal{I}(0, 0)$.
- Now we suppose that there exists $x_0, y_0 \neq 0$ such that $f(x_0, 0) \neq 0$ and $f(0, y_0) \neq 0$. As we just did for the previous case, we can define a mapping f' such that f' is injective over $(V \setminus \{0\})^2$, $f'(x_0, 0) = f(x_0, 0) \neq 0$, and $f'(0, y_0) = f'(0, y_0) \neq 0$. Note that we can assume that there is no zero on at least one axis. Indeed, if it was not the case, we would be back in the first case of this proof. Hence, we can apply Lemma 5.2.7 and conclude that f' together with $\mathrm{Aut}(V; +)$ locally generates an operation $h \in \mathcal{I}(\mathrm{inj}, \mathrm{inj})$.

□

Lemma 5.3.3. *Let Γ be a reduct of $(V; +)$ such that $0, \mathrm{Ind}_1 \in \langle\Gamma\rangle_{\mathrm{pp}}$, and such that there exists a non-injective endomorphism g of Γ. Then Γ is homomorphically equivalent to a two-element structure.*

Proof. Let g be an non-injective endomorphism of Γ. There exists $x_0 \neq x_1$ such that $g(x_0) = g(x_1)$. Since 0 and $\neq 0$ are preserved by g, we have $x_0, x_1 \neq 0$, and $g(x_0) \neq 0$. We now prove that g together with $\mathrm{Aut}(V;+)$ locally generates the operation h defined as follows: $h(0) = 0$ and $h(x) = g(x_0)$ for all $x \neq 0$. For every relation R of Γ, we define a relation R' on $\{0, g(x_0)\}$ such that for all $a_1, \dots, a_n \in \{0, g(x_0)\}$:

$$R'(a_1, \dots, a_n) \Leftrightarrow \exists b_1, \dots, b_n \in V \text{ such that } h(b_i) = a_i \text{ for all } i \leq n \text{ and } R(b_1, \dots, b_n)$$

Hence, Γ will be homomorphically equivalent to the two element structure of domain $\{0, g(x_0)\}$ and whose relations are the R' defined previously. Let us show that $h \in \overline{\langle \{g\} \cup \mathrm{Aut}(V;+) \rangle}$. Suppose that for every subset S of $V \setminus \{0\}$ of cardinal n, there exists an operation g_S in $\langle \{g\} \cup \mathrm{Aut}(V;+) \rangle$ such that $h(x) = g_S(x)$ for all $x \in S \cup \{0\}$. Let S' be a finite subset of $V \setminus \{0\}$ of cardinal $n+1$. There exists $y \neq 0$ such that $S' = S \cup \{y\}$. So by induction, there exists g_S such that $h(x) = g_S(x)$ or all $x \in S \cup \{0\}$. If $g_S(y) = g(x_0)$, then we take $g_{S'} := g_S$ and we conclude the induction. If $g_S(y) \neq g(x_0)$, let α be a self-embedding of $(V;+)$ such that $\alpha(g_S(y)) = x_1$ and $\alpha(g_S(x)) = x_0$ for all $x \in S$. Such an α exists because $(V;+)$ is 2-transitive on $V \setminus \{0\}$. Indeed, g_S is constant on S, because h is constant, and $x_0 \neq x_1$. We now define $g_{S'} := g \circ \alpha \circ g_S$. It is straightforward to check that $g_{S'}(x) = g(x_0)$ for all $x \in S'$. Consequently, $h \in \overline{\langle \{g\} \cup \mathrm{Aut}(V;+) \rangle}$. □

Lemma 5.3.4. *Let Γ be a reduct of $(V;+)$ such that $0, \mathrm{Ind}_1 \in \langle \Gamma \rangle_{\mathrm{pp}}$, and such that there exists no binary polymorphism of Γ which is injective on $(V \setminus \{0\})^2$. Then either Γ is homomorphically equivalent to a two-element structure, or $(V \setminus \{0\}; P_3^{V\setminus\{0\}}, \neq)$ has a primitive positive interpretation in Γ.*

Proof. For every f in $\mathrm{Pol}(\Gamma)$, we define the operation $\phi(f)$ as the restriction of f on $V \setminus \{0\}$. Note that since f preserves 0 and Ind_1, the operation $\phi(f)$ is an operation of $V \setminus \{0\}$, i.e., if f has arity n, then $\phi(f)\colon (V \setminus \{0\})^n \to V \setminus \{0\}$. Let $\mathcal{C} := \{\phi(f) \mid f \in \mathrm{Pol}(\Gamma)\}$. The mapping ϕ is a surjective homomorphism of algebras between $(V; \mathrm{Pol}(\Gamma))$ and $(V \setminus \{0\}; \mathcal{C})$, hence $\mathcal{C}$ is in the pseudo-variety generated by $(V; \mathrm{Pol}(\Gamma))$.

First suppose that every binary operation of $\mathcal{C}$ is essentially unary. By Lemma 4.4.15, $\mathcal{C} \subseteq \mathrm{Pol}(V \setminus \{0\}; P_3^{V\setminus\{0\}})$. If every unary operation of Γ is injective, $\mathcal{C} \subseteq \mathrm{Pol}(V \setminus \{0\}; P_3^{V\setminus\{0\}}, \neq)$. Hence, by Theorem 4.3.3, $(V \setminus \{0\}; P_3^{V\setminus\{0\}}, \neq)$ has a primitive positive interpretation in Γ. Else, Γ has a non-injective endomorphism. So by Lemma 5.3.3, Γ is homomorphically equivalent to a two-element structure. Now suppose that there exists $f' = \phi(f) \in C$ such that $\phi(f)$ depends on both argument. We consider now the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ by pointwise convergence. Note that $\mathcal{C}$ is already closed by composition and contains all the projections. Hence, $\overline{C}$ is the smallest locally closed clone containing $\mathcal{C}$. By Theorem 4.1.10, we have $\overline{\mathcal{C}} = \mathrm{Pol}(\mathrm{Inv}(\mathcal{C}))$. Since 0 and Ind_1 are first-order definable on Γ, the automorphisms of $(V \setminus \{0\}; (\mathrm{Eq}_i)_{i\geq 2})$ are exactly the restriction of the automorphisms of $(V;+)$ over $V \setminus \{0\}$, so we have $\mathrm{Aut}(V \setminus \{0\}; (\mathrm{Eq}_i)_{i\geq 2}) \subseteq \mathrm{Pol}(\mathrm{Inv}(\mathcal{C}))$. Hence, $\mathrm{Inv}(\mathcal{C})$ is a reduct of $(V \setminus \{0\}; (\mathrm{Eq}_i^{\neq 0})_{i\geq 2})$ by Theorem 4.1.11, and by Lemma 3.2.39, $\mathrm{Inv}(\mathcal{C})$ is 2-transitive, as a reduct of a 2-transitive structure.

Since $\phi(f) \in \overline{\mathcal{C}}$, all operations of $\overline{\mathcal{C}}$ are not essentially unary, so $\overline{\mathcal{C}}$ contains either an injective binary operation, or a constant operation by Theorem 4.4.35. Suppose that $\overline{\mathcal{C}}$

contains an injective binary operation h'. Let $(v_0 = 0, v_1, \dots, v_n, \dots)$ be an enumeration of V. We denote by V_n the set $\{v_0, \dots, v_n\}$. Then there exists a family $(h_n)_{n\geq 0}$ of elements of $\mathrm{Pol}(\Gamma)$ such that for all n, $h'\restriction(V_n \setminus \{0\})^2 = h_n\restriction(V_n \setminus \{0\})^2$. We now prove by a König construction that there exists $h \in \mathrm{Pol}(\Gamma)$ which is injective over $(V \setminus \{0\})^2$. This will contradict the fact that there exists no binary polymorphism of Γ which is injective over $(V \setminus \{0\})^2$ by assumption.

Consider the following infinite tree $\mathcal{T}$ whose vertices lie on levels $1, 2, \dots$. The vertices at the n-th level are all the complete types over $(V; +)$ of tuples of the form $(f(v_i, v_j))_{0\leq i,j\leq n}$, with f being a binary polymorphism of Γ which satisfies $f(0, 0) = 0$, $f(x, y) \neq 0$ for all $x, y \neq 0$, and f injective on $(V_n \setminus \{0\})^2$. We say that a node N corresponding to a map f_N is a descendant of a node M corresponding to a map f_M if:

$$\mathrm{tp}_{(V;+)}(f_M(v_i, v_j))_{0\leq i,j\leq n} \subseteq \mathrm{tp}_{(V;+)}(f_N(v_i, v_j))_{0\leq i,j\leq n}$$

Note that $\mathcal{T}$ has finitely many vertices at each level since $(V; +)$ is ω-categorical, and so by Theorem 2.1.24, there is a finite number of distinct $(n + 1)$-types for all n. We have to prove that at each level n, there exists at least one node S, with a corresponding polymorphism g_S of Γ, which is injective over $(V_n \setminus \{0\})^2$, and such that $g_s(x, y) \neq 0$ for all $x, y \neq 0$.

The initialization step is straightforward. Now we prove that at each level n of the tree, there exists at least one node S, with a corresponding operation $h_S \in \mathrm{Pol}(\Gamma)$ from V^2 to V, which is injective on $(V \setminus \{0\})^2$, and such that $h_S(x, y) \neq 0$ for all $x, y \neq 0$. But this is straightforward since for all n, h_n is injective over $(V_n \setminus \{0\})^2$. Hence the tree $\mathcal{T}$ has an infinite branch $(h_{S_1}, \dots, h_{S_n}, \dots)$ by Theorem 2.4.14. Note that we can assume that h_{S_n} is not a type but an operation which satisfies the following: $h_{S_n}\restriction V_n = h_{S_{n+1}}\restriction V_n$ for all $n \geq 1$. Indeed, let g be an operation whose type over V_n belongs to a node S_n, and let h be an operation whose type over V_{n+1} belongs to a child node of S_ni.e., we have:

$$\mathrm{tp}_{(V;+)}(g(v_i, v_j))_{0\leq i,j\leq n} \subseteq \mathrm{tp}_{(V;+)}(h(v_i, v_j))_{0\leq i,j\leq n}$$

By ω-categoricity of $(V; +)$, there exists an automorphism α such that $\alpha \circ g(x) = h(x)$ for all $x \in V_n$ by Proposition 2.1.26. Hence, we define h as follows: $h(x) := \cup_{n\geq 1} h_{S_n}\restriction V_n$. Note that h belongs to $\mathrm{Pol}(\Gamma)$ as locally generated by polymorphisms of Γ. It is now straightforward to see that h is injective over $(V \setminus \{0\})^2$.

Consequently, $\overline{\mathcal{C}}$ contains a constant operation h'. Hence there exists a family $(h_n)_{n\geq 0}$ of elements of $\mathrm{Pol}(\Gamma)$ such that for all n, $h'\restriction(V_n \setminus \{0\})^2 = h_n\restriction(V_n \setminus \{0\})^2$. This choice is made possible by Proposition 2.1.26 since $(V; +)$ is ω-categorical. Then by a similar proof using König's Lemma, we prove that there exists $h \in \mathrm{Pol}(\Gamma)$ such that h is constant over $(V \setminus \{0\})^2$. Hence, $x \mapsto h(x, x)$ is an endomorphism of Γ whose image has exactly two elements, and Γ is homomorphically equivalent with a two-element structure by Lemma 5.3.3. $\square$

Definition 5.3.5. We denote by $\mathcal{Q}_m$ the set of all ternary operations h such that for all x, y, z, x', y', z' in V, we have the following:

- h is injective on $(V \setminus \{0\})^3$

- $h(x, y, z) = 0$ if and only if $x = y = z = 0$ or exactly one of x, y, z equals 0
- $h(x, y, z) = h(x', y', z')$ if and only if: $(x = x' \wedge y = y' \wedge z = z')$ or $h(x, y, z) = h(x', y', z') = 0$

Definition 5.3.6. We denote by $\mathcal{Q}_M$ the set of all ternary operations h such that for all x, y, z, x', y', z' in V, we have the following:

- h is injective on $(V \setminus \{0\})^3$
- $h(x, y, z) = 0$ if and only at least two of x, y, z are set to 0
- $h(x, y, z) = h(x', y', z')$ if and only if: $(x = x' \wedge y = y' \wedge z = z')$ or $h(x, y, z) = h(x', y', z') = 0$

Proposition 5.3.7. *Let Γ be a first-order reduct of $(V; +)$ such that $0, \mathrm{Ind}_1$, Ind_2, and E are* pp*-definable over Γ. Then at least one of the following cases holds:*

- *Γ is homomorphically equivalent to a two element structure;*
- 1IN3 *has a primitive positive interpretation in Γ;*
- *$(V \setminus \{0\}; P_3^{V\setminus\{0\}}, \neq)$ has a primitive positive interpretation in Γ;*
- *Γ has a polymorphism which belongs to $\mathcal{I}(\mathrm{inj}, \mathrm{inj})$;*
- *Γ has a polymorphism which belongs to $\mathcal{I}(0, 0)$;*
- *Γ has a polymorphism which belongs to $\mathcal{Q}_m$;*
- *Γ has a polymorphism which belongs to $\mathcal{Q}_M$.*

Proof. Since E is an equivalence relation pp-definable in Γ, E is preserved by every polymorphism of Γ. Hence, we can associate to every $f \in \mathrm{Pol}(\Gamma)$ the mapping $\overline{f}$ such that $\overline{f}(cl(x_1), \ldots, cl(x_n)) = \mathrm{cl}(f(x_1, \ldots, x_n))$ for all $x_1, \ldots, x_n \in V$. Note that we can consider the mapping $\overline{f}$ as an operation on $\{0, 1\}$, 0 being the class of 0, and 1 being the class of all the non-zero elements. Let $\mathcal{C}$ be the set defined as follow: $\mathcal{C} := \{g \in \mathrm{Op}(\{0, 1\}) \mid \exists f \in \mathrm{Pol}(\Gamma),\ g = \overline{f}\}$. One can easily see that since $\mathrm{Pol}(\Gamma)$ is a clone over V, $\mathcal{C}$ is a clone over $\{0, 1\}$. Since 0 and Ind_1 are *pp*-definable in Γ, every operation of $\mathcal{C}$ is idempotent, since every polymorphism of Γ has to preserve 0 and $\neq 0$.

