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EMBEDDING SEMILATTICES OF SUBSPACES OF VECTOR SPACES

MÁTÉ L. JUHÁSZ and ANDRÁS PONGRÁCZ

Eötvös Loránd University, Department of Algebra and Number Theory, 1117 Budapest,
Pázmány Péter sétány 1/c, Hungary
e-mails: jmatel@cs.elte.hu, pongeee@cs.elte.hu

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Abstract

Let V be a left vector space over the division ring D and let $\dim_D V = \kappa$. Let $(\operatorname{Sub} V, \cap)$ and $(\operatorname{Sub} V, +)$ denote the meet-semilattice and join-semilattice of the subspaces of V, respectively. We prove that for κ finite $(\operatorname{Sub} V, \cap)$ embeds into $(\operatorname{Sub} V, +)$ if and only if D embeds into D^{op}. We show that the similar statement holds for every infinite κ if and only if it holds for $\kappa = \aleph_0$. For vector spaces over fields we obtain that $(\operatorname{Sub} V, \cap)$ always embeds into $(\operatorname{Sub} V, +)$.

1. Introduction

In [1] the following statement is proved (Corollary 4.5): Let V be an infinite-dimensional vector space over the division ring D. Then there is no embedding from $(\operatorname{Sub}^{\mathrm{fin}} V, +)$ into $(\operatorname{Sub} V, \cap)$. Here, $(\operatorname{Sub}^{\mathrm{fin}} V, +)$ denotes the semilattice of finite-codimensional subspaces of V under the operation $+$ and $(\operatorname{Sub} V, \cap)$ is the meet-semilattice of the subspaces of V under $\cap$. More generally it is shown that there is an embedding from $(\operatorname{Sub}^{\mathrm{fin}} V, +)$ into $(\operatorname{Sub} W, \cap)$ if and only if $\dim W \geqq (\operatorname{card} D)^{\dim V}$. The dual of the above problem is asked (Problem 1): For an infinite-dimensional vector space V does $(\operatorname{Sub} V, \cap)$ have an embedding into $(\operatorname{Sub} V, +)$? In this paper we investigate this question for both finite- and infinite-dimensional vector spaces. Let V be a left vector

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space over the division ring D and let $\dim_D V = \kappa$. We prove that for κ finite $(\operatorname{Sub} V, \cap)$ embeds into $(\operatorname{Sub} V, +)$ if and only if D embeds into D^{op}.

For κ infinite we show that $(\operatorname{Sub} V, \cap)$ embeds into $(\operatorname{Sub} V, +)$ if and only if for every finite-dimensional vector space W over D the semilattice $(\operatorname{Sub} W, \cap)$ embeds into $(\operatorname{Sub} V, +)$. This settles the case of vector spaces over fields: $(\operatorname{Sub} V, \cap)$ embeds into $(\operatorname{Sub} V, +)$.

Finally, we discuss the general case and give some necessary conditions for the problem. We show that if there is a vector space V with no such embedding then there is also such a vector space W of countable dimension over the same division ring.

2. Vector spaces over fields

In this section, V will denote a vector space of dimension κ over the field F. Let V^* denote the *dual* of V. Note that $\dim V^* = \kappa$ if κ is finite and $\dim V^* = (\operatorname{card} F)^{dim V}$, otherwise. This result is proved in [2] for fields, but the explanation works for division rings, as well.

Recall the most important properties of the Galois connection between V and V^*. For a subspace $U \leqq V$ let $U^\sharp = \left\{ f \in V^* \mid f(u) = 0 \text{ for every } u \in U \right\}$. For a subspace $W \leqq V^*$ let $W^\flat = \left\{ v \in V \mid f(v) = 0 \text{ for every } f \in W \right\}$. For $U \leqq V$ we have $U^{\sharp\flat} = U$ and for a subspace $U^\sharp = W \leqq V^*$ the relationship $W^{\flat\sharp} = W$ holds. The subspaces of V^* of the form $U^\sharp$ are called closed subspaces. Finite-dimensional subspaces are closed.

Denote by $\operatorname{Sub} V$ the lattice of subspaces of V and by $\operatorname{ClSub} V^*$ the lattice of closed subspaces of V^*. Now, $\operatorname{Sub} V$ is dually isomorphic to $\operatorname{ClSub} V^*$, that is $(\operatorname{Sub} V, +, \cap)$ and $(\operatorname{ClSub} V^*, \cap, +)$ are isomorphic as lattices. Hence the map $W \to W^\flat$ is a semilattice isomorphism from $(\operatorname{ClSub} V^*, \cap)$ to $(\operatorname{Sub} V, +)$.

