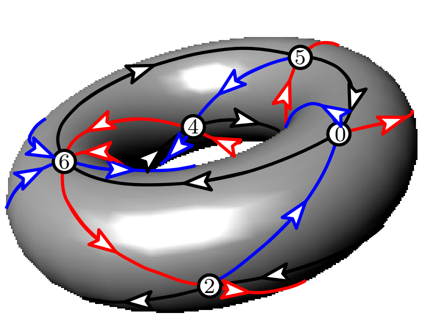

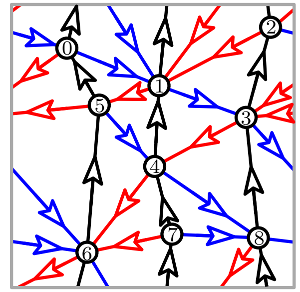

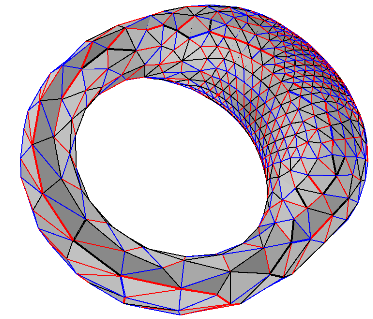

Computation of toroidal Schnyder woods made simple and fast: from theory to practice

june 26th 2025, SoCG (Kanazawa)

Luca Castelli Aleardi
(LIX, Ecole Polytechnique, IP Paris)

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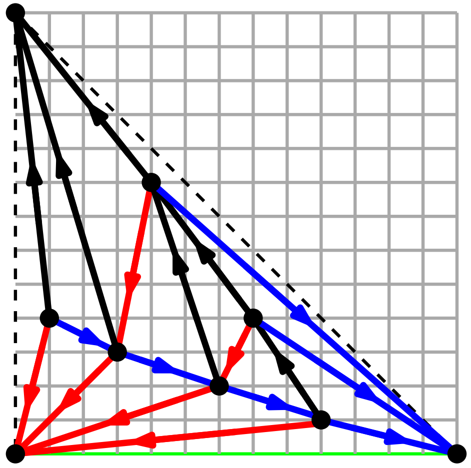
Razvan S. Puscasu
(EPFL)

Main goals of this talk

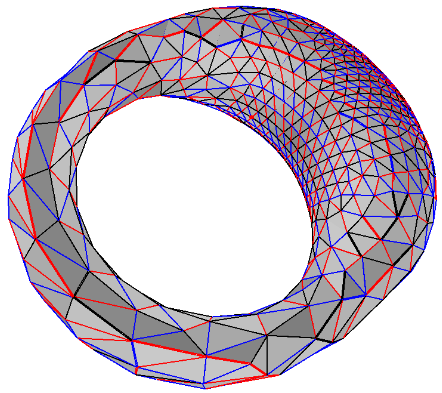
- either you do not know Schnyder woods

I will make you discover the amazing world of Schnyder woods



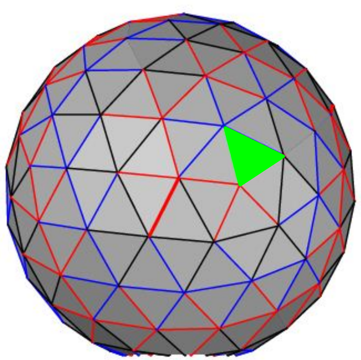
- or you already encountered Schnyder woods

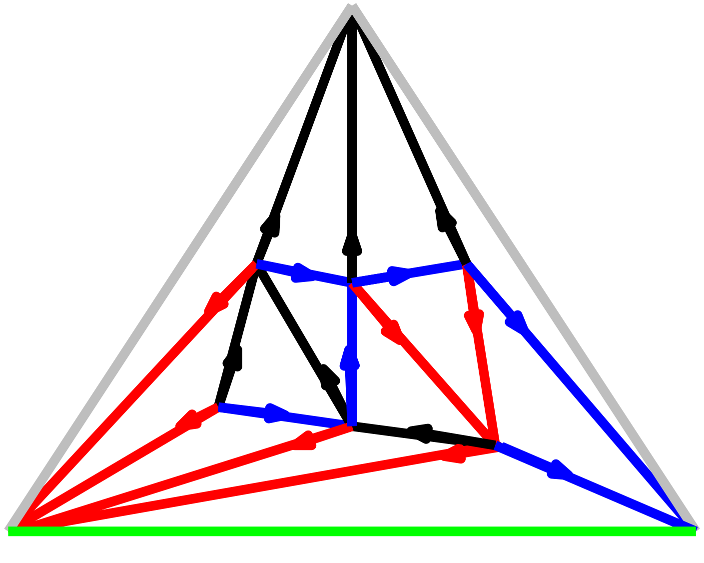
I will explain how to efficiently compute Schnyder woods for toroidal triangulations



(Planar) Schnyder woods

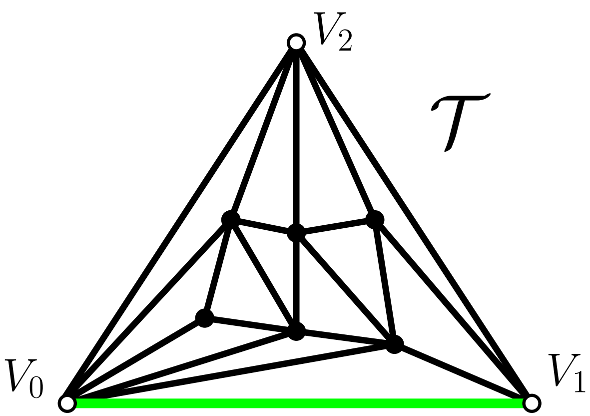
(definitions and main properties)



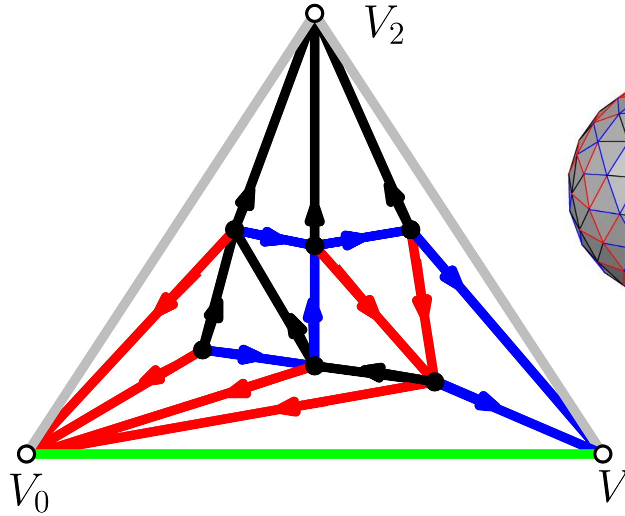


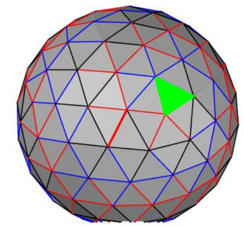
Schnyder woods for genus 0 (plane) triangulations: definition

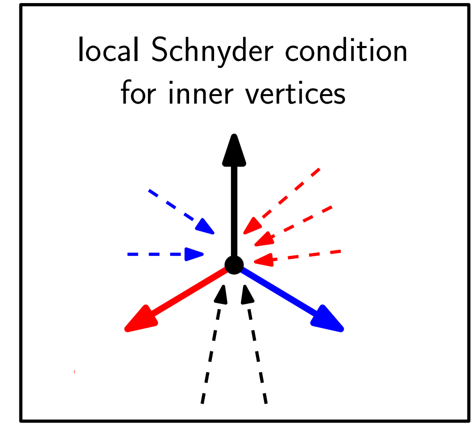
[W. Schnyder SoDA'90]



input: a genus 0 triangulation $\mathcal{T}$ with a marked **root** face $\{V_0, V_1, V_2\}$







Definition [Schnyder '90]

A **Schnyder wood** of a (rooted) planar triangulation $\mathcal{T}$ is partition of all inner edges into three sets T_0, T_1 and T_2 such that

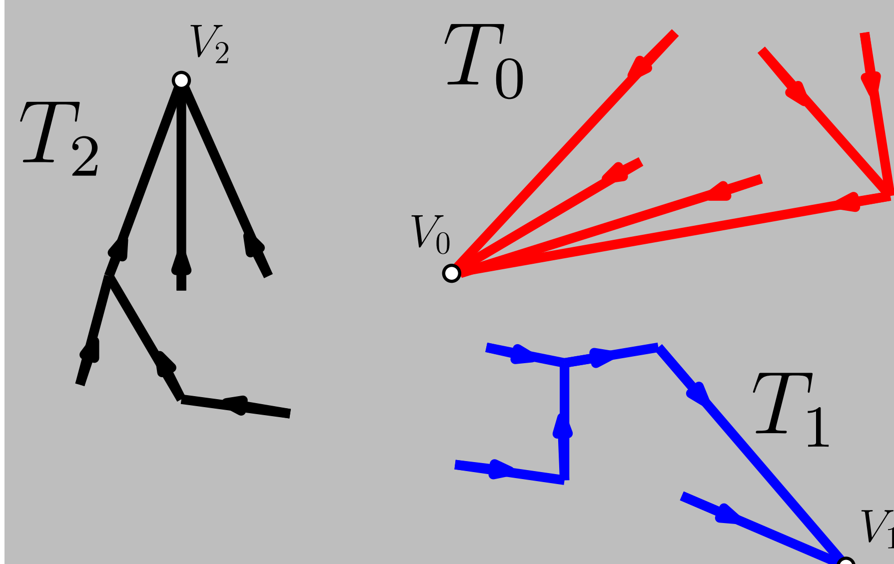
i) edges are colored and oriented in such a way that each inner node **has exactly one outgoing edge of each color**

ii) colors and orientations around each inner node must respect the local Schnyder condition

iii) inner edges incident to V_i are of color i and oriented toward V_i

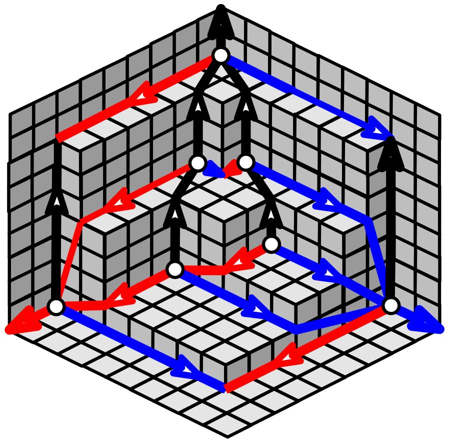
Theorem [global spanning property]
The three sets T_0, T_1, T_2 are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex v_i)

$T_i :=$ digraph defined by directed edges of color i

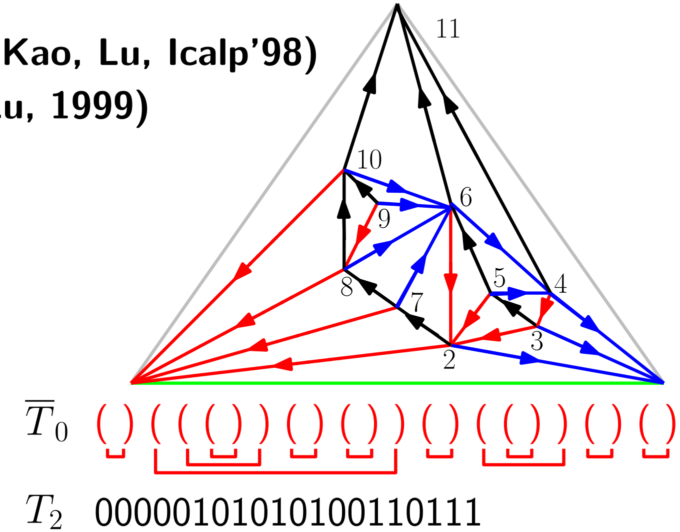


Schnyder woods: some (classical) applications

[Felsner, Bonichon et al. '10, ...]
geodesic embeddings on coplanar orthogonal surfaces, TD-Delaunay graphs and Half-Θ_6-graphs

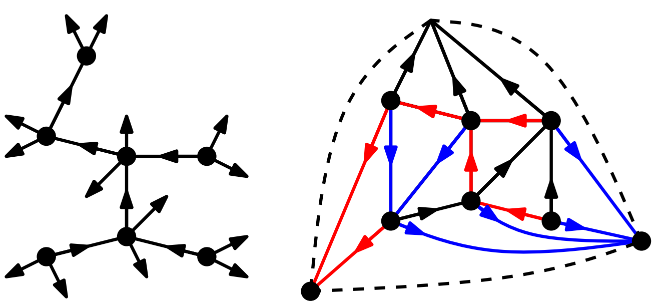


(Chuang, Garg, He, Kao, Lu, Icalp'98)
(He, Kao, Lu, 1999)



Graph encoding

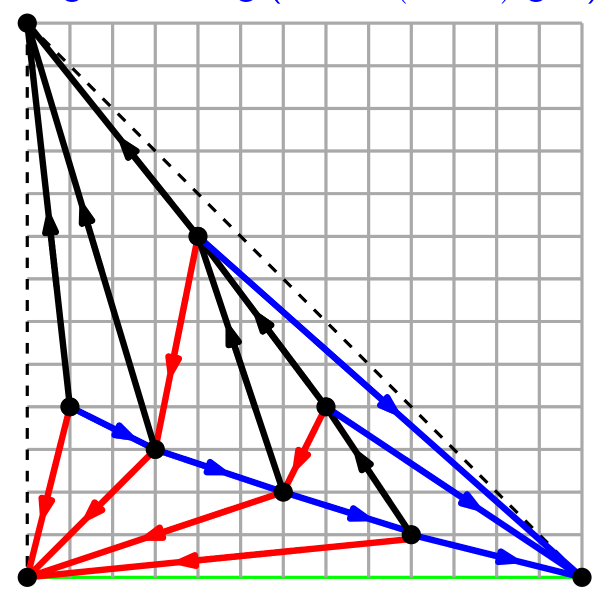
(Poulalhon-Schaeffer, Icalp 03)
bijective counting, random generation



$$c_n = \frac{2(4n+1)!}{(3n+2)!(n+1)!}$$

$\Rightarrow$ optimal encoding ≈ 3.24 bits/vertex

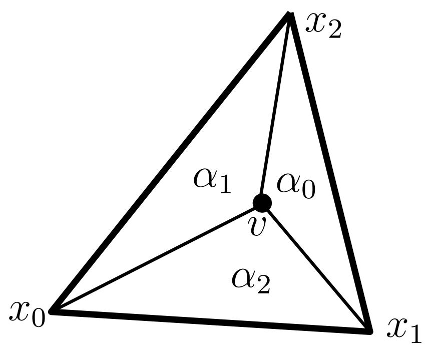
(Schnyder SoDA'90)
Planar straight-line grid drawing (on a $O(n \times n)$ grid)



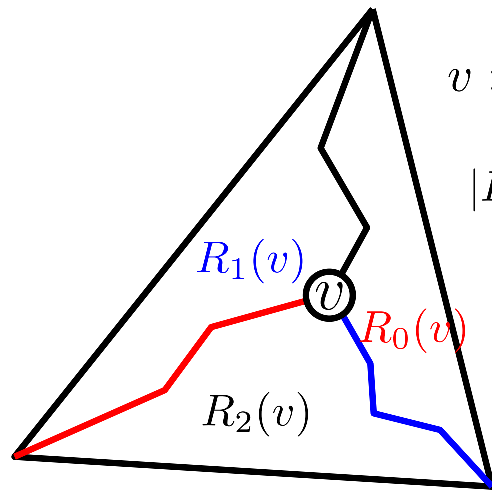
Schnyder grid drawing: face counting algorithm

Theorem (Schnyder, Soda '90)
For a triangulation $\mathcal{T}$ having n vertices, we can draw it (with no edge crossings) on a grid of size $(2n-5) \times (2n-5)$, by setting $x_0 = (2n-5, 0)$, $x_1 = (0,0)$ and $x_2 = (0, 2n-5)$.

$$v = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$$



α_i :=normalized area of (x_{i-1}, x_{i+1}, v)
(barycentric coordinates of v)