First suppose that every operations of $\mathcal{C}$ preserve 1IN3. First note that the mapping ϕ which sends $f \in \mathrm{Pol}(\Gamma)$ to $\overline{f}$ such that $\overline{f}(cl(x_1), \ldots, cl(x_n)) = \mathrm{cl}(f(x_1, \ldots, x_n))$ for all $x_1, \ldots, x_n \in V$, is a homomorphism of algebras from $\mathrm{Pol}(\Gamma)$ to C. So C belongs to the pseudo-variety generated by $\mathrm{Pol}(\Gamma)$. Hence by Theorem 4.3.3, 1IN3 is has a primitive positive interpretation in Γ.

Suppose now that there exists an operation $\overline{f} \in \mathcal{C}$ which violates 1IN3. Since $\overline{f}$ is in $\mathcal{C}$, $\overline{f}$ is idempotent. Then by Lemma 4.4.9, $\overline{f}$ locally generates either min, max, minority, or majority. We now make a case distinction:

- Suppose that $\overline{f}$ locally generates min. Then $\min \in \mathcal{C}$, so there exists $h \in \text{Pol}(\Gamma)$ such that $\overline{h} = \min$. Recall that such a h must preserve 0, Ind_1, and E. So $h(x,0) = h(0,x) = 0$ for all $x \in V$, and $h(x,y) \neq 0$ for all $x,y \neq 0$. If there is no binary operation of $\text{Pol}(\Gamma)$ which is injective on $(V \setminus \{0\})^2$, then by Lemma 5.3.4, either Γ is homomorphically equivalent to a two element structures, or $(V \setminus \{0\}; P_3^{V\setminus\{0\}}, \neq)$ has a primitive positive interpretation in Γ. Suppose that there exists $g \in \text{Pol}(\Gamma)$ which is injective on $(V \setminus \{0\})^2$. Then by Proposition 5.2.4, there are three possibilities: if g belongs to $\mathcal{I}(\text{inj}, \text{inj})$ or $\mathcal{I}(0,0)$, we are done. The last remaining case is: g belongs to $\mathcal{I}(\text{inj}, 0)$ (the case $g \in \mathcal{I}(0, \text{inj})$ is analogous), i.e., $g(x,0) = 0$ and g is injective over $V \times V \setminus \{0\}$. In this case, let $h'(x,y) := h(g(x,y), g(y,x))$ for all x,y. It is straightforward to check that h' belongs to $\mathcal{I}(0,0)$ since Ind_2 is pp-definable in Γ and g is injective on $(V \setminus \{0\})^2$.

- Suppose now that $\overline{f}$ locally generates max. Similarly to the previous case, there exists $h \in \text{Pol}(\Gamma)$ such that $\overline{h} = \max$. Recall that such a h must preserve 0, Ind_1, and E. So $h(x,0) \neq 0$, and $h(0,x) \neq 0$ for all $x \in V$, and $h(x,y) \neq 0$ for all $x,y \neq 0$. As we did in the previous case, we can consider that there exists a binary polymorphism g such that $g(x,0) = 0$ for all $x \in V$, and g is injective on $V \times V \setminus \{0\}$. Then, if we define h' as follows: $h'(x,y) := h(g(x,y), g(y,x))$ for all x,y, we have h' is injective over $(V \setminus \{0\})$. Hence, by Corollary 5.2.5, h' together with $\text{Aut}(V;+)$ locally generates an operation of $\mathcal{I}(\text{inj}, \text{inj})$.

- Suppose now that $\overline{f}$ locally generates minority. Similarly to the previous cases, there exists $h \in \text{Pol}(\Gamma)$ such that $\overline{h} = \text{minority}$. Recall that such a h must preserve 0, Ind_1, and E. So $h(x,0,0) \neq 0$, $h(0,y,0) \neq 0$, $h(0,0,z) \neq 0$, $h(x,y,0) = h(x,0,z) = h(0,y,z) = 0$, and $h(x,y,z) \neq 0$ for all $x,y,z \neq 0$. As we did in the previous cases, we can consider that there exists a binary polymorphism g such that $g(x,0) = 0$ for all $x \in V$, and g is injective on $V \times V \setminus \{0\}$. Let $g'(x,y,z) := g(g(x,y),z)$ for all $x,y,z \in V$. The operation g' is a ternary polymorphism of Γ which is injective on $(V \setminus \{0\})^3$. We define h' as follows:

$$h'(x,y,z) := h(g(g(x,y),z), g(g(y,z),x), g(g(z,x),y)) \text{ for all } x,y,z.$$

  Note that we still have $\overline{h'} = \text{minority}$. Further more, h' is injective on $(V \setminus \{0\})^3$, since h' preserves Ind_2 and g is injective on $(V \setminus \{0\})^2$. We will prove that h' together with $\text{Aut}(V;+)$ locally generates a mapping h'' which belongs to $\mathcal{Q}_m$, i.e., a ternary operation such that $h''(0,0,0) = h''(x,y,0) = h''(x,0,z) = h''(0,y,z) = 0$ for all $x,y,z \neq 0$, and $x \mapsto h''(x,0,0)$, $y \mapsto h''(0,y,0)$, $z \mapsto h''(0,0,z)$ are injective, and h'' injective on $(V \setminus \{0\})^3$.

  Suppose that one of the following sets is infinite: $h'(V,0,0), h'(0,V,0)$, $h'(0,0,V)$, let say $h'(V,0,0)$. Then, composing h' with a well-chosen self-embedding of $(V;0)$, we have $x \mapsto h'(\alpha(x),0,0)$ is injective. So let h'' be the operation defined as follows:

$$h''(x,y,z) := h'(h'(x,y,z), h'(y,z,x), h'(z,x,y)) \text{ for all } x,y,z \in V$$

Then it is straightforward to check that h'' has all the required properties. Suppose now that $h'(V,0,0), h'(0,V,0), h'(0,0,V)$ are finite. Then there exists $k_1, k_2, k_3 \in \mathbb{N}$ and three self-embeddings $\alpha_1, \alpha_2, \alpha_3$ of $(V;0)$ such that $h'(\alpha_1(x),0,0) = k_1$, $h'(0,\alpha_2(x),0) = k_2$, and $h'(0,0,\alpha_3(x)) = k_3$ for all $x \in V$. Let $j(x,y,z) := h'(h'(x,y,z),x,x)$ for all $x,y,z \in V$. It is straightforward to check that $\overline{j} =$ minority, and $x \mapsto j(x,0,0)$ is injective, and j is injective on $(V \setminus \{0\})^3$, so we are back in the previous case, since $j(V,0,0)$ is now infinite.

- Finally suppose that $\overline{f}$ locally generates majority. Similarly to the previous cases, there exists $h \in \mathrm{Pol}(\Gamma)$ such that $\overline{h} =$ majority. Recall that such a h must preserve 0, Ind_1, and E. So $h(x,0,z) \neq 0$, $h(x,y,0) \neq 0$, $h(0,y,z) \neq 0$, $h(x,0,0) = h(0,0,z) = h(0,y,0) = 0$, and $h(x,y,z) \neq 0$ for all $x,y,z \neq 0$. As we did in the previous cases, we can consider that there exists a binary polymorphism g such that $g(x,0) = 0$ for all $x \in V$, and g is injective on $V \times V \setminus \{0\}$. Let $g'(x,y,z) := g(g(x,y),z)$ for all $x,y,z \in V$. The operation g' is a ternary polymorphism of Γ which is injective on $(V \setminus \{0\})^3$. Let h' be the following operation:

  $$h'(x,y,z) := h(g(g(x,y),z), g(g(y,z),x), g(g(z,x),y)) \text{ for all } x,y,z \in V$$

  Note that we still have $\overline{h'} =$ majority. Further more, h' is injective on $(V \setminus \{0\})^3$, since h' preserves Ind_2 and g is injective on $(V \setminus \{0\})^2$.

  We will use h' together with $\mathrm{Aut}(V;+)$ to locally generate an operation $h'' \in \mathcal{Q}_M$. Recall that such an operation h'' must satisfy the following: $\overline{h''} =$ majority, h'' is injective on $(V \setminus \{0\})^3$, and $(x,y) \mapsto h''(x,y,0)$, $(x,z) \mapsto h''(x,0,z)$, and $(y,z) \mapsto h''(0,y,z)$ are all three injective. Indeed, it is possible that, for instance, $(x,y) \mapsto h'(x,y)$ is not injective. So let us define the following operation:

  $$h'_1(x,y,z) := h'(h'(x,y,z), g(x,y), h'(y,x,z)) \text{ for all } x,y,z \in V$$

  It is straightforward to check that $\overline{h'_1} =$ majority, and that h'_1 is injective on $(V \setminus \{0\})^3$. But now, it is also true that $(x,y) \mapsto h'_1(x,y,0)$ is injective. But we also have to check that if $(x,z) \mapsto h'(x,0,z)$, or $(y,z) \mapsto h'(0,y,z)$ were injective, it is still true for h'_1. This is a straightforward verification which mainly use the fact h' is injective on $(V \setminus \{0\})^3$.

  Now, we only have to iterate this process two more times. Suppose indeed that $(y,z) \mapsto h'_1(0,y,z)$ is not injective. Note we are sure that $(x,y) \mapsto h'_1(x,y,0)$ is injective. Then we define $h'_2(x,y,z) := h'_1(h'_1(x,y,z), h'_1(x,z,y), g(y,z))$. Following the previous remark, having built h'_2 symmetrically with respect to h'_1, we have that $(x,y) \mapsto h'_2(x,y,0)$ is still injective. But this time, $(y,z) \mapsto h'_2(0,y,z)$ is also injective. It is straightforward to verify that the following operation belongs to $\mathcal{Q}_M$:

  $$h''(x,y,z) := h'_2(g(x,z), h'_2(x,y,z), h'_2(z,y,x))$$

$\square$

5.4 The $(V;0)$ Case

Notation 5.4.1. We denote by $(V;0)$ the structure of domain V with the constant 0.

This section opens the series of classification of reducts of $(V;+)$ which follows from Theorem 3.6.20. As we wrote in the introduction, CSPs of reducts of equality have already been classified by Bodirsky and Kára in [BK08a]. Hence it is a very natural continuation to add a constant and see how this new classification problem can be solved.

The P/NP-complete dichotomy result we establish for CSPs of reducts of $(V;0)$ is a joint work with Mottet. Intuitively, this classification is at least as rich as Schaefer's classification, since the boolean CSPs can be encoded in $(V;0)$. Interestingly, we prove that the fact that the domain is infinite does not bring other hardness cases than those originally present in Schaefer's Theorem and in the classification of reducts of equality.

5.4.1 Key Polymorphisms and Algorithms

Data: a pp-sentence $\Phi = \exists \overline{x}.\varphi$ in the language of Γ
Result: *accepts* if $\Gamma \models \Phi$, *rejects* otherwise
$\Psi := \emptyset$;
repeat
 $\Psi := \{C \in \Phi \mid C$ is a clause without negative literals$\}$;
 forall the *clauses of* Φ *of the type* $x = y$ **do**
  Replace every occurrence of y by x in Φ;
  Delete all the literals $x \neq x$ in Φ;
  Delete all the clauses containing a literal $x = x$ in Φ;
 end
 forall the *clauses of* Φ *of the type* $x = 0$ **do**
  Replace every occurrence of x by 0 in Φ;
  Delete all the literals $x \neq 0$ in Φ;
  Delete all the clauses containing a literal $0 = 0$ in Φ;
 end
 if Φ *is empty* **then**
  accept
 end
 if *one of the clauses of* Φ *is empty* **then**
  reject
 end
until Φ *doesn't change*;
accept

Algorithm 1: solves CSP(Γ) where $\mathrm{Pol}(\Gamma) \cap \mathcal{I}(\mathrm{inj}, \mathrm{inj}) \neq \emptyset$

Proposition 5.4.2. *Let* Γ *be a reduct of* $(V;0)$ *such that* $\mathrm{Pol}(\Gamma) \cap \mathcal{I}(\mathrm{inj},\mathrm{inj}) \neq \emptyset$. *Then Algorithm 1 solves* $\mathrm{CSP}(\Gamma)$ *in polynomial time.*

Proof. Let f be an element of $\mathrm{Pol}(\Gamma) \cap \mathcal{I}(\mathrm{inj},\mathrm{inj}) \neq \emptyset$. The operation f is clearly an embedding from $(V;0)^2$ to $(V;0)$. Since Γ is a reduct of $(V;0)$, every relation of Γ has a first-order definition in $(V;0)$. And since $(V;0)$ has quantifier elimination by Remark 2.2.12, and this definition can be assumed to be quantifier-free. Hence, by Theorem 4.1.17, every relation R of Γ has a quantifier-free Horn-definition over the language $(V;0)$, i.e., there exists $(\phi_{i,1}(\overline{x}),\ldots,\phi_{i,k_i}(\overline{x}))_{i\leq k}$ such that for all $\overline{v}$, $\Gamma \models R(v_1,\ldots,v_n)$ if and only if $(V;0) \models \bigwedge_{i\leq k}\bigvee_{j\leq k_i}\phi_{i,j}(\overline{v})$, with $\phi_{i,j}$ atomic formulas or negations of atomic formulas of $(V;0)$, such that for all $i \leq k$, at most one of the $\phi_{i,j}$ is positive.