For a subset $X \subseteqq V$, let $\langle X \rangle$ denote the subspace of V generated by X. For an index-set I and vector spaces V_i, $i \in I$ let $\bigoplus_{i \in I} V_i$ denote the (discrete) direct sum and $\prod_{i \in I} V_i$ denote the complete direct product of these vector spaces, moreover, let ι_i denote the canonical embedding of V_i into the complete direct product, that is

$$\iota_i(v)_j = \begin{cases} v & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

for every $i \in I$.

THEOREM 1. *Let V be a vector space over a field. Then there is an embedding of $(\operatorname{Sub} V, \cap)$ into $(\operatorname{Sub} V, +)$.*

Proof. For finite-dimensional vector spaces the statement follows from $V^* \cong V$ and $\mathrm{ClSub}\, V^* = \mathrm{Sub}\, V^*$. For the infinite-dimensional case the embedding will be constructed in several steps. According to the above mentioned Galois-connection, it is enough to construct an embedding into $(\mathrm{ClSub}\, V^*, \cap)$, instead of $(\mathrm{Sub}\, V, +)$. For this we show an embedding from $(\mathrm{Sub}\, V, \cap)$ into $(\mathrm{Sub}\, V^*, \cap)$, such that each subspace in V corresponds to a closed subspace of V^*. Denote by $(v_i)_{i\in I}$ a basis of V and by $v_i^* \in V^*$ the projection to the i-th coordinate, that is $v_i^*(\sum_j \alpha_j v_j) = \alpha_i$. Note that $V^* = \prod \langle v_i^* \rangle$. For each $\lambda \subseteqq I$ let $V_\lambda = \langle v_i \mid i \in \lambda \rangle \leqq V$, and $V_{\lambda^*} = \prod_{i\in\lambda} \langle v_i^* \rangle \leqq V^*$. Denote by $\mathbb{P}$ the set of finite subsets of I, that is $\mathbb{P} = \left\{ \lambda \subset I \mid |\lambda| < \infty \right\}$.

Now, using our notations, for every $\lambda \subseteqq I$ and $i \in I$ we have that ${v_j}^* \in {V_\lambda}^\sharp$ if and only if $j \notin \lambda$ and $v_i \in {V_{\lambda^*}}^\flat$ if and only if $i \notin \lambda$. Hence $(V_{\lambda^*})^\flat = \langle v_i \mid i \in I \setminus \lambda \rangle$ and $(V_\lambda)^\sharp = \prod_{i \in I\setminus\lambda} \langle v_i^* \rangle$ for any $\lambda \subset I$ and so

$$V = V_\lambda \oplus (V_{\lambda^*})^\flat \quad \text{and} \quad V^* = V_{\lambda^*} \oplus (V_\lambda)^\sharp. \tag{1}$$

Let $\overline{V} = \prod_{\lambda\in\mathbb{P}} V_\lambda$, and for every $\lambda \in \mathbb{P}$ and $i \in \lambda$ let $v_{(i,\lambda)} = \iota_\lambda(v_i)$. Moreover, let $\mathcal{B} = \left\{ v_{(i,\lambda)} \mid \lambda \in \mathbb{P}, i \in \lambda \right\}$, the disjoint union of the bases of the subspaces $\iota_\lambda(V_\lambda)$. The cardinality of $\mathcal{B}$ is equal to the cardinality of I, therefore there is a bijection $f : \mathcal{B} \to \{v_i^* \mid i \in I\}$. This f extends naturally to an isomorphism $\zeta : \overline{V} \to V^*$. Note that $\dim \overline{V} = \dim V^* = (\mathrm{card}\, F)^{|I|} = (\mathrm{card}\, F)^{\dim V}$. For a $\lambda \in \mathbb{P}$ define $\lambda' = \left\{ j \in I \mid v_j^* = f\big(v_{(i,\lambda)}\big),\, i \in \lambda \right\}$.

The map ζ provides an internal complete direct product structure on V^*. By the definition of λ' we have $\overline{\zeta}\big(\iota_\lambda(V_\lambda)\big) = V_{\lambda'^*}$ for every $\lambda \in \mathbb{P}$, hence $V^* = \prod_{\lambda\in\mathbb{P}} V_{\lambda'^*}$. The linear map ζ induces an isomorphism $\overline{\zeta}$ between the semilattices $(\mathrm{Sub}\, \overline{V}, \cap)$ and $(\mathrm{Sub}\, V^*, \cap)$.