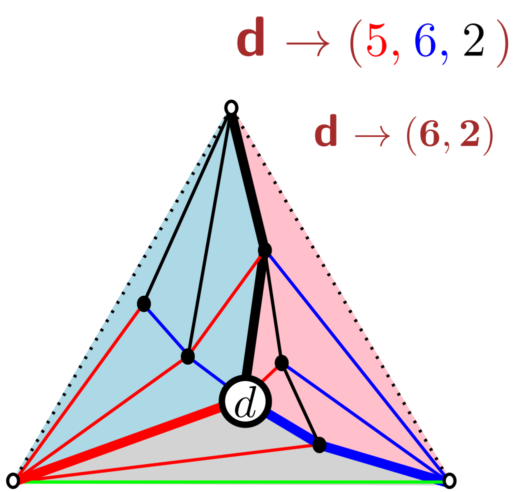


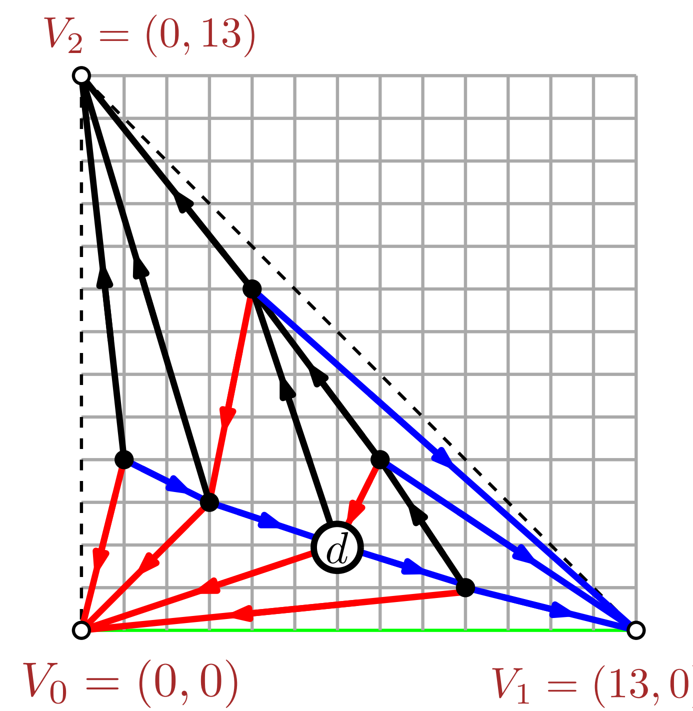
$$v := \frac{|R_0(v)|}{|F|-1} V_0 + \frac{|R_1(v)|}{|F|-1} V_1 + \frac{|R_2(v)|}{|F|-1} V_2$$

$|R_i(v)|$ is the number of triangles in $R_i(v)$

$|F| - 1 = 2n - 5$
(number of inner triangles)

Lemma
For each inner vertex v the three monochromatic paths P_0, P_1, P_2 directed from v toward each vertex V_i are vertex disjoint (except at v) and partition the inner faces into three sets $R_0(v), R_1(v), R_2(v)$





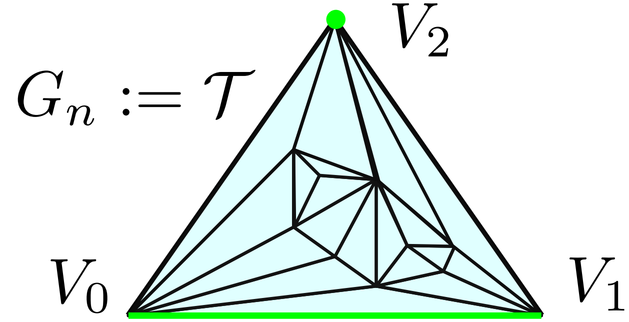
Linear-time computation of (planar) Schnyder woods

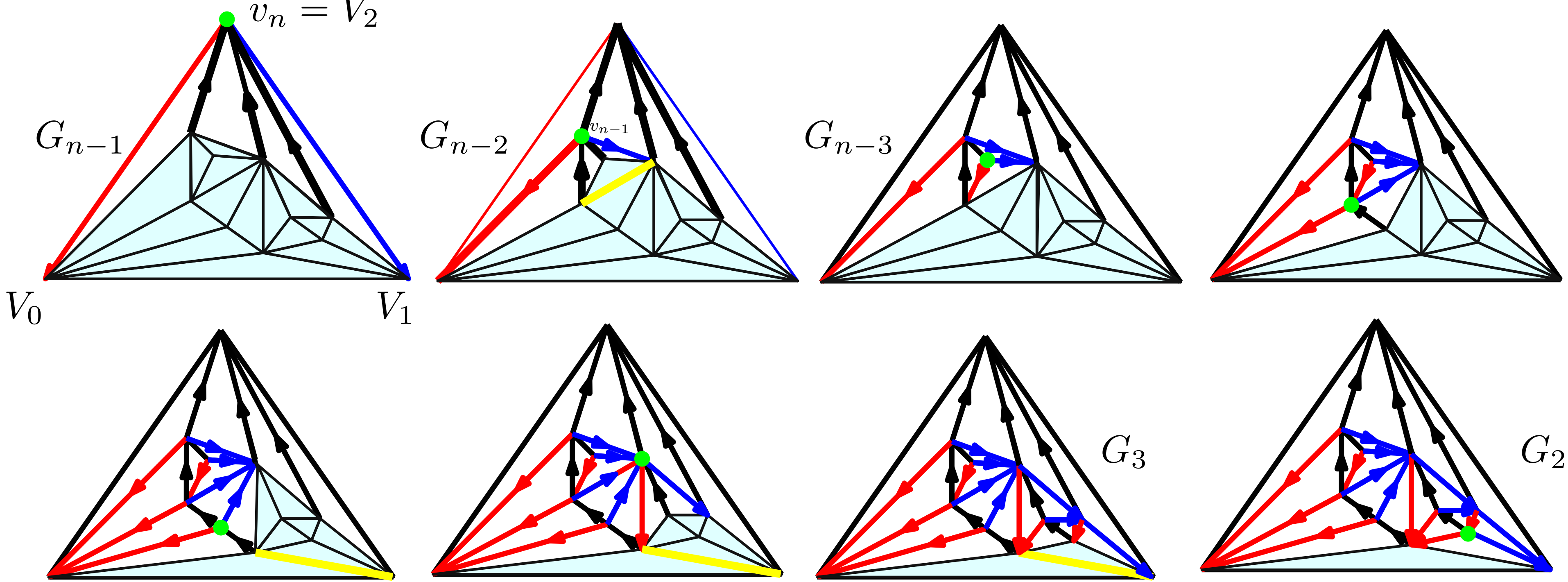
use Canonical Orderings [De Fraysseix, Pach, Pollack '89]

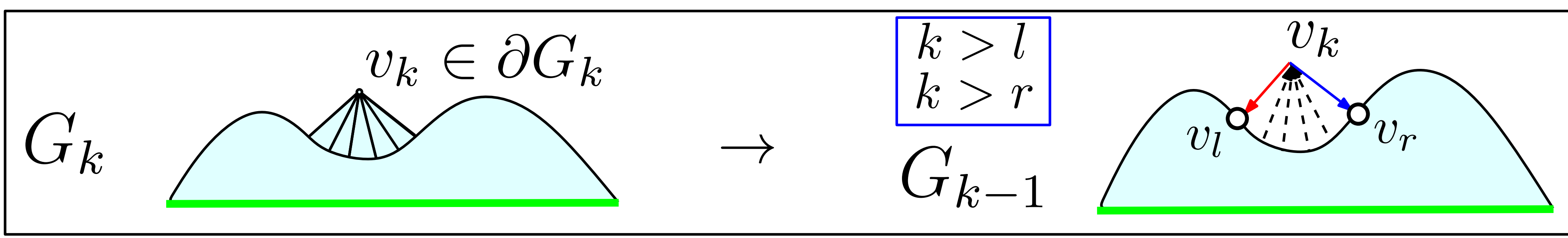
Theorem (Brehm, 2000)

A Schnyder wood can be computed in linear-time
(via a sequence of $n-2$ vertex shellings)

Remove at each step a vertex v on the boundary ∂G_k
(with no incident chordal edges in the gray region)



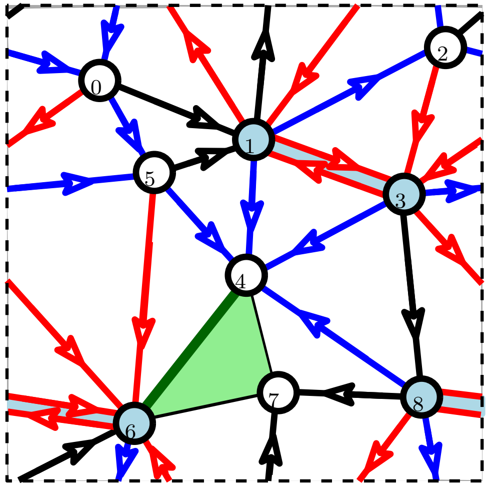




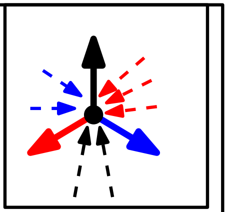
Schnyder woods for higher genus surfaces

g-**Schnyder woods** (for genus g surfaces)
[Castelli Aleardi, Fusy, Lewiner, SoCG'08]

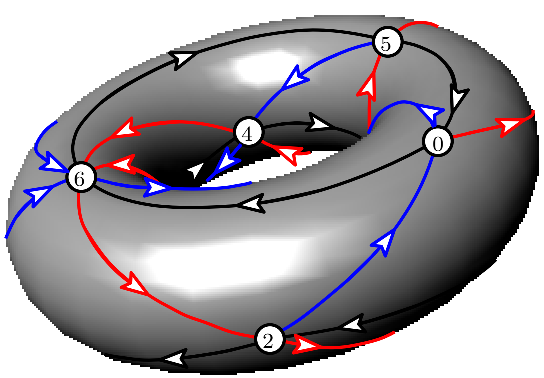
Schnyder local rule valid **only almost everywhere** (except $O(g)$ vertices)

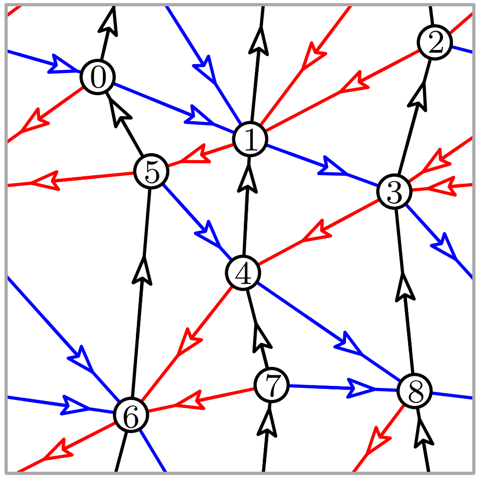


Toroidal Schnyder woods (g=1)
[Goncalves Lévêque, DCG'14]



Schnyder local rule **valid at each vertex**





Toroidal Schnyder woods: definition

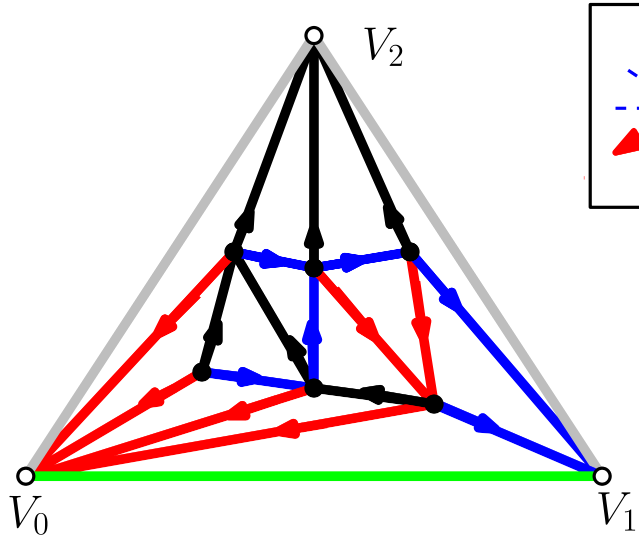
[Goncalves Lévêque, DCG'14]

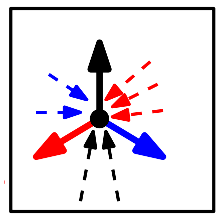
$n - e + f = 2$

$$e = 3n - 6$$

Def. **Planar Schnyder woods**

3-orientation + Schnyder local rule valid at each **inner** vertex



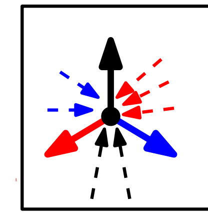


$n - e + f = 2 - 2g$

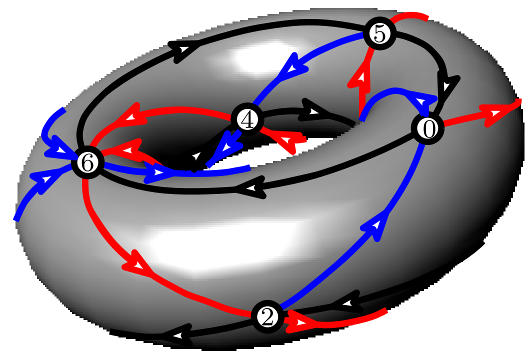
$$e = 3n$$

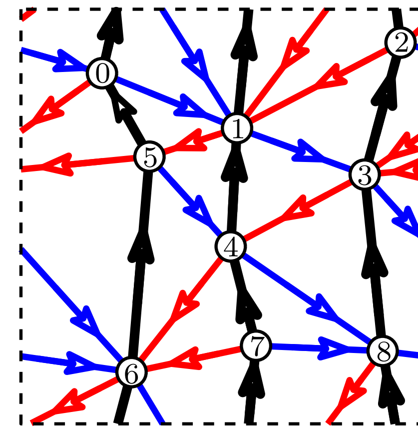
Def. **Toroidal Schnyder woods**

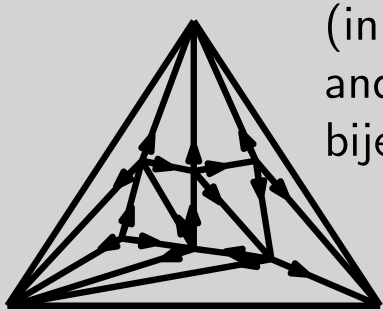
3-orientation + Schnyder local rule valid at each vertex



toroidal Schnyder wood



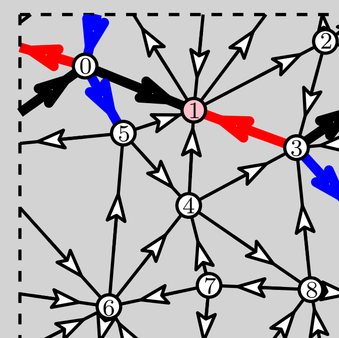




(in the plane 3-orientations and Schnyder woods are in bijection)

3-**orientation** of $\mathcal{T}$

In the toroidal case a 3-orientation does not necessarily yield a valid toroidal Schnyder wood

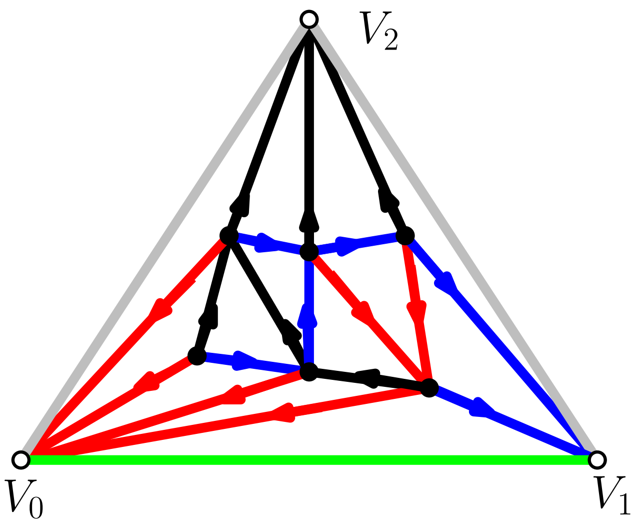


Cycles in Toroidal Schnyder woods

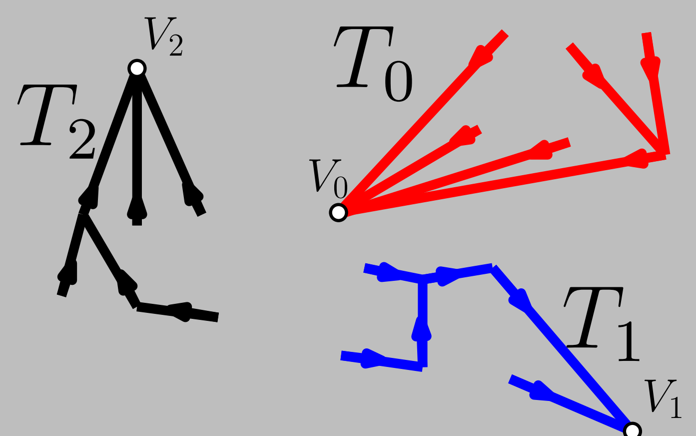
[Goncalves Lévêque, DCG'14]