Let $\phi : \exists\overline{x}. \bigwedge \psi_i(\overline{x})$ be an instance of $\mathrm{CSP}(\Gamma)$, and suppose that the algorithm rejects ϕ. Then, at the last step of the algorithm, one of the clauses is empty. But since we obtain a formula equi-satisfiable with ϕ at each step of the algorithm, then ϕ is not satisfiable.

Suppose now that $\mathrm{CSP}(\Gamma)$ accepts ϕ. Then ϕ is equi-satisfiable with a formula of the form:

$$\bigwedge_{j\leq n}(\bigwedge_{i\leq k_j} p_{i,j}(\overline{x}) \to p'(\overline{x})) \text{ with } p_{i,j}, p' \text{ being positive literals.}$$

Let Var be the set of free variables which appear in one of the $p_{i,j}$, and let δ be any injection from Var to $V \setminus \{0\}$. Then δ satisfies $\psi(\overline{x})$. Indeed, suppose for a contradiction that one of the clauses, indexed for instance by j_0, is not satisfied by δ. Then all the p_{i,j_0} are satisfied by δ, which contradicts the fact that all the $p_{i,j}$ are positive constraints of the form $x = 0$ or $x = y$, so they can not be satisfied by an injective mapping on $V \setminus \{0\}$. $\square$

Proposition 5.4.3. *Let* Γ *be a reduct of* $(V;0)$ *such that* $\mathrm{Pol}(\Gamma) \cap \mathcal{I}(0,0) \neq \emptyset$. *Then Algorithm 2 solves* $\mathrm{CSP}(\Gamma)$ *in polynomial time.*

Proof. Recall that $\mathrm{Ieq}_2(x,y)$ if and only if $x = y \neq 0$. Let f be an element of $\mathrm{Pol}(\Gamma) \cap \mathcal{I}(0,0) \neq \emptyset$. The operation f is clearly an embedding from the structure $(V;\mathrm{Ind}_1,\mathrm{Ieq}_2)^2$ to $(V;\mathrm{Ind}_1,\mathrm{Ieq}_2)$. Since $(V;0)$ has quantifier elimination by Remark 2.2.12, and since for all $x \in V$, $x = 0 \Leftrightarrow \neg\,\mathrm{Ind}_1(x)$, the structure $(V;\mathrm{Ind}_1,\mathrm{Ieq}_2)$ also has quantifier elimination. Hence, by Theorem 4.1.17, every relation R of Γ has a quantifier-free Horn definition, i.e., a formula in conjunctive normal form such that every clause has at most one literal of the form $x \neq 0$ or $x = y \neq 0$, and all the other literals are of the form $x \neq y$ or $x = 0$.

First note that it is allowed to use $\mathsf{Solve}\,\mathrm{CSP}((V,\mathrm{Inv}(\mathcal{I}(\mathrm{inj},\mathrm{inj}))))$ (i.e. Algorithm 1) on the entry $(\exists\overline{x}. \bigwedge \Psi)$, since every clause of Ψ is preserved by $\mathcal{I}(\mathrm{inj},\mathrm{inj})$.

Let $\phi : \exists\overline{x}. \bigwedge \psi_i(\overline{x})$ be an instance of $\mathrm{CSP}(\Gamma)$, and let Φ be initialized to the corresponding set of clauses. Note that at each step of the algorithm, the set of clauses Φ is equi-satisfiable with the original set of clauses Φ given as the instance of the CSP.

Suppose that the algorithm rejects ϕ. Then, at the last step of the algorithm, the set of clauses Φ is not empty, equi-satisfiable with ϕ, and not satisfiable. Hence, ϕ is not satisfiable.

Data: a pp-sentence $\Phi = \exists \overline{x}.\varphi$ in the language of Γ
Result: *accepts* if $\Gamma \models \Phi$, *rejects* otherwise
$\Psi := \emptyset$;
repeat
  $\Psi := \{C \in \Phi \mid C$ has no litteral of the form $x = 0\}$;
  if $\mathsf{Solve}\,\mathrm{CSP}(V, \mathrm{Inv}(\mathcal{I}(\mathrm{inj}, \mathrm{inj}))(\exists \overline{x}. \bigwedge \Psi)$ *rejects* **then**
    reject;
  end
  forall the *clauses* C *of* Φ *of the form* $q \vee \bigvee n_i \vee \bigvee z_j$ *with* q *of the form* $x \neq 0$ *or* $x = y \neq 0$, n_i *of the form* $x \neq y$, *and* z_j *of the form* $x = 0$ **do**
    forall the *literals* $x = 0$ *of the clause* C **do**
      if $\mathsf{Solve}\,\mathrm{CSP}((V, \mathrm{Inv}(\mathcal{I}(\mathrm{inj}, \mathrm{inj})))(\exists \overline{x}. \bigwedge \Psi \wedge x = 0)$ *rejects* **then**
        Delete all the clauses which contain the literal $x \neq 0$ from Φ;
        Delete all the literals $x = 0$ from Φ;
      end
    end
  end
  forall the *clauses* C *of* Φ *of the form* $x = y \neq 0$ **do**
    Replace every occurence of y by x in Φ;
    Add the clause $x \neq 0$ to Φ;
    Delete all the literals $x \neq x$ in Φ;
    Delete all the clauses containing a literal $x = x$ in Φ;
  end
until Φ *doesn't change*;
accept;

Algorithm 2: solves $\mathrm{CSP}(\Gamma)$ where $\mathrm{Pol}(\Gamma) \cap (\mathcal{I}(0, 0)) \neq \emptyset$

Suppose now that CSP(Γ) accepts ϕ. Then ϕ is equi-satisfiable with a CNF formula ϕ', which clauses are exactly the one contained in Φ at the end of the algorithm. The formula ϕ' is of the form:

$$\bigwedge M(\overline{x}) \wedge \bigwedge (\bigvee M'(\overline{x}) \vee \bigvee x_i = 0)$$

with M, M' being formulas of the form $p \vee \bigvee x_i \neq y_i$ with p of the form $x \neq 0$ or $x = y \neq 0$. We also know that, since Φ' is a fixed point of the process, for every clause C of the form $\bigvee M'(\overline{x}) \vee \bigvee x_i = 0$, there exists a mapping $\delta_C\colon \mathrm{Var} \to V$ such that one of the literal $x_i = 0$ is satisfied by δ_C, and δ also satisfies $\bigwedge M(\overline{x})$.

Let $g \in \mathrm{Pol}(\Gamma) \cap (\mathcal{I}(0,0))$, let Var be the set of free variables of Φ', and let δ be the mapping from Var to V defined as follows: $\delta := g(\delta_{C_1}, g(\delta_{C_2}, g(\delta_{C_3}, \ldots, g(\delta_{C_{n-1}}, \delta_{C_n}) \ldots))$. Then δ satisfies Φ'. Indeed, since every clause of the form $M(\overline{x})$ is preserved by g (indeed, each one of these clauses is pp-definable on Γ and $g \in \mathrm{Pol}(\Gamma)$), δ satisfies all the $M(\overline{x})$. Then since $g(0, x) = 0$ for all x, for each clause C of the form $\bigvee M'(\overline{x}) \vee \bigvee x_i = 0$, there exists i such that $\delta(x_i) = 0$. Consequently, all clauses of Φ' are satisfied by δ. And since Φ' is equi-satisfiable with Φ, the algorithm is valid. □

Definition 5.4.4. Let R be a n-ary relation over V. We define the equality-horn characterization of $R(\overline{x})$ as the set $\tau(R(\overline{x}))$ of formulas over the language of $(V, =)$ such that:

- for every $\phi(\overline{x}) \in \tau(R(\overline{x}))$, the free variables of ϕ belong to $\overline{x}$
- for every $\phi(\overline{x}) \in \tau(R(\overline{x}))$, ϕ is Horn, i.e., of the form:

$$(x_1 = x'_1 \wedge \cdots \wedge x_k = x'_k) \Rightarrow x_{k+1} = x'_{k+1}$$

- for every $\phi(\overline{x}) \in \tau(R(\overline{x}))$, we have: $(V, =) \models \forall \overline{x}.(R(\overline{x}) \Rightarrow \phi(\overline{x}))$
- for every Horn formula $\phi(\overline{x})$ over $(V, =)$, if $(V, =) \models \forall \overline{x}.(R(\overline{x}) \Rightarrow \phi(\overline{x}))$, then $\phi(\overline{x}) \in \tau(R)$.

Note that $\tau(R(\overline{x}))$ is finite for all R, since there is a finite number of free variables in $R(\overline{x})$.

Notation 5.4.5. Let $\zeta\colon V \to \{0,1\}$ be such that: $\zeta(x) = 0$ if $x = 0$, and $\zeta(x) = 1$ if $x \neq 0$. For any relation $R \subseteq V^n$, we denote by $\rho(R)$ the relation over $\{0,1\}$ defined as follows:

$$\rho(R) := \{(\zeta(x_1), \ldots, \zeta(x_n)) \mid (x_1, \ldots, x_n) \in R\}$$

Recall that we give the definitions of the classes of operations $\mathcal{Q}_m$ and $\mathcal{Q}_M$ in Definitions 5.3.5 and 5.3.6.

Proposition 5.4.6. *Let* Γ *be a reduct of* $(V; 0)$ *such that:*

$$\mathrm{Pol}(\Gamma) \cap (\mathcal{Q}_m \cup \mathcal{Q}_M) \neq \emptyset$$

Then the Algorithm 3 solves CSP(Γ) *in polynomial time.*

Data: a pp-sentence $\Phi = \exists \overline{x}.\varphi$ in the language of Γ
Result: *accepts* if $\Gamma \models \Phi$, *rejects* otherwise
$\Psi_H := \{\tau(R_i(\overline{x})) \mid R_i(\overline{x}) \text{ appears in } \Phi\}$;
repeat
  forall the *couples of variables* (x, y) *of* Φ **do**
    if $\mathsf{Solve}\,\mathrm{CSP}(V,=)(\exists \overline{x}. \bigwedge \Psi_H \wedge x \neq y)$ *rejects* **then**
      Replace every occurence of y by x in Ψ_H;
      Replace every occurence of y by x in Φ;
    end
  end
until Ψ_H *doesn't change*;
$\rho(\Phi) := \{S_i(\overline{x}) \mid S_i = \rho(R_i) \text{ for } R_i(\overline{x}) \text{ appearing in } \Phi\}$;
if $\mathsf{Solve}\,\mathrm{CSP}(\rho(\Gamma))(\exists \overline{x}'. \bigwedge \rho(\Phi)))$ *rejects* **then**
  reject;
end
accept;

Algorithm 3: solves CSP(Γ) where $\mathcal{Q}_m$ or $\mathcal{Q}_M$ in Pol(Γ)

Proof. First note that since $\mathrm{Pol}(\Gamma) \cap (\mathcal{Q}_m \cup \mathcal{Q}_M) \neq \emptyset$, either the majority function or minority function preserves $\rho(\Gamma)$ (here, $\rho(\Gamma)$ denotes the structure over the boolean domain obtained by projecting every relation of Γ on $\{0, 1\}$ using the mapping ρ). Hence, by Theorem 4.4.10, there exists an algorithm to solve $\rho(\Phi)$ over $\rho(\Gamma)$ in polynomial time. That is why we can have the following sub-routine in our algorithm: $\mathsf{Solve}\,\mathrm{CSP}(\rho(\Gamma))(\exists \overline{x}'. \bigwedge \rho(\Phi)))$ Also note that Φ is equi-satisfiable at each step of the algorithm. Hence, we suppose now that Φ is the set of clauses obtained after the first step of the algorithm.