From (1) we can obtain the following two decompositions:

$$x = x_\lambda + (x - x_\lambda) \quad \text{where} \quad x_\lambda \in V_{\lambda'},\ \text{and}\ x - x_\lambda \in (V_{\lambda'^*})^\flat \tag{2}$$

for every $x \in V$, and

$$w = w_\lambda + (w - w_\lambda) \quad \text{where} \quad w_\lambda \in V_{\lambda'^*},\ \text{and}\ w - w_\lambda \in (V_{\lambda'})^\sharp \tag{3}$$

for every $w \in V^*$. Moreover, $w_\lambda(x) = w_\lambda\big(x_\lambda + (x - x_\lambda)\big) = w_\lambda(x_\lambda) = \big(w_\lambda + (w - w_\lambda)\big)(x_\lambda) = w(x_\lambda)$, hence

$$w_\lambda(x) = w(x_\lambda) \tag{4}$$

holds for every $x \in V$ and $w \in V^*$.

For a subspace $U \leqq V$, let $\psi(U) = \prod_{\lambda \in \mathbb{P}} (U \cap V_\lambda)$. Now, $\psi(U)$ is a subspace of $\overline{V}$, thus $\overline{\zeta}\big(\psi(U)\big)$ is a subspace of V^*.

Claim. For any $U \leqq V$ the subspace $\overline{\zeta}\big(\psi(U)\big)$ is closed.

Proof of Claim. We show that if $U_\lambda \leqq V_{\lambda'^*}$ for $\lambda \in \mathbb{P}$ then we have $\prod_\lambda U_\lambda = \big(\bigcap_\lambda U_\lambda^\flat\big)^\sharp$.

For every $w \in \prod_\lambda U_\lambda$ and $v \in \bigcap U_\lambda^\flat$, we shall prove that $w(v) = 0$. Now, v can be written as a sum, $v = \sum v_\lambda$ with $v_\lambda \in V_{\lambda'}$. Then $w(v) = \sum w(v_\lambda) = \sum w_\lambda(v)$, according to (4). Since U_λ is finite dimensional, $w_\lambda \in U_\lambda = (U_\lambda^\flat)^\sharp$ holds, therefore $w_\lambda(v) = 0$ and so $\sum w_\lambda(v) = w(v) = 0$, as well.

For the other direction assume that $w \in \big(\bigcap_{\lambda \in \mathbb{P}} U_\lambda^\flat\big)^\sharp$. Now, $U_\lambda \leqq V_{\lambda'^*}$ for each $\lambda \in \mathbb{P}$, hence $(V_{\lambda'^*})^\flat \leqq (U_\lambda)^\flat$, and so $(x - x_\lambda) \in (U_\lambda)^\flat$ for any $x \in V$ and $\lambda \in \mathbb{P}$. Thus $x_\lambda = x - (x - x_\lambda) \in (U_\lambda)^\flat$ for every $x \in (U_\lambda)^\flat$. Moreover, $x_\lambda \in V_{\lambda'} \leqq \bigcap_{\alpha \in \mathbb{P},\ \alpha \neq \lambda} (V_{\alpha'^*})^\flat \leqq \bigcap_{\alpha \in \mathbb{P},\ \alpha \neq \lambda} U_\alpha^\flat$, hence $x_\lambda \in \bigcap_{\alpha \in \mathbb{P}} U_\alpha^\flat$ for every $\lambda \in \mathbb{P}$ and $x \in (U_\lambda)^\flat$. Since $w \in \big(\bigcap_{\alpha \in \mathbb{P}} U_\alpha^\flat\big)^\sharp$, for any $\lambda \in \mathbb{P}$ and $x \in (U_\lambda)^\flat$ we have that $w(x_\lambda) = 0$. Hence by (4) we obtain that $w_\lambda(x) = 0$. Thus for every $\lambda \in \mathbb{P}$ we have $w_\lambda \in (U_\lambda^\flat)^\sharp$. As U_λ is finite dimensional, $w_\lambda \in U_\lambda$, hence $w \in \prod_{\lambda \in \mathbb{P}} U_\lambda$, and this is what we wanted to prove.

