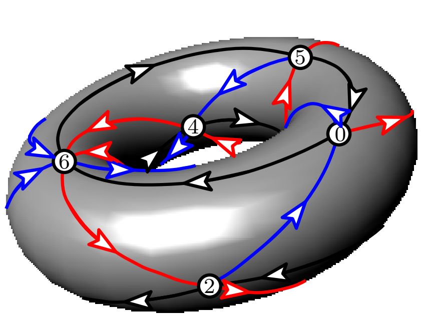

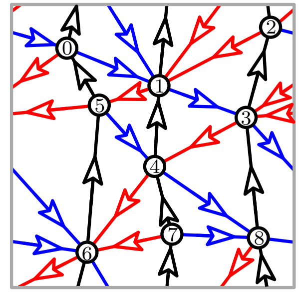

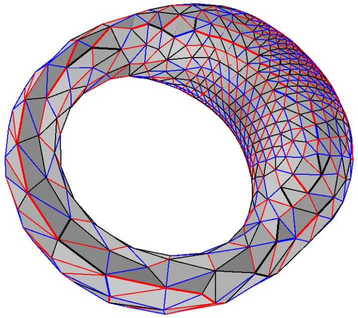

Computation of toroidal Schnyder woods made simple and fast: from theory to practice

april 1st 2025, LIGM (Univ. Gustave Eiffel)

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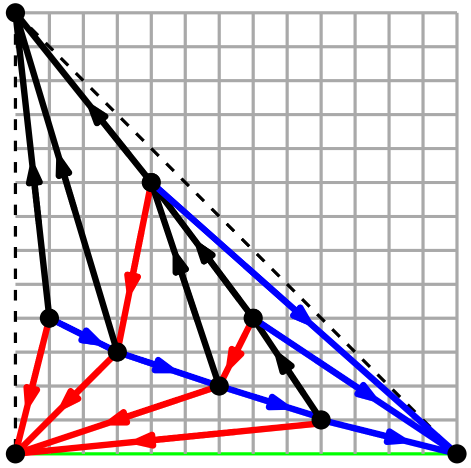
(LIX, Ecole Polytechnique)

Razvan S. Puscasu
(EPFL)

Main goals of this talk

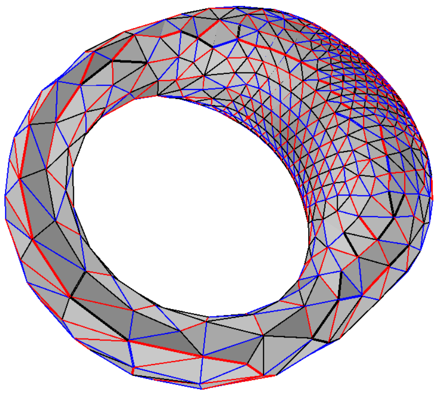


- either you do not know Schnyder woods

I will make you discover the magic world of Schnyder woods

- or you already encountered Schnyder woods

I will explain how to efficiently compute Schnyder woods for toroidal triangulations



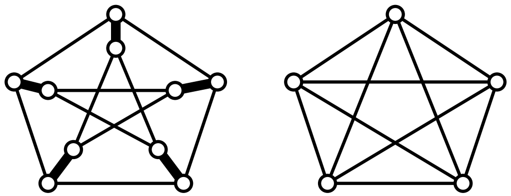
Some facts about (planar) maps

("As I have known them")

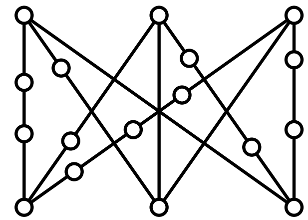
Let us review some major results on planar graphs

Kuratowski theorem (1930) (cfr Wagner's theorem, 1937)

- G contains neither K_5 nor $K_{3,3}$ as minors (or no subdivisions of K_5 nor $K_{3,3}$)



K_5 is a minor of the Petersen graph



subdivision of $K_{3,3}$

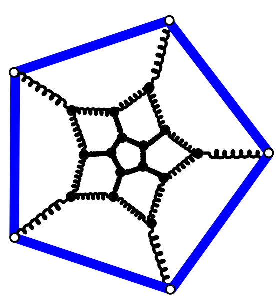
Thm (Colin de Verdière, 1990) Colin de Verdiere invariant (multiplicity of λ_2 eigenvalue of a generalized laplacian)

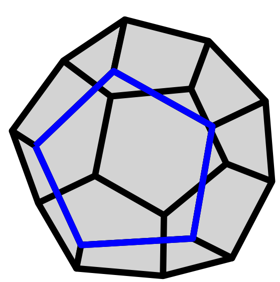
- $\mu(G) \leq 3$

$$\begin{bmatrix} 4 & -1 & \dots & \dots & 0 \\ -1 & 5 & \dots & & \\ \dots & & \dots & & \\ \dots & & & & \dots \\ 0 & \dots & & & 3 \end{bmatrix} \qquad L_G[i,k] = \begin{cases} deg(v_i) \\ -A_G[i,j] \end{cases}$$

Thm (Tutte barycentric method, 1963)

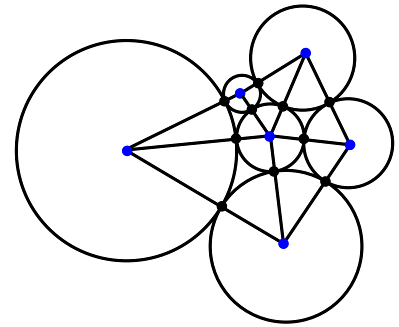
Every 3-connected planar graph G admits a convex representation in R^2.

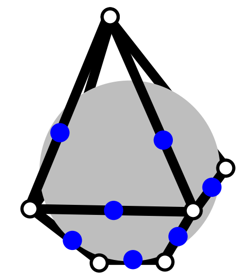




Thm (Koebe-Andreev-Thurston)

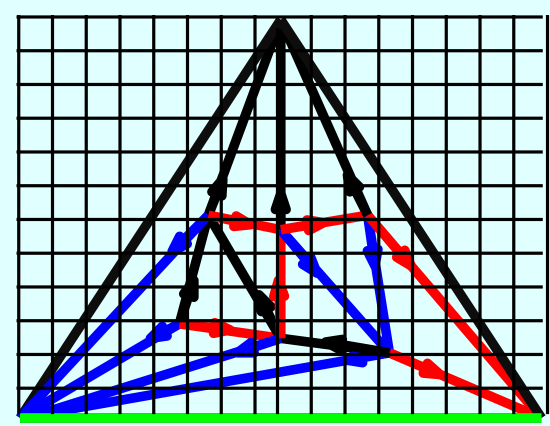
Every planar graph with n vertices is isomorphic to the intersection graph of n disks in the plane.





Schnyder woods (Walter Schnyder '89)

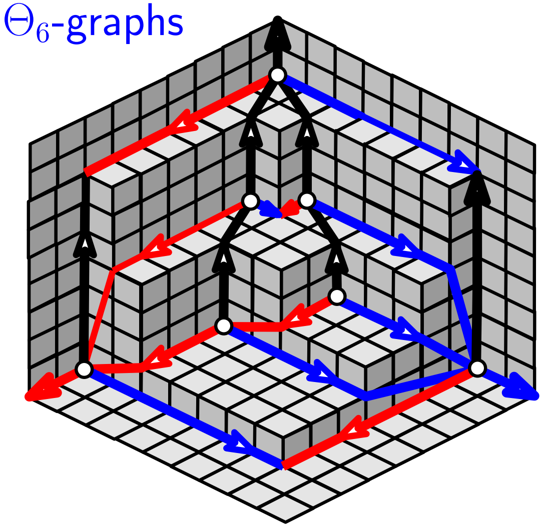
- planarity criterion via dimension of partial orders: $dim(G) \leq 3$
- linear-time grid drawing, with $O(n) \times O(n)$ resolution



Schnyder woods: some (classical) applications

[Felsner, Bonichon et al. '10, ...]

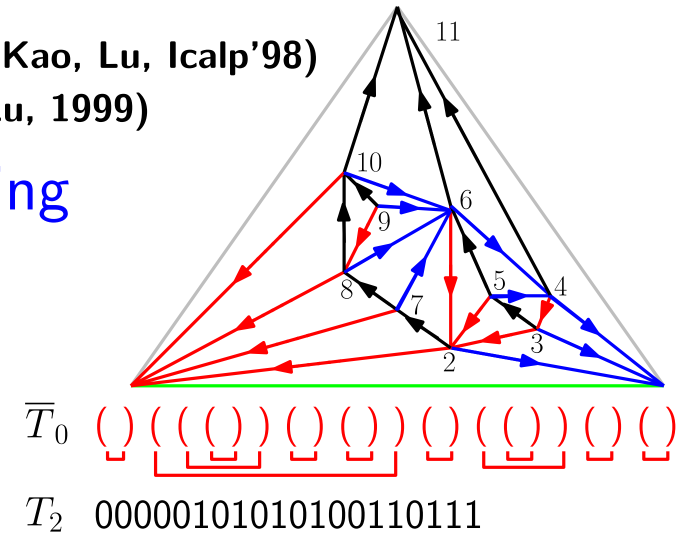
geodesic embeddings on coplanar orthogonal surfaces, TD-Delaunay graphs and Half-Θ_6-graphs



(Chuang, Garg, He, Kao, Lu, Icalp'98)

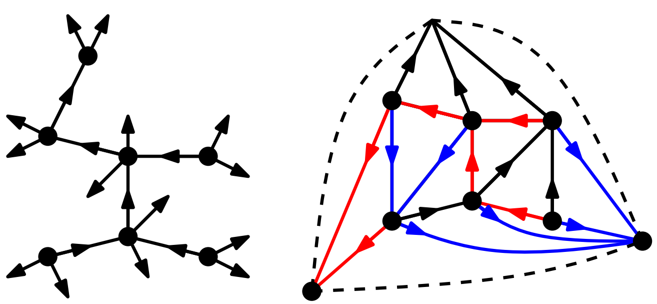
(He, Kao, Lu, 1999)

Graph encoding



(Poulalhon-Schaeffer, Icalp 03)

bijective counting, random generation

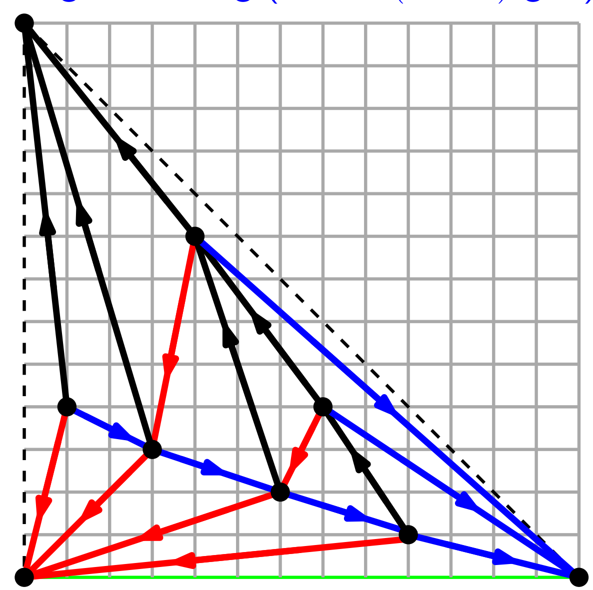


$$\boxed{c_n = \frac{2(4n+1)!}{(3n+2)!(n+1)!}}$$

$\Rightarrow$ optimal encoding ≈ 3.24 bits/vertex

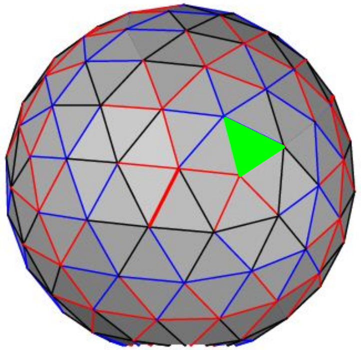
(Schnyder '90)

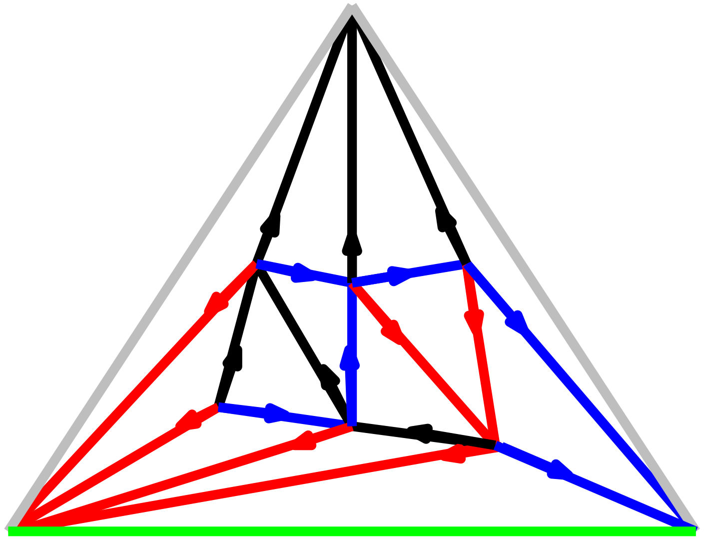
Planar straight-line grid drawing (on a $O(n \times n)$ grid)



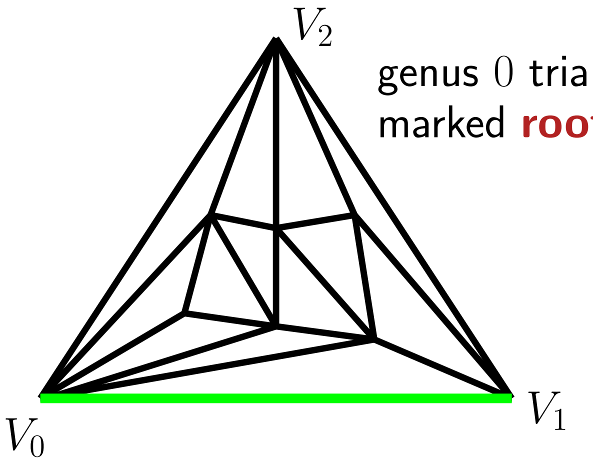
(Planar) Schnyder woods

(definitions and main properties)