$n - e + f = 2$

$$e = 3n - 6$$

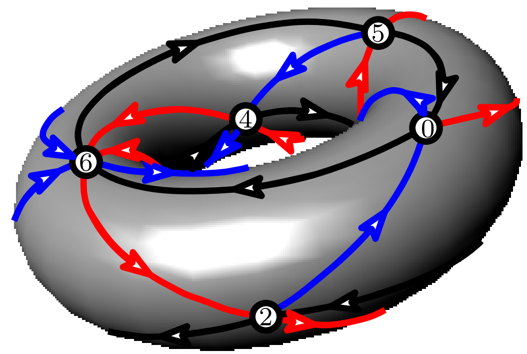


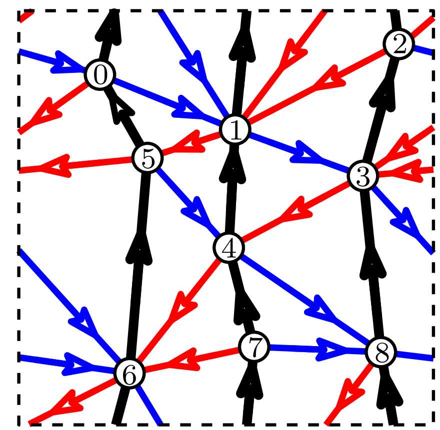
mono-chromatic components are trees: connected graphs without cycles



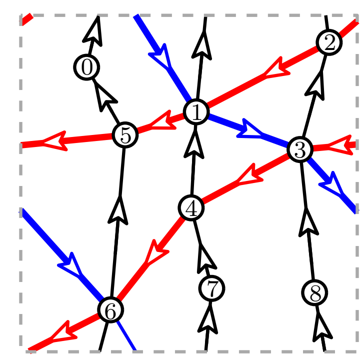
$$e = 3n$$

→ toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color



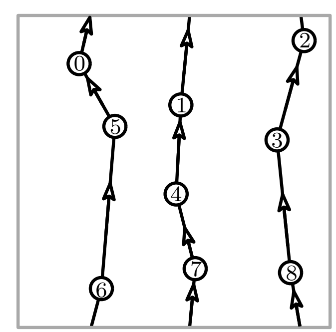


mono-chromatic cycles are non-contractibles



some colors may define disconnected components

all mono-chromatic cycles of the same color are:
homotopic and disjoint (parallel) and oriented in one direction

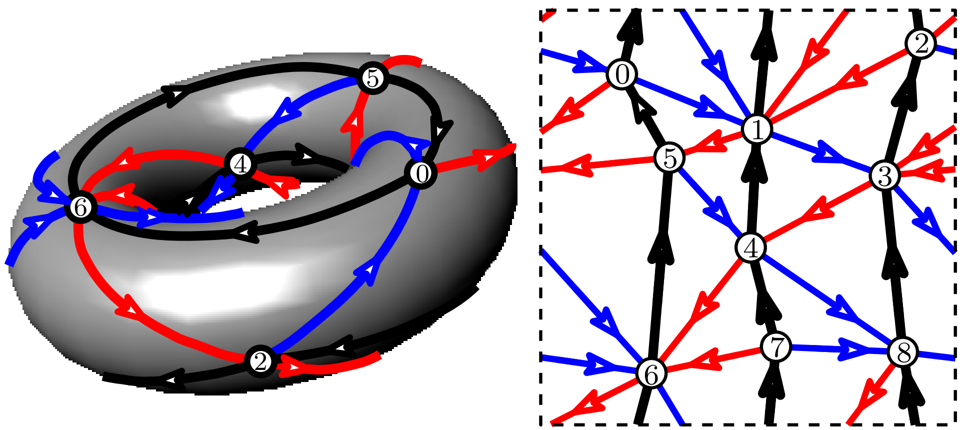


Crossing cycles: a hierarchy of Schnyder woods

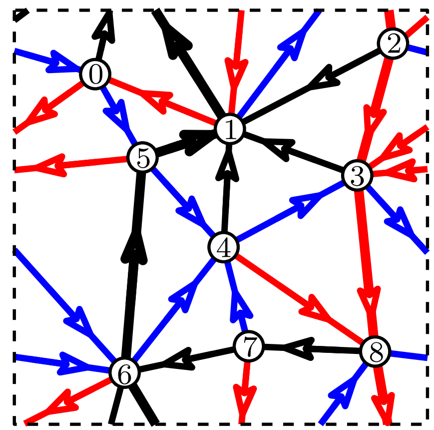
Toroidal Schnyder woods [Goncalves Lévêque, DCG'14]

Toroidal Schnyder woods can be:

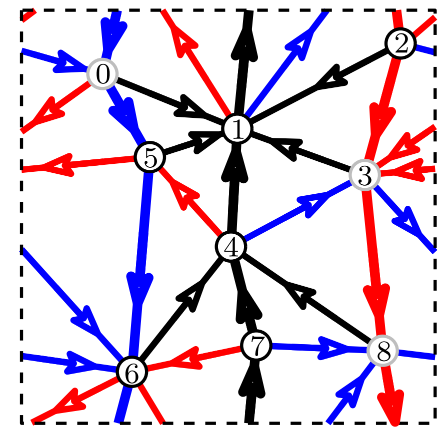
- **crossing:** every monochromatic cycle intersects at least one monochromatic cycle of each color
- only **half-crossing:** only two mono-chromatic cycles are pairwise crossing
- **non-crossing:** all mono-chromatic i-cycles are parallel (non crossing)



crossing Schnyder wood
(required for xy-periodic drawing)

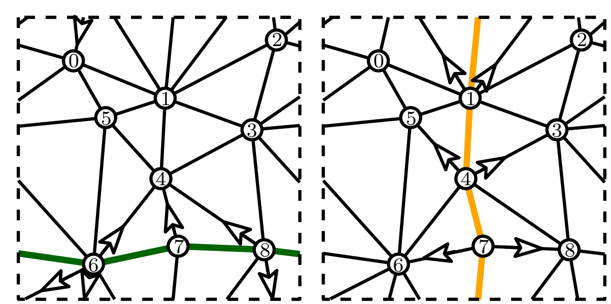


half-crossing
Schnyder wood



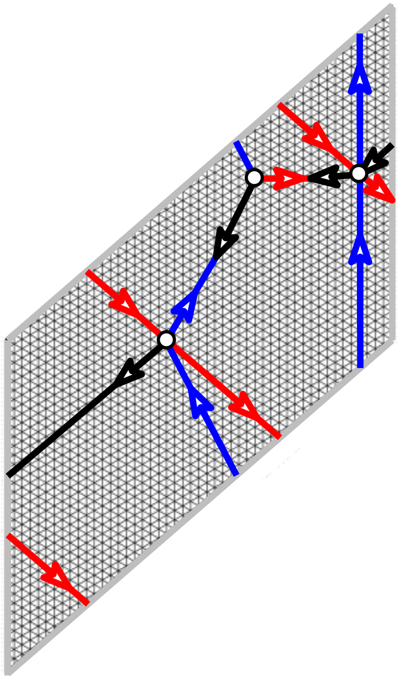
the Schnyder wood is
non-crossing but at
least **balanced**

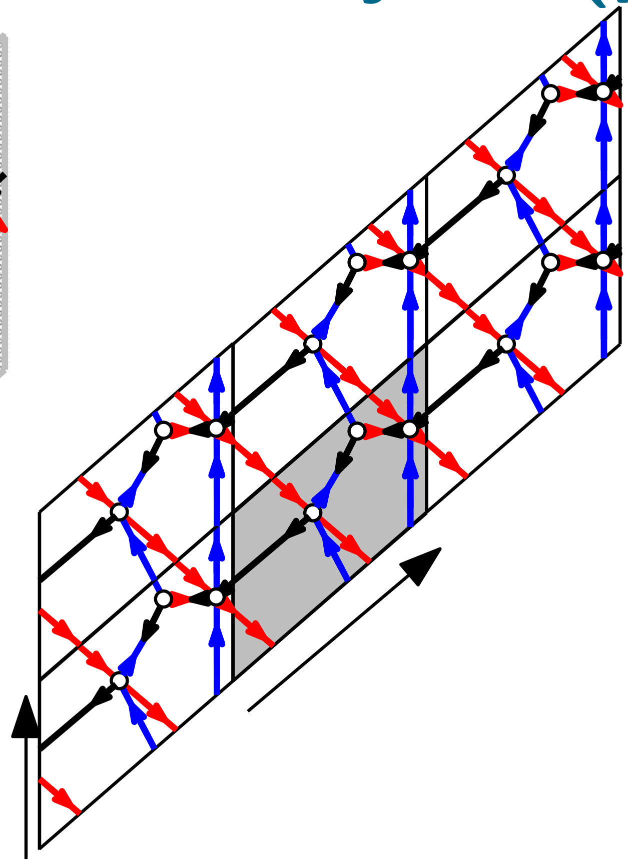
balanced Schnyder woods
useful for bijective encoding
[Despré, Goncalves, Lévêque DCG'17]



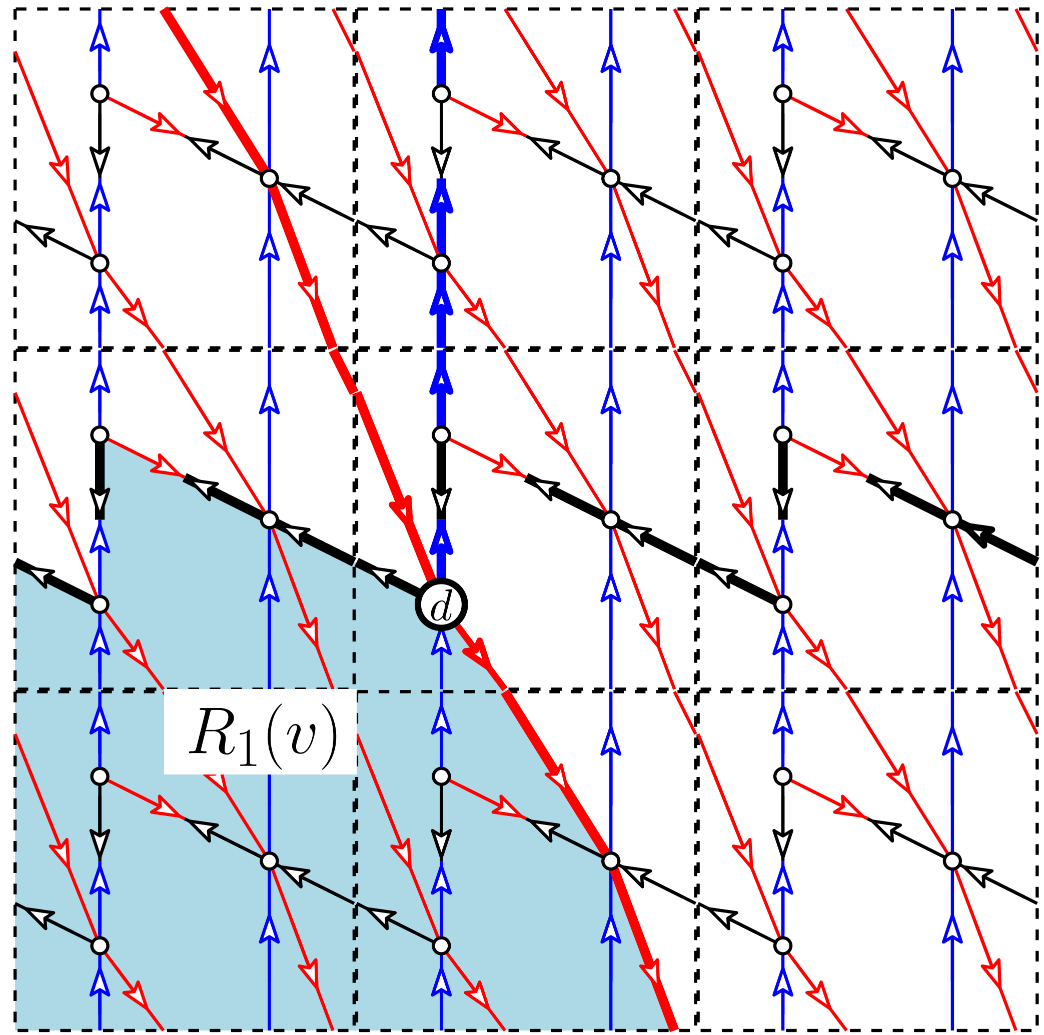
Toroidal Schnyder (periodic) drawings

[Goncalves and Lévêque, DCG'14]





xy-periodic grid drawing on a grid of size $O(n^2) \times O(n^2)$



Idea: use the face-counting method in the universal cover to assign (relative) coordinates

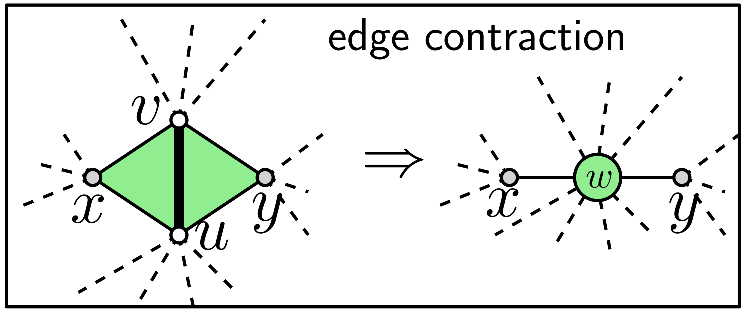
In the toroidal case: regions are unbounded

Warning: regions can be defined if the Schnyder wood is crossing

Toroidal Schnyder woods: existence

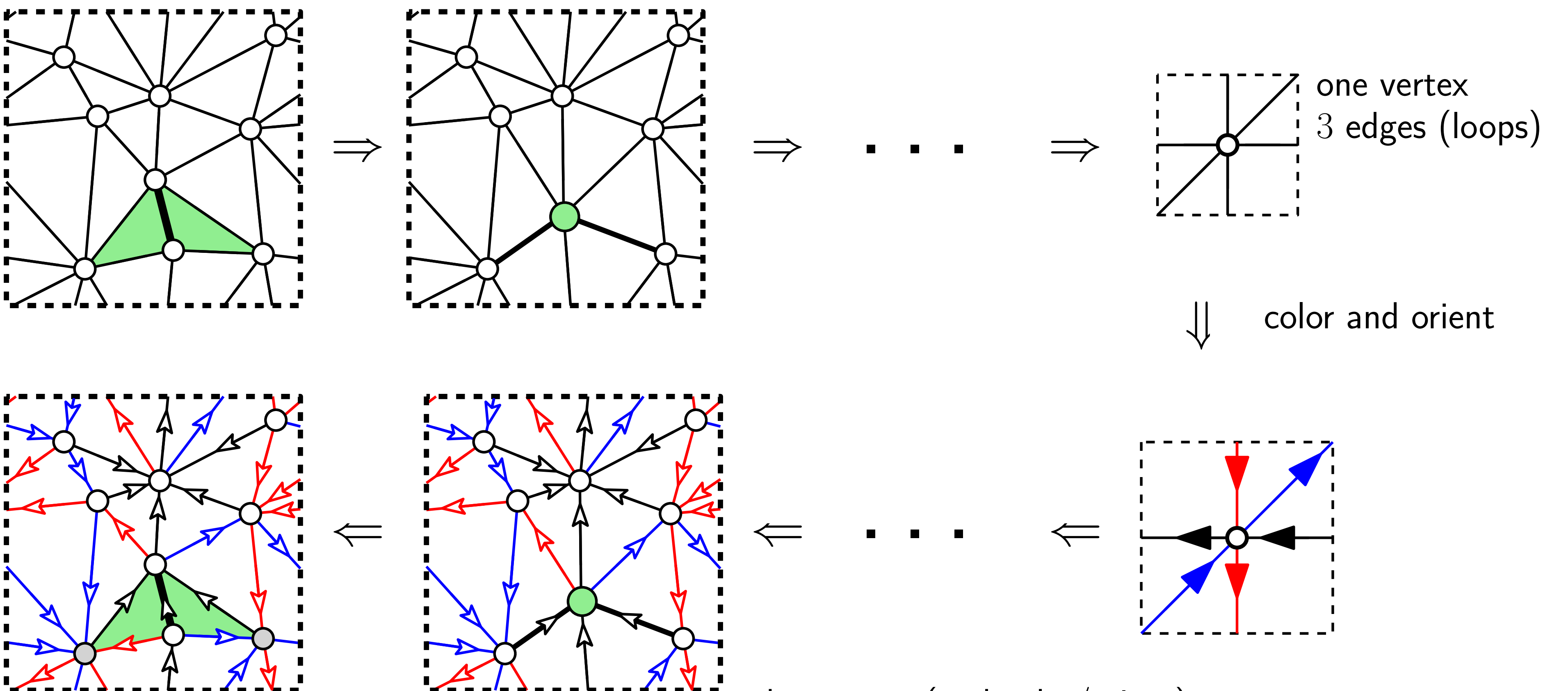
Thm[Goncalves Lévêque, DCG'14] (for general toroidal triangulations and maps)