The mapping that takes U to $\overline{\zeta}\big(\psi(U)\big)$ is an embedding that preserves intersections. Since $\overline{\zeta}\big(\psi(U)\big) = \prod_\lambda \overline{\zeta}\big(\psi(U) \cap V_\lambda\big)$, is closed by the claim, the map $\overline{\zeta} \circ \psi$ is a semilattice-embedding from $(\operatorname{Sub} V, \cap)$ to $(\operatorname{ClSub} V^*, \cap)$. $\square$

3. Vector spaces over division rings

We use the notation ${}_D V$ for a vector space V over a division ring D (the elements of D are left multipliers). Our aim is to characterize those finite-dimensional vector spaces such that the semilattice $(\operatorname{Sub} V, \cap)$ is embeddable into the semilattice $(\operatorname{Sub} V, +)$. The question will be answered a bit more generally as we will characterize those pairs of vector spaces ${}_F A$ and ${}_G B$ of the same finite dimension such that $(\operatorname{Sub} A, \cap)$ is embeddable into $(\operatorname{Sub} B, +)$. For this we start with a few definitions. Let ${}_F A$ and ${}_G B$ be vector spaces over the division rings F and G. A *duality* is a one-to-one and order-reversing mapping from $\operatorname{Sub} A$ to $\operatorname{Sub} B$, a *projectivity* is a one-to-one and order-preserving mapping from $\operatorname{Sub} A$ to $\operatorname{Sub} B$. For the moment, we will consider only finite-dimensional vector spaces, now. Both a duality and a projectivity can exist only if $\dim A = \dim B$. A projectivity preserves the dimension of a subspace, while a duality "reverses" it, meaning that if δ is a duality and $U \leqq A$ with $\dim U = k$ and $\dim A = n$ then $\dim(\delta U) = n - k$. For a vector space ${}_DV$ we denote by $L(V)$ the space of linear forms over V. The right vector space $L(V)_D$ is called the *adjoint space* of ${}_DV$. As we committed ourselves to use left vector spaces we have to convert the adjoint space into a left vector space. We denote by ${}_{D^{\mathrm{op}}}V^*$ the *canonical dual space* of ${}_DV$. Here $V^* = L(V)$ and $(D, *) = D^{\mathrm{op}}$ with the multiplication $f, g \in D^{\mathrm{op}}$ and $a \in V^*$: $f * g = gf$ and $f * a := af$. In [2] (Chapter IV.1) the following is proved: If $\dim {}_DV$ is finite then there is a duality from ${}_DV$ to ${}_{D^{\mathrm{op}}}V^*$. Hence the construction of Theorem 1 is unattainable in the non-commutative case, as the vector spaces that we use in the proof are over distinct division rings. Or, if we copy the arguments we embed $(\operatorname{Sub} {}_FA, \cap)$ into $(\operatorname{Sub} A^*_F, +)$. We show a way of clarifying the situation.

THEOREM 2. *Let ${}_FA$ and ${}_GB$ be left vector spaces of equal finite dimension at least* 3. *Then the following are equivalent:*

(1) *there is a meet-semilattice embedding from* $(\operatorname{Sub} {}_FA, \cap)$ *into* $(\operatorname{Sub} {}_GB, \cap)$.

(2) *there is a lattice embedding from* $\operatorname{Sub} {}_FA$ *into* $\operatorname{Sub} {}_GB$.

(3) *there exists a join-semilattice embedding from* $(\operatorname{Sub} {}_FA, +)$ *into* $(\operatorname{Sub} {}_GB, +)$.

(4) *there is an embedding from F into G.*

PROOF. (1) $\iff$ (2) $\iff$ (3). Let us assume that we have a meet-semilattice embedding α. It is an order-preserving map, hence it preserves the height of an element. Thus for every $U_1, U_2 \leqq A$ we have $\dim \alpha(U_1) = \dim U_1$, $\dim \alpha(U_2) = \dim U_2$ and $\dim\big(\alpha(U_1) \cap \alpha(U_2)\big) = \dim \alpha(U_1 \cap U_2) = \dim(U_1 \cap U_2)$. Thus $\dim\big(\alpha(U_1) + \alpha(U_2)\big) = \dim(U_1 + U_2)$. The embedding is order-preserving, hence $\alpha(U_1) + \alpha(U_2) \leqq \alpha(U_1 + U_2)$. The dimension of these two subspaces are equal, so $\alpha(U_1) + \alpha(U_2) = \alpha(U_1 + U_2)$. Thus a meet-semilattice embedding is always a lattice embedding. Similarly, a join-semilattice embedding is always a lattice embedding, hence (1) and (2) and (3) are equivalent.

