The number of monounary algebras

Gábor Horváth, Kamilla Kátai-Urbán, Péter Pál Pach, Gabriella Pluhár, András Pongrácz, and Csaba Szabó

Abstract. In this note we give an asymptotic estimate for the number of monounary algebras of given size.

1. Introduction

A monounary algebra is an algebra with a single unary operation. The theory of monounary algebras is well-developed, for a recent monograph see [1]. Let $\mathcal{A} = (A, f)$ be a monounary algebra. The function f defines a directed graph on A. Let $G_A = (A, E)$, the vertex set is A and the edges are $E = \{(a, f(a)) \mid a \in A\}$. In G_A every vertex has outdegree 1, and every graph G with outdegree 1 defines a monounary algebra on its vertex set, where $f(a)$ is the single vertex such that $(a, f(a))$ is an edge in G. Hence, there is a bijection between monounary algebras and directed graphs, where each vertex has outdegree 1. At first, we investigate the number of connected monounary algebras.

Theorem 1.1. *Let C_n denote the number of connected monounary algebras of size n. Then there is an $\alpha > 1$ such that $\log_\alpha C_n \sim n$.*

Proof. A directed graph corresponding to a connected monounary algebra is a directed cycle with a (possibly one-element) rooted tree at each vertex. If there is a loop in the graph we say that the length of the cycle is 1. Each edge of each rooted tree is directed towards the cycle. Hence, a connected graph of a monounary algebra of size n is built up from a cycle of length k, where $1 \le k \le n$ and to each vertex of the cycle we glue a rooted tree such that the sum of the sizes of the rooted trees is n. Every monounary algebra of size n can be obtained this way. Note that these graphs are not necessarily non-isomorphic. Naturally, each rooted tree is a monounary algebra. Let T_n denote the number of rooted trees on n vertices. By the above arguments we have

$$T_n \le C_n \le \sum_{k=1}^{n} \sum_{i_1+\cdots+i_k=n} T_{i_1} T_{i_2} \cdots T_{i_k} \tag{1}$$

Date: October 14, 2010.

2000 *Mathematics Subject Classification*: buli.

Key words and phrases: monounary algebra, generating function.

The research of the authors was supported by the Hungarian National Foundation for Scientific Research Grant K67870.

Let $T_0 = C_0 = 1$. Let $\psi(x) = \sum_{n=1}^{\infty} T_n x^{n-1}$ and $\Gamma(x) = \sum_{n=0}^{\infty} C_n x^n$. Now Γ is the generating function of the number of connected monounary algebras, and $x\psi(x)+1$ is the generating function of the number of rooted trees. By formula (1) the following holds:

$$x\psi(x)+1 \le \Gamma(x) \le 1 + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\sum_{i_1+\cdots i_k=n} T_{i_1}T_{i_2}\dots T_{i_k}\right)x^n = \sum_{n=0}^{\infty}(x\psi(x))^n \quad (2)$$

In [2] the function $\psi(x)$ is well analyzed. It is proved that there is a constant c_T such that $T_n \sim c_T\alpha^n n^{-\frac{3}{2}}$, where $\alpha \sim 2.955765$. Moreover the power series $\psi(x)$ has radius of convergence $\frac{1}{\alpha}$ for the same α, and $\psi(\frac{1}{\alpha}) = \alpha$, hence $\frac{1}{\alpha}\psi(\frac{1}{\alpha}) = 1$ holds. As $\sum_{n=0}^{\infty}(x\psi(x))^n = \dfrac{1}{1-x\psi(x)}$ is strictly monotonically increasing in $\mathbb{R}^+$, the radius of convergence of this power series is the unique positive solution of the equation $x\psi(x) = 1$, which is $\frac{1}{\alpha}$. Therefore for each power series in (2) the radius of convergence is $\frac{1}{\alpha}$. This holds for $\Gamma(x)$, as well. Thus for the coefficients of $\Gamma(x)$ we have $\limsup_{n\to\infty}\sqrt[n]{C_n} = \alpha$. From $T_n \le C_n$ we obtain $\liminf_{n\to\infty}\sqrt[n]{C_n} \ge \alpha$, thus $\lim_{n\to\infty}\sqrt[n]{C_n} = \alpha$, and $\log_\alpha C_n \sim n$ is gained.

□

Theorem 1.2. *Let M_n denote the number of monounary algebras of size n. Then $\log_\alpha M_n \sim n$.*

Proof. A monounary algebra is the disjoint union of connected monounary algebras. Let $\mathcal{A}$ be a monounary algebra of size n containing μ_i connected algebras of size i. Then $\sum_{i=1}^{n} i\mu_i = n$. Note that μ_i can be 0. The number of ways of picking k indistinguishable objects of p type is the coefficient of x^k in the generating function $\dfrac{1}{(1-x)^p}$. Hence, the number of ways picking μ_i indistinguishable connected monounary algebras of C_i type is the coefficient of $x^{i\mu_i}$ in $\dfrac{1}{(1-x^i)^{C_i}}$. Thus the generating function for the number of monounary algebras is

$$1 + \sum_{n=1}^{\infty} M_n x^n = \prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{C_k}} \quad (3)$$

For $x > 0$ this series converges if and only if $\log\left(\prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{C_k}}\right) = \sum_{k=1}^{\infty} -C_k\log(1-x^k)$ is convergent. If $x \in (0,\frac{1}{\alpha})$ then $x^k \in (0,\frac{1}{\alpha})$ for all $k \ge 1$. As $\log x$ is concave we have $\log(1-t) \ge -ct$ for all $t \in (0,\frac{1}{\alpha})$ with $c = -\alpha\log(1-\frac{1}{\alpha})$. Therefore $\sum_{k=1}^{\infty} -C_k\log(1-x^k) \le \sum_{k=1}^{\infty} C_k c x^k = c(\Gamma(x)-1)$. Hence this power series – and consequently $\sum_{n=0}^{\infty} M_n x^n$ – is convergent in $(0,\frac{1}{\alpha})$. This yields $\limsup_{n\to\infty}\sqrt[n]{M_n} \le$

α. The lower bound $\liminf_{n\to\infty} \sqrt[n]{M_n} \geq \alpha$ can be derived from $C_n \leq M_n$. Thus $\lim_{n\to\infty} \sqrt[n]{M_n} = \alpha$ and $\log_\alpha M_n \sim n$.

$\square$

References

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(1) Institute of Mathematics, University of Debrecen, Pf. 12, Debrecen, 4010, Hungary

E-mail address: ghorvath@math.unideb.hu

(2) Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

E-mail address: katai@math.u-szeged.hu

(3,4,6) Eötvös Loránd University, Department of Algebra and Number Theory, 1117 Budapest, Pázmány Péter sétány 1/c, Hungary

E-mail address: ppp24@cs.elte.hu

E-mail address: plugab@cs.elte.hu

E-mail address: csaba@cs.elte.hu

(5) Central European University, Budapest, Hungary

E-mail address: pongeee@cs.elte.hu