Any toroidal triangulation admits a toroidal **crossing Schnyder wood**



remark: maintaining the crossing property can require quadratic time

perform (carefully) a sequence of $n-1$ edge contractions



Toroidal Schnyder woods: existence

Thm[Goncalves Lévêque, DCG'14] (for general toroidal triangulations and maps)

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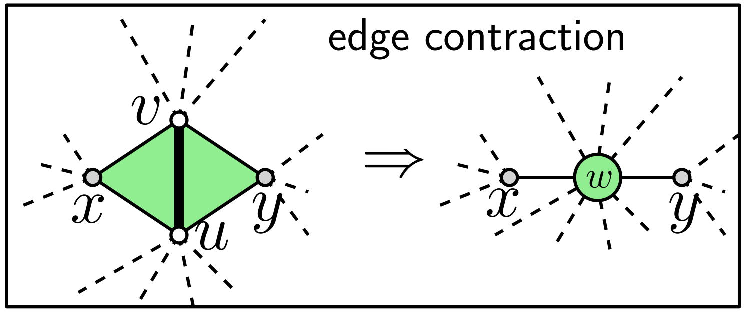
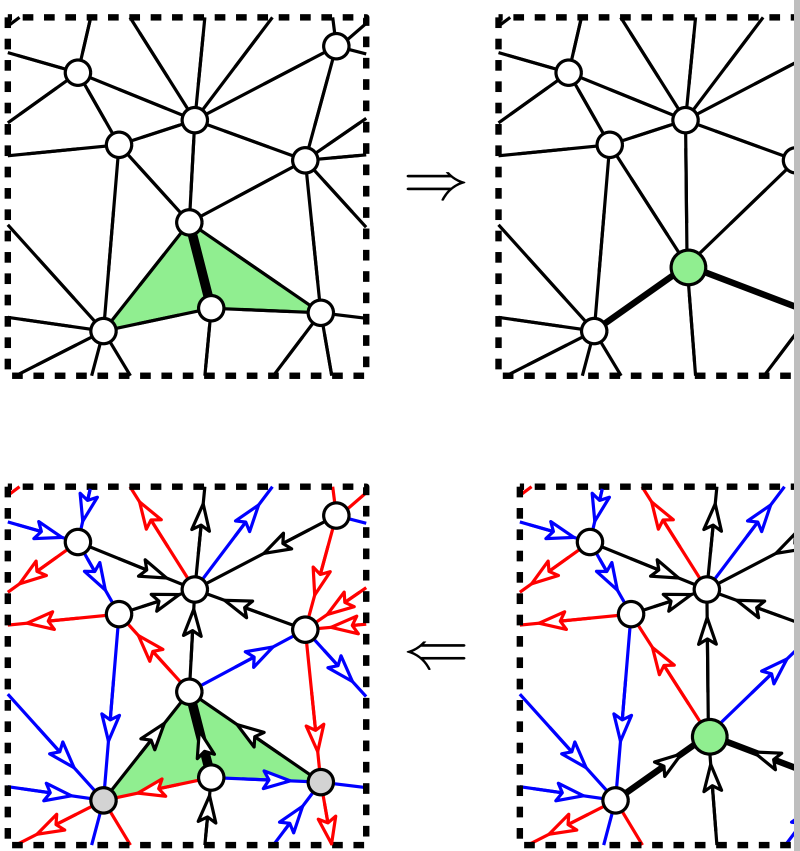
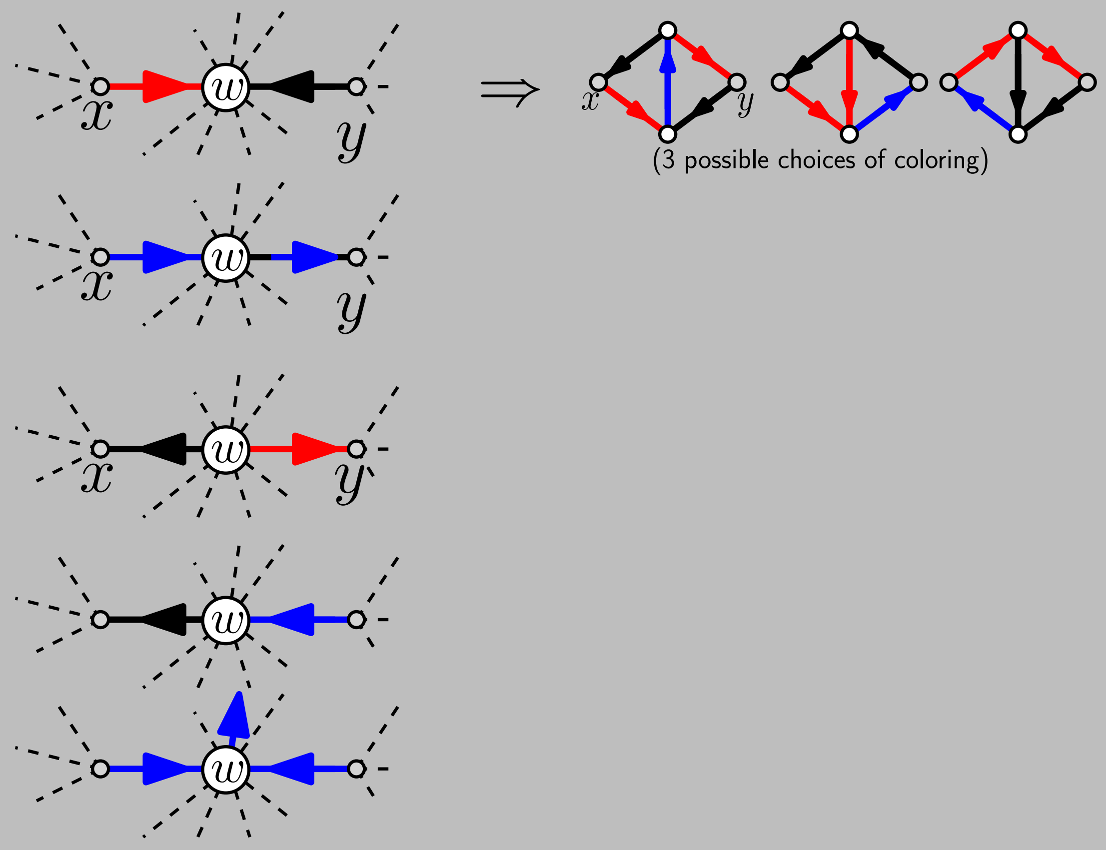


perform (carefully) a se



proof: (involved) case analysis

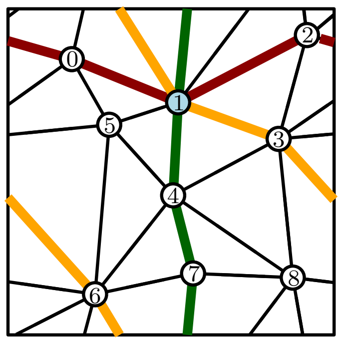


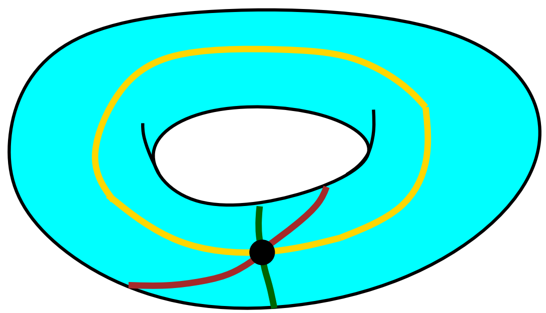
Toroidal Schnyder woods: existence II

Thm[Fijavz, unpublished]

[for simple toroidal triangulations]
(no multiple edges, no loops)

A simple toroidal triangulation contains three non-contractible and non-homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.

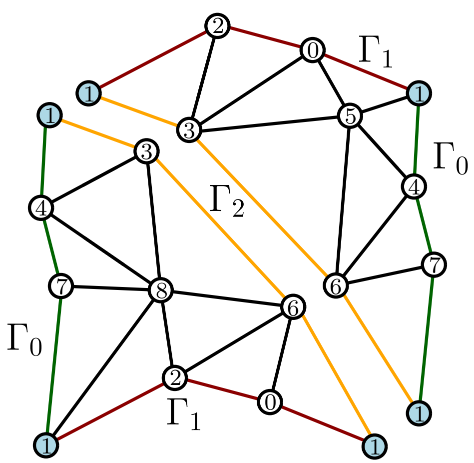




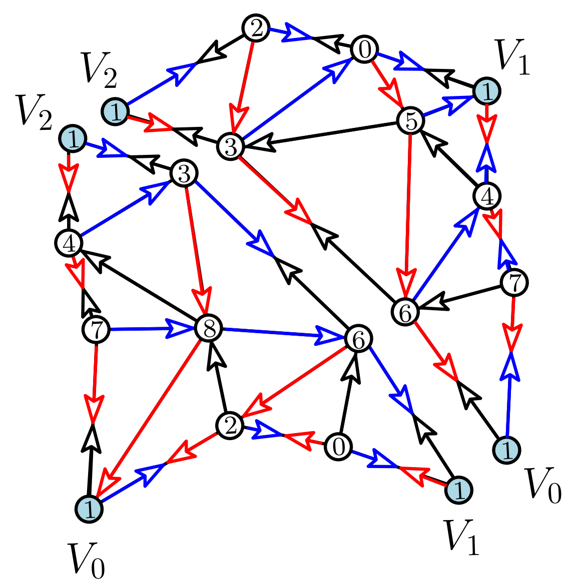
Corollary[Goncalves Lévêque, DCG'14]

Any simple toroidal triangulation admits a toroidal **crossing Schnyder wood**

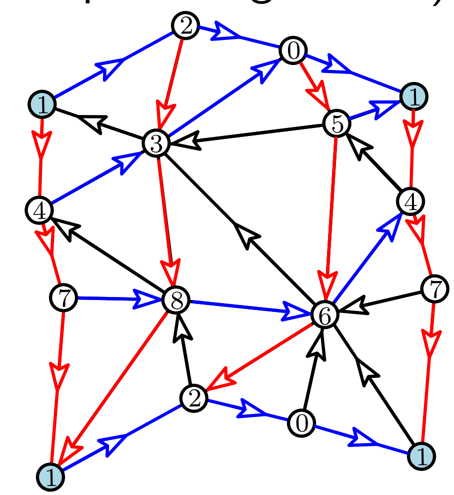
split along Γ_0, Γ_1, Γ_2



(two planar quasi-triangulations)



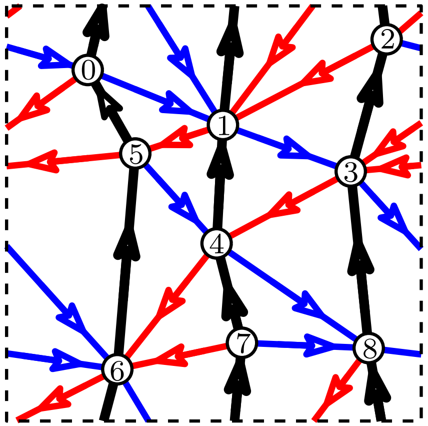
crossing toroidal Schnyder wood
(for simple triangulations)



Open problems

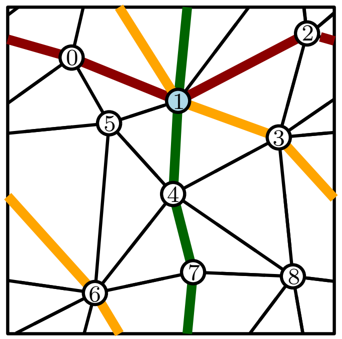
Open problem [Lévêque, 2015]
Is it possible to compute in linear-time crossing toroidal Schnyder woods via vertex shellings?

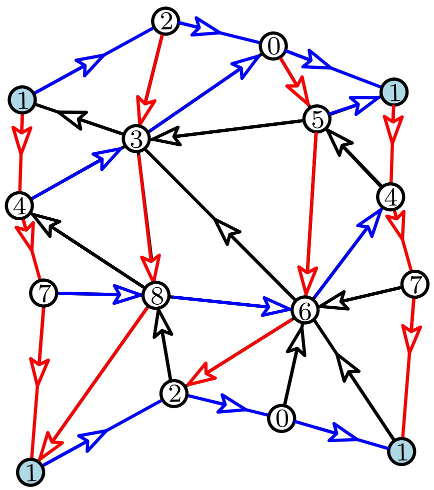
Open problem:[Goncalves Lévêque, DCG'14] is it possible to find (at least) one toroidal Schnyder wood which is crossing and with connected mono-chromatic components (one for each color)?



3 disjoint mono-chromatic cycles of color 2
Mono-chromatic cycles of color 0 and 1 are connected

Open problem:[Goncalves Lévêque, DCG'14] is it possible to find a toroidal Schnyder wood with connected mono-chromatic components and such the intersection of the three cycles is a single vertex (and with connected components)?

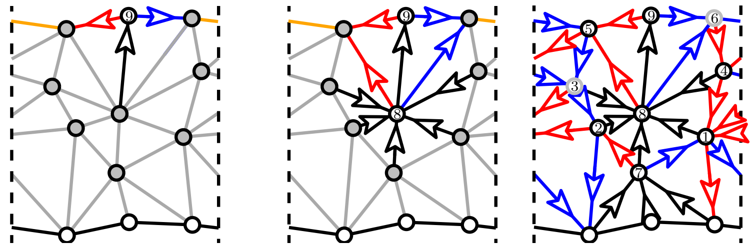


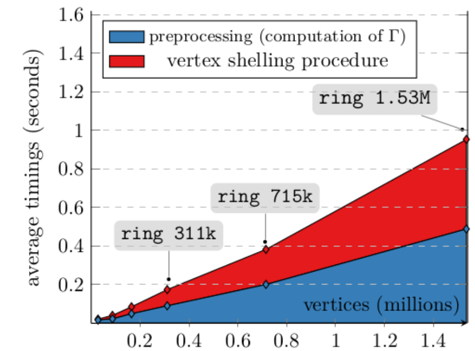


Our contributions

Open problem [Lévêque, 2015]
Is it possible to compute in linear-time crossing toroidal Schnyder woods via vertex shellings?

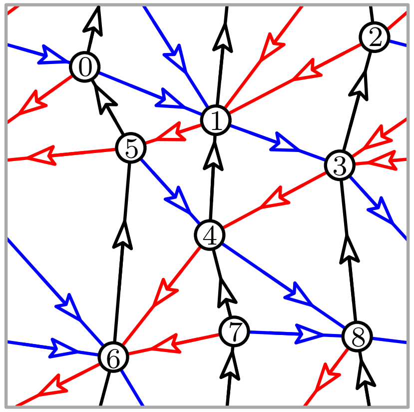
Yes, our implementation can process more than $1M$ vertices/second

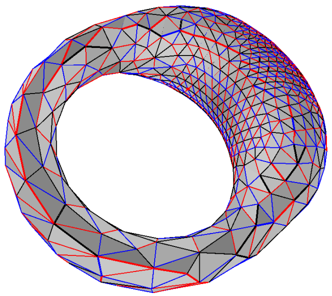




Open problem:[Goncalves Lévêque, DCG'14] is it possible to find (at least) one toroidal Schnyder wood which is crossing and with connected mono-chromatic components (one for each color)?

Almost Yes, the connectness is true for at least two colors

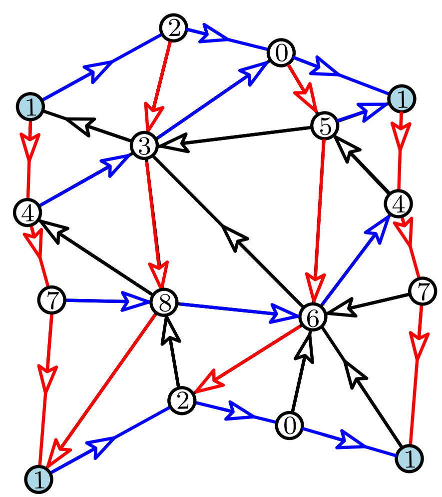




crossing Schnyder wood
Mono-chromatic cycles of color red and blue are connected

Open problem:[Goncalves Lévêque, DCG'14] is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components and such the intersection of the three cycles is a single vertex?