(4) $\Longrightarrow$ (1). Let $F \leqq G$ and $A = F^n$. Then, $B = G^n$. Let $U \leqq A$ be a subspace of A. The set of vectors U is naturally a subset of B. Let $f(U) = \langle U \rangle_B$, the subspace generated by $U \subseteqq B$ of B. We claim that f is a meet-semilattice embedding. The rank of a system of vectors does not depend on the field. Thus for $u, v_1, v_2, \ldots, v_n \in A$ it is obvious that $u \in \langle v_1, \ldots, v_n \rangle_A$ is equivalent to $u \in \langle v_1, \ldots, v_n \rangle_B$. Also the independence of the system of

vectors $\{v_1, \dots, v_n\}$ over A is equivalent to the independence of this system over B. Thus $f(U \oplus V) = f(U) \oplus f(V)$. Let $U_1, U_2 \leqq A$, $U_1 \cap U_2 = W$, $U_1 = W \oplus \tilde{U_1}$, $U_2 = W \oplus \tilde{U_2}$. $\tilde{U_1} \cap \tilde{U_2} = W \cap \tilde{U_1} = W \cap \tilde{U_2} = \{0\}$, hence $f(\tilde{U_1}) \cap f(\tilde{U_2}) = f(W) \cap f(\tilde{U_1}) = f(W) \cap f(\tilde{U_2}) = \{0\}$. Thus $f(U_1) \cap f(U_2) = \big(f(W) \oplus f(\tilde{U_1})\big) \cap \big(f(W) \oplus f(\tilde{U_2})\big) = f(W) = f(U_1 \cap U_2)$, so f is indeed a meet-semilattice embedding.

(2) $\implies$ (4). By the Projective Structure Theorem in [2] (Chapter III.1) we have that in the case of $\dim_F A = \dim_G B = 3$ there is a lattice isomorphism (projectivity) from ${}_F A$ to ${}_G B$ if and only if F and G are isomorphic division rings. Now we consider this theorem with the following approach: if we use any process of coordinatization of a projective plane, the resulting division ring of the coordinates is uniquely determined up to isomorphism. There are several ways to construct the division ring (see e.g. [3]). We will use the following one, because it is based on the lattice of the projective plane: Take four geometrically independent points, two of them will be 0 and 1. Geometrical independence is a property that we can check using purely the lattice of the projective plane, and it depends only on the sublattice generated by the four points. We can define an addition and a multiplication on the points of the line through 0 and 1 using only lattice operations.

Let $f: {}_F A \to {}_G B$ be a lattice embedding. Now consider four arbitrary independent points in ${}_F A$ and take the images of these four points at f. The latter will form an independent quadruple of points in $\operatorname{Sub} B$, because the sublattice they generate is isomorphic to the one, generated by the original four points of ${}_F A$. At first, we construct the coordinates for the image of ${}_F A$ in ${}_G B$. We obtain F, of course (up to isomorphism), as a subset of a line in B. When we proceed for the division ring for the whole line of B, we obtain the division ring G. The coordinates of the points corresponding to F remain unchanged. Hence F will be a sub-division ring of G. By the Projective Structure Theorem both F and G are uniquely determined, hence F is embeddable into G.

If $\dim_F A = \dim_G B > 3$, then let $A' \leqq A$ be a subspace of dimension 3. The image of A' is B'. Since f is order-preserving it also preserves the dimension of a subspace, hence $\dim A' = \dim B' = 3$. An order-preserving map is restrictable to a subspace, hence $f|_{A'}$ is a lattice embedding from $\operatorname{Sub}_F A'$ into $\operatorname{Sub}_G B'$. Thus $F \leqq G$ holds, again. $\square$

Now we can easily characterize the finite-dimensional vector spaces A over a division ring F with the property that $(\operatorname{Sub}_F A, \cap)$ is embeddable into $(\operatorname{Sub}_F A, +)$.

THEOREM 3. *Let ${}_F A$ be a finite-dimensional vector space over a division ring F. There exists an embedding from* $(\operatorname{Sub}_F A, \cap)$ *into* $(\operatorname{Sub}_F A, +)$ *if and only if* $\dim_F A \leqq 2$ *or* $\dim_F A > 2$ *and there is an embedding from F into F^{op}.*

Proof. If $\dim_F A = 1$ then the lattice is the two element lattice, so the statement is true. If $\dim_F A = 2$ then $\mathrm{Sub}_F A \simeq M_{|F|}$, the lattice of height 2, where there are $|F|$ elements in the middle.