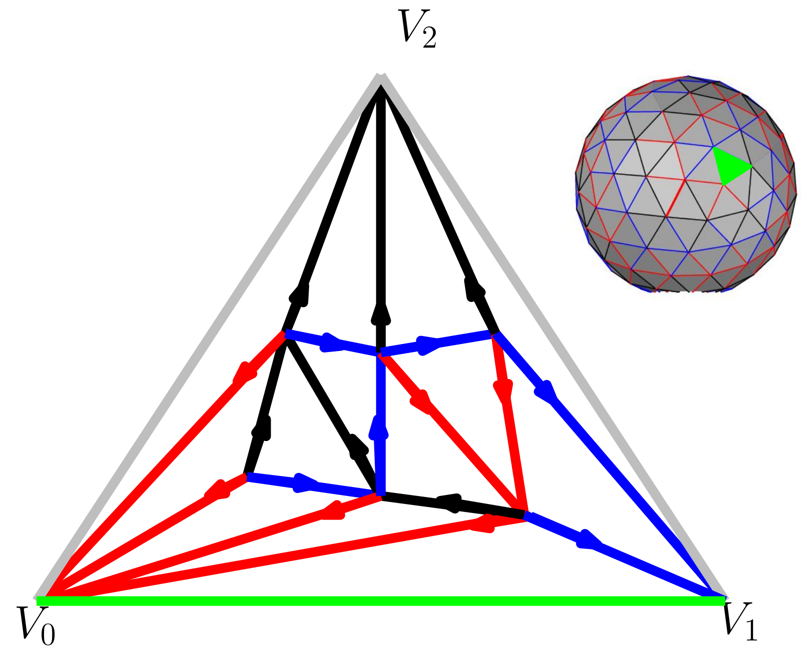




Schnyder woods for genus 0 (plane) triangulations: definition

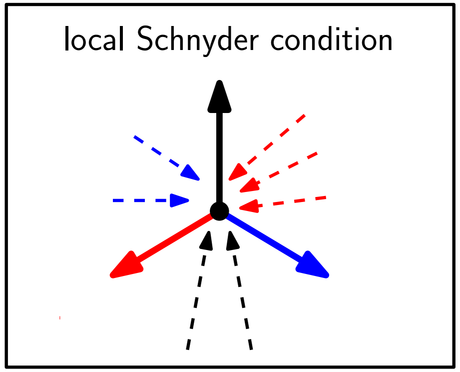


genus 0 triangulation with a marked **root** face $\{v_0, v_1, v_2\}$



Definition [Schnyder '90]

A **Schnyder wood** of a (rooted) planar triangulation is partition of all inner edges into three sets T_0, T_1 and T_2 such that



i) edge are colored and oriented in such a way that each inner node has exaclty one outgoing edge of each color

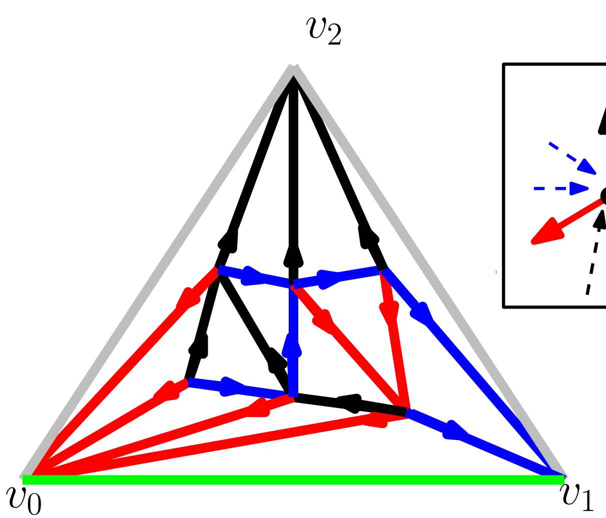
ii) colors and orientations around each inner node must respect the local Schnyder condition

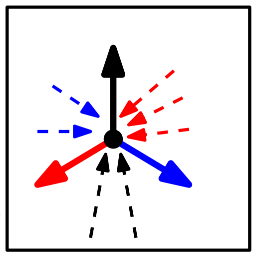
iii) inner edges incident to V_i are of color i and oriented toward V_i

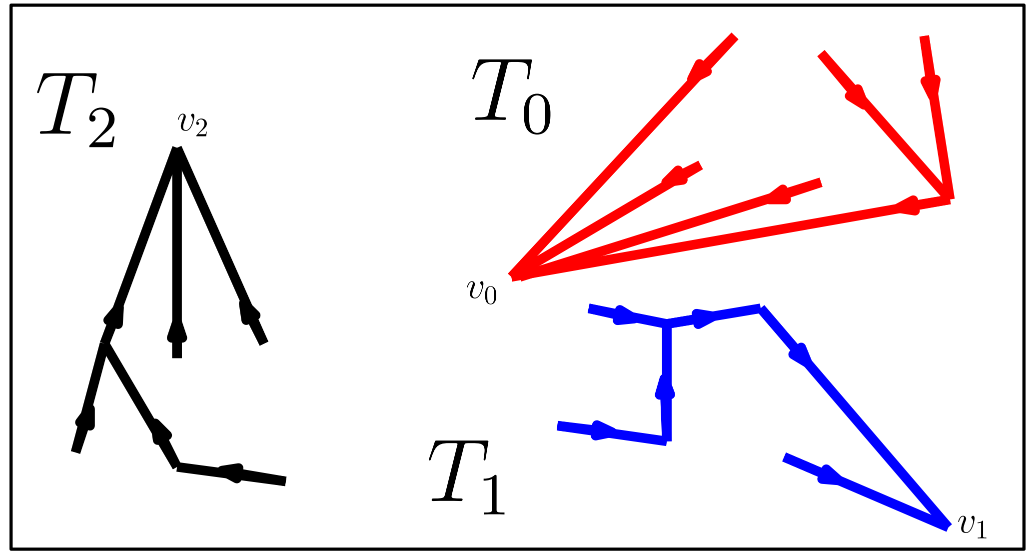
Spanning property of Schnyder woods

Theorem **[Schnyder '90]** $T_i :=$ digraph defined by directed edges of color i

The three sets T_0, T_1, T_2 are spanning trees of the inner vertices of $\mathcal{T}$ (each rooted at vertex v_i)



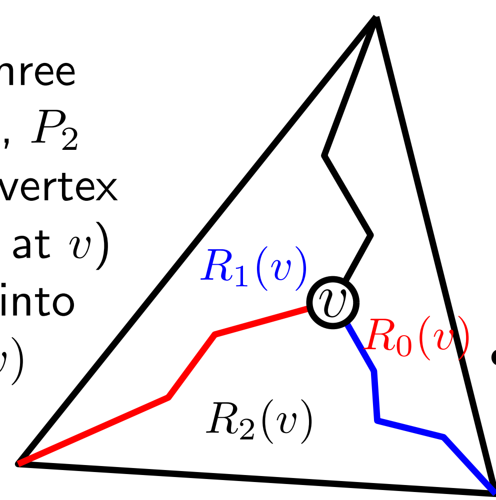


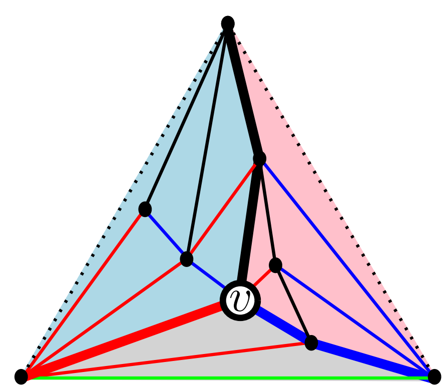


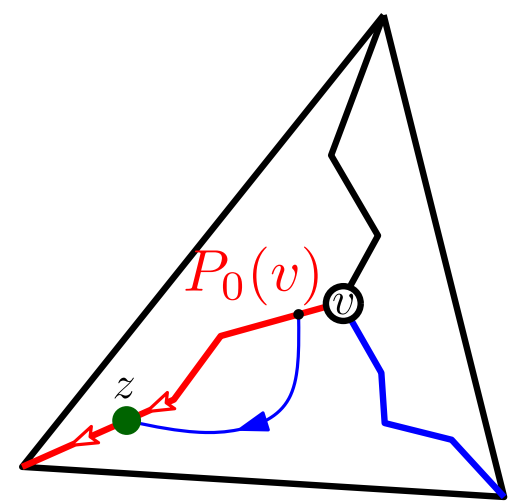
Mono-chromatic paths

Lemma

For each inner vertex v the three monochromatic paths P_0, P_1, P_2 directed from v toward each vertex V_i are vertex disjoint (except at v) and partition the inner faces into three sets $R_0(v), R_1(v), R_2(v)$





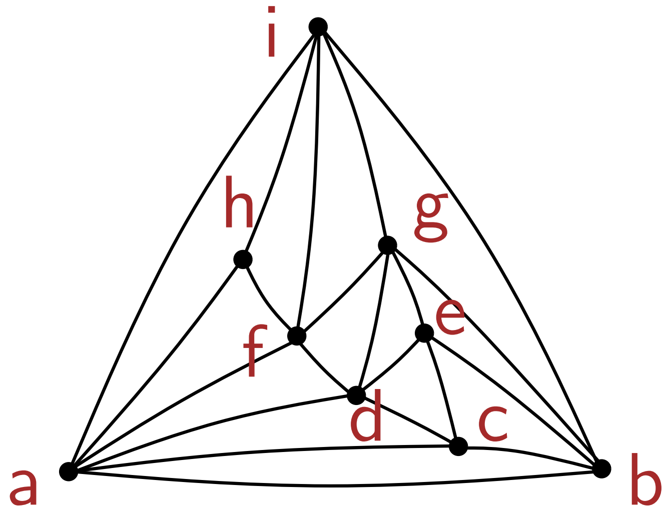


each path $P_i(v)$ is chord-free

Schnyder drawings: face counting algorithm

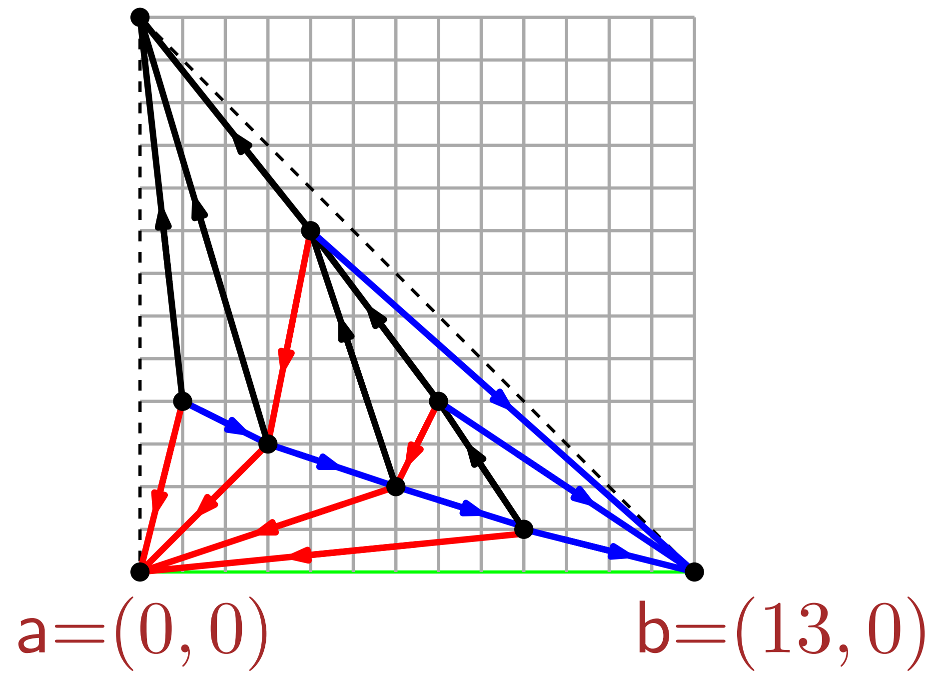
Theorem (Schnyder, Soda '90)
For a triangulation $\mathcal{T}$ having n vertices, we can draw it (with no edge crossings) on a grid of size $(2n-5) \times (2n-5)$, by setting $x_0 = (2n-5, 0)$, $x_1 = (0,0)$ and $x_2 = (0, 2n-5)$.