Suppose that Φ is satisfiable, and let $(a_1, \ldots, a_n)$ be such that $\Gamma \models \Phi(a_1, \ldots, a_n)$. Then $(\zeta(a_1), \ldots, \zeta(a_n))$ is a solution of $\Psi_{0,1}$. Indeed, let $R(\overline{x})$ be any clause of Φ. Then $\Gamma \models R(\overline{a})$. Hence, the tuple $(\zeta(a_1), \ldots, \zeta(a_n))$ is in $\tau(R(\overline{x}))$. So Algorithm 3 accepts Φ

Conversely, suppose that Algorithm 3 accepts Φ. We show that there exists an assignation of the variables which satisfies Φ. Let $x_1, \ldots, x_n$ be the free variables of $\rho(\Phi)$. Since Algorithm 3 accepts Φ, then:

$$\mathsf{Solve}\,\mathrm{CSP}(\rho(\Gamma))(\exists \overline{x}'. \bigwedge \rho(\Phi)) \text{ accepts.}$$

So there exists $(b_1, \ldots, b_n) \in \{0, 1\}^n$ such that $\rho(\Gamma) \models \rho(\Phi)(\overline{b})$. Let $I := \{i \mid b_i \neq 0\}$, and let g be any injection from I to $V \setminus \{0\}$. We now define $\overline{a} \in V^n$ as follows: if $i \in I$, $a_i = g(i)$, and if $i \notin I$, $a_i = 0$. We now show that $\overline{a}$ satisfies Φ. Let $S(x_{i_1}, \ldots, x_{i_k})$ be a clause of Φ. Note that the free variables $x_{i_1}, \ldots, x_{i_k}$ are included in $x_1, \ldots, x_n$. We know that $\rho(\Gamma) \models \rho(S(b_{i_1}, \ldots, b_{i_k}))$, so there exists $(c_{i_1}, \ldots, c_{i_k}) \in S$ such that $(b_{i_1}, \ldots, b_{i_k}) = (\zeta(c_{i_1}), \ldots, \zeta(c_{i_k}))$. For the sake of the notations, we suppose that $c_{i_{j+1}} = \cdots = c_{i_k} = 0$ and $c_{i_1}, \ldots, c_{i_j}$ are all distinct from 0. Assume that $c_{i_1} = c_{i_2}$. Thanks to the first subroutine of the algorithm, we know that there exists a tuple $(c'_{i_1}, c'_{i_2}, \dots, c'_{i_k}) \in S$ such that $c'_{i_1} \neq c'_{i_2}$. We now make a case distinction:

- either there exists $h \in \mathrm{Pol}(\Gamma) \cap \mathcal{Q}_M \neq \emptyset$. In this case, since h is a polymorphism of S, we have $\Gamma \models S(h(c_{i_1}, c_{i_1}, c'_{i_1}), h(c_{i_2}, c_{i_2}, c'_{i_2}), \dots, h(c_{i_k}, c_{i_k}, c'_{i_k}))$. But since $h \in \mathcal{Q}_M$, we have $h(c_{i_l}, c_{i_l}, c'_{i_l}) = 0$ for all $j+1 \leq l \leq k$. Since h is also injective over $(V \setminus \{0\})^2$, we have $h(c_{i_1}, c_{i_1}, c'_{i_1}) \neq h(c_{i_2}, c_{i_2}, c'_{i_2})$. Consequently, we have:

$$(b_{i_1}, \dots, b_{i_k}) = (\zeta(h(c_{i_1}, c_{i_1}, c'_{i_1})), \dots, \zeta(h(c_{i_k}, c_{i_k}, c'_{i_k})))$$

  So iterating this process of "injectivization" of the tuple $\overline{c}$, we conclude that $\overline{a}$ satisfies $S(x_{i_1}, \dots, x_{i_k})$, by homogeneity of $(V; 0)$.

- or there exists $h \in \mathrm{Pol}(\Gamma) \cap \mathcal{Q}_m \neq \emptyset$. In this case, since h is a polymorphism of S, we have $\Gamma \models S(h(c_{i_1}, c'_{i_1}, c'_{i_1}), h(c_{i_2}, c'_{i_2}, c'_{i_2}), \dots, h(c_{i_k}, c'_{i_k}, c'_{i_k}))$. But since $h \in \mathcal{Q}_m$, we have $h(c_{i_l}, c'_{i_l}, c'_{i_l}) = 0$ for all $j+1 \leq l \leq k$. Since h is also injective over $(V \setminus \{0\})^2$, we have $h(c_{i_1}, c'_{i_1}, c'_{i_1}) \neq h(c_{i_2}, c'_{i_2}, c'_{i_2})$. Consequently, we have:

$$(b_{i_1}, \dots, b_{i_k}) = (\zeta(h(c_{i_1}, c'_{i_1}, c'_{i_1})), \dots, \zeta(h(c_{i_k}, c'_{i_k}, c'_{i_k})))$$

  So iterating this process of "injectivization" of the tuple $\overline{c}$, we conclude that $\overline{a}$ satisfies $S(x_{i_1}, \dots, x_{i_k})$, by homogeneity of $(V; 0)$.

Hence, in both cases, the tuple $\overline{a}$ satisfies the $S(x_{i_1}, \dots, x_{i_k})$. But this has been proven without any assumption on $S(x_{i_1}, \dots, x_{i_k})$. Consequently, $\overline{a}$ satisfies every clause of Φ. □

5.4.2 Classification for Reducts of $(V; 0)$

Recall that we already gave the following definitions:

- Let Ind_1 be the unary relation such that for all $x \in V$:

$$x \in \mathrm{Ind}_1 \text{ if and only if } x \neq 0$$

- Let Ind_2 be the binary relation such that for all $x, y \in V$:

$$(x, y) \in \mathrm{Ind}_2 \text{ if and only if } 0 \neq x \neq y \neq 0$$

- Let N be the binary relation such that for all $x, y \in V$:

$$(x, y) \in N \text{ if and only if } x = 0 \Leftrightarrow y \neq 0$$

Note that both Ind_1 and Ind_2 consist in only one orbit, and N consists in two orbits.

Proposition 5.4.7. *Let Γ be a first-order reduct of $(V; 0)$. Then at least one of the following holds:*

- Γ *is homomorphically equivalent to a first-order reduct of* $(V, =)$
- Γ *is homomorphically equivalent to an at most two elements structure*
- $0, \mathrm{Ind}_1, \mathrm{Ind}_2$*, and* E *are* pp*-definable over* Γ
- $0, \mathrm{Ind}_1$*, and* Ind_2 *are* pp*-definable over* Γ *but* N *is not.*

Proof. Let Γ be a first-order reduct of $(V; 0)$. It is a tautology to say that at least one of the following holds:

- 0 is not primitive positive definable on Γ
- $0 \in \langle\Gamma\rangle_{\mathrm{pp}}$ but Ind_1 is not pp-definable over Γ
- $0, \mathrm{Ind}_1 \in \langle\Gamma\rangle_{\mathrm{pp}}$ but Ind_2 is not pp-definable over Γ
- $0, \mathrm{Ind}_1, \mathrm{Ind}_2$, and N are pp-definable on Γ
- $0, \mathrm{Ind}_1, \mathrm{Ind}_2$ have pp-definitions over Γ, but $N \notin \langle\Gamma\rangle_{\mathrm{pp}}$

We use that tautology to distinguish the following cases:

- First suppose that 0 is not primitive positive definable on Γ. In this case, there exist an endomorphism γ of Γ which violates 0.
  - If $\gamma(V)$ is infinite, there exists a self-embedding α of $(V; 0)$ such that $\gamma \circ \alpha$ is an injective endomorphism of Γ which violates 0, and such that $0 \notin \gamma \circ \alpha(V)$. Let Γ' be the structure defined on $\gamma \circ \alpha(V)$ such that the relations R' of Γ' are the restriction of the relations R of Γ on $\mathrm{Dom}(\Gamma')$. Note that the identity is a homomorphism from Γ' to Γ by definition of Γ'. Also note that $\gamma \circ \alpha$ is a homomorphism from Γ to Γ', so Γ and Γ' are homomorphically equivalent. We now prove that Γ' is highly transitive, which is equivalent to: Γ' is a reduct of $(\mathrm{Dom}(\Gamma'), =)$. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be two n-tuples of distinct points of $\mathrm{Dom}(\Gamma')$. Let σ be a permutation of $\mathrm{Dom}(\Gamma')$ such that $\sigma(a_i) = b_i$ for all $i \leq n$ (such a permutation exists since $\mathrm{Dom}(\Gamma')$ is infinite). Since $0 \notin \mathrm{Dom}(\Gamma') \subseteq V$, there exists an automorphism β of $(V; 0)$ such that $\sigma \subseteq \beta$. Note that since Γ is a first-order reduct of $(V; 0)$, β is also an automorphism of Γ. It is now straightforward to prove that σ is an automorphism of Γ' by definition of Γ'. Indeed, if $(x_1, \ldots, x_n) \in R'$, then $(x_1, \ldots, x_n) \in R$, so $(\beta(x_1), \ldots, \beta(x_n)) \in R$ because β is an automorphism of Γ. But since $(\beta(x_1), \ldots, \beta(x_n)) = (\sigma(x_1), \ldots, \sigma(x_n))$, we have $(\sigma(x_1), \ldots, \sigma(x_n)) \in R$. Finally, since $\sigma(\mathrm{Dom}(\Gamma')) \subseteq \mathrm{Dom}(\Gamma')$, we have $(\sigma(x_1), \ldots, \sigma(x_n)) \in R'$. Now suppose that $(x_1, \ldots, x_n) \notin R'$. Following the same reasoning, we conclude that $(\sigma(x_1), \ldots, \sigma(x_n)) \notin R'$.

– Now, if $\gamma(V)$ is finite, there exists $k \in V$ such that $\gamma^{\langle -1 \rangle}(k)$ is infinite. Let α be any self-embedding of $(V;0)$ such that $\alpha(V \setminus \{0\}) \subseteq \gamma^{\langle -1 \rangle}(k)$. Then $\gamma \circ \alpha$ is an endomorphism of Γ whose image has at most 2 elements. Let Γ' be the structure defined on $\gamma \circ \alpha(V)$ such that the relations R' of Γ' are the restriction of the relations R of Γ on $\mathrm{Dom}(\Gamma')$. Note that the identity is a homomorphism from Γ' to Γ by definition of Γ'. Also note that $\gamma \circ \alpha$ is a homomorphism from Γ to Γ', so Γ and Γ' are homomorphically equivalent. Hence, Γ is homomorphically equivalent to an at most two elements structure.

- Now suppose that $0 \in \langle \Gamma \rangle_{\mathrm{pp}}$ but $\mathrm{Ind}_1 \notin \langle \Gamma \rangle_{\mathrm{pp}}$. In this case, there exists an endomorphism γ of Γ, and $x_0 \in V \setminus \{0\}$ such that $\gamma(x_0) = \gamma(0) = 0$. Then it is straightforward to prove that γ together with $\mathrm{Aut}(V;0)$ locally generates the constant function $\gamma_0(x) = 0$ for all $x \in V$. We give the first step of the induction. Let $x_1 \in V$ such that $\gamma(x_1) \neq 0$. There exists an automorphism of $(V;0)$ such that $\beta(x_1) = x_0$. Then $\gamma \circ \beta(x_1) = \gamma \circ \beta(x_0) = \gamma \circ \beta(0) = 0$. Consequently, Γ is homomorphically equivalent to a one element structure.

- Now suppose that $0, \mathrm{Ind}_1$ are in $\langle \Gamma \rangle_{\mathrm{pp}}$, but Ind_2 is not pp-definable in Γ. Since Ind_2 consists in only one orbit and is not in $\langle \Gamma \rangle_{\mathrm{pp}}$, there exists an endomorphism g of Γ which violates Ind_2. Hence, there exists two distinct non-zero elements of V x, x' such that $g(x) = g(x')$ or $0 \in \{g(x), g(x')\}$. Since Ind_1 is preserved by g, we have: $g(x) = g(x') \neq 0$. Hence, Γ has a non injective endomorphism. Then by Lemma 5.3.3, Γ is homomorphically equivalent to a 2-element structure.

- We can now suppose that $0, \mathrm{Ind}_1, \mathrm{Ind}_2 \in \langle \Gamma \rangle_{\mathrm{pp}}$. Suppose also that N is pp-definable over Γ. Then E is *pp*-definable over Γ since $E(x,y)$ if and only if $\exists z(N(x,z) \wedge N(y,z))$.

- Finally, $0, \mathrm{Ind}_1, \mathrm{Ind}_2 \in \langle \Gamma \rangle_{\mathrm{pp}}$, but $N \notin \langle \Gamma \rangle_{\mathrm{pp}}$.

□

Proposition 5.4.8. *Let* Γ *a first-order reduct of* $(V;0)$ *which is homomorphically equivalent to a first-order reduct* Γ' *of* $(V,=)$. *Then* $\mathrm{CSP}(\Gamma)$ *is either in P or NP-complete.*

Proof. By Proposition 4.4.3, $\mathrm{CSP}(\Gamma)$ equals $\mathrm{CSP}(\Gamma')$. But since Γ' is a reduct of $(V,=)$, $\mathrm{CSP}(\Gamma)$ is either in P or NP-complete, by Theorem 4.4.24. □

Theorem 5.4.9. *Let* Γ *be a reduct of* $(V;0)$, *then* $\mathrm{CSP}(\Gamma)$ *is either in P or NP-complete.*

Proof. By Proposition 5.4.7, one of the following holds:

- Γ is homomorphically equivalent to a first-order reduct of $(V,=)$. Then by Proposition 5.4.8, we have the dichotomy result for $\mathrm{CSP}(\Gamma)$.

- Γ is homomorphically equivalent to an at most two elements structure. Then by Proposition 4.4.12, we have: $\mathrm{CSP}(\Gamma)$ is in P or NP-complete.