Finally, let $\dim_F A \geqq 3$. Let f denote an embedding of $(\mathrm{Sub}_F A, \cap)$ into $(\mathrm{Sub}_F A, +)$ and δ be a projectivity from ${}_F A$ to ${}_{F^{\mathrm{op}}} A^*$. The map δf is an embedding of ${}_F A$ to ${}_{F^{\mathrm{op}}} A^*$. Hence, by Theorem 2, $F \leqq F^{\mathrm{op}}$ holds.

If $F \leqq F^{\mathrm{op}}$, then by Theorem 2 ${}_F A$ embeds to ${}_{F^{\mathrm{op}}} A^*$. □

Hence, in order to show a finite-dimensional vector space that has the lattice of subspaces not embeddable into the dual of the lattice it is enough to construct a division ring not embeddable into its opposite. These division rings are well-studied. For a detailed account on this and other questions on division rings see [4] (Chapter 2.5). Here we recall a construction for such division rings. Consider a Galois extension $F|\mathbb{Q}$ with cyclic Galois group Z_n. Let $3 \leqq n$ and denote by σ a generator of Z_n. Let $q \in \mathbb{Q}$ be a nonzero element. The division ring D is generated by F and an element y satisfying the following relations: $y^n = q$ and for any $x \in F$ $xy = y\sigma(x)$. This division ring is a finite-dimensional vector space over $\mathbb{Q}$, hence an embedding from D into D^{op} is always an isomorphism. Note that in the Brauer group the class of G is the inverse of the class of G^{op} for a division ring G. Hence if G is isomorphic to G^{op} then the order of the class of G in the Brauer group is 2. The order of the class of D in the Brauer group is n, hence D and D^{op} are not isomorphic. Thus D is not embeddable into D^{op}.

It is worth mentioning the following corollary of Theorems 2 and 3.

Corollary 4. *Let ${}_F A$ and ${}_G B$ be vector spaces of equal finite dimension at least* 3. *There is an embedding from* $(\mathrm{Sub}_F A, +)$ *into* $(\mathrm{Sub}_G B, \cap)$ *if and only if there is an embedding from F into G^{op}.*

Finally, we consider infinite dimensional vector spaces.

Theorem 5. *Let D be a division ring and let U be a left D-vector space with countable infinite dimension.*

(1) *There is an embedding of* $(\mathrm{Sub}\, V, \cap)$ *into* $(\mathrm{Sub}\, V, +)$ *for every V over D if and only if $D \leqq D^{\mathrm{op}}$.*

(2) *There is an embedding of* $(\mathrm{Sub}\, V, \cap)$ *into* $(\mathrm{Sub}\, V, +)$ *for every infinite-dimensional vector space V over D if and only if* $(\mathrm{Sub}\, U, \cap)$ *embeds into* $(\mathrm{Sub}\, U, +)$.

(3) *There is an embedding of* $(\mathrm{Sub}\, U, \cap)$ *into* $(\mathrm{Sub}\, U, +)$ *if and only if* $(\mathrm{Sub}\, V, \cap)$ *embeds into* $(\mathrm{Sub}\, U, +)$ *for every finite-dimensional V.*

Proof. For item 1 first let $D \leqq D^{\mathrm{op}}$. Then $(\mathrm{Sub}_D V, +) \leqq (\mathrm{Sub}_{D^{\mathrm{op}}} V, +)$ and the construction of Theorem 1 combined with this embedding gives the desired result. For the other direction, by Theorem 3 the embedding of the 3-dimensional case implies that $D \leqq D^{\mathrm{op}}$.

For items 2 and 3 if $(\mathrm{Sub}\, U, \cap)$ embeds to $(\mathrm{Sub}\, U, +)$, then for every finite-dimensional vector space V the semilattice $(\mathrm{Sub}\, V, \cap)$ embeds to $(\mathrm{Sub}\, U, +)$, as well. Replacing V_λ by U with an embedded copy of V_λ, copying the proof of Theorem 1 gives the desired result. $\square$

Although the original question of [1] remains open, by the following consequence of item 2 of Theorem 5 we got closer to the answer.

COROLLARY 6. *Let* $\dim_D U = \aleph_0$. *If there is no embedding of* $(\mathrm{Sub}\, V, \cap)$ *into* $(\mathrm{Sub}\, V, +)$ *for some infinite-dimensional vector space* V *over* D*, then* $(\mathrm{Sub}\, U, \cap)$ *does not embed into* $(\mathrm{Sub}\, U, +)$*, either.*

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