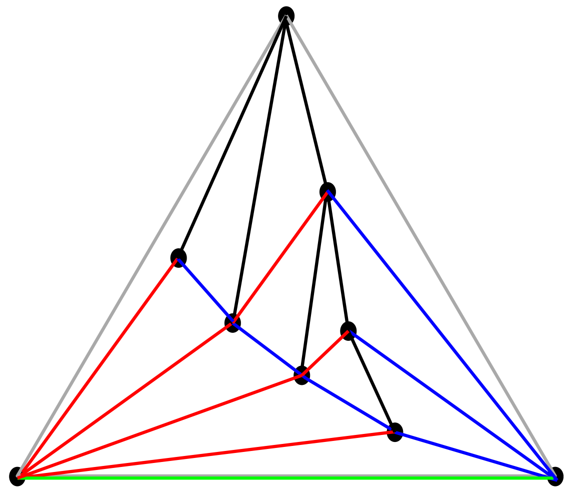
Input: a planar triangulation $\mathcal{T}$



Output:
a straight-line planar grid-drawing of $\mathcal{T}$

$\Rightarrow$





$\mathcal{T}$ endowed with a Schnyder wood

Schnyder drawings: face counting algorithm

Theorem (Schnyder, Soda '90)

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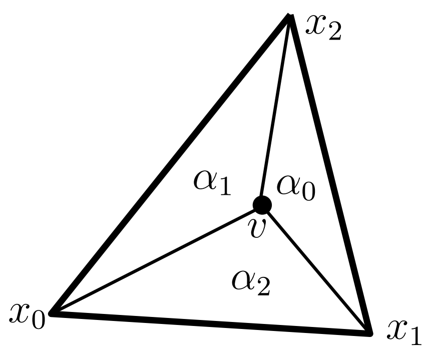
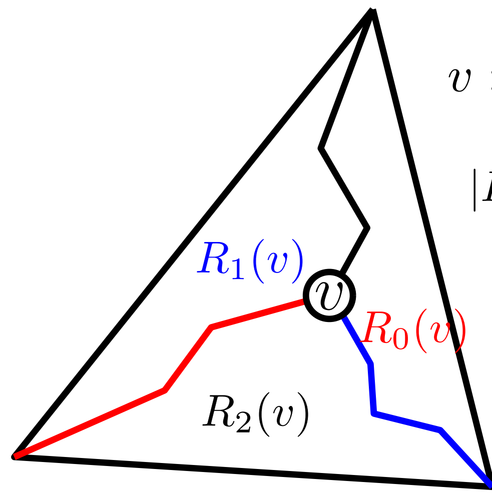
$$v = \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2$$



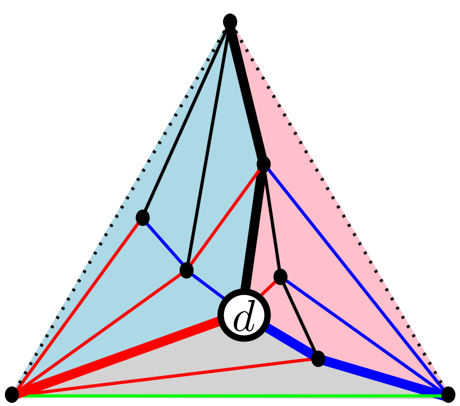
α_i is the normalized area of the triangle (x_{i-1}, x_{i+1}, v)



$$v := \frac{|R_0(v)|}{|F|-1} x_0 + \frac{|R_1(v)|}{|F|-1} x_1 + \frac{|R_2(v)|}{|F|-1} x_2$$

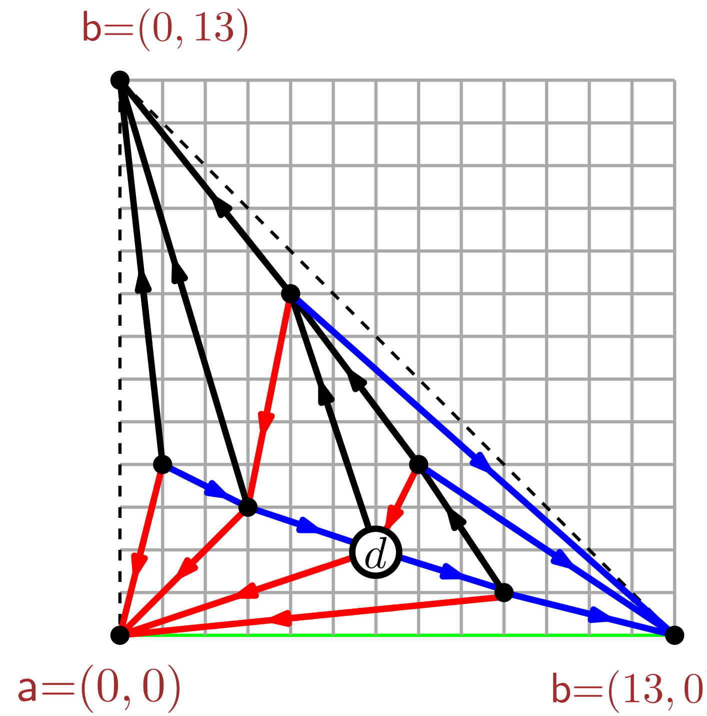
$|R_i(v)|$ is the number of triangles in $R_i(v)$

$|F| - 1 = 2n - 5$ is the number of inner triangles



a $\to (13, 0, 0)$
b $\to (0, 13, 0)$
c $\to (9, 3, 1)$
d $\to (5, 6, 2)$
e $\to (2, 7, 4)$
f $\to (7, 3, 3)$
g $\to (1, 4, 8)$
h $\to (8, 1, 4)$
i $\to (0, 0, 13)$

a $\to (0, 0)$
b $\to (13, 0)$
c $\to (9, 1)$
d $\to (\mathbf{6}, \mathbf{2})$
e $\to (7, 4)$
f $\to (3, 3)$
g $\to (4, 8)$
h $\to (1, 4)$
i $\to (0, 13)$



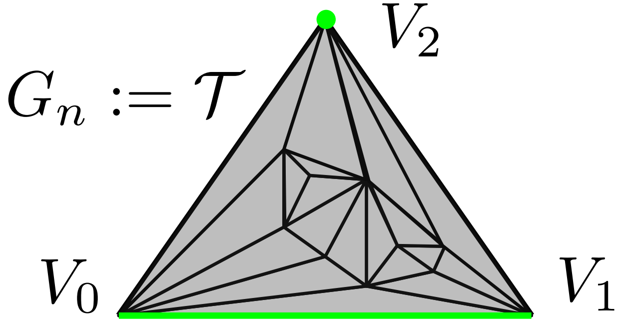
Linear-time computation of (planar) Schnyder woods

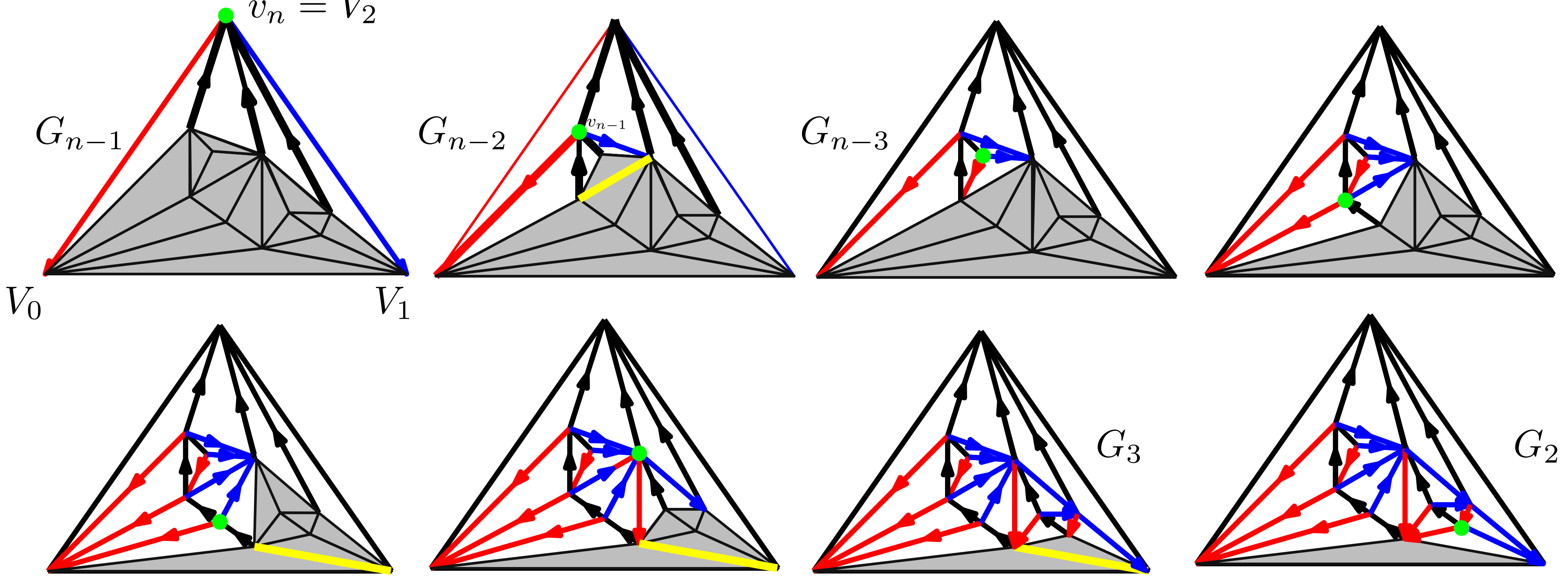
use Canonical Orderings [De Fraysseix, Pach, Pollack '89]

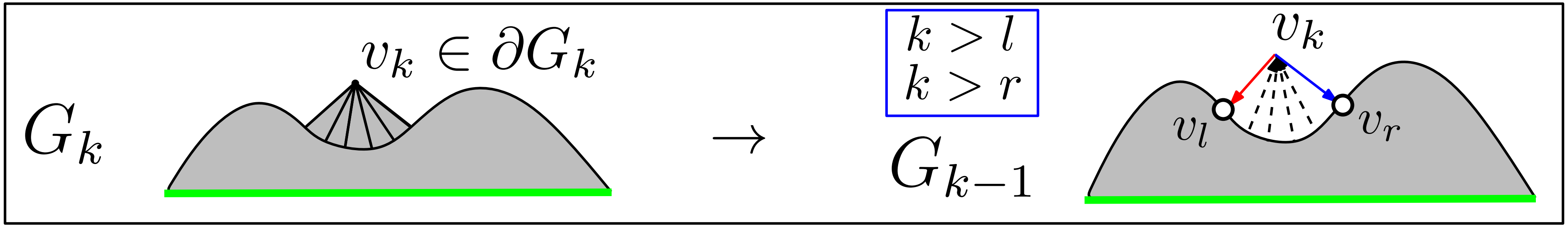
Theorem (Brehm, 2000)

A Schnyder wood can be computed in linear-time (via a sequence of $n-2$ vertex shellings)

Remove at each step a vertex v on the boundary ∂G_k (with no incident chordal edges in the gray region)





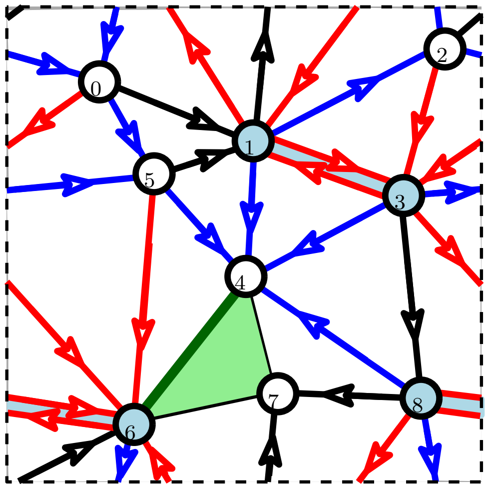


Schnyder woods for higher genus surfaces

g-Schnyder woods

[Castelli Aleardi, Fusy, Lewiner, SoCG'08]

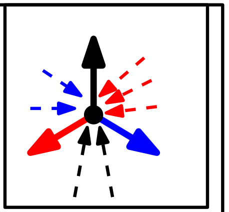
Schnyder local rule valid **almost everywhere** (except $O(g)$ vertices)

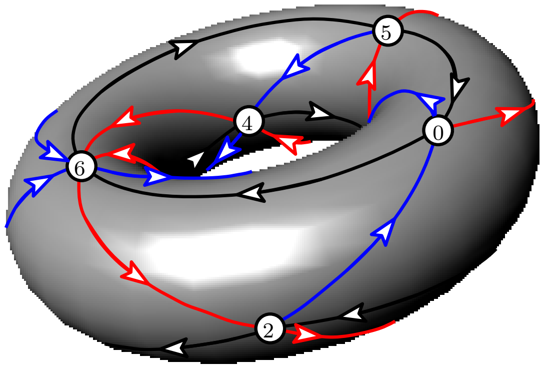


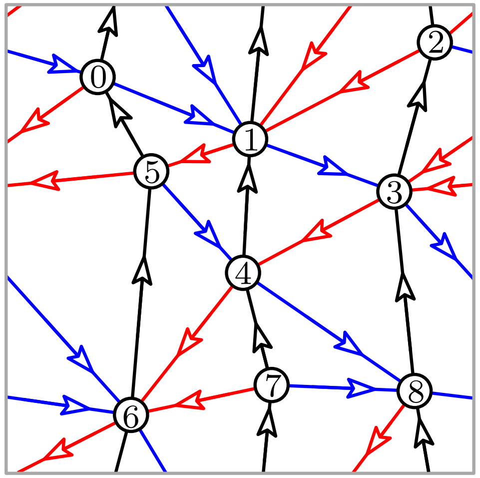
Toroidal Schnyder woods (g=1)

[Goncalves Lévêque, DCG'14]

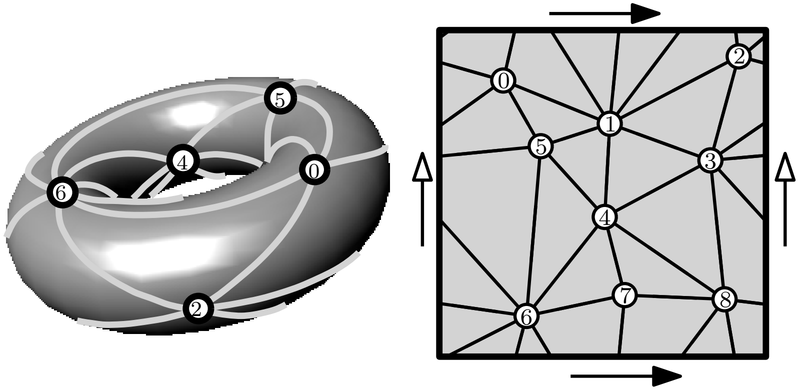
Schnyder local rule valid at each vertex