- $0, \mathrm{Ind}_1, \mathrm{Ind}_2$ and E are pp-definable over Γ. Then by Proposition 5.3.7, one of the following cases holds:
  - Γ is homomorphically equivalent to a two element structure. In this case, CSP(Γ) is either in P or NP-complete by Proposition 4.4.12.
  - 1IN3 has a primitive positive interpretation in Γ; in this case, by Theorem 4.4.4 combined with Corollary 4.4.11, CSP(Γ) is NP-hard. Finally, since Γ is a reduct of $(V;+)$, CSP(Γ) is NP-complete by Proposition 5.1.2.
  - $(V \setminus \{0\}; P_3^{V\setminus\{0\}}, \neq)$ has a primitive positive interpretation in Γ; in this case, by Theorem 4.4.4 and Corollary 4.4.20, CSP(Γ) is NP-hard. Finally, since Γ is a reduct of $(V;+)$, CSP(Γ) is NP-complete by Proposition 5.1.2.
  - Γ has a polymorphism which belongs to $\mathcal{I}(\mathrm{inj}, \mathrm{inj})$; in this case, we solve CSP(Γ) in polynomial time with Algorithm 1.
  - Γ has a polymorphism which belongs to $\mathcal{I}(0,0)$; in this case, we solve CSP(Γ) in polynomial time with Algorithm 2.
  - Γ has a polymorphism which belongs to $\mathcal{Q}_m$; in this case, we solve CSP(Γ) in polynomial time with Algorithm 3.
  - Γ has a polymorphism which belongs to $\mathcal{Q}_M$; in this case, we solve CSP(Γ) in polynomial time with Algorithm 3.
- $0, \mathrm{Ind}_1$, and Ind_2 are pp-definable over Γ but N is not. In this case, by Proposition 5.3.2, one of the following cases holds:
  - Γ has no injective binary polymorphism on $(V \setminus \{0\})^2$; then by Lemma 5.3.4, $(V \setminus \{0\}; P_3^{V\setminus\{0\}}, \neq)$ has a primitive positive interpretation in Γ. And by Theorem 4.4.4 and Corollary 4.4.20, CSP(Γ) is NP-hard. Finally, since Γ is a reduct of $(V;+)$, CSP(Γ) is NP-complete by Proposition 5.1.2.
  - Γ has a polymorphism which belongs to $\mathcal{I}(\mathrm{inj}, \mathrm{inj})$; in this case, we solve CSP(Γ) in polynomial time with Algorithm 1.
  - Γ has a polymorphism which belongs to $\mathcal{I}(0,0)$; in this case, we solve CSP(Γ) in polynomial time with Algorithm 2.

$\square$

5.5 Solving Equations on $V \setminus \{0\}$

In this section, we give a classification of CSPs over reducts Γ of $(V;+)$ whose model-complete core Γ' satisfies $\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$. We start by stating a general lemma which helps us understanding binary polymorphism of $(V;+)$. In particular, a binary operation $g(x,y)$ preserving Ieq_3 is in fact the sum $\alpha(x) + \beta(y)$ of two self-embeddings α, β of $(V;+)$.

Proposition 5.5.1. *Let g be a binary operation from $V^2 \longrightarrow V$ which preserves* Ieq_3*. Then there exist two endomorphisms α, β of $(V;+)$ such that:*

$$g(x,y) = \alpha(x) + \beta(y) \textit{ for all } x, y \neq 0$$

Furthermore, we have $\alpha(V) \cap \beta(V) = \{0\}$, and either α or β belongs to $\mathrm{Emb}(V;+)$.

Proof. Let a, b be two non-zero elements of V. We define $\alpha(x) := g(x+a,b) + g(a,b)$ for all $x \neq 0$, and $\alpha(0) = 0$. We first prove that α does not depend on the choice of a, b. Indeed, let $\alpha'(x) := g(x+a',b') + g(a',b')$ for all x, with $a' \notin \{0,a\}$ and $b' \notin \{0,b\}$, we show that $\alpha(x) = \alpha'(x)$ for all $x \in V \setminus \{0,a,a'\}$.

$$\begin{aligned}
\alpha(x) + \alpha'(x) &= g(x+a,b) + g(a,b) + g(x+a',b') + g(a',b') \\
&= g(x+a,b) + g(x+a',b') + g(a,b) + g(a',b') \\
&= g(a+a',b+b') + g(a+a',b+b') \text{ since } g \text{ preserves } \mathrm{Eq}_3^{\neq 0} \\
&= 0
\end{aligned}$$

Now we show that α is a self-embedding of $(V;+)$. Recall that α does not depend on the choice of a, b. Hence, we have the following:

$$\begin{aligned}
\alpha(x_1) + \alpha(x_2) &= g(x_1+a,b) + g(a,b) + g(x_2+a+a',b') + g(a+a',b') \\
&= g(x_1+a,b) + g(x_2+a+a',b') + g(a,b) + g(a+a',b') \\
&= g(x_1+x_2+a',b+b') + g(a',b+b') \\
&= \alpha(x_1+x_2) \text{ since } \alpha \text{ does not depend on the choice of } a, b
\end{aligned}$$

Consequently, α is an endomorphism of $(V;+)$. The proof for β can be handled the same way. We only have to define β as follows: $\beta(y) := g(a, y+b) + g(a,b)$ for all $y \in V$. We now check that $g(x,y) = \alpha(x) + \beta(y)$ for all $x, y \neq 0$. We have:

$$\begin{aligned}
\alpha(x) + \beta(y) &= g(x+a,b) + g(a,b) + g(a,y+b) + g(a,b) \\
&= g(x+a,b) + g(a,y+b) \\
&= g(x,y) \text{ since } g \text{ preserves } \mathrm{Ieq}_3
\end{aligned}$$

Finally, since Ieq_3 pp-defines Ind_2, we have $\alpha(x_1) + \beta(y_1) \neq \alpha(x_2) + \beta(y_2)$ whenever $\mathrm{Ind}_2(x_1,x_2)$ and $\mathrm{Ind}_2(y_1,y_2)$. So $\alpha(x_1+x_2) \neq \beta(y_1+y_2)$. As a consequence, $\alpha(x) \neq \beta(y)$ for all $x, y \neq 0$. Consequently, $\alpha(V) \cap \beta(V) = \{0\}$. Since g preserves Ieq_3, we have $\mathrm{Ieq}_3(g(x_1,y_1), g(x_2,y_2), g(x_1+x_2,y_1+y_2))$ for all $x_1 \neq x_2, y_1 \neq y_2$ elements of $V \setminus \{0\}$. Hence, if there exists $x_1 \neq x_2$ such that $\alpha(x_1) = \alpha(x_2)$, we must have $\beta(y_1) \neq \beta(y_2)$ for all $y_1 \neq y_2$. Consequently, if α is not injective, then β is injective, and conversely, if β is not injective, then α is. So either α or β belongs to $\mathrm{Emb}(V;+)$. □

The following corollary generalizes the Proposition 5.5.1 to arbitrary finite arities:

Corollary 5.5.2. *Let g be a n-ary operation which preserves* Ieq_3*, then there exist n endomorphisms $\alpha_1, \ldots, \alpha_n$ of V such that:*

$$g(x_1, \dots, x_n) = \alpha_1(x_1) + \cdots + \alpha_n(x_n) \textit{ for all } \overline{x} \in (V \setminus \{0\})^n$$

Furthermore, $\alpha_1(V \setminus \{0\}), \dots, \alpha_n(V \setminus \{0\})$ *are pairwise disjoint and at least one of the* α_i *belongs to* $\mathrm{Emb}(V; +)$*.*

Proof. The proof is a straightforward generalization of the proof of Proposition 5.5.1. We only have to define α_i as follows:

$$\text{for all } x \in V,\ \alpha_i(x) := g(a_1, \dots, a_{i-1}, x + a_i, a_{i+1}, \dots, a_n) + g(a_1, \dots, a_n)$$

□

The following result is unlocked by the corollary we just stated. Recall that we already stated it without proving it in Remark 3.2.20. It echoes Lemma 3.2.19 but allows us to drop the hypothesis of Ind_j being also preserved.

Corollary 5.5.3. *The relation* Ieq_3 pp*-defines* Ieq_n *for all* $n \geq 2$*.*

Proof. By Theorem 4.1.11, we only have to prove that every polymorphism of V which preserves Ieq_3 also preserves Ieq_n for all $n \geq 2$. Let g be an k-ary operation from $V^k \longrightarrow V$ which preserves Ieq_3. By Corollary 5.5.2, there exist $\alpha_1, \dots, \alpha_k \in \mathrm{End}(V; +)$ such that $g(x_1, \dots, x_k) = \sum_{i \leq k} \alpha_i(x_i)$ for all $\overline{x} \in (V \setminus \{0\})^k$, and at least one of the α_i belongs to $\mathrm{Emb}(V; +)$. Furthermore, $\alpha_1(V \setminus \{0\}), \dots, \alpha_n(V \setminus \{0\})$ are pairwise disjoint. It now straightforward to check that g preserves Ieq_n for all $n \geq 2$. □

Recall that we defined the relation $\mathrm{Eq}_n^{\neq 0}$ as follows:

$$\mathrm{Eq}_n^{\neq 0}(x_1, \dots, x_n) \text{ if and only if } \mathrm{Eq}_n(x_1, \dots, x_n) \text{ and } x_j \neq 0 \text{ for all } 1 \leq j \leq n$$

Corollary 5.5.4. *Let* g *be a binary injection from* $(V \setminus \{0\})^2 \longrightarrow V \setminus \{0\}$ *which preserves* Ieq_3*. Then* g *preserves* $\mathrm{Eq}_n^{\neq 0}$ *for all* $n \geq 2$*, and:*

$$g \textit{ is an embedding from } (V \setminus \{0\}); (\mathrm{Eq}_n^{\neq 0})_{n \geq 2})^2 \longrightarrow (V \setminus \{0\}); (\mathrm{Eq}_n^{\neq 0})_{n \geq 2})$$

Proof. We already know that g preserves $\mathrm{Ieq}_3 = \mathrm{Eq}_3^{\neq 0}$. Hence by Lemma 3.2.6, g preserves $\mathrm{Eq}_n^{\neq 0}$ for all $n \geq 2$. We now prove that g is an embedding from $(V \setminus \{0\}); (\mathrm{Eq}_n^{\neq 0})_{n \geq 2})^2$ to $(V \setminus \{0\}); (\mathrm{Eq}_n^{\neq 0})_{n \geq 2})$. Let n be an integer, and $\overline{x}$ be a tuple such that $\neg\, \mathrm{Eq}_n^{\neq 0}(\overline{x})$. We prove that for all n-tuple $\overline{y}$ of non-zero elements of V, we have:

$$\neg\, \mathrm{Eq}_n^{\neq 0}(g(x_1, y_1), \dots, g(x_n, y_n))$$

By Proposition 5.5.1, there exist $\alpha, \beta \in \mathrm{End}(V; +)$ such that $g(x, y) = \alpha(x) + \beta(y)$ for all $x, y \neq 0$, and such that $\alpha(V) \cap \beta(V) = \{0\}$. But since g is injective, we have $\alpha, \beta \in \mathrm{Emb}(V; +)$. Indeed, if $\alpha(x_1) = \alpha(x_2)$ for some $x_1 \neq x_2$, then $g(x_1, y) = g(x_2, y)$, a contradiction with the injectivity of g. Since $\sum_{i \leq n} x_i \neq 0$, we have $\sum_{i \leq n} \alpha(x_i) = \alpha(\sum_{i \leq n} x_i) \neq 0$. And since $\alpha(V) \cap \beta(V) = \{0\}$ and $g(x_1, y_1) + \cdots + g(x_n, y_n) = \alpha(\sum_{i \leq n} x_i) + \beta(\sum_{i \leq n} y_i)$, we have: $\sum_{i \leq n} g(x_i, y_i) \neq 0$. Hence:

$$\neg\, \mathrm{Eq}_n^{\neq 0}(g(x_1, y_1), \dots, g(x_n, y_n))$$

□

Algorithm: Horn-Ieq$_3$ without 0

Data: a conjunction of Horn $\{(\mathrm{Eq}_n^{\neq 0})_{n\geq 2}\}$-formulas $\Phi := \bigwedge H_k$

$\Psi := \emptyset$;

repeat

  $\Psi := \{C \in \Phi \mid C$ is a clause without negative literals$\}$;

  forall the *negative literals* l *of* Φ **do**

    if *Gauss method gives no solution on* $V \setminus \{0\}$ *to the set of equations* $\{\Psi, l\}$ **then**

      Delete all literals l appearing in Φ

    end

  end

  if *one of the clauses of* Φ *is empty* **then**

    reject

  end

until Φ *doesn't change*;

accept

Proposition 5.5.5. *Let Γ' be a finitely relational structure such that Γ' has a binary injective polymorphism f, and such that:*

$$\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$$

Then $\mathrm{CSP}(\Gamma')$ *is polynomial-time tractable using Algorithm 4.*

Proof. First note that since $\mathrm{End}(\Gamma') = \mathrm{End}(V\setminus\{0\}; \mathrm{Ieq}_3)$, we have by Proposition 3.6.14:

$$\mathrm{Aut}(\Gamma') = \mathrm{Aut}(V \setminus \{0\}; \mathrm{Ieq}_3) = \mathrm{Aut}(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$$

By Proposition 3.6.14, $(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$ is homogeneous and ω-categorical as a reduct of an ω-categorical structure and hence, it admits quantifier elimination by Theorem 2.2.13. Consequently, every relation R of Γ' has a quantifier free definition over $(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$ by Theorem 4.1.11.

Since f is an injective binary polymorphism of Γ' which preserves Ieq_3, f is an embedding from $(V\setminus\{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})^2 \to (V\setminus\{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$ by Corollary 5.5.4. Hence, by Theorem 4.1.17, every relation R of Γ' has a quantifier-free Horn definition on the language of $(V\setminus\{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$. And if ϕ is a primitive positive formula on the language of Γ', then its translation in the language of $(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$ is a conjunction of quantifier-free Horn formulas preceded by existential quantifiers. We denote this last formula by $\Lambda(\phi)$. We can thus treat an instance of $\mathrm{CSP}(\Gamma')$ as a conjunctions of quantifier free Horn formulas on the language of $(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$, and give it as an input to Algorithm 4. Note that since the quantifier-free Horn definition of each relation R of Γ' is pre-calculated, we can build $\Lambda(\phi)$ in linear time.