True for all toroidal triangulations of size at most $n = 11$ (experimental)

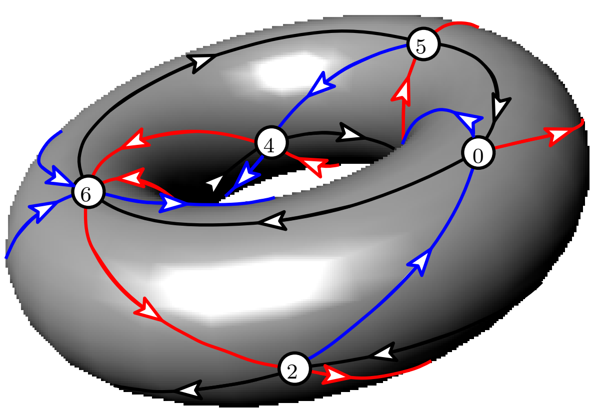


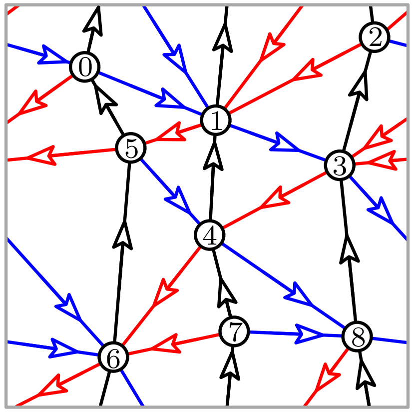
n	# irreducible triangulations	#triangulations (g = 1)
7	1	1
8	4	7
9	15	112
10	1	2109
11	–	37867

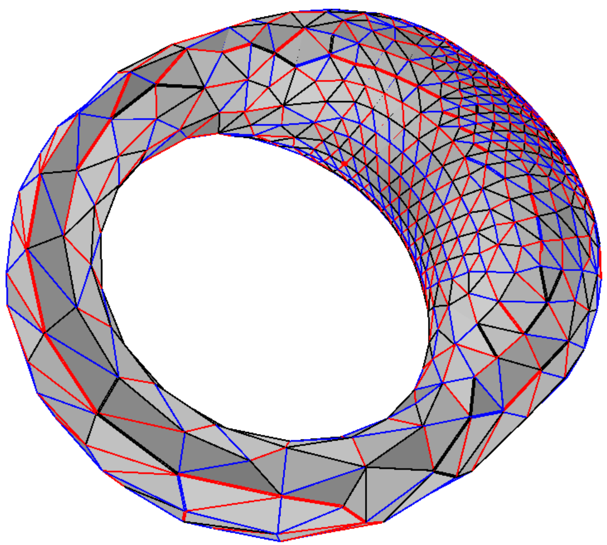
triangulations generated using **surftri** tool (by T. Sulanke)

Our contribution:

Computing in linear time (crossing) Schnyder woods with at least two monochromatic connected components (via vertex shellings)



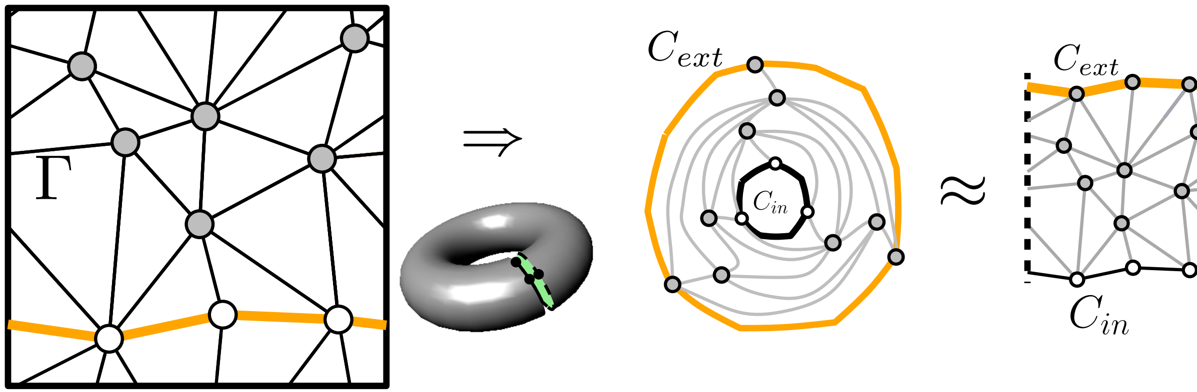




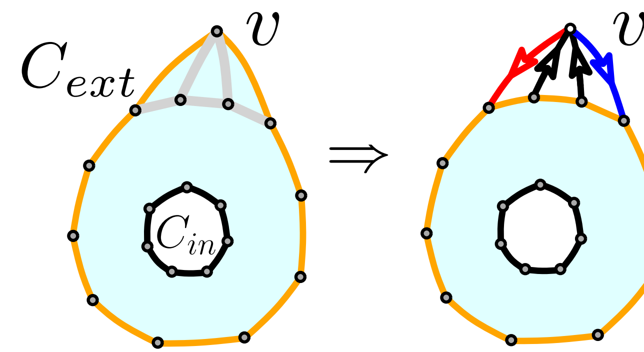
Algo 1: Toroidal Schnyder woods via (cylindric) canonical orderings

Pre-processing: cut along a non-contractible cycle Γ

Γ is split into two copies: C_{ext} and C_{in}

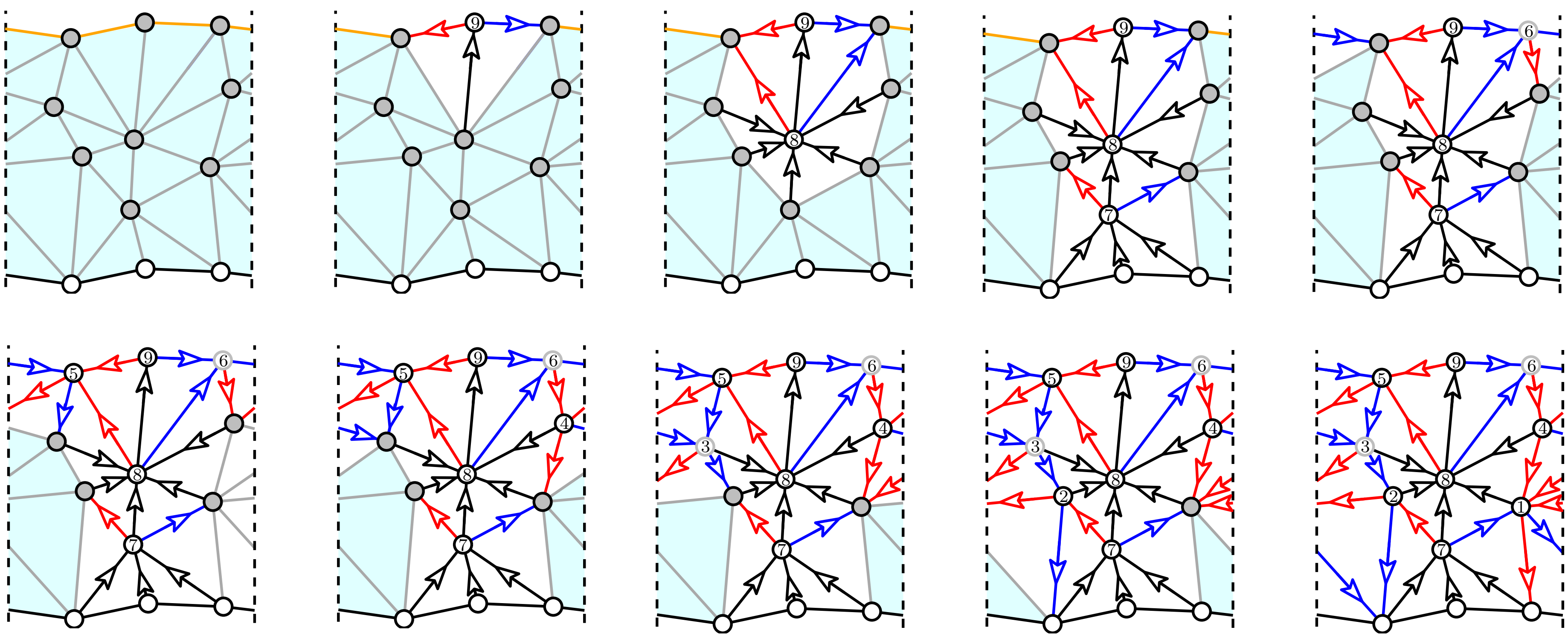


Compute a **cylindric canonical ordering**
[Castelli Aleardi, Fusy, Devillers, GD2012]



At each step remove a vertex and color/orient edges

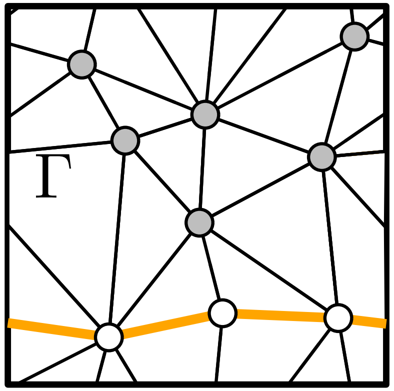
input: cylindric triangulation



output: cylindric Schnyder wood

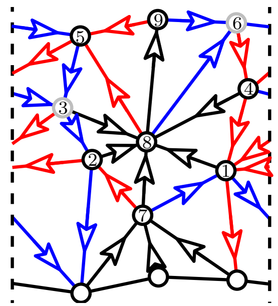
Algo 1: Toroidal Schnyder woods via (cylindric) canonical orderings

(**not necessarily crossing** Schnyder woods)



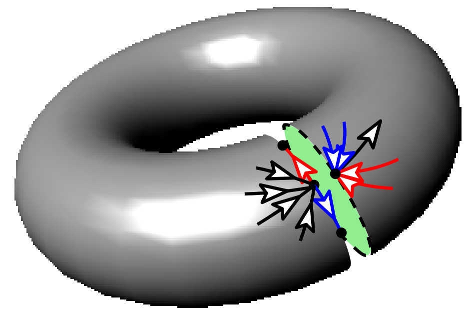
⇒

cylindric Schnyder wood



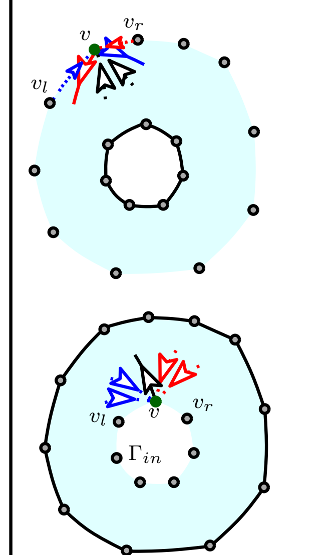
⇒

toroidal Schnyder wood

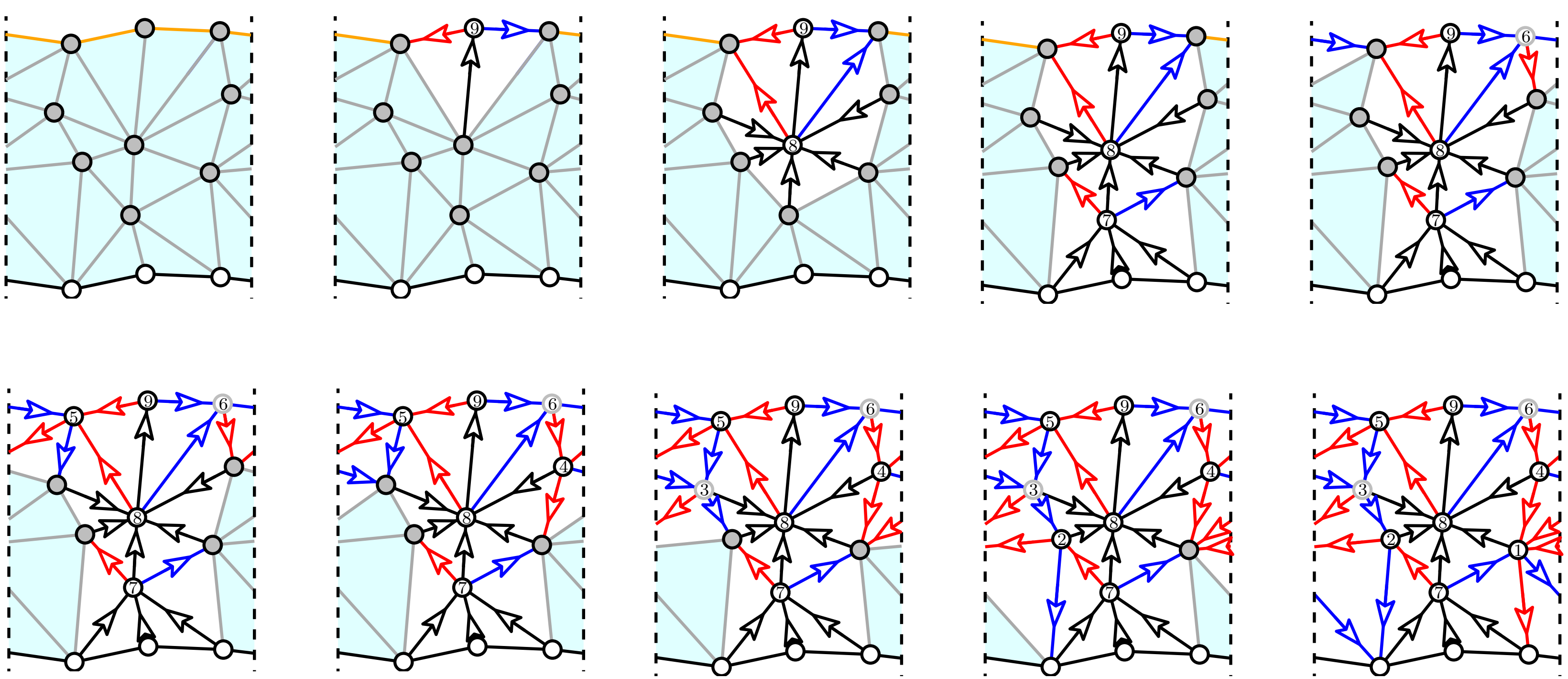


glue together the two boundaries

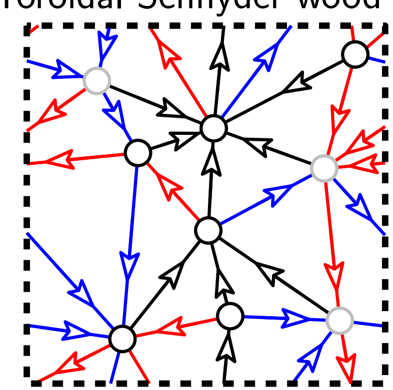
(the local Schnyder woods remains satisfied on Γ)



local invariant



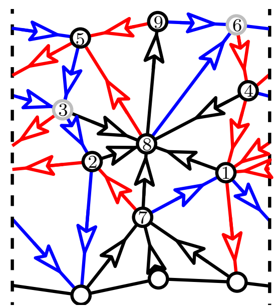
Toroidal Schnyder wood



Algo 1: Toroidal (and cylindric) Schnyder woods : properties

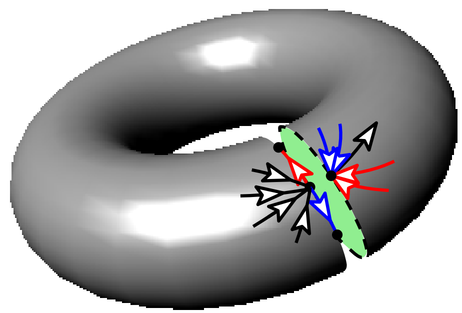
(**not necessarily crossing** Schnyder woods)