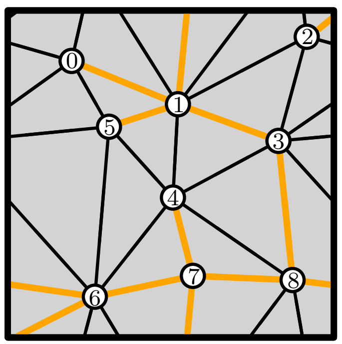






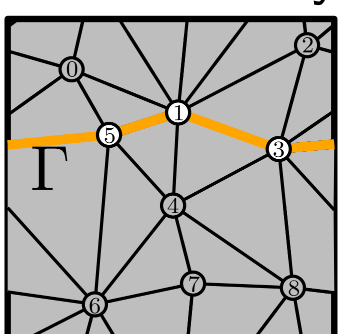
Planarization: from the torus back to the plane





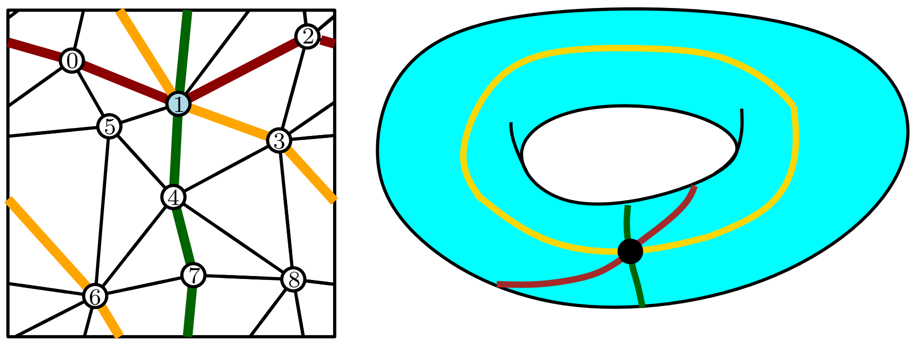
G cut-graph

Γ non-contractible cycle



Thm[Fijavz, unpublished]

A simple toroidal triangulation contains **three non-contractible and non-homotopic cycles that all intersect on one vertex** and that are pairwise disjoint otherwise.



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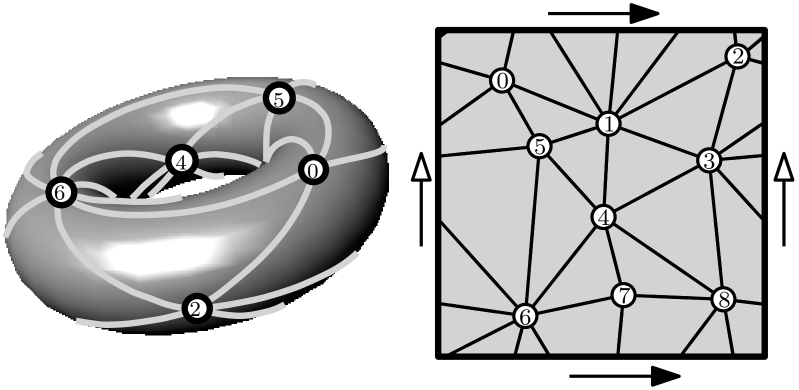

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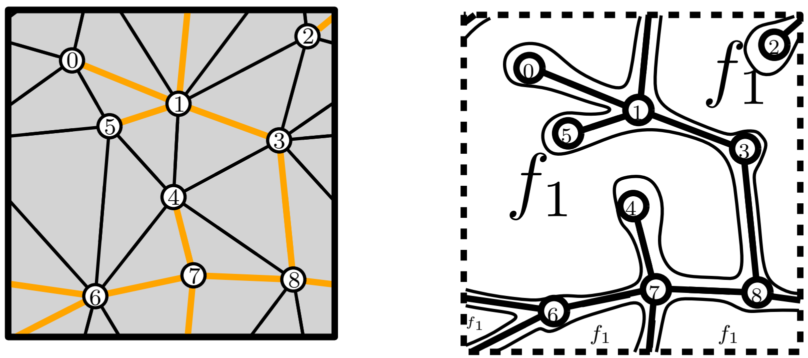
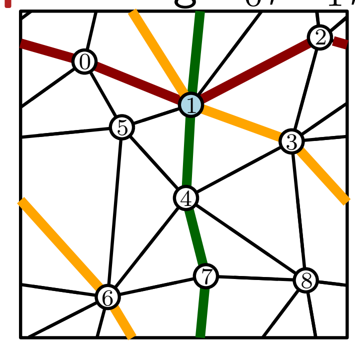
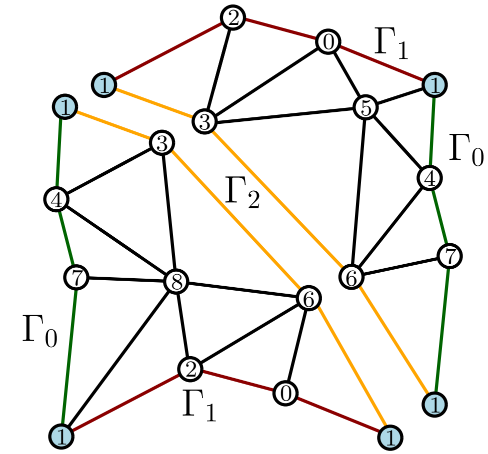


G cut-graph

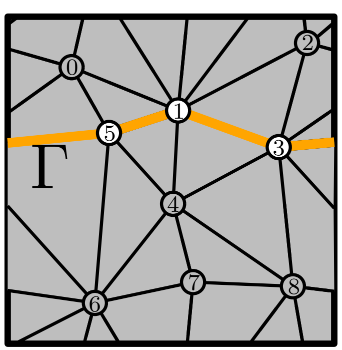
split along Γ_0, Γ_1, Γ_2



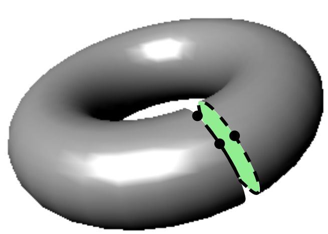


(two planar quasi-triangulations)

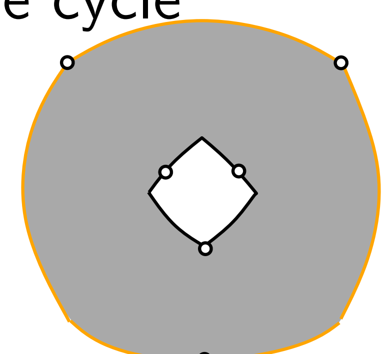
Γ non-contractible cycle

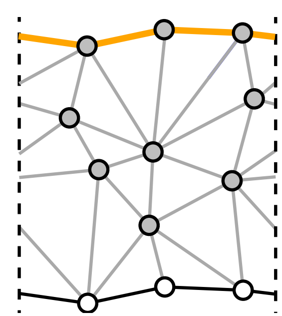


$\Rightarrow$



$\Rightarrow$





cylindric triangulation: planar triangulation with two boundaries

Toroidal Schnyder woods: definition

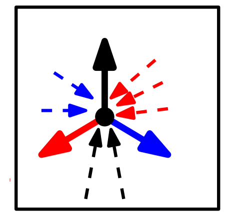
[Goncalves Lévêque, DCG'14]

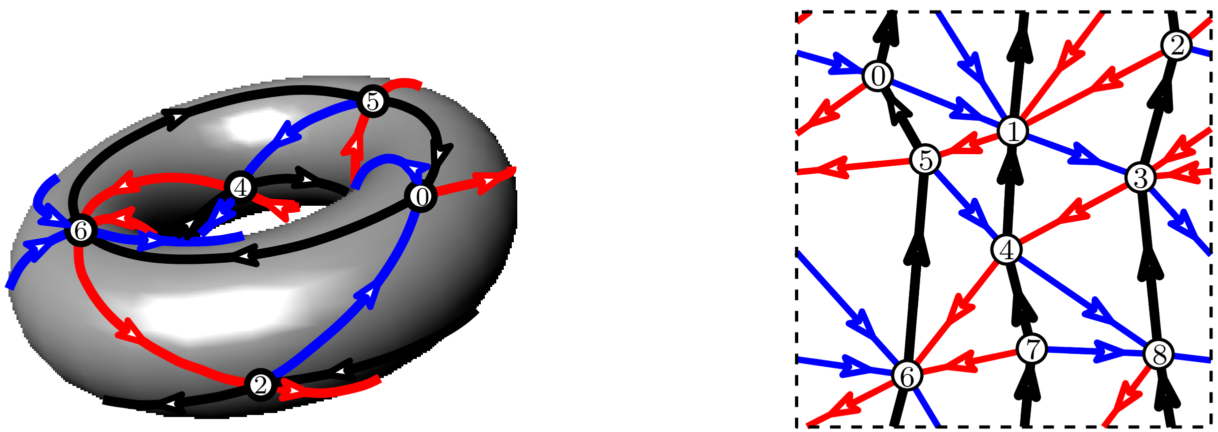
Remark: in the toroidal case $(g = 1)$
$n - e + f = 2 - 2g$

$$e = 3n$$

Def. **Toroidal Schnyder woods** [Goncalves Lévêque, DCG'14]

- 3-orientation + Schnyder local rule valid at each vertex





toroidal Schnyder wood

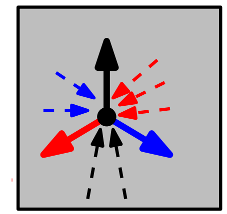
Toroidal Schnyder woods vs. 3-orientations

Remark: in the toroidal case $(g = 1)$
$n - e + f = 2 - 2g$

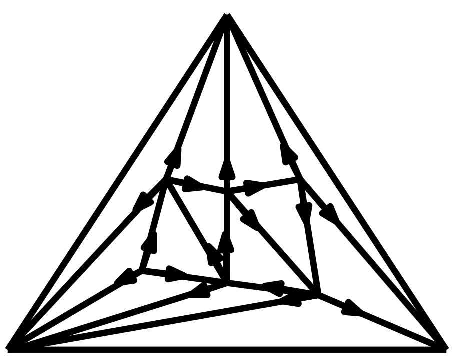
$$\boxed{e = 3n}$$

Def. **Toroidal Schnyder woods** [Goncalves Lévêque, DCG'14]

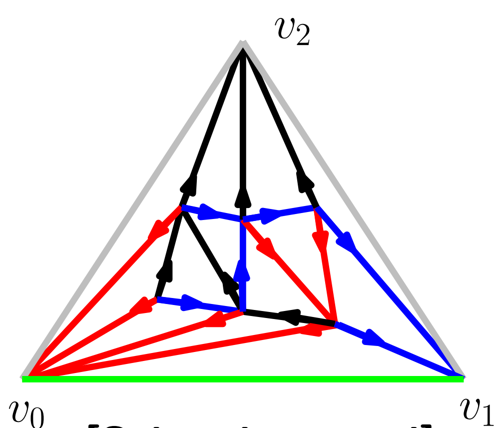
- 3-orientation $+$ $\boxed{\text{Schnyder local rule valid at each vertex}}$



Remark: unlike the planar case, some 3-orientations do not lead to valid Schnyder woods



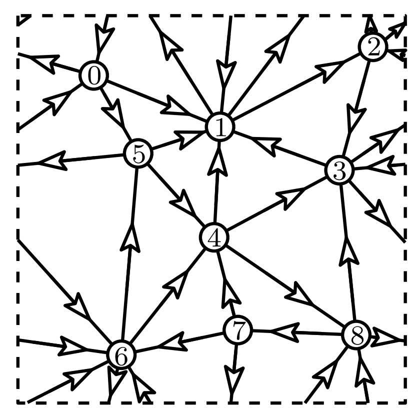
[3-orientation]



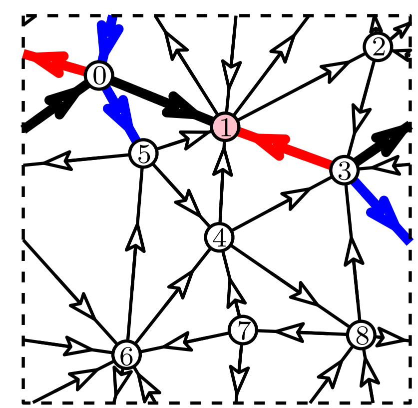
[Schnyder wood]

(in the plane there is a bijection between 3-orientations and Schnyder woods)

valid 3-orientation

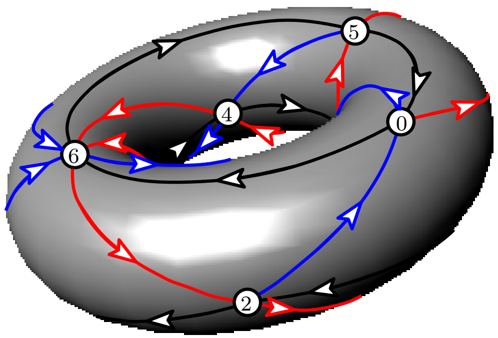


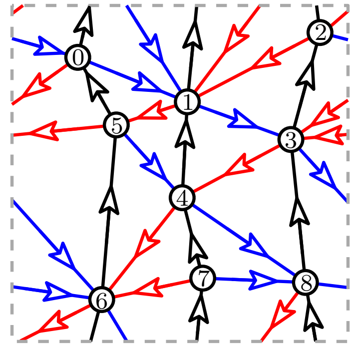
not valid toroidal Schnyder wood



(the local Schnyder rule cannot be propagated everywhere)