We now prove that, given a primitive positive formula ϕ on the language of Γ', Algorithm 4 runs in polynomial time and accepts $\Lambda(\phi)$ if and only if $\Gamma' \models \phi$. First note that Φ is equi-satisfiable at each step of the algorithm. Suppose that $\Gamma' \models \phi$. Then there exists a tuple $\overline{a}$ of elements of $V \setminus \{0\}$ such that $(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2}) \models \Lambda(\phi)(\overline{a})$, by definition of $\Lambda(\phi)$. But since Φ is equi-satisfiable at each step of the algorithm, the algorithm accepts $\Lambda(\phi)$.

Conversely, suppose that Algorithm 4 accepts $\Lambda(\phi)$. Let us consider the sets Φ and Ψ at the last step of the algorithm. For every negative literal l in a clause $C \in \Phi$, we know that the system of equations $\{\Psi, l\}$ has solution thanks to Gauss polynomial method, i.e., there exists an assignation $\overline{a}_l$ of the variables of Φ such that $(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2}) \models \Psi(\overline{a}_l) \wedge l(\overline{a}_l)$. Let $l_1, \dots, l_j$ be all the distinct negative literals appearing in clauses of Φ. We now consider the assignation $\overline{b} := g(\overline{a}_{l_1}, g(\overline{a}_{l_2}, g(\overline{a}_{l_3}, \dots, g(\overline{a}_{l_{j-1}}, \overline{a}_{l_j}) \dots)$. Since g strongly preserves $\mathrm{Eq}_n^{\neq 0}$ for all $n \geq 2$, we easily see that $\overline{b}$ satisfies every clause C of Φ. So $\Lambda(\phi)$ is satisfiable, and $\Gamma' \models \phi$. □

Theorem 5.5.6. *Let Γ be a first-order reduct of $(V;+)$ such that the model-complete core Γ' of Γ satisfies:* $\mathrm{End}(\Gamma') = \mathrm{End}(V \setminus \{0\}; \mathrm{Ieq}_3)$*. Then* $\mathrm{CSP}(\Gamma)$ *is either polynomial-time tractable, or NP-complete.*

Proof. By Proposition 4.4.3, $\mathrm{CSP}(\Gamma)$ equals $\mathrm{CSP}(\Gamma')$. So we can focus on studying the complexity of $\mathrm{CSP}(\Gamma')$. Note that since $\mathrm{Aut}(\Gamma') = \mathrm{Aut}(V \setminus \{0\}; (\mathrm{Eq}_n^{\neq 0})_{n\geq 2})$, Γ' is 2-transitive by Lemma 3.2.39. So by Theorem 4.4.35, one of the following cases holds:

- Γ' has a constant polymorphism. In this case, $\mathrm{CSP}(\Gamma')$ is in P since the associated constant tuple is a solution of every instance.

- Every polymorphism of Γ' is essentially unary, and $\neq \in \langle \Gamma' \rangle_{\mathrm{pp}}$. In this case, P_3^B is also in $\langle \Gamma' \rangle_{\mathrm{pp}}$ by Proposition 4.4.16, so $\mathrm{Pol}(\Gamma') \subseteq \mathrm{Pol}(V; P_3^B, \neq)$. Hence, by Proposition 4.4.6 and Corollary 4.4.20, $\mathrm{CSP}(\Gamma')$ is NP-hard. Finally, since Γ is a reduct of $(V;+)$, $\mathrm{CSP}(\Gamma)$ is NP-complete by Proposition 5.1.2.

- Γ' has a binary injective polymorphism. Then by Proposition 5.5.5, $\mathrm{CSP}(\Gamma)$ is in P, and can be solved using Algorithm 4.

□

5.6 The Affine Case

In this section, we classify CSPs of reducts of $(V;+)$ whose model-complete core Γ' satisfies $\mathrm{End}(\Gamma') = \mathrm{End}(V; \mathrm{Ieq}_4, \neq)$. We then state a more general corollary classifying the complexity of CSPs of reducts of $(V; \mathrm{Ieq}_4)$.

We start by establishing the 3-transitivity property of $\mathrm{Aut}(V; \mathrm{Ieq}_4)$.

Lemma 5.6.1. *The automorphism group of* $(V; \mathrm{Ieq}_4^{\mathrm{inj}})$ *is equal to the automorphism group of* $(V; \mathrm{Ieq}_4)$*, and it is* 3*-transitive.*

Proof. First note that $(V; \mathrm{Ieq_4}^{\mathrm{inj}})$ and $(V; \mathrm{Ieq_4})$ are first-order interdefinable, using $\neq$. So by Theorem 4.1.11, $\mathrm{Aut}(V; \mathrm{Ieq_4}^{\mathrm{inj}}) = \mathrm{Aut}(V; \mathrm{Ieq_4})$. Now let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two pairs of pairwise distinct elements of V. Let c be a vector of V such that:

$$c \notin \mathrm{Vect}(x_1, x_2, y_1, y_2, z_1, z_2)$$

Such a c exists because $(V; +)$ has infinite dimension. Then the two families $(x_1 + c, y_1 + c, z_1 + c)$ and $(x_2 + c, y_2 + c, z_2 + c)$ are both linearly independent. So there exists $\alpha \in \mathrm{Aut}(V; +)$ such that $\alpha.(x_1+c, y_1+c, z_1+c) = (x_2+c, y_2+c, z_2+c)$, and consequently, if we denote by t_c the translation of vector c, we have $(t_c \circ \alpha \circ t_c).(x_1, y_1, z_1) = (x_2, y_2, z_2)$. By Theorem 3.5.8, $\mathrm{Aut}(V; \mathrm{Ieq_4}) = \langle \mathrm{Aut}(V; +) \cup \{t_a\} \rangle$ for any $a \neq 0$, hence $t_c \circ \alpha \circ t_c \in \mathrm{Aut}(V; \mathrm{Ieq_4})$. □

Algorithm: Horn-Aff

Data: a conjunction of Horn formulas $\Phi := \bigwedge H_k$ in the language of $(V; (\mathrm{Eq}_{2i})_{i \geq 1})$

$\Psi := \emptyset$;

repeat

  $\Psi := \{C \in \Phi \mid C$ is a clause without negative literals$\}$;

  forall the *negative literals* l *of* Φ **do**

    if *Gauss method gives no solution on* V *to the set of equations* $\{\Psi, l\}$ **then**

      Delete all literals l appearing in Φ

    end

  end

  if *one of the clauses of* Φ *is empty* **then**

    reject

  end

until Φ *doesn't change*;

accept

Proposition 5.6.2. *Let* Γ *be a first-order reduct of* $(V; +)$ *such that* Γ *has a binary injective polymorphism* f *and such that:*

$$\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq_4}^{\mathrm{inj}})$$

Then $\mathrm{CSP}(\Gamma)$ *is polynomial-time tractable using Algorithm 5.*

Proof. First note that $\mathrm{End}(V; (\mathrm{Eq}_{2i}^{\mathrm{inj}})_{i \geq 1}) = \mathrm{End}(V; (\mathrm{Eq}_{2i})_{i \geq 1}, \neq)$ by Lemmas 3.2.10 and 3.2.11, and Theorem 4.1.11. Hence, $\mathrm{End}(\Gamma) = \mathrm{End}(V; (\mathrm{Eq}_{2i}^{\mathrm{inj}})_{i \geq 1})$ by Proposition 3.6.18, and since for all $i \geq 1$, $\mathrm{Eq}_{2i}^{\mathrm{inj}}$ is only one orbit of the action of $\mathrm{Aut}(V; \mathrm{Ieq_4})$ on V, $\mathrm{Eq}_{2i}^{\mathrm{inj}}$ has a primitive positive definition over Γ by Corollary 4.1.13. Hence, every polymorphism of Γ, and f in particular, preserves $\mathrm{Eq}_{2i}^{\mathrm{inj}}$ for all $i \geq 1$. But since f preserves $\mathrm{Eq}_{2i}^{\mathrm{inj}}$ for all $i \geq 1$, it preserves in particular $\mathrm{Eq}_{8}^{\mathrm{inj}}$. Consequently, f preserves Eq_{2i}

for all $i \geq 1$, by Lemma 3.2.11. Moreover, $\neq$ is also preserved by every polymorphism of Γ, since $\neq$ is pp-definable over $(V; \mathrm{Eq}_4^{\mathrm{inj}})$ as follows, :

$$x \neq y \Leftrightarrow \exists u, v.\, \mathrm{Eq}_4^{\mathrm{inj}}(x, y, u, v)$$

Since $\mathrm{End}(\Gamma) = \mathrm{End}(V; (\mathrm{Eq}_{2i})_{i\geq 1}, \neq)$, every relation of Γ has an existential positive definition in $(V; (\mathrm{Eq}_{2i})_{i\geq 1}, \neq)$, and since $(V; (\mathrm{Eq}_{2i})_{i\geq 1}, \neq)$ has quantifier elimination by Theorem 2.2.13 and Proposition 3.6.18, every relation R of Γ also has a quantifier-free definition over $(V; (\mathrm{Eq}_{2i})_{i\geq 1}, \neq)$ denoted by Q_R. The clauses of Q_R are all disjunctions of atomic formulas (or negations of atomic formulas) on the language of $(V; (\mathrm{Eq}_{2i})_{i\geq 1}, \neq)$. We assume from now on that for every relation R, the corresponding Q_R is in conjunctive reduced normal form.

We now prove that at most one literal of the form $\mathrm{Eq}_{2i}(\overline{x})$ can appear in a clause of Q_R. Indeed, assume that a clause C of Q_R contains two literals of the form $\mathrm{Eq}_{2i}(\overline{x})$, for instance: C is $\mathrm{Eq}_{2i}(x_1, \ldots, x_{2i}) \vee \mathrm{Eq}_{2j}(y_1, \ldots, y_{2j}) \vee \ldots$. Since Q_R is in reduced CNF, there exists two assignations s and t of the variables such that the only literal of clause C satisfied by s is $\mathrm{Eq}_{2i}(\overline{x})$, and the only literal of C satisfied by t is $\mathrm{Eq}_{2j}(y_1, \ldots, y_{2j})$. Then we have $\mathrm{Eq}_{2i}(s(x_1), \ldots, s(x_{2i})) \wedge \neg\, \mathrm{Eq}_{2j}(s(y_1), \ldots, s(y_{2j}))$, and $\neg\, \mathrm{Eq}_{2i}(t(x_1), \ldots, t(x_{2i})) \wedge \mathrm{Eq}_{2j}(t(y_1), \ldots, t(y_{2j}))$. Since $\neg\, \mathrm{Eq}_{2i}(t(x_1), \ldots, t(x_{2i}))$, there exists $a_1 \in V$ such that $\mathrm{Eq}_{2i}(a_1, t(x_2), \ldots, t(x_{2i}))$. So, since f preserves Eq_{2i}, we have:

$$\mathrm{Eq}_{2i}(f(s(x_1), a_1), f(s(x_2), t(x_2)), \ldots, f(s(x_{2i}), t(x_{2i})))$$

so $f(s(x_2), t(x_2)) + \cdots + f(s(x_{2i}), t(x_{2i})) = f(s(x_1), a_1)$. Hence, we have:

$$\neg\, \mathrm{Eq}_{2i}(f(s(x_1), t(x_1)), \ldots, f(s(x_{2i}), t(x_{2i}))$$

since f is injective and $t(x_1) \neq a_1$. For the same reason, we have:

$$\neg\, \mathrm{Eq}_{2j}(f(s(y_1), t(y_1)), \ldots, f(s(y_{2j}), t(y_{2j}))$$

And all the other literals of C are not satisfied, since neither s nor t satisfy them (this is true since they are of the form $\neg\, \mathrm{Eq}_i(\overline{z})$). But this contradicts the fact that f preserves C, as C is a clause of Q_R, and f preserves R. From there, it is straightforward to see that f is an embedding from $(V; (\mathrm{Eq}_{2i})_{i\geq 1})^2$ to $(V; (\mathrm{Eq}_{2i})_{i\geq 1})$, since f preserves $\neg\, \mathrm{Eq}_{2i}$ for all $i \geq 1$. Indeed, $\neg\, \mathrm{Eq}_{2i}$ can be pp-defined over $(V; (\mathrm{Eq}_{2i})_{i\geq 1})^2, \neq)$ as follows:

$$\neg\, \mathrm{Eq}_{2i}(x_1, \ldots, x_{2i}) \Leftrightarrow \exists u.\, \mathrm{Eq}_{2i}(x_1, \ldots, x_{2i-1}, u) \wedge x_{2i} \neq u$$

Consequently, by Theorem 4.1.17, every clause of Q_R has at most one literal of the form $\mathrm{Eq}_{2i}(\overline{x})$, i.e., every clause of Q_R is a Horn formula on the language of $(V; (\mathrm{Eq}_{2i})_{i\geq 1})$, and if ϕ is a primitive positive formula on the language of Γ, then its translation in the language of $(V; (\mathrm{Eq}_{2i})_{i\geq 1})$ (using the translations Q_R), is a conjunction of Horn formulas preceded by existential quantifiers. We denote it by $\Lambda(\phi)$. We can thus treat an instance of $\mathrm{CSP}(\Gamma)$ as a conjunctions of Horn formulas on the language of $(V; (\mathrm{Eq}_{2i})_{i\geq 1})$, and give it as an input to Algorithm 5. Note that since Q_R is pre-calculated for all relation R of Γ, we can build $\Lambda(\phi)$ in linear time.