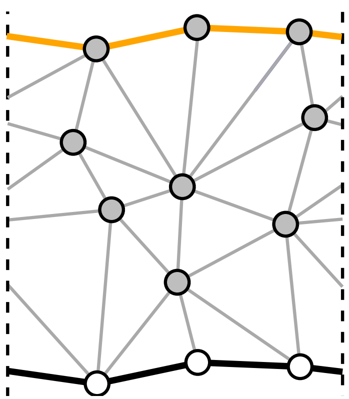
cylindric Schnyder wood

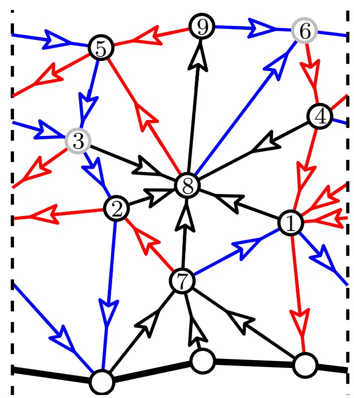


⇒

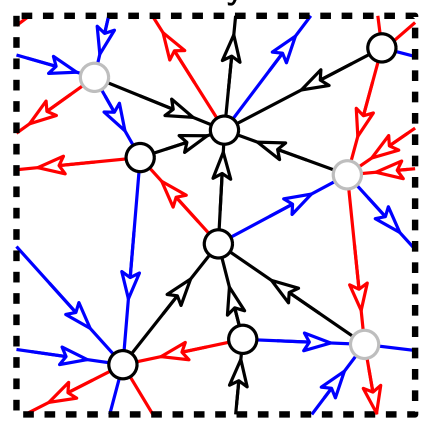
toroidal Schnyder wood

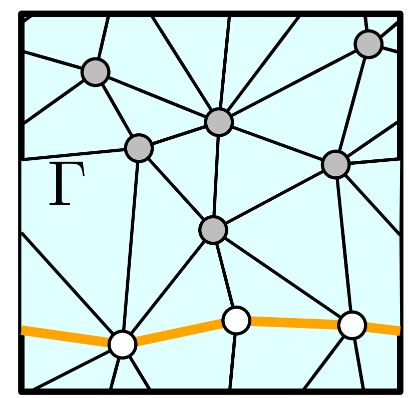






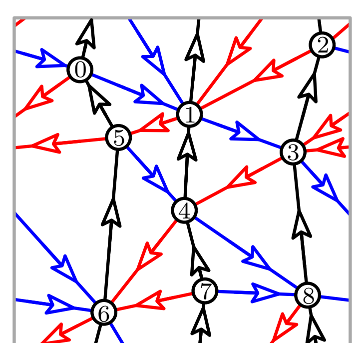
Toroidal Schnyder wood





mono-chromatic cycles are never homotopic to Γ

- edges of Γ are either 0 or 1
- 0 and 1-paths are oriented downward
- 2-paths are oriented upward
- 0, 1 and 2-paths cross the cycle Γ



If the Schnyder wood is (at least) half-crossing then the 0-cycles and 1-cycles are pairwise crossing

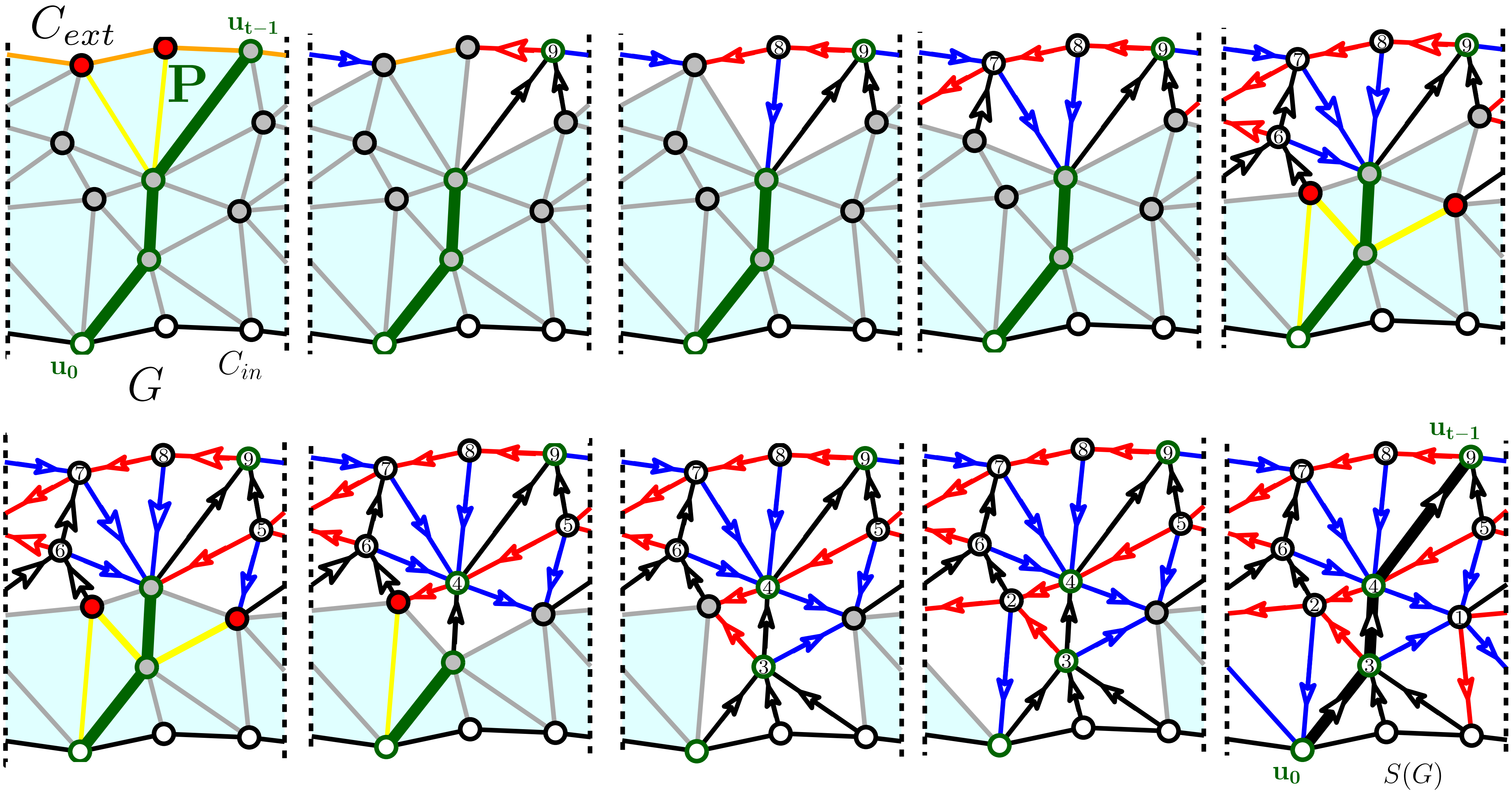
P-constrained (cylindric) Schnyder woods

Input: a **cylindric triangulation** G and a **chord-free path** $P := \{u_0, \dots, u_{t-1}\}$

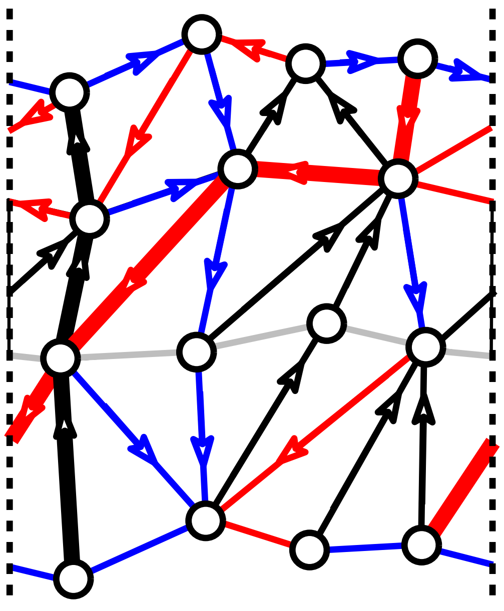
the path P must intersect the two boundary cycles only at u_0 and u_{t-1}

Output: a Schnyder wood $S_P(G)$ such that the edges of P are of black

Solution: perform vertex shellings only for (boundary) vertices which are not adjacent to an inner vertex of P

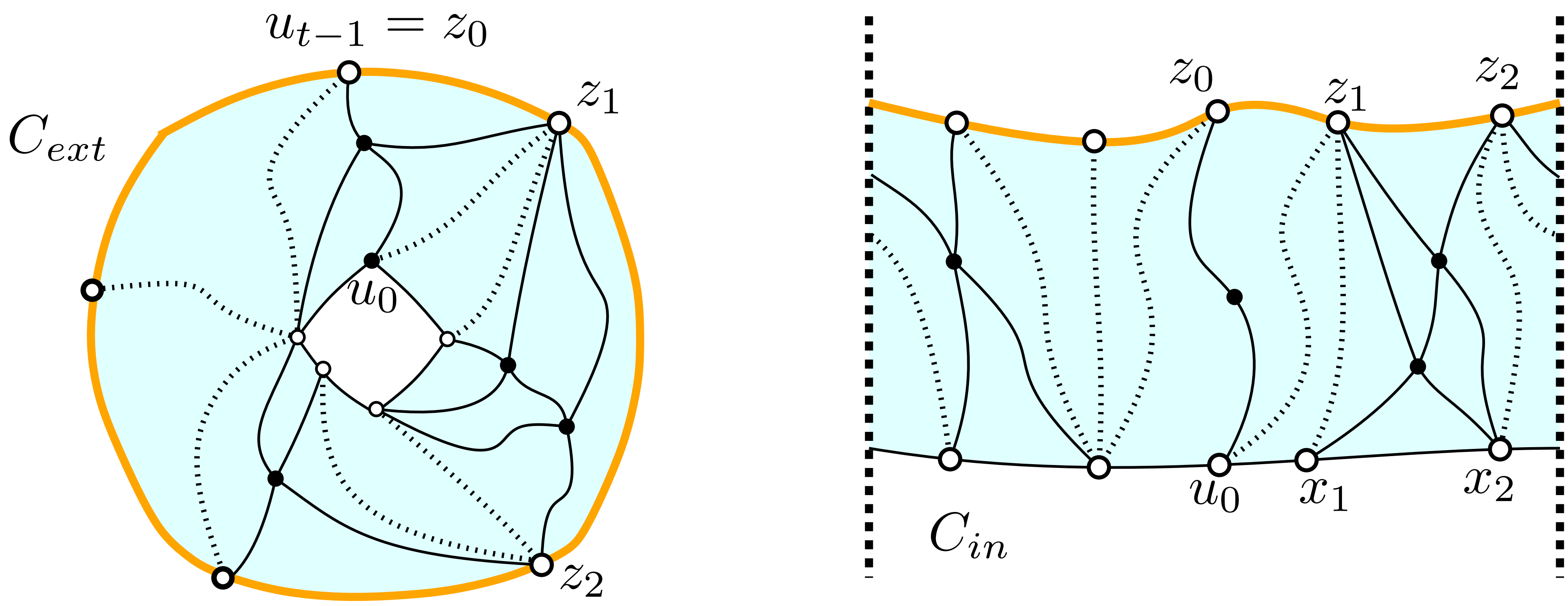


Toward half-crossing Schnyder woods
(with one connected mono-chromatic component)



Rivers: definition

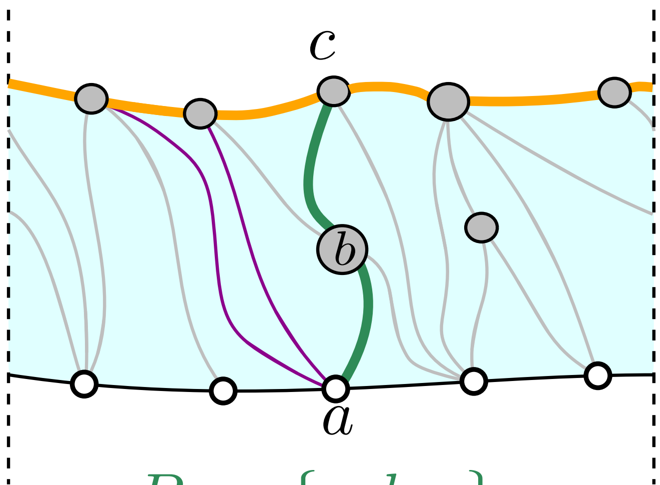
Def: a **river** is a thin cylindric triangulation such that the two boundaries are disjoint and chordless and such every vertex is incident to a non-trivial chord (connecting the two boundaries)



P-constrained right-most traversal of a river

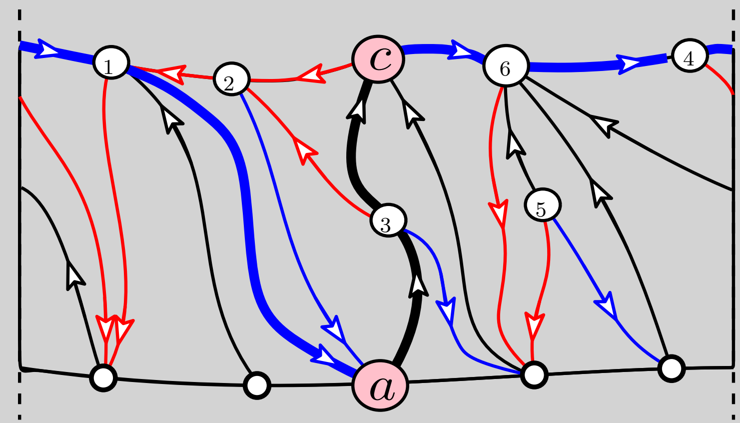
input: a river and a chord-free path P

assume there some chords at the left of a

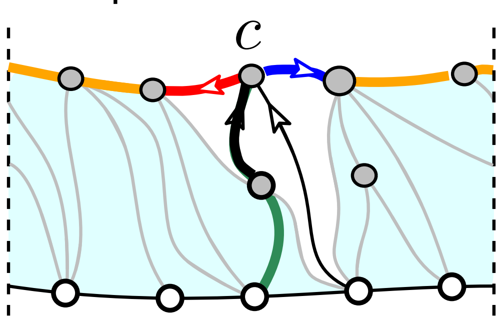


output: a Schnyder wood

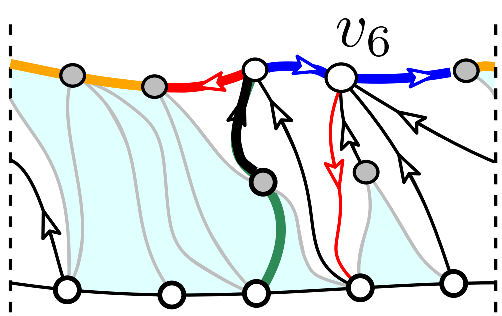
containing a blue path $P_1 = \{c, v_6, v_4, v_1, a\}$, starting at c and ending at a

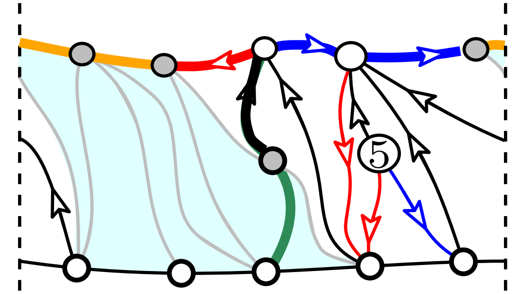


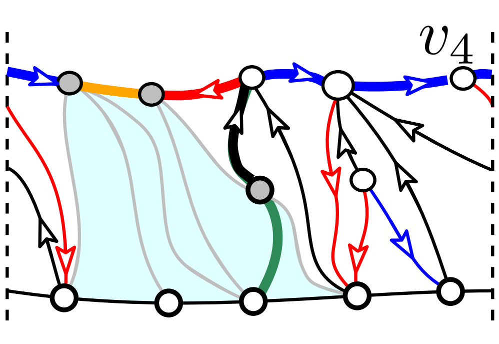
first step: remove c

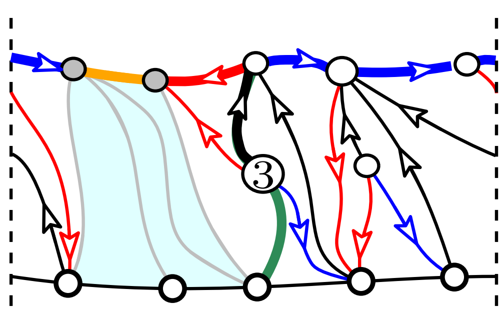


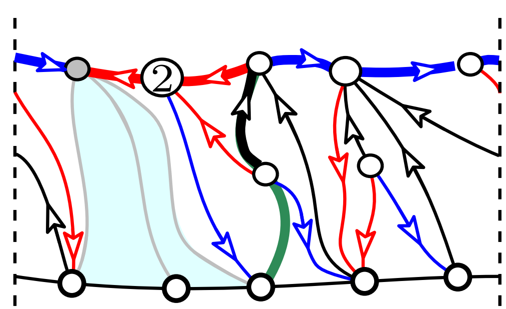
remove vertices without chords in right-most manner (at the right of P)