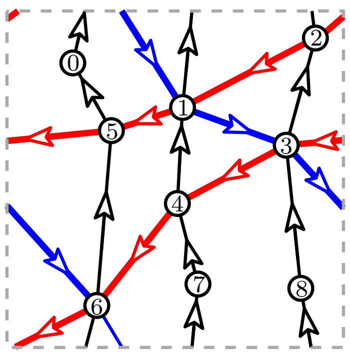
Toroidal Schnyder woods: cycles

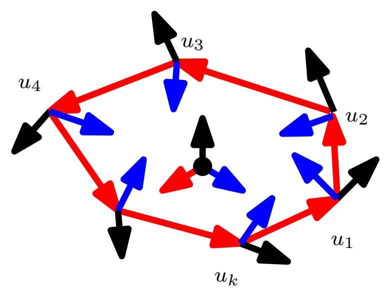




toroidal Schnyder wood

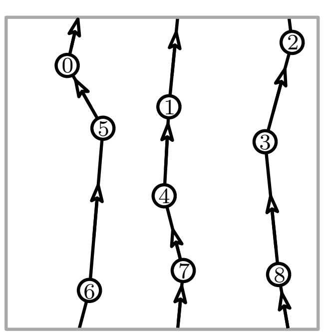
- toroidal Schnyder woods must contain a (mono-chromatic) cycle in each color: $$e = 3n$$ (n edges in each color)
- mono-chromatic cycles are non-contractibles





Remark: the inner region of a contractible mono-chromatic cycle is a (planar) topological disk

- some colors may define disconnected components



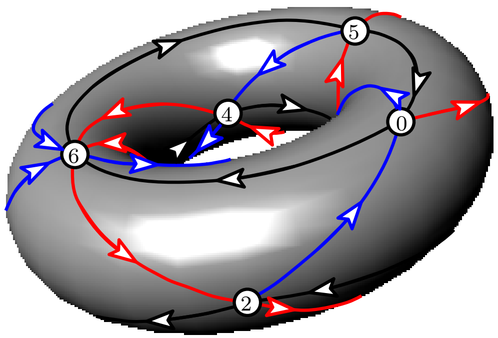
(there are 3 disjoint mono-chromatic cycles of color 2)

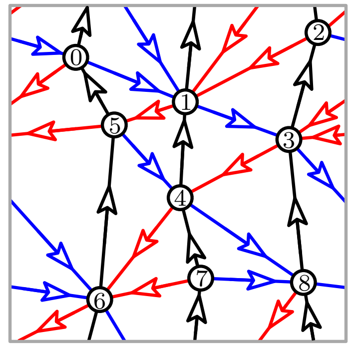
Open problem: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components

n	# irreducible triangulations	#triangulations (g = 1)
7	1	1
8	4	7
9	15	112
10	1	2109
11	–	37867

(true for all triangulations of size at most $n = 11$)

Toroidal Schnyder woods: cycles





toroidal Schnyder wood

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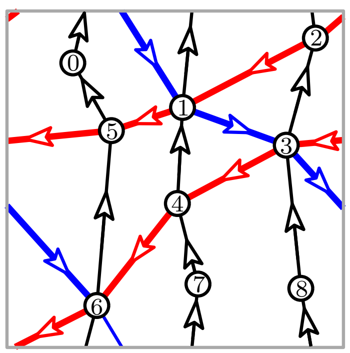
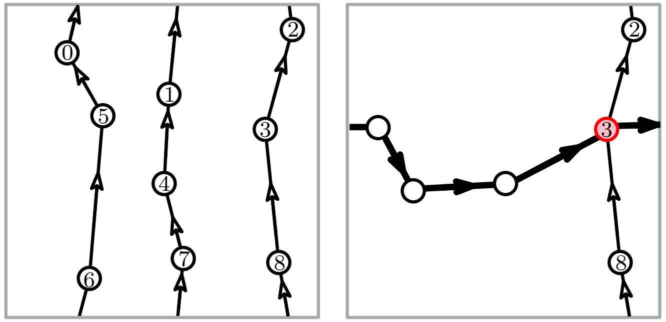
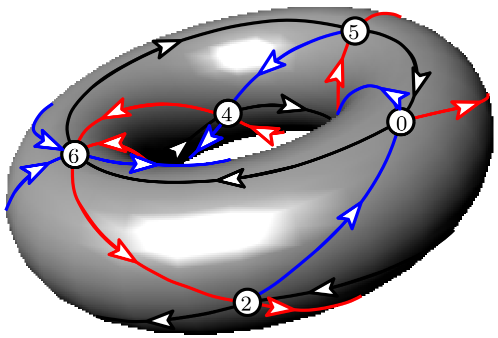
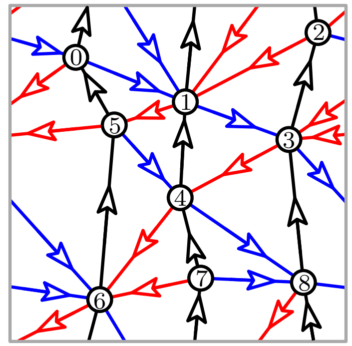


all mono-chromatic cycles of the same color are: homotopic and disjoint (parallel) and oriented in one direction



Toroidal Schnyder woods: cycles





toroidal Schnyder wood

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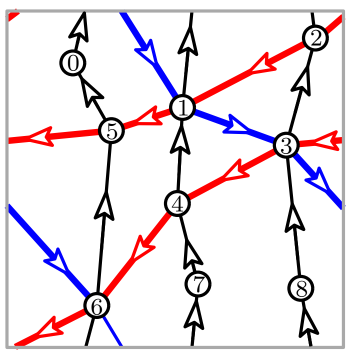
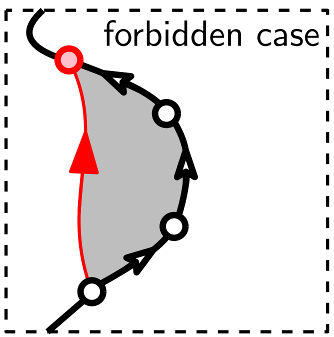
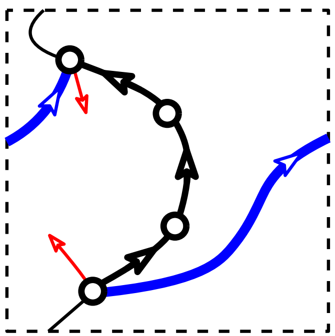
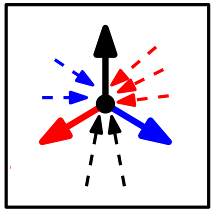
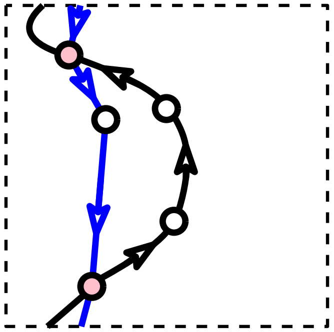
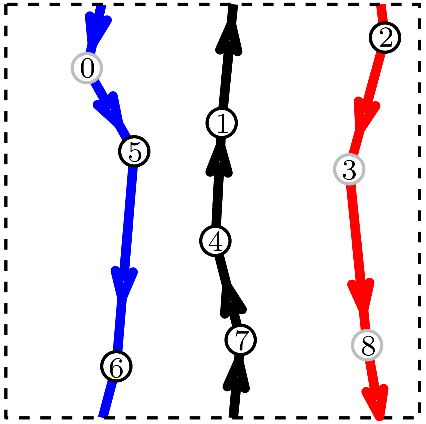


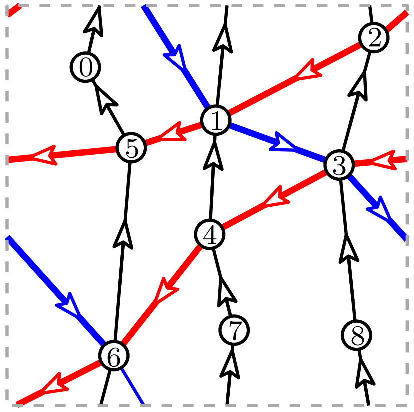




mono-chromatic paths $P_i(v)$ may have incident chords

- all mono-chromatic cycles of different colors are:
  either homotopic and disjoint (parallel) or crossing

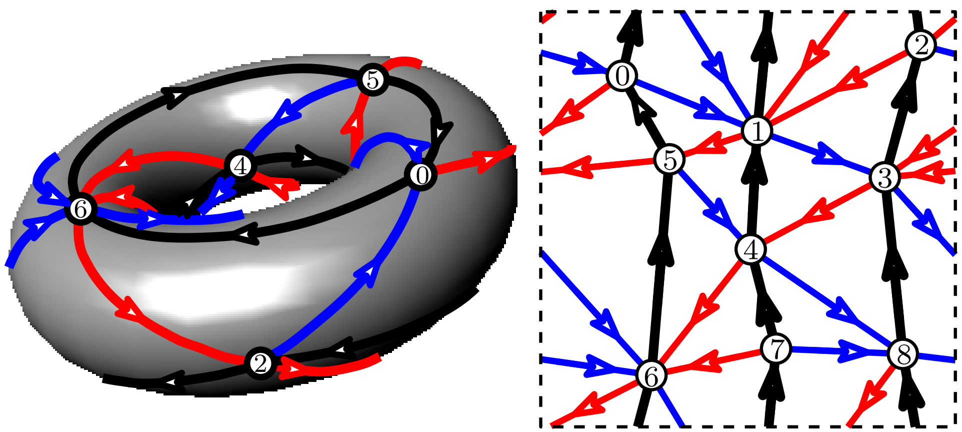


Crossing cycles: a hierarchy of Schnyder woods

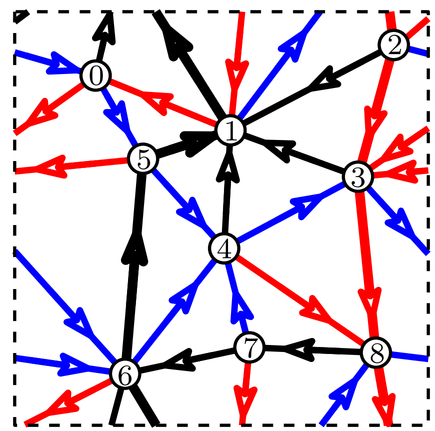
Toroidal Schnyder woods [Goncalves Lévêque, DCG'14]

Toroidal Schnyder woods can be:

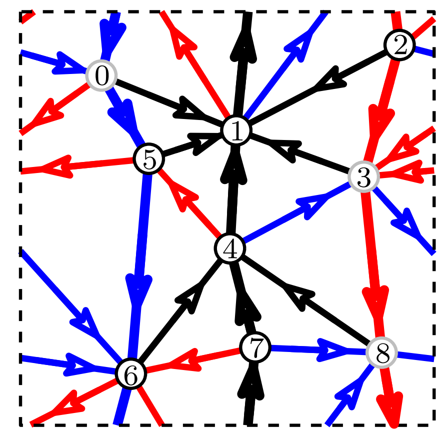
- **crossing:** every monochromatic cycle intersects at least one monochromatic cycle of each color
- only **half-crossing:** only two mono-chromatic cycles are pairwise crossing
- **non-crossing:** all mono-chromatic i-cycles are parallel (non crossing)



crossing Schnyder wood

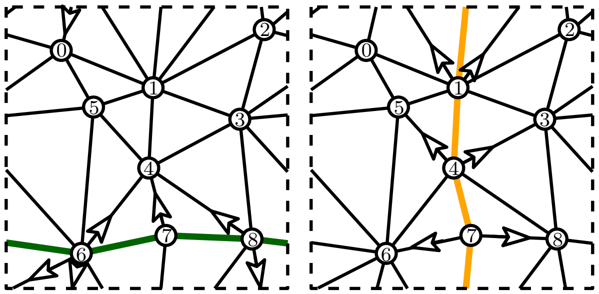


half-crossing Schnyder wood



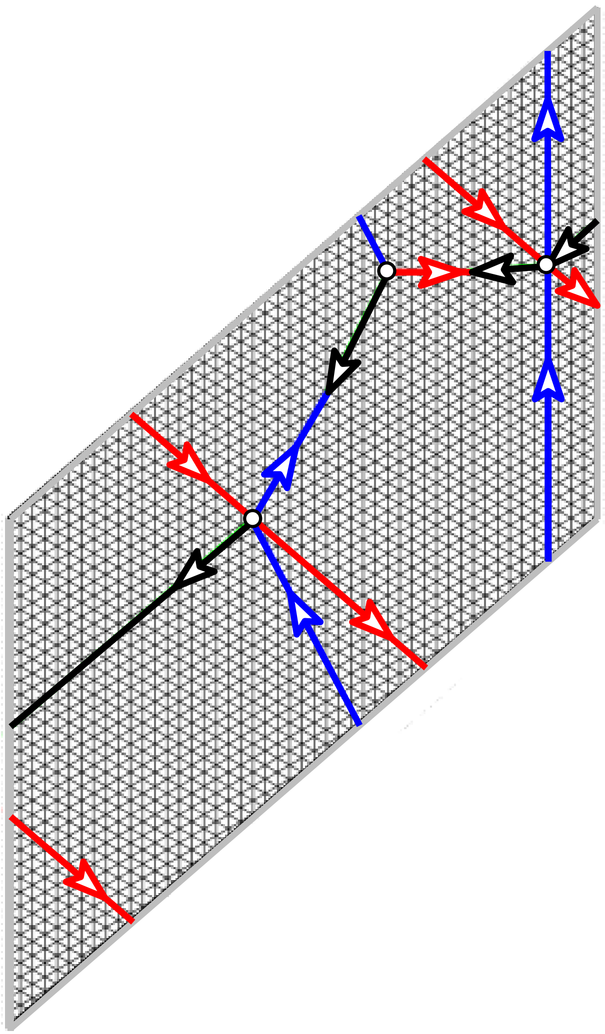
the Schnyder wood is **non-crossing**

the Schnyder wood is **balanced**



Crossing Schnyder woods are relevant for defining toroidal Schnyder (periodic) drawings