We now prove that, given a primitive positive formula ϕ on the language of Γ, Algorithm 5 runs in polynomial time and accepts $\Lambda(\phi)$ if and only if $\Gamma \models \phi$. First note that Φ is equi-satisfiable at each step of the algorithm. Suppose that $\Gamma \models \phi$. Then there exists a tuple $\overline{a}$ of elements of V such that $(V; (\mathrm{Eq}_{2i})_{i\geq 1}) \models \Lambda(\phi)(\overline{a})$, by definition of $\Lambda(\phi)$. But since Φ is equi-satisfiable at each step of the algorithm, the algorithm accepts $\Lambda(\phi)$.

Conversely, suppose that Algorithm 5 accepts $\Lambda(\phi)$. Let us consider the sets Φ and Ψ at the last step of the algorithm. For all negative literal l in a clause $C \in \Phi$, we know that the system of equations $\{\Psi, l\}$ has solution thanks to Gauss polynomial method, i.e., there exists an assignation $\overline{a}_l$ of the variables of Φ such that $(V; (\mathrm{Eq}_{2i})_{i\geq 1}) \models \Psi(\overline{a}) \wedge l(\overline{a})$. Let $l_1, \ldots, l_j$ be all the distinct negative literals appearing in clauses of Φ. We now consider the assignation $\overline{b} := f(\overline{a}_{l_1}, f(\overline{a}_{l_2}, f(\overline{a}_{l_3}, \ldots, f(\overline{a}_{l_{j-1}}, \overline{a}_{l_j}) \ldots)$. Since f is an embedding from $(V; (\mathrm{Eq}_{2i})_{i\geq 1})^2$ to $(V; (\mathrm{Eq}_{2i})_{i\geq 1})$, we easily see that $\overline{b}$ satisfies every clause C of Φ. So $\Lambda(\phi)$ is satisfiable, and $\Gamma \models \phi$. $\square$

Proposition 5.6.3. *Let Γ be a first-order reduct of $(V; +)$ which satisfies:*

$$\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, \neq)$$

Then $\mathrm{CSP}(\Gamma)$ *is either polynomial-time tractable, or NP-complete.*

Proof. First note that since $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V; \mathrm{Ieq}_4, \neq)$, Γ is 3-transitive and a fortiori 2-transitive by Lemma 5.6.1. So by Theorem 4.4.35, one of the following cases holds:

- Γ has a constant polymorphism. In this case, $\mathrm{CSP}(\Gamma)$ is in P, since the associated constant tuple is a solution of every instance.
- Every polymorphism of Γ is essentially unary, and $\neq \in \langle\Gamma\rangle_{\mathrm{pp}}$. In this case, P_3^B is also in $\langle\Gamma\rangle_{\mathrm{pp}}$ by Proposition 4.4.16, so $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(V; P_3^B, \neq)$. Hence, by Proposition 4.4.6 and Corollary 4.4.20, $\mathrm{CSP}(\Gamma)$ is NP-hard. Finally, since Γ is a reduct of $(V; +)$, $\mathrm{CSP}(\Gamma)$ is NP-complete by Proposition 5.1.2.
- Γ has a binary injective polymorphism. Then by Proposition 5.6.2, $\mathrm{CSP}(\Gamma)$ is in P, and can be solved using Algorithm 5.

$\square$

Theorem 5.6.4. *Let Γ be a reduct of $(V; \mathrm{Ieq}_4)$. Then either* $\mathrm{CSP}(\Gamma)$ *is polynomial-time tractable, or* $\mathrm{CSP}(\Gamma)$ *is NP-complete.*

Proof. Since Γ is a reduct of $(V; \mathrm{Ieq}_4)$, the translation t_b belongs to $\mathrm{End}(\Gamma)$ for every $b \in V$. By Theorem 3.5.21, as we already have a functional description of each endomorphism monoid listed in the theorem statement, we have that at least one of the following cases holds:

- $\mathrm{End}(\Gamma) = \mathrm{End}(V; \mathrm{Ieq}_4, \neq)$ in which case we are done by Proposition 5.6.3.
- $\mathrm{End}(\Gamma) \cap \{\mathrm{gen}, \mathrm{gen}^*\} \neq \emptyset$. In this case, we have $\mathrm{gen}^* \in \mathrm{End}(\Gamma)$. Indeed, if gen belongs to $\mathrm{End}(\Gamma)$, we also have $\mathrm{gen}^* \in \mathrm{End}(\Gamma)$ since Γ contains t_b for all $b \neq 0$, and $\mathrm{gen}^* = t_b \circ \mathrm{gen}$, for a well-chosen b. Consequently, Γ is homomorphically equivalent to a reduct of $(V; =)$ by Proposition 3.5.1, and we are done by Theorem 4.4.24.
- Γ is homomorphically equivalent to a structure with at most two elements, in which case we are done by Proposition 4.4.12.

$\square$

Corollary 5.6.5. *Let Γ be a first-order reduct of $(V; +)$ such that the model-complete core Γ' of Γ satisfies: $\mathrm{End}(\Gamma') = \mathrm{End}(V; {\mathrm{Ieq}_4}^{\mathrm{inj}})$. Then $\mathrm{CSP}(\Gamma)$ is either polynomial-time tractable, or NP-complete.*

Proof. By Proposition 4.4.3, $\mathrm{CSP}(\Gamma)$ reduces to $\mathrm{CSP}(\Gamma')$ in polynomial time, and vice versa. So we can focus on studying the complexity of $\mathrm{CSP}(\Gamma')$. Furthermore, it is straightforward to see that $\mathrm{End}(V; {\mathrm{Ieq}_4}^{\mathrm{inj}}) = \mathrm{End}(V; \mathrm{Ieq}_4, \neq)$. We then conclude by Proposition 5.6.3. $\square$

5.7 The Full Case

Proposition 5.7.1. *Let Γ be a first-order reduct of $(V; +)$ such that:*

$$\mathrm{End}(\Gamma) = \mathrm{End}(V; +, \neq)$$

Then $0, \mathrm{Ind}_1, \mathrm{Ind}_2$ are pp-definable over Γ. Furthermore, exactly one of the following cases holds:

- *E is pp-definable over Γ;*
- *N is not pp-definable over Γ.*

Proof. Every endomorphism of Γ is injective and preserves 0, hence $0, \mathrm{Ind}_1, \mathrm{Ind}_2 \in \langle\Gamma\rangle_{\mathrm{pp}}$. Assume now that N is pp-definable over Γ. Then E is *pp*-definable over Γ since $E(x, y)$ if and only if $\exists z(N(x, z) \wedge N(y, z))$. $\square$

Notation 5.7.2. In the following, if f is a binary operation, we denote by f_x the operation $x \mapsto f(x, 0)$, and f_y the operation $x \mapsto f(0, y)$.

Lemma 5.7.3. *Let Γ be a reduct of $(V; +)$ such that $\mathrm{End}(\Gamma) = \mathrm{End}(V; +, \neq)$, and assume that there exists a polymorphism f of Γ which belongs to $\mathcal{I}(\mathrm{inj}, \mathrm{inj})$. Then f together with $\mathrm{Aut}(V; +)$ locally generates a canonical binary injection g from $(V; +, <)$ to $(V; +)$ such that exactly one of the following holds:*

- *g_x and g_y both behave as id over V;*

- g_x *and* g_y *both behave as* af *over* V*;*

- g_x *and* g_y *both behave as* gen *over* V*.*

Proof. By Corollary 4.2.10, since $(V;+,<)$ is an ω-categorical totally ordered Ramsey structure by Corollary 3.4.5, f locally generates a canonical binary injection f' from $(V;+,<)$ to $(V;+)$. Consequently, f'_x and f'_y are both canonical injective operations from $(V;+,<)$ to $(V;+)$. Recall that we classify them in 3.4.17. Consequently, since $f'(0,0)=0$ (as 0 is preserved by every polymorphism of Γ), we can assume that f'_x and f'_y behave as either id, af, or gen.

We first give the proof in the case where one of them behaves like gen. Indeed, assume for instance that f'_x behaves as gen, and consider the operation $g(x,y):=f'(f'(x,y),x)$ for all $x,y\in V$. The operation g is clearly a canonical injective operation from $(V;+,<)$ to $(V;+)$ as a composition of such operations. We now prove that g_x and g_y both behave as gen over V. Let $\overline{a}\in V^n$ such that $\mathrm{Ieq}_n(\overline{a})$. Since f'_y is canonical, we have either $\mathrm{Ieq}_n(f'(0,a_1),\ldots,f'(0,a_n))$ or $\mathrm{Ind}_n(f'(0,a_1),\ldots,f'(0,a_n))$. But since f'_x behaves like gen, we have $\mathrm{Ind}_n(g(0,a_1),\ldots,g(0,a_n))$ in both cases, so g_y behaves like gen. Similarly, since f'_x behaves like gen, we have $\mathrm{Ind}_n(f'(a_1,0),\ldots,f'(a_n,0))$, and since f'_y is canonical, it preserves Ind_n. Hence, we have $\mathrm{Ind}_n(g(a_1,0),\ldots,g(a_n,0))$, and g_x also behaves like gen.

Two cases remain:

- none of g_x and g_y behave like gen, and at least one of the them behaves as af. In this case, we can prove exactly as we did in the previous case that g_x and g_y both behave as af over V;

- g_x and g_y behave like id, and there is nothing to do.

□

Lemma 5.7.4. *Let* Γ *be a reduct of* $(V;+)$ *such that* $\mathrm{End}(\Gamma)=\mathrm{End}(V;+,\neq)$*, and assume that there exists a binary polymorphism* g *of* Γ *which is canonical from* $(V;+,<)$ *to* $(V;+)$*, and which belongs to* $\mathcal{I}(\mathrm{inj},\mathrm{inj})$ *and such that* g_x *and* g_y *both behave as* gen *over* V*. Then* g *strongly preserves* $\mathrm{Eq}_i^{\neq 0}$ *for all* $i\geq 2$*.*

Proof. First note that since $\mathrm{End}(\Gamma)=\mathrm{End}(V;+,\neq)=\mathrm{End}(V;\mathrm{Ieq}_3,0)$ by Lemma 3.2.23, and since $\mathrm{Eq}_i^{\neq 0}\in\langle(V;\mathrm{Ieq}_3)\rangle_{\mathrm{pp}}$ by Corollary 3.2.15, $\mathrm{Eq}_i^{\neq 0}$ is preserved by g for all $i\geq 2$. Since $g\in\mathcal{I}(\mathrm{inj},\mathrm{inj})$, we also have that $g_x(V)\cap g_y(V)=\{0\}$. By Proposition 5.5.1, there exist α,β two endomorphisms of $(V;+)$ such that $g(x,y)=\alpha(x)+\beta(y)$ for all $x,y\neq 0$ and $\alpha(V)\cap\beta(V)=\{0\}$. Furthermore, since g is injective, we have $\alpha,\beta\in\mathrm{Emb}(V;+)$. Since g_x and g_y both behave as gen over V with g is canonical, it is straightforward to check that $\alpha(V\setminus\{0\}),\beta(V\setminus\{0\}),g_x(V\setminus\{0\}),g_y(V\setminus\{0\})$ are pairwise disjoint. We now prove that g strongly preserves $\mathrm{Eq}_i^{\neq 0}$ for all $i\geq 2$.

Let $x_1,\ldots,x_n$ be n distinct elements of $V\setminus\{0\}$, and let $y_1,\ldots,y_n,y_{n+1}$ be elements of V. Assume for the sake of the notations that $y_1=\cdots=y_k\neq 0$, and $y_{k+1}=\cdots=$

$y_{n+1} = 0$. We set $x_{n+1} := x_1 + \cdots + x_n$. Then:

$$\begin{aligned} g(x_1, y_1) + \cdots + g(x_n, y_n) + g(x_{n+1}, y_{n+1}) &= \sum_{i \leq k} (\alpha(x_i) + \beta(y_i)) + \sum_{i \geq k+1} g_x(x_i) \\ &= \alpha(\sum_{i \leq k} x_i) + \beta(\sum_{i \leq k} y_i) + \sum_{i \geq k+1} g_x(x_i) \end{aligned}$$

Since $k + 1 \leq n + 1$ (i.e., $(y_1, \dots, y_{n+1})$ contains some 0), we have: $\sum_{i \geq k+1} g_x(x_i) \neq 0$, and since $\alpha(V \setminus \{0\})$ and $\beta(V \setminus \{0\})$ are disjoint from $g_x(V \setminus \{0\})$, we have $g(x_1, y_1) + \cdots + g(x_n, y_n) + g(x_{n+1}, y_{n+1}) \neq 0$.

If $k = n + 1$ (i.e., $(y_1, \dots, y_{n+1})$ are all distinct from 0), assume that:

$$\neg \operatorname{Eq}^{\neq 0}_{n+1}(y_1, \dots, y_{n+1})$$

(indeed, we already know that g preserves $\operatorname{Eq}^{\neq 0}_{n+1}$). In this case, $\beta(\sum_{i \leq k} y_i) \neq 0$, and since $\alpha(V \setminus \{0\})$ and $\beta(V \setminus \{0\}$ are disjoint, we have $g(x_1, y_1) + \cdots + g(x_n, y_n) + g(x_{n+1}, y_{n+1}) \neq 0$.