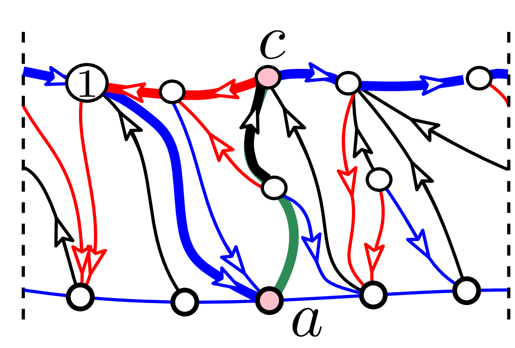












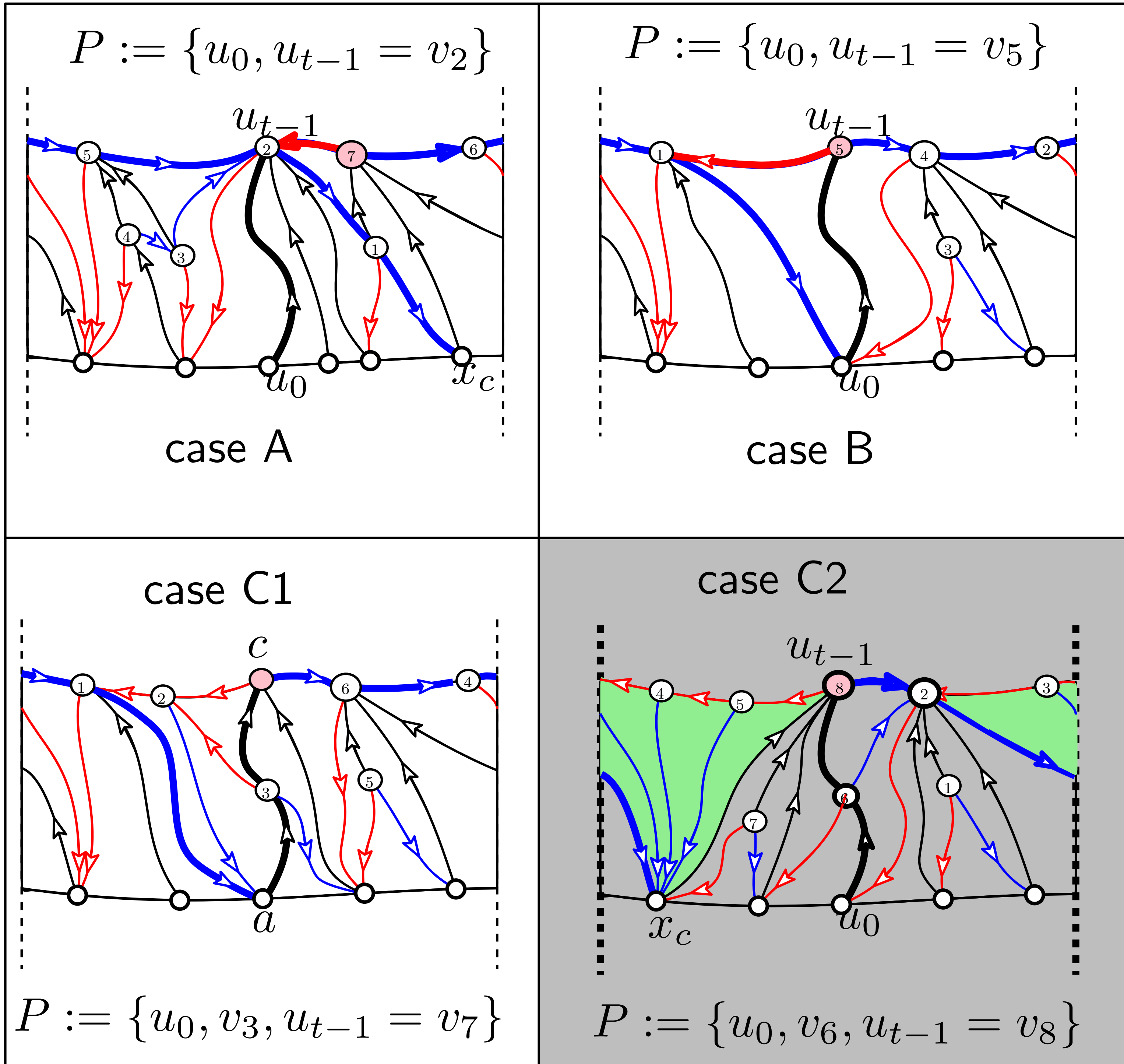
Right-most traversal of a river

Right-most traversal: remove at each step the left-most vertex without chords

Lemma

In cases (A), (B) and (C_1), the blue path P_1 visits all vertices on the top boundary and crosses P either at u_0 or at u_{t-1}

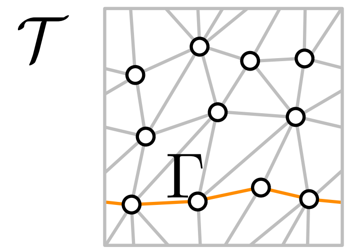
In case (C_2), the blue path P_1 may not cover all top boundary vertices (not crossing P), but then there exists a ccw-oriented (contractible) cycle (green region)



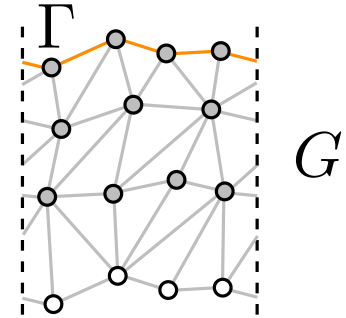
An algorithm for half-crossing Schnyder woods

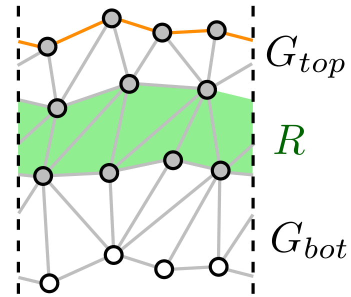
Algo 2 f-crossing Schnyder woods (with a connected mono-chromatic component)

Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood
// Pre-processing step
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
// First pass
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
// Run a second pass on R
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; // Remark: the r-cycle and 2-cycles are parallel
$P_2 \leftarrow \gamma_2 \cap R$; // restriction of γ_2 to the river R
$u \leftarrow \partial^e R \cap P_2$;
if *u has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
// Remark: $S(G_{bottom})$ and $S(G_{top})$ are P_2-constrained
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;

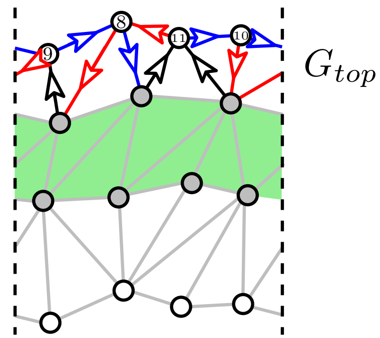


$\Rightarrow$ cut along Γ

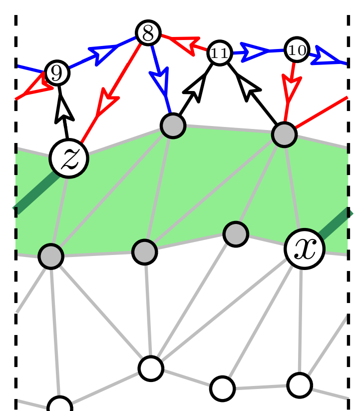




compute a river R



compute a Schnyder wood for G_{top}



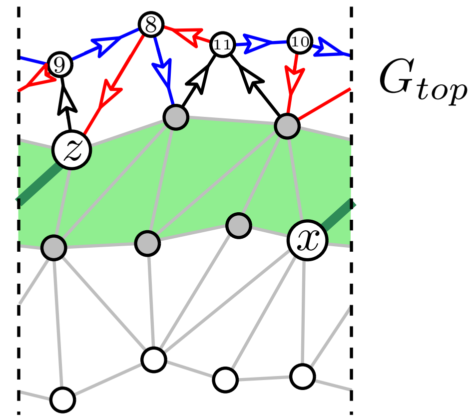
choose an arbitrary chordal edge in the river and take a path of length 1:

$P = \{x, z\}$

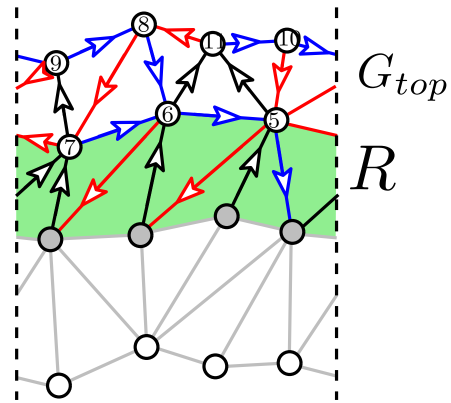
An algorithm for half-crossing Schnyder woods

Algo 2 Half-crossing Schnyder woods (with a connected mono-chromatic component)

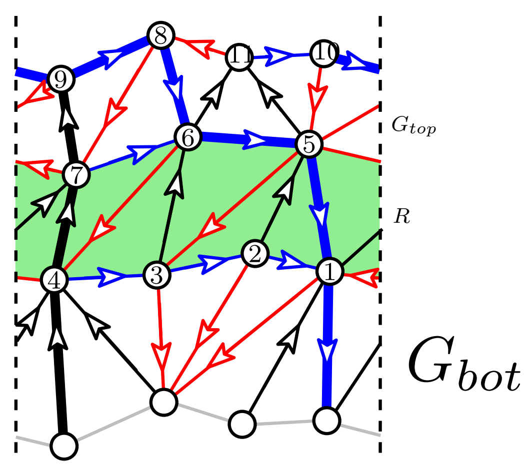
Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood
// Pre-processing step
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
// First pass
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
// Run a second pass on R
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; // Remark: the r-cycle and 2-cycles are parallel
$P_2 \leftarrow \gamma_2 \cap R$; // restriction of γ_2 to the river R
$u \leftarrow \partial^e R \cap P_2$;
if *u has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
// Remark: $S(G_{bottom})$ and $S(G_{top})$ are P_2-constrained
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;



$P = \{x, z\}$



compute a P-constrained rightmost traversal of the river



compute $S(G_{bot})$

blue and black cycles are crossing
there is one connected blue cycle

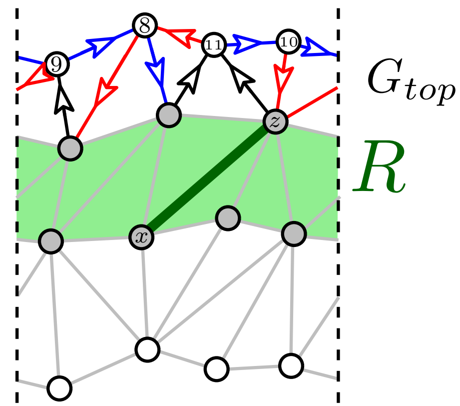
return the Schnyder wood

One pass suffices!

An algorithm for half-crossing Schnyder woods

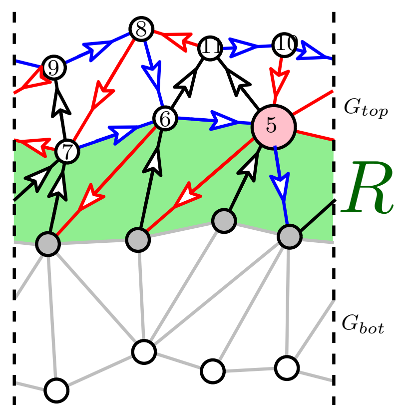
Algo 2 Half-crossing Schnyder woods (with a connected mono-chromatic component)

Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood
`// Pre-processing step`
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
`// First pass`
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has **type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
`// Run a second pass on` R
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; `// Remark: the` r`-cycle and 2-cycles are parallel`
$P_2 \leftarrow \gamma_2 \cap R$; `// restriction of` γ_2 `to the river` R
$u \leftarrow \partial^e R \cap P_2$;
if *u has **type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
`// Remark:` $S(G_{bottom})$ `and` $S(G_{top})$ `are` P_2`-constrained`
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;

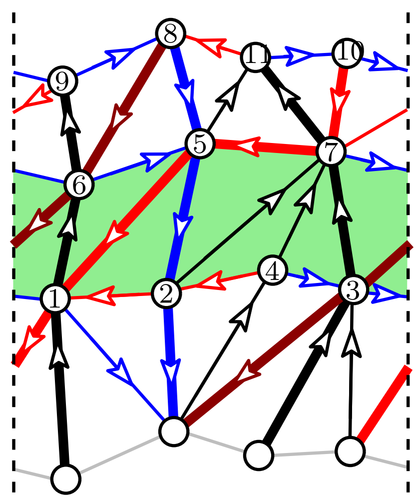


Sometimes two passes are required

$P = \{x, z\}$



compute a P-constrained rightmost traversal of R



compute $S(G_{bot})$

the blue cycle and black cycles are NOT crossing

red cycles cross black cycles but have 2 components

bad news: we need more work

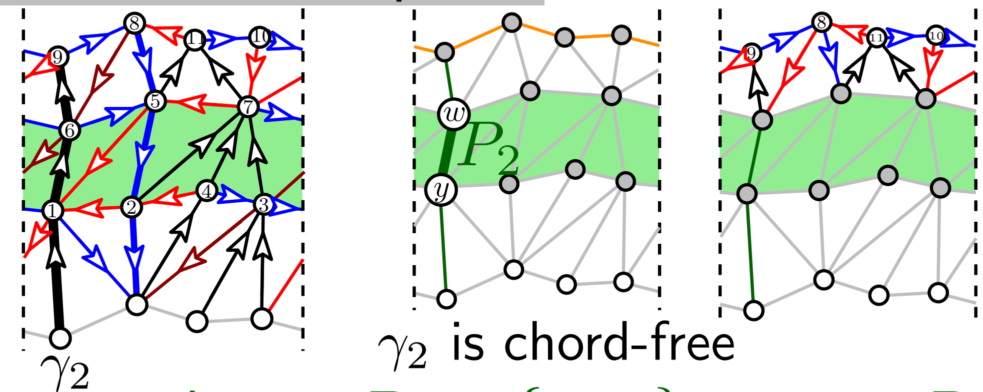
good news: black cycles are chord-free, we can run a second pass

An algorithm for half-crossing Schnyder woods

Algo 2 Half-crossing Schnyder woods (with a connected mono-chromatic component)

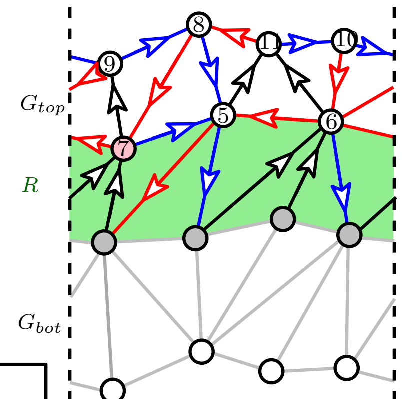
Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood
// Pre-processing step
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
// First pass
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
// Run a second pass on R
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; // Remark: the r-cycle and 2-cycles are parallel
$P_2 \leftarrow \gamma_2 \cap R$; // restriction of γ_2 to the river R
$u \leftarrow \partial^e R \cap P_2$;
if *u has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
// Remark: $S(G_{bottom})$ and $S(G_{top})$ are P_2-constrained
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;

Run the second pass



γ_2 is chord-free

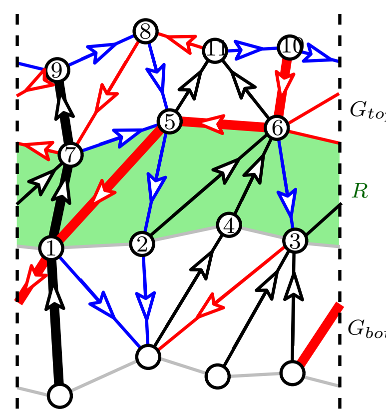
choose $P_2 = \{y, w\} := \gamma_2 \cap R$



w has type C_2

compute a constrained leftmost traversal of R

compute $S(G_{bot})$

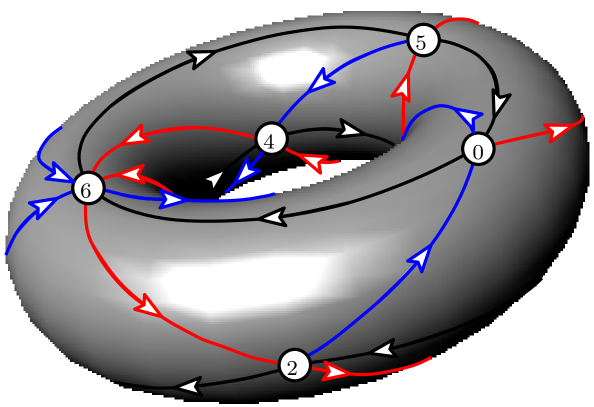


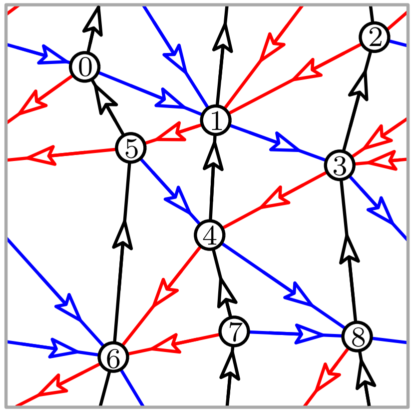
there is only one connected red cycle

the red cycle and the black cycle are crossing

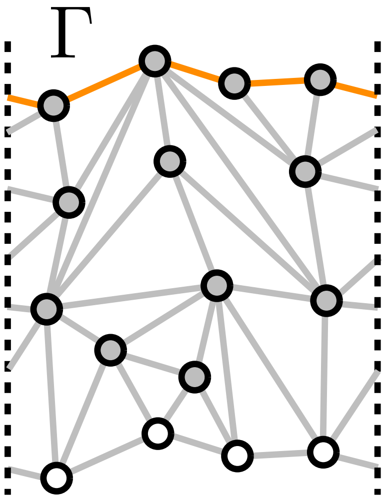
return the Schnyder wood

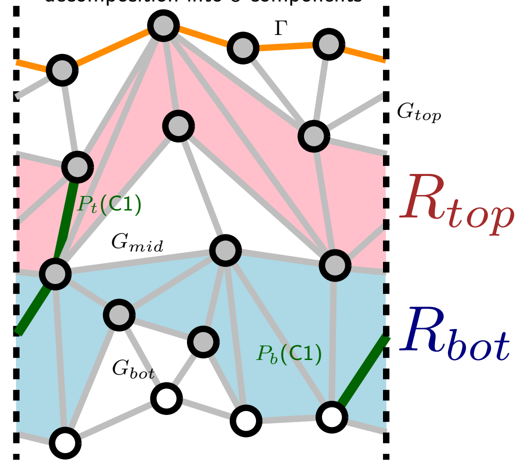
Toward crossing Schnyder woods
(with two connected mono-chromatic components)





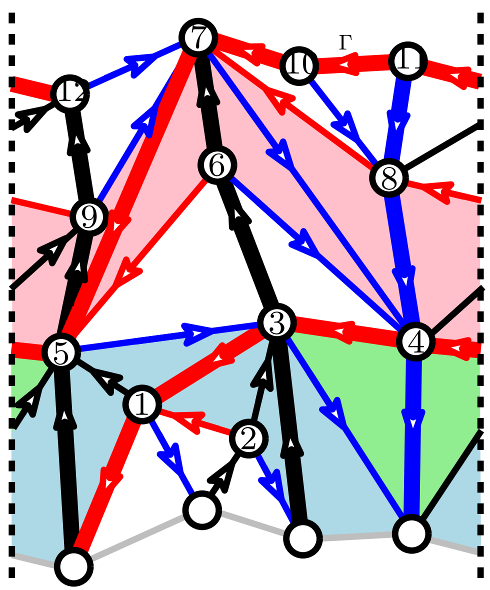
An algorithm for crossing Schnyder woods





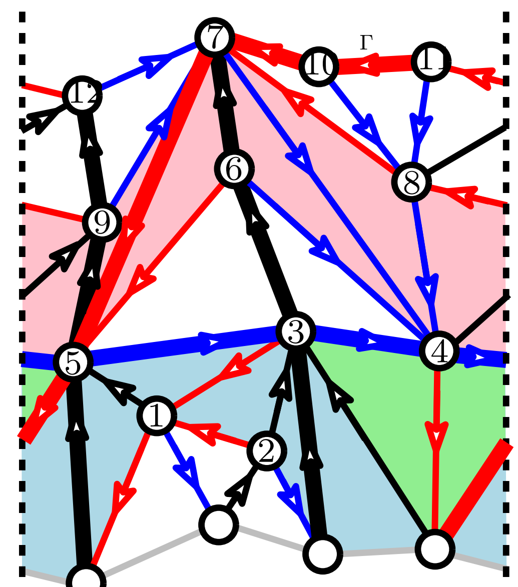
compute two non overlapping rivers

the 2-cycles and the 1-cycle are NOT crossing



half-crossing before reversing

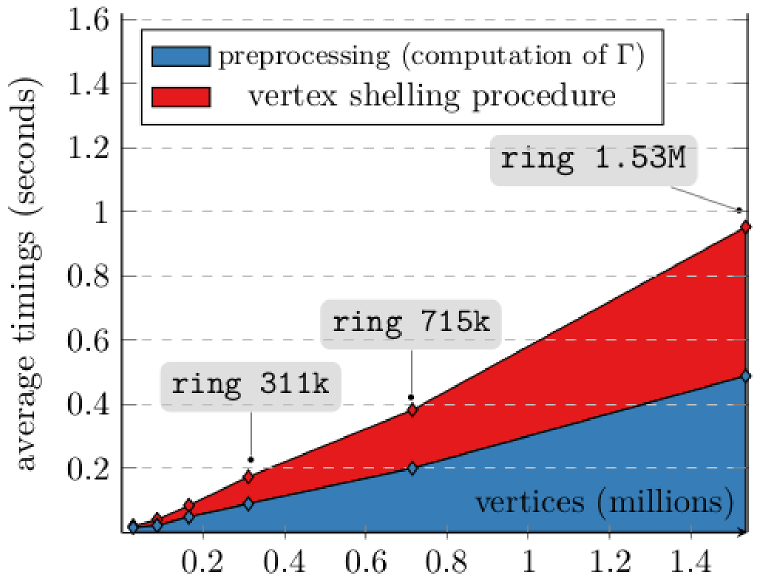
an oriented cycle in R_{bot} to be reversed

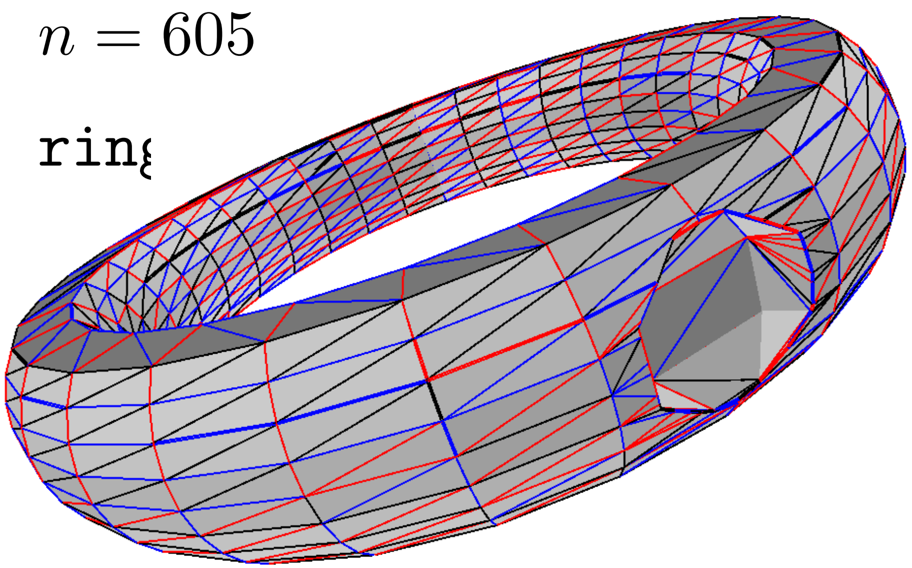


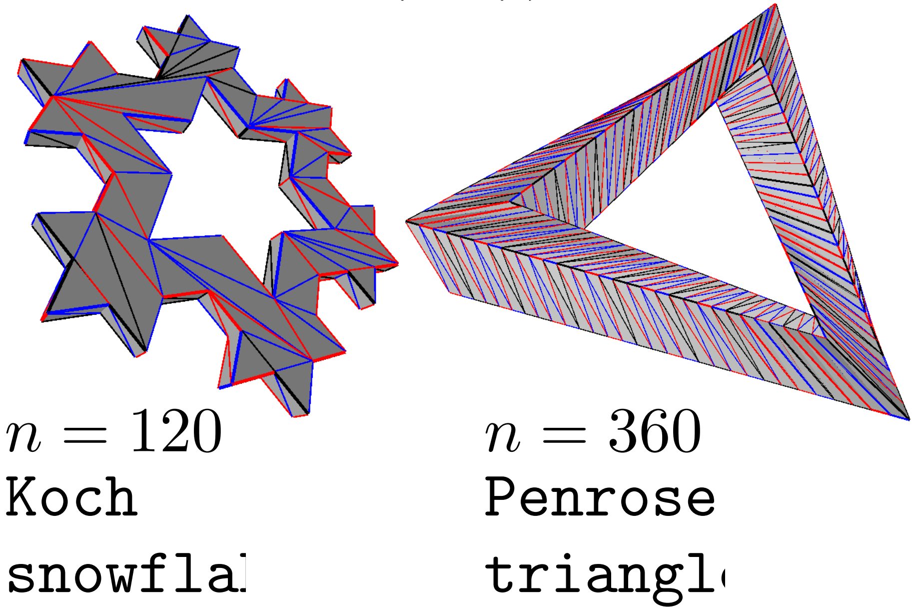
crossing after reversing

Experimental results

Fast linear-time implementation

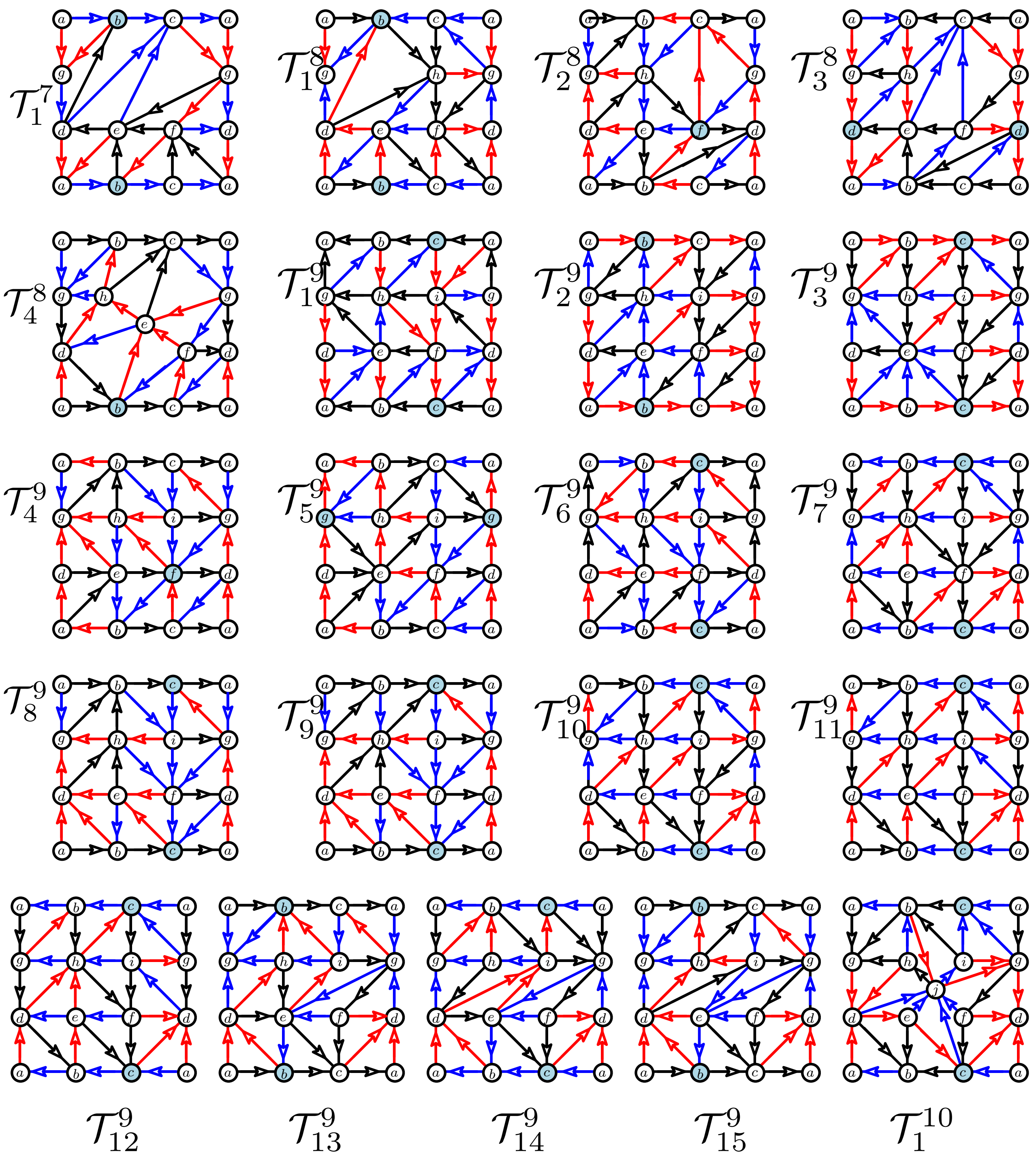






(with Java 1.8, on a Dell Laptop, Intel core i7 2.6GHz, 8GB RAM)

Conjectures on toroidal Schyder woods: experimental confirmation



Open problem: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components and such the intersection of the three cycles is a single vertex?

(true for all triangulations of size at most $n = 11$)

n	# irreducible triangulations	#triangulations (g = 1)
7	1	1
8	4	7
9	15	112
10	1	2109
11	–	37867

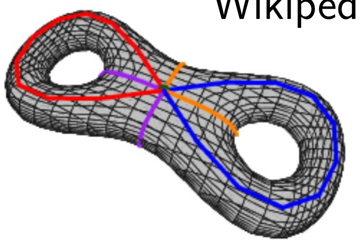
triangulations are generated with **surftri** software [Sulanke, 2006]

Schyder woods for $g \geq 2$

Thm (3-orientations for graphs on surfaces, of arbitrary genus)
[Albar Goncalves Knauer, 2014]

Any triangulation of a surface (the sphere and the projective plane) admits a '3-orientation': orientation without sinks s.t. every vertex has outdegree divisible by three

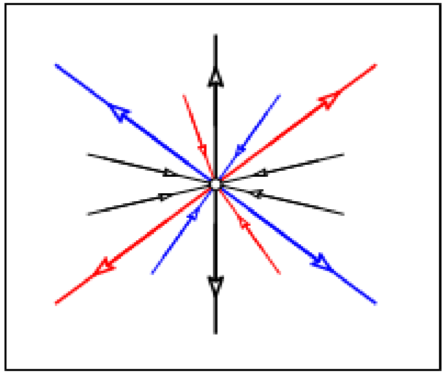
Wikipedia picture



Open problem [Goncalves Knauer Lévêque, 2016]
Existence of Schnyder woods for higher genus triangulations

Multiple local Schnyder condition: the outdegree of every vertex is a **positive** multiple of 3.

(there are no **sinks**)



Thm [Suagee, 2021]

Simple triangulations of genus $g \geq 1$ having "large" **edgewidth** do admit Schnyder woods

$$\textbf{edgewidth} \geq 40(2^g - 1)$$

(size of the smallest non contractible cycle)

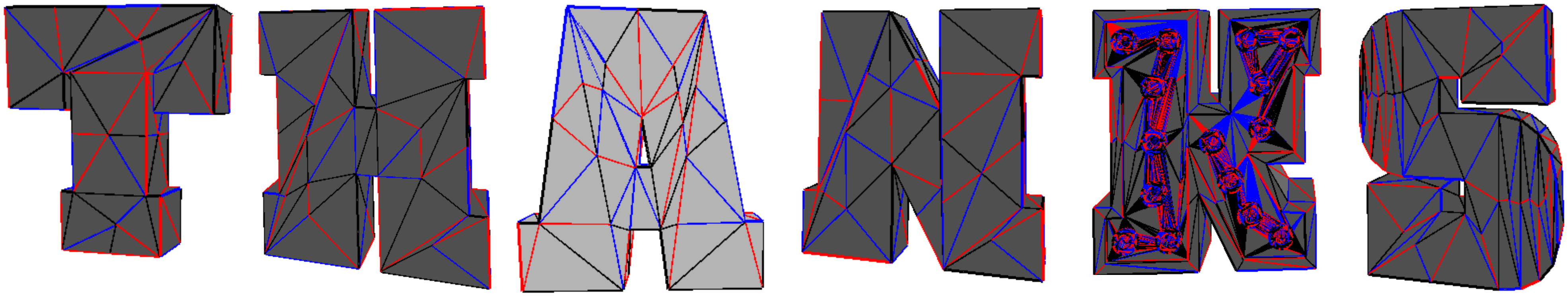
Experimental confirmation $(g = 2)$

exaustive generation of all 3-orientations for all triangulations with $g = 2$, $n \leq 11$

All simple triangulations of genus $g = 2$ and size ≤ 11 admit Schnyder woods

n	# irreducible triangulations	#triangulations (g = 2)
7	–	–
8	–	–
9	–	–
10	865	865
11	26276	113506

surftri software [Sulanke, 2006]


THANKS