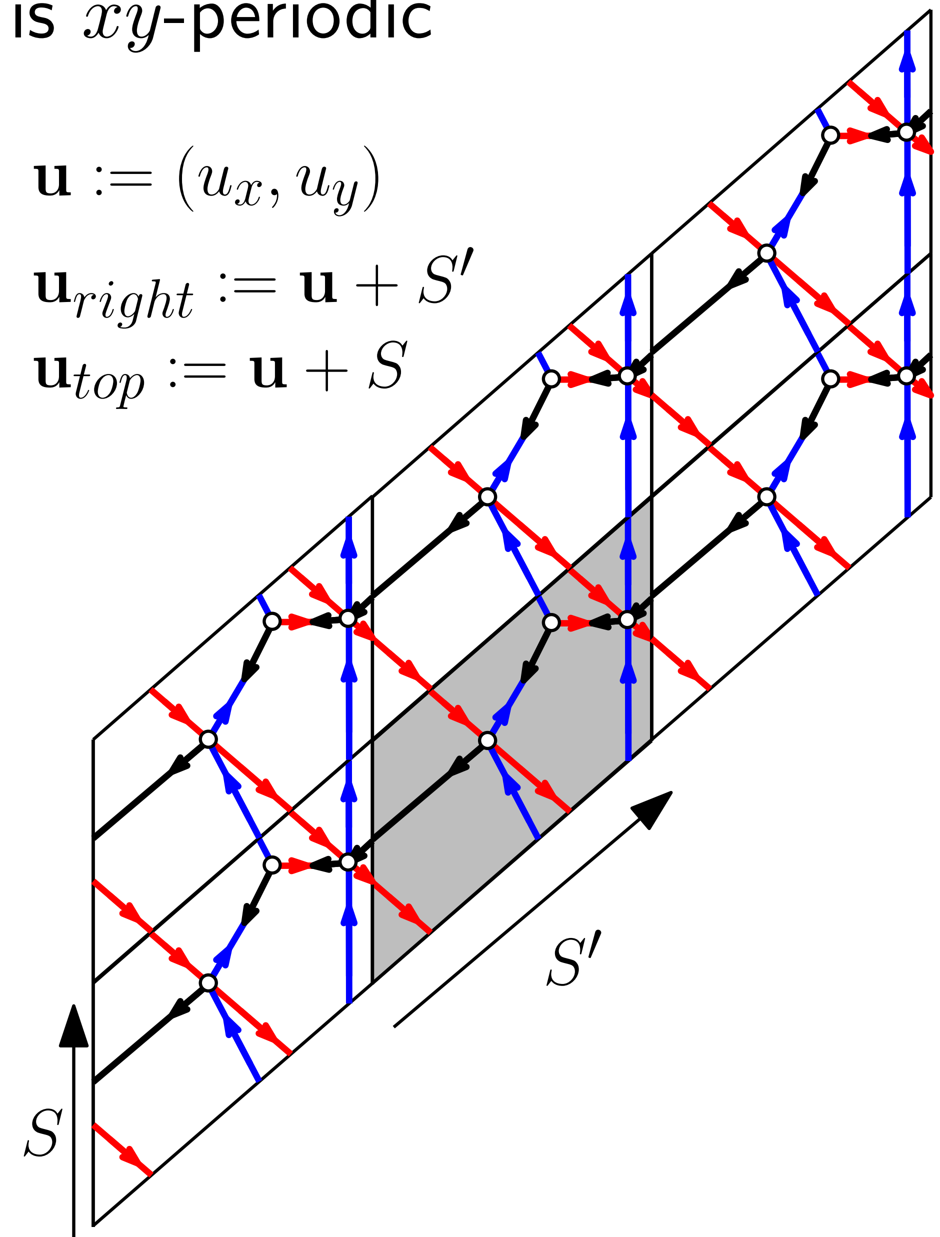
Goal: try to generalize the region counting method to obtain a straight-line grid drawing which is xy-periodic



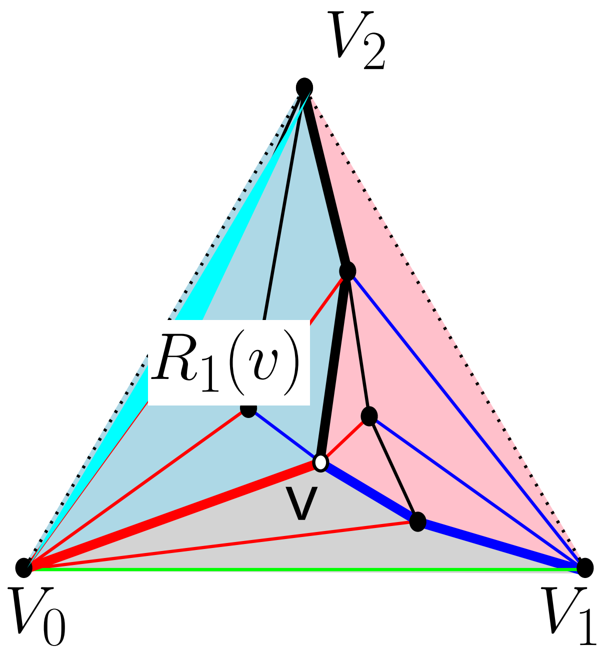
$\mathbf{u} := (u_x, u_y)$

$\mathbf{u}_{right} := \mathbf{u} + S'$

$\mathbf{u}_{top} := \mathbf{u} + S$



Regions are defined by crossing cycles

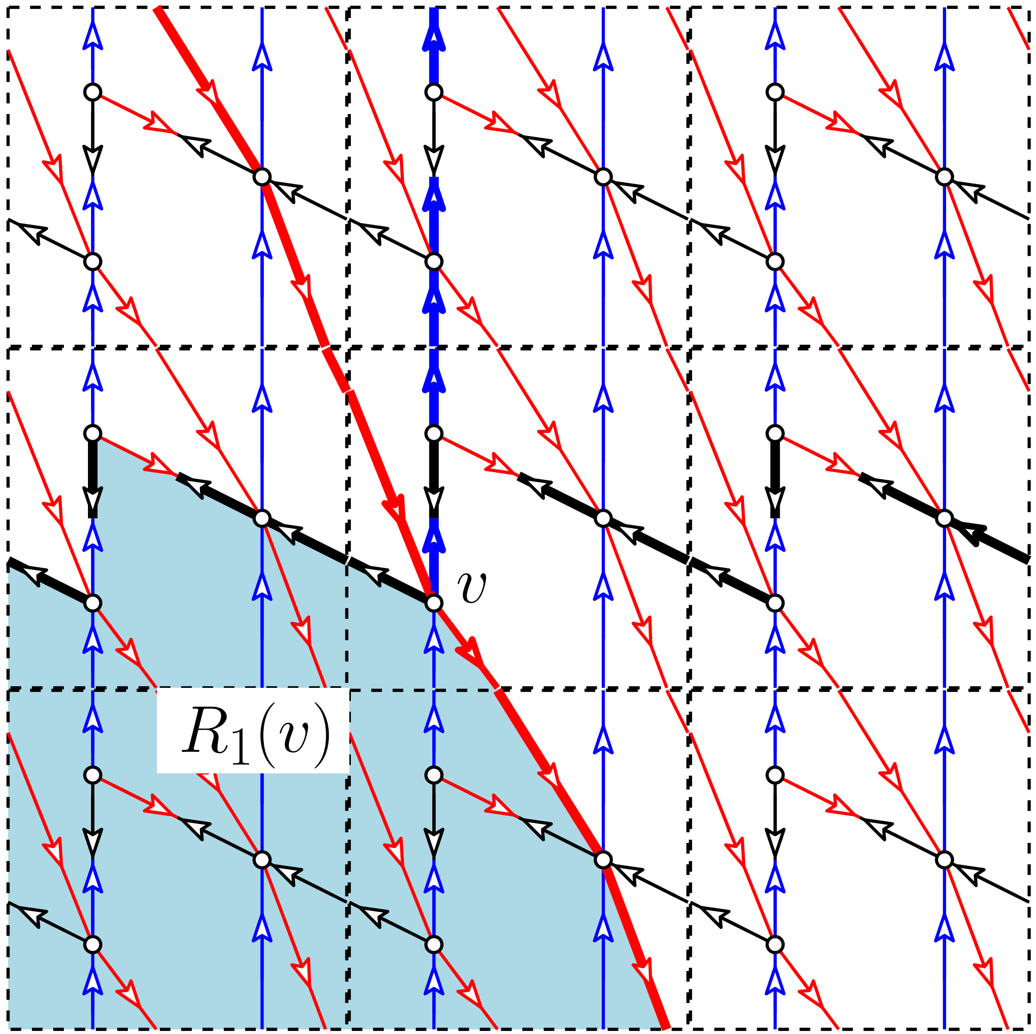


region $R_i(v)$ is defined by two (crossing) paths outgoing from vertex v

$v =: \frac{|R_0(v)|}{|F|-1} V_0 + \frac{|R_1(v)|}{|F|-1} V_1 + \frac{|R_2(v)|}{|F|-1} V_2$

In the toroidal case: regions are unbounded

(in the universal cover)



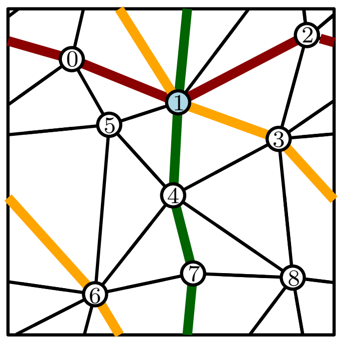
Regions are defined if cycles are crossing

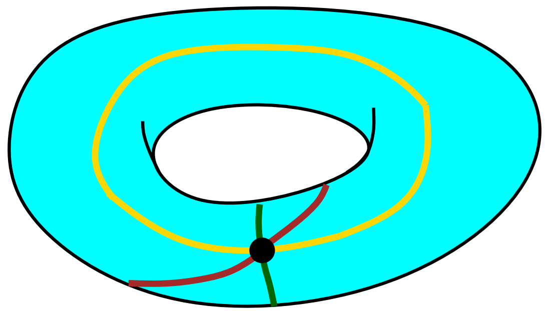
Toroidal Schnyder woods: existence I

Thm[Fijavz, unpublished]

[for simple toroidal triangulations]
(no multiple edges, no loops)

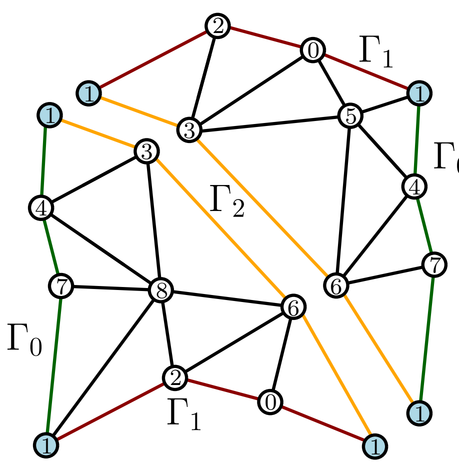
A simple toroidal triangulation contains three non-contractible and non-homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.



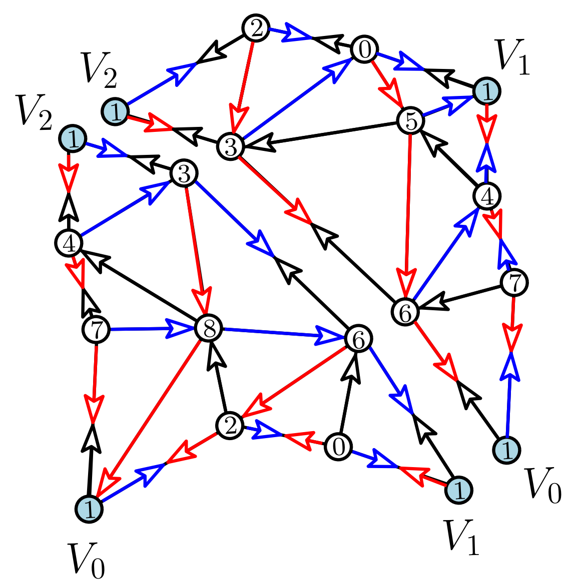


Corollary[Goncalves Lévêque, DCG'14]
Any simple toroidal triangulation admits a toroidal **crossing Schnyder wood**

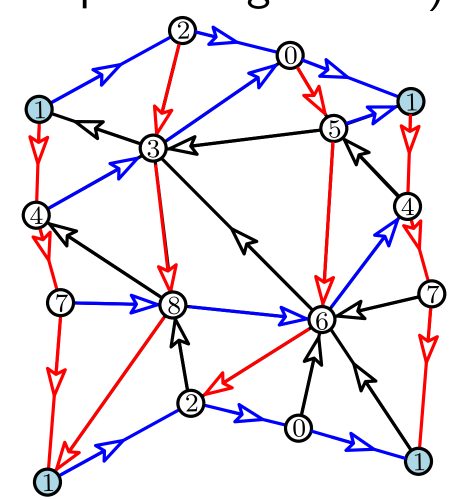
split along Γ_0, Γ_1, Γ_2



(two planar quasi-triangulations)



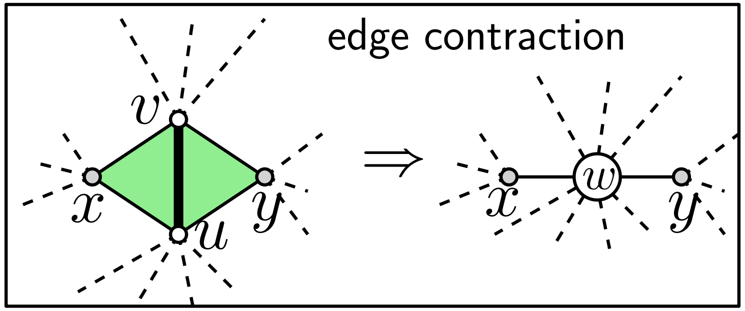
crossing toroidal Schnyder wood
(for simple triangulations)



Toroidal Schnyder woods: existence II

Thm[Goncalves Lévêque, DCG'14] (for general toroidal triangulations and maps)

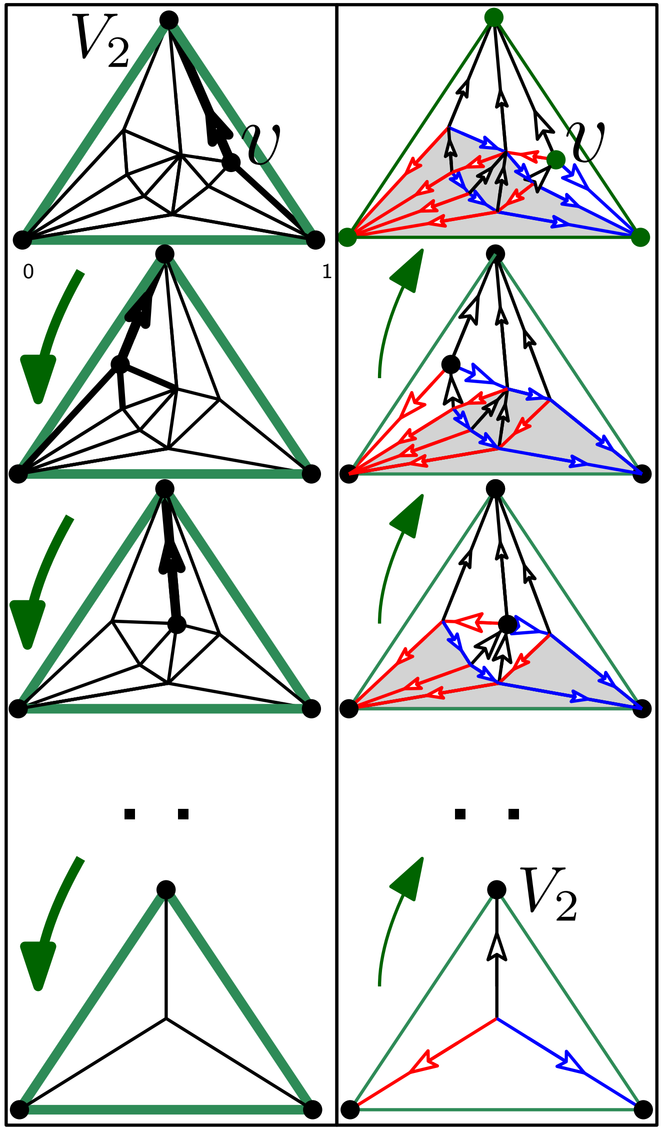
Any toroidal triangulation admits a toroidal **crossing Schnyder wood**



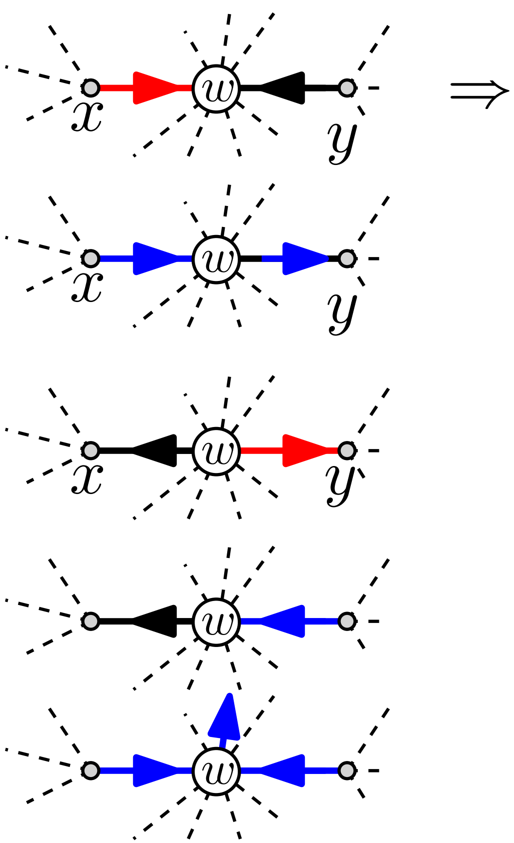
computation of (planar) Schnyder woods

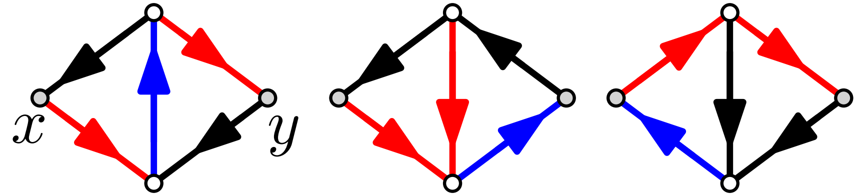
first phase: perform edge contractions

second phase: perform edge expansion+edge coloring



(several cases to distinguish during the decontraction)



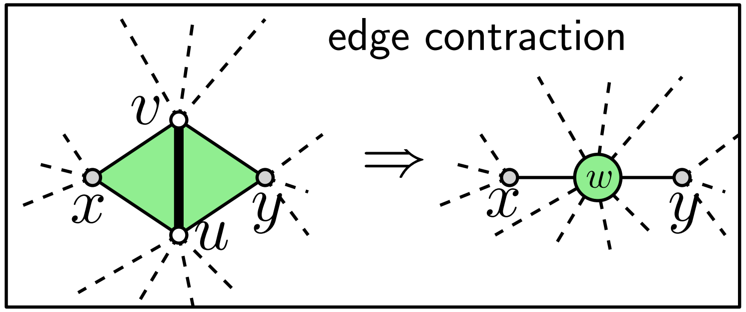


(3 possible choices of coloring)

Toroidal Schnyder woods: existence II

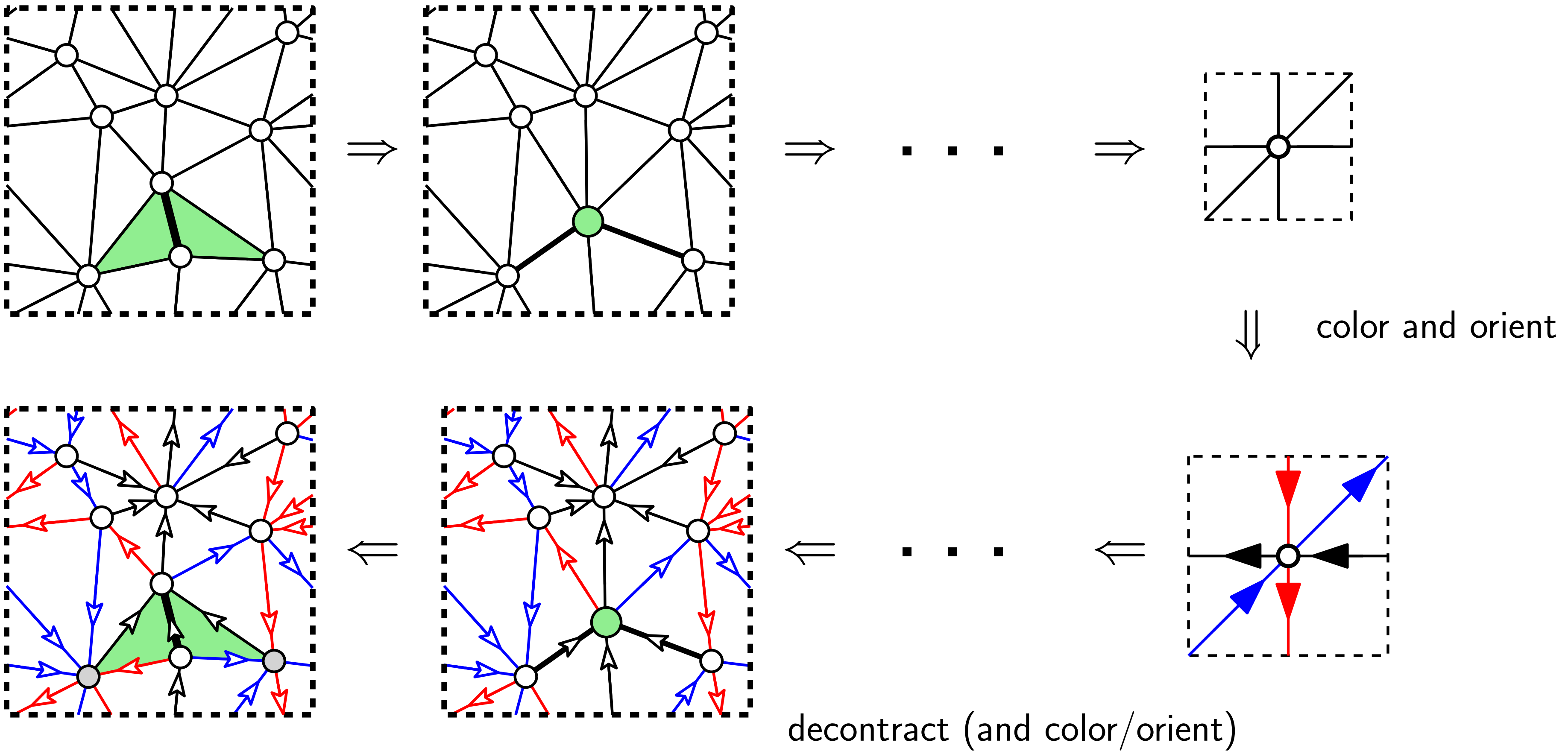
Thm[Goncalves Lévêque, DCG'14] (for general toroidal triangulations and maps)

Any toroidal triangulation admits a toroidal **crossing Schnyder wood**



remark: maintaining the crossing property can require quadratic time

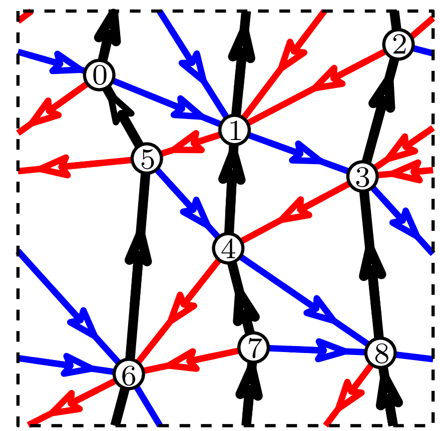
perform (carefully) a sequence of $n-1$ edge contractions



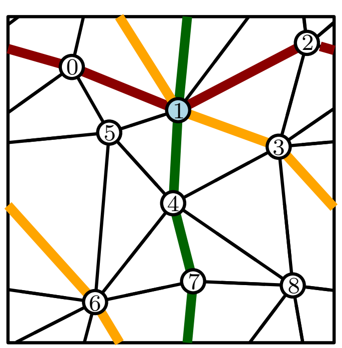
Open problems

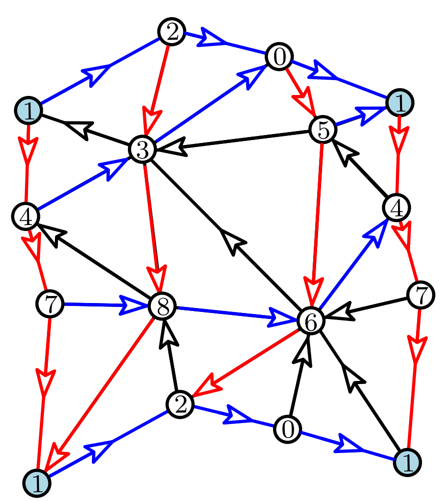
Open problem[Lévêque, 2015]
Is it possible to compute crossing toroidal Schnyder woods via vertex shellings?

Open problem:[Goncalves Lévêque, DCG'14] is it possible to find (at least) one toroidal Schnyder wood which is crossing and with connected mono-chromatic components (one for each color)?



3 disjoint mono-chromatic cycles of color 2
Mono-chromatic cycles of color 0 and 1 are connected

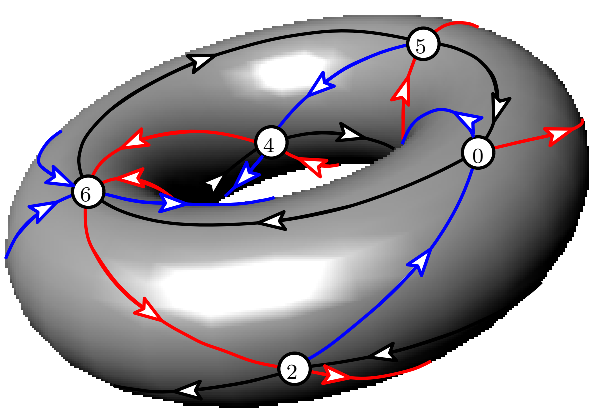


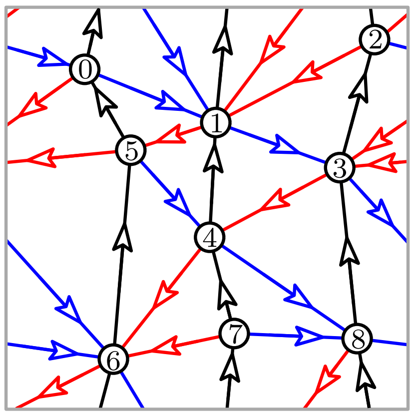


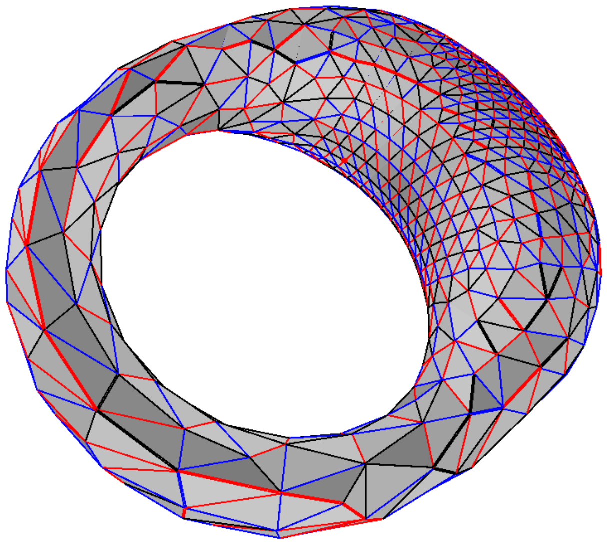
Open problem: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components and such the intersection of the three cycles is a single vertex?

Our contribution:

Computing in linear time (crossing) Schnyder woods with at least two monochromatic connected components (via vertex shellings)





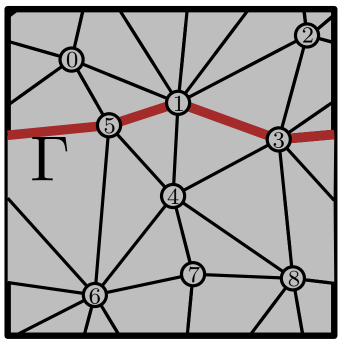


Toroidal Schnyder woods via (cylindric) canonical orderings

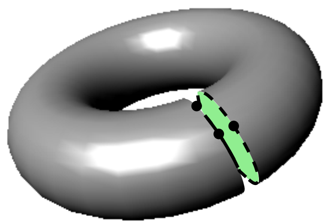
(not necessarily crossing Schnyder woods)

First step: compute a (chord-free) non-contractible cycle Γ

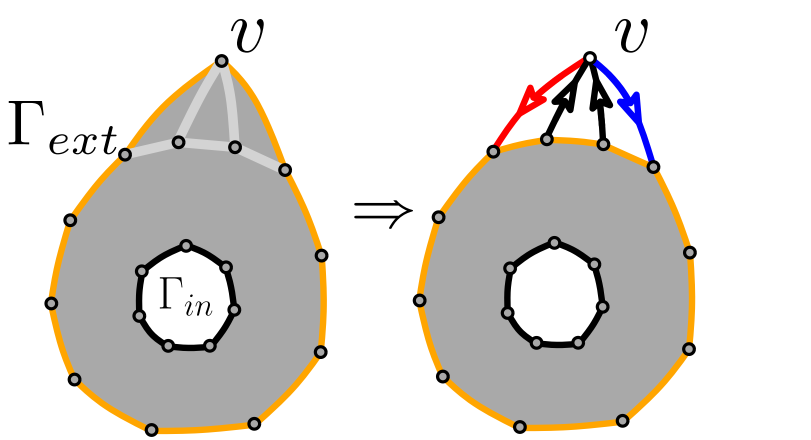
Cut along the cycle Γ



(cylindric triangulation)



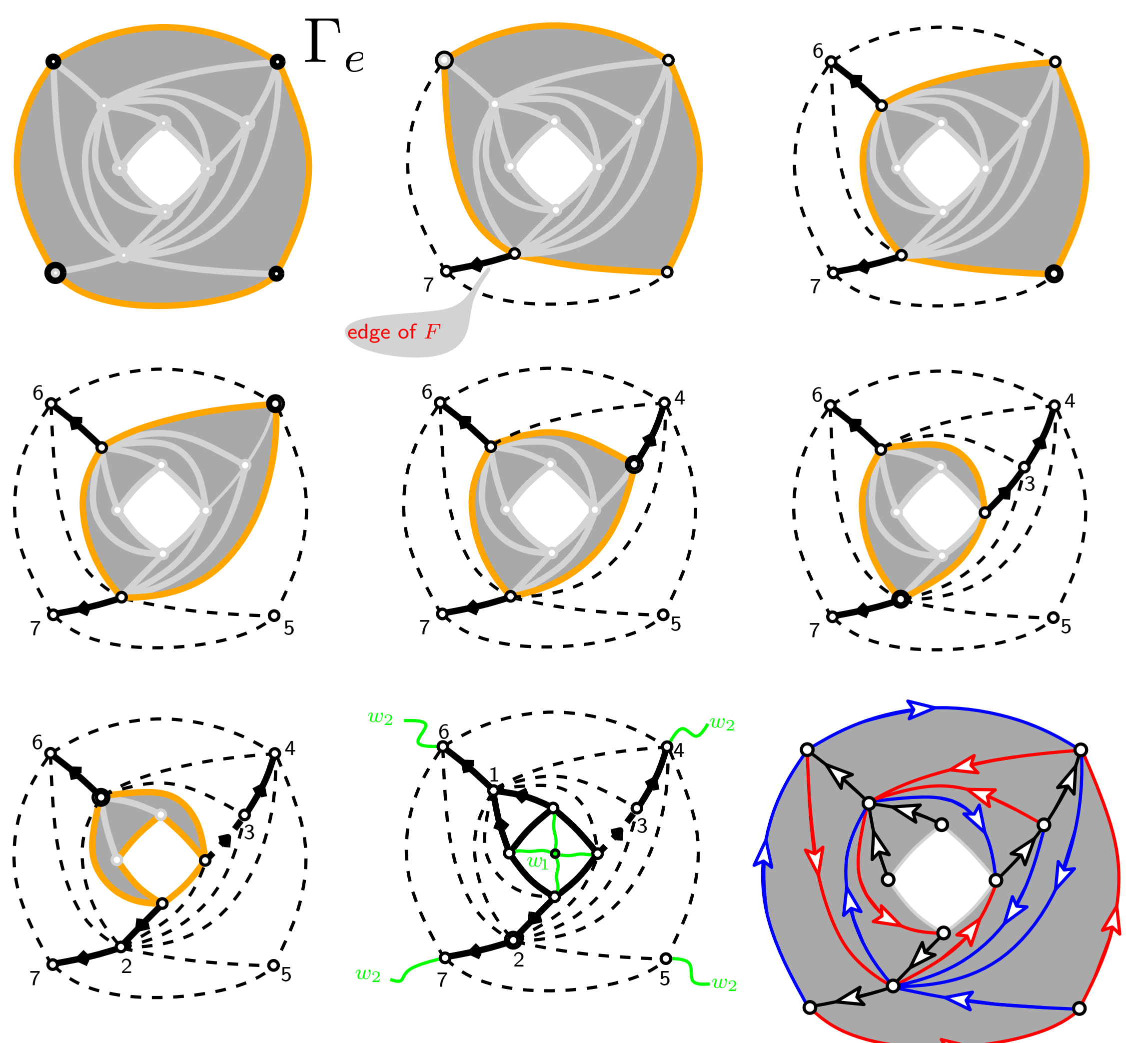
Γ is split into two copies: Γ_{ext} and Γ_{in}



At each step color and orient edges

Compute a **cylindric canonical ordering**
[Castelli Aleardi, Fusy, Devillers, GD2012]

Perform an incremental vertex shelling, starting from Γ_{ext}

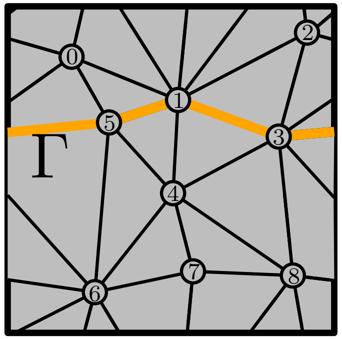


Corollary

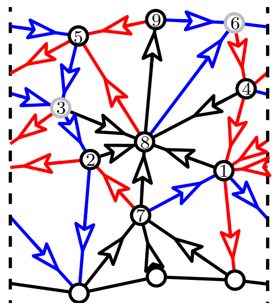
Any simple toroidal triangulation admits a toroidal **(not necessarily crossing) Schnyder wood**

Algo 1: Toroidal Schnyder woods via (cylindric) canonical orderings

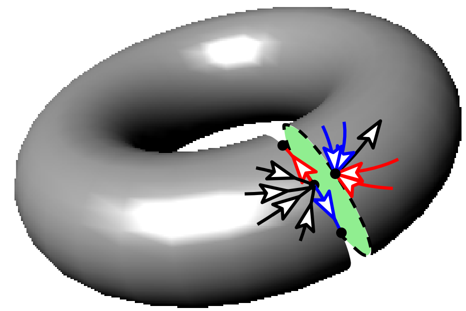
(not necessarily crossing Schnyder woods)



⇒

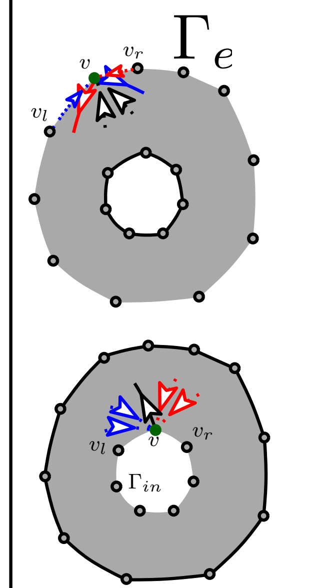


⇒



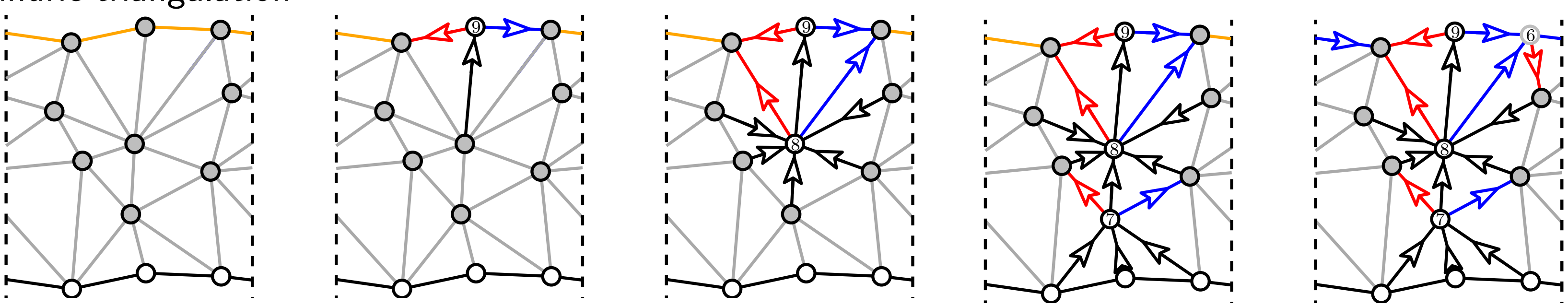
glue together the two boundaries

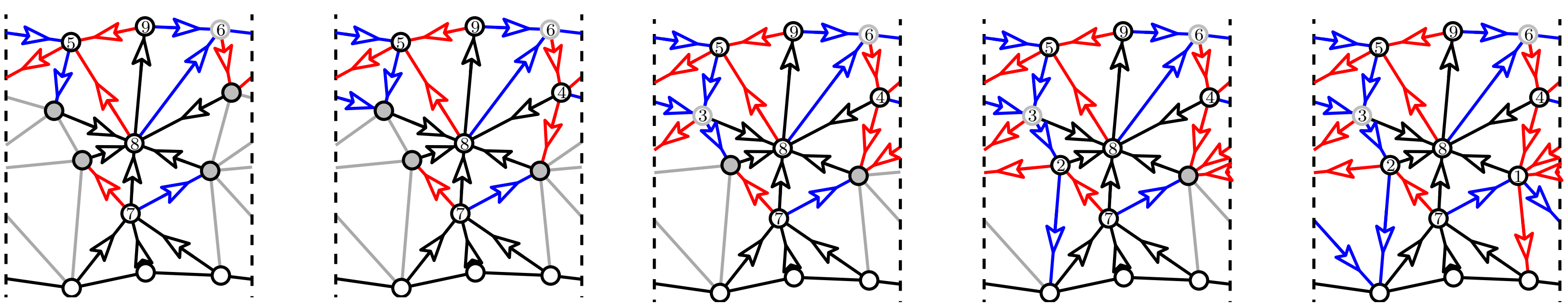
(the local Schnyder woods remains satisfied on Γ)



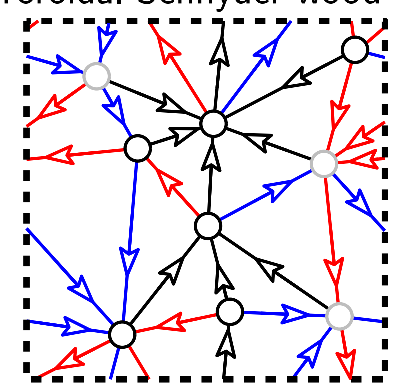
local invariant

Cylindric triangulation

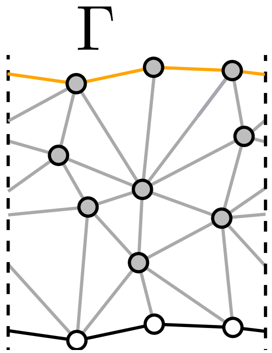




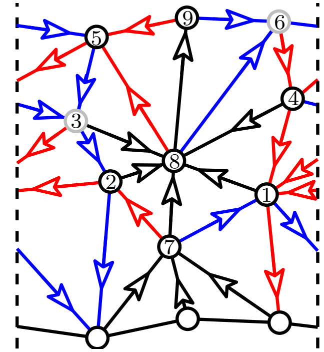
Toroidal Schnyder wood