Now let $x_1, y_1, \dots, x_{n+1}, y_{n+1}$ be elements of V such that $\neg \operatorname{Eq}^{\neq 0}_{n+1}(x_1, \dots, x_n)$ and $\neg \operatorname{Eq}^{\neq 0}_{n+1}(y_1, \dots, y_n)$. We prove that $\neg \operatorname{Eq}^{\neq 0}_{n+1}(g(x_1, y_1), \dots, g(x_n, y_n))$. If $(x_i, y_i)_{i \leq n}$ are all distinct from 0, the result is clear since $g(x, y) = \alpha(x) + \beta(y)$ for all $x, y \neq 0$, with $\alpha, \beta \in \operatorname{Emb}(V; +)$. If some of the x_i and y_i are set to 0, since g_x and g_y behave like gen, it is easy to see that we will always have $\neg \operatorname{Eq}^{\neq 0}_{n+1}(g(x_1, y_1), \dots, g(x_n, y_n))$. Consequently, g strongly preserves $\operatorname{Eq}^{\neq 0}_i$ for all $i \geq 2$. $\square$

Lemma 5.7.5. *Let Γ be a reduct of $(V; +)$ such that $\operatorname{End}(\Gamma) = \operatorname{End}(V; +, \neq)$, and assume that there exists a binary polymorphism g of Γ which is canonical from $(V; +, <)$ to $(V; +)$, and which belongs to $\mathcal{I}(\text{inj}, \text{inj})$ and such that g_x and g_y both behave as* id *over V. Then either g strongly preserves $\operatorname{Eq}^{\neq 0}_3$ for all $i \geq 2$, or g strongly preserves Eq_i for all $i \geq 1$.*

Proof. First note that since $\operatorname{End}(\Gamma) = \operatorname{End}(V; +, \neq) = \operatorname{End}(V; \operatorname{Ieq}_3, 0)$ by Lemma 3.2.23, and since $\operatorname{Eq}^{\neq 0}_i \in \langle (V; \operatorname{Ieq}_3) \rangle_{\text{pp}}$ by Corollary 3.2.15, $\operatorname{Eq}^{\neq 0}_i$ is preserved by g for all $i \geq 2$. Since $g \in \mathcal{I}(\text{inj}, \text{inj})$, we also have that $g_x(V) \cap g_y(V) = \{0\}$. By Proposition 5.5.1, there exist α, β two endomorphisms of $(V; +)$ such that $g(x, y) = \alpha(x) + \beta(y)$ for all $x, y \neq 0$ and $\alpha(V) \cap \beta(V) = \{0\}$. Furthermore, since g is injective, $\alpha, \beta \in \operatorname{Emb}(V; +)$.

Since g_x and g_y behave like id, g_x and g_y are self-embeddings of $(V; +)$. We now show that either $g_x = \alpha$, or $g_x(V) \cap \alpha(V) = \{0\}$. Similarly, either $g_y = \beta$, or $g_y(V) \cap \beta(V) = \{0\}$.

Assume that g does not strongly preserve $\operatorname{Eq}^{\neq 0}_3$. Since g preserves $\operatorname{Eq}^{\neq 0}_3 \cup \{(0, 0, 0)\}$, there exist $a_1, a_2 \in V$ such that $\operatorname{Eq}^{\neq 0}_3(a_1, a_2, a_1 + a_2)$, and such that at least one of the following cases holds:

- there exists $b_1 \neq 0$ such that $g(a_1, b_1) + g(a_2, b_1) + g(a_1 + a_2, 0) = 0$. In this case, $g_x(a_1 + a_2) = \alpha(a_1 + a_2)$. Since g is canonical, we have $g_x(a) = \alpha(a)$ for all $a \in V$.

- there exist $b_1, b_2 \neq 0$ such that $g(a_1, b_1) + g(a_2, b_2) + g(a_1 + a_2, 0) = 0$. In this case, $g_x(a_1 + a_2) = \alpha(a_1 + a_2) + \beta(b_1 + b_2)$. Making vary the values of b_1, b_2, we get a contradiction.

- there exists $b_1 \neq 0$ such that $g(a_1, 0) + g(a_2, 0) + g(a_1 + a_2, b_1) = 0$. In this case, $g_x(a_1 + a_2) = \alpha(a_1 + a_2) + \beta(b_1)$. Making vary the values of b_1, we get a contradiction.

Note that we forgot three symmetric cases in the previous case distinction, which correspond to the same cases with the x and the y axes reversed. Hence, we conclude that g does not strongly preserve $\mathrm{Eq}_3^{\neq 0}$ if and only if $\alpha = g_x$ or $\beta = g_y$.

Finally it is straightforward to see that if exactly one of the two previous equalities is satisfied, then considering the operation $(x, y) \mapsto g(g(x, y), g(y, x))$, we are back in a case where none of these two equalities is satisfied anymore.

We now prove that if $\alpha = g_x$ and $\beta = g_y$, g strongly preserves Eq_i for all $i \geq 1$, but this is straightforward, knowing that $g(x, y) = \alpha(x) + \beta(y)$ for all $x, y \in V$. $\square$

Chapter 6

Conclusion: Achievements and Perspectives

We first state the main results proved in this thesis.

6.1 Achievements

Notation 6.1.1. Let $(V;+)$ be the (unique) countably infinite vector space over $\mathbb{F}_2$.

We first defined important relations which in fact isolate model-theoretic types of $(V;+)$.

Definition 6.1.2. For $n \geq 1$, and for all $(x_1,\dots,x_n) \in V^n$, we define the following relations:

- $(x_1,\dots,x_n) \in \mathrm{Eq}_n$ iff $\sum_{i\leq n} x_i = 0$;
- $(x_1,\dots,x_n) \in \mathrm{Ind}_n$ iff $x_1,\dots,x_n$ is linearly independent;
- $(x_1,\dots,x_n) \in \mathrm{Ieq}_n$ iff $\mathrm{Eq}_n(x_1,\dots,x_n)$ and every strict subfamily of $(x_1,\dots,x_n)$ belongs to Ind_n.

We then proved that $(V;+)$ is not only ω-categorical, but homogeneous on a functional or an infinite relational signature.

Theorem 6.1.3. $(V;+)$ *is not first-order interdefinable with any homogeneous structure on a finite relational language. Nevertheless,* $(V;+)$ *is first-order definable with the homogeneous structure* $(V;(\mathrm{Ieq}_i)_{i\geq 1})$.

We then defined very important classes of functions which generate endomorphism monoids of reducts.

Definition 6.1.4. We define the following functions:

- f is a id-function iff $f(x) = h(x)$ for all $x \neq 0$, for some $h \in \mathrm{Emb}(V;+)$;

- f is an af-function iff $f(x) = h(x) + a$ for all $x \neq 0$, for some $h \in \mathrm{Emb}(V;+)$ and $a \notin h(V)$;
- f is a gen-function iff f sends any injective tuple of vectors to a linearly independent family of vectors.

Definition 6.1.5. Given an infinite domain D and a subset $\mathcal{F}$ of D^D, we denote by $\overline{\langle \mathcal{F} \rangle_1}$ the closure of $\mathcal{F}$ under composition and pointwise convergence.

Definition 6.1.6. Let Δ_1, Δ_2 be two structures and let f be a function from $\mathrm{Dom}(\Delta_1) \to \mathrm{Dom}(\Delta_2)$. We say that f is canonical from Δ_1 to Δ_2 if for all n and all $(a_1, \ldots, a_n) \in \mathrm{Dom}(\Delta_1)^n$, the orbit of $(f(a_1), \ldots, f(a_n))$ under the natural action of $\mathrm{Aut}(\Delta_2)$ on $\mathrm{Dom}(\Delta_2)^n$ only depends on the orbit of $(a_1, \ldots, a_n)$ under the natural action of $\mathrm{Aut}(\Delta_1)$ on $\mathrm{Dom}(\Delta_1)^n$.

We then gave a list of the canonical functions from $(V;+)$ to $(V;+)$.

Theorem 6.1.7. *Let f be an injective canonical function from $(V;+)$ to $(V;+)$. Then one of the following holds:*

- *f is an* id*-function;*
- *f is an* af*-function;*
- $\overline{\langle \{f\} \cup \mathrm{Aut}(V;+) \rangle_1}$ *contains a* gen*-function.*

Definition 6.1.8. A reduct Γ of a structure Δ is a relational structure with same domain as Δ whose relations are first-order definable over Δ.

Proposition 6.1.9. *Let Γ be a reduct of $(V;+)$. If $\mathrm{End}(\Gamma)$ contains a* gen*-function, then Γ is homomorphically equivalent to a reduct of $(V;0)$.*

The classification of canonical functions we just gave unlocked non trivial classification results for endomorphism monoids of reducts, self-embedding monoids, and automorphism groups.

Theorem 6.1.10. *Let Γ be a reduct of $(V;+)$ which is not homomorphically equivalent to a reduct of $(V;0)$. Then $\mathrm{End}(\Gamma)$ is one of the 27 monoids identified in Theorem 3.5.21.*

Corollary 6.1.11. *Let Γ be a reduct of $(V;+)$. Exactly one of the following holds:*

- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;+)$*;*
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Ieq}_3)$*;*
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Ieq}_4, 0)$*;*
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Ieq}_4)$*;*
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;\mathrm{Eq}_4)$*;*

- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;0)$*;*
- $\mathrm{Emb}(\Gamma) = \mathrm{Emb}(V;=)$.

Corollary 6.1.12. *Let Γ be a reduct of $(V;+)$. Exactly one of the following holds:*

- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;+)$*;*
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;\mathrm{Eq}_4)$*;*
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;0)$*;*
- $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(V;=)$.

These classification results also open the door to classification of model-complete cores of reducts.

Definition 6.1.13. A structure Δ is a model-complete core of a structure Γ iff Δ is homomorphically equivalent to Γ and:

$$\overline{\langle \mathrm{Aut}(\Delta)\rangle_1} = \mathrm{End}(\Delta)$$

By [Bod07], every ω-categorical structure has a model-complete core, which is unique up to isomorphism. By definition, the model-complete core Δ of a reduct Γ is homomorphically equivalent to Γ. Hence, the complexity of $\mathrm{CSP}(\Gamma)$ is the same as $\mathrm{CSP}(\Delta)$.

Theorem 6.1.14. *Let Γ be a reduct of $(V;+)$. Exactly one of the following holds:*

- $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$,
- $\mathrm{End}(\Gamma) = \mathrm{End}(V;\mathrm{Ieq}_4,0)$, *or*
- *the model-complete core of Γ is isomorphic to a structure Γ' s.t.:*
  - $\mathrm{End}(\Gamma') = \mathrm{End}(V\setminus\{0\};\mathrm{Ieq}_3)$,
  - $\mathrm{End}(\Gamma') = \mathrm{End}(V;\mathrm{Eq}_4,\neq)$,
  - Γ' *is a reduct of* $(V;0)$, *or*
  - Γ' *is a 2-element structure.*

Corollary 6.1.15. *Let Γ be a reduct of $(V;+)$. There exists a structure Γ' with same CSP as Γ and s.t. one of the following holds:*

1. $\mathrm{End}(\Gamma') = \mathrm{End}(V;+,\neq)$*;*
2. $\mathrm{End}(\Gamma') = \mathrm{End}(V\setminus\{0\};\mathrm{Ieq}_3)$*;*
3. $\mathrm{End}(\Gamma') = \mathrm{End}(V;\mathrm{Eq}_4,\neq)$*;*
4. $\mathrm{End}(\Gamma') = \mathrm{End}(V;\mathrm{Ieq}_4,0)$*;*

5. Γ' *is a reduct of* $(V;0)$*;*

6. Γ' *is a 2-element structure.*

Theorem 6.1.16. *Let* Γ *be a reduct of* $(V;+)$ *such that we are in Case 2,3,5,6 of Corollary 6.1.15. Then* $\mathrm{CSP}(\Gamma)$ *is either in P or NP-complete.*

Remark 6.1.17. If we prove that P/NP-complete dichotomy in Case 1, then the proof can easily be adapt to Case 4.

6.2 Perspectives

By far the most important remaining task, which is already half done, is to complete the proof of the conjecture by solving Case 1 of Corollary 6.1.15. Note that if we are in Case 1, i.e., $\mathrm{End}(\Gamma) = \mathrm{End}(V;+,\neq)$, we already know a great deal on $\mathrm{Pol}(\Gamma)$. For instance, any n-ary polymorphism of Γ is of the form $f(x_1,\dots,x_n) = \sum_{i\leq n} \alpha_i(x_i)$ with $\alpha_i \in \mathrm{Aut}(V;+)$, for all $(x_1,\dots,x_n) \in (V \setminus \{0\})^n$. We also designed algorithms for some cases, depending on the value of f on tuples with at least one coordinate equal to 0. Nevertheless, some new ideas should be introduce, especially in the case where a binary polymorphism of Γ is of the form $f(x_1,0) = \beta_1(x_1)$ and $f(0,x_2) = \beta_2(x_2)$ for all x_1, x_2, with automorphisms β_1, β_2 such that they are not related with α_i for all $i \leq n$.

Another interesting would be to assume that the vector space $(V;+)$ is not over $\mathbb{F}_2$, but over any finite field $\mathbb{F}$. Sadly, we had no time to tackle this problem, but we are convinced our method could apply in this setting too.

Initially, we were brought to $(V;+)$ by the atomless boolean algebra. This structure is, as $(V;+)$, homogeneous and ω-categorical. It is the Fraïssé limit of the class of finite boolean algebras. It is also more general than $(V;+)$ since it contains $(V;+)$ as a reduct. Indeed, we can define $x+y=z$ as follows: $(x \cup y) \setminus (x \cap y) = z$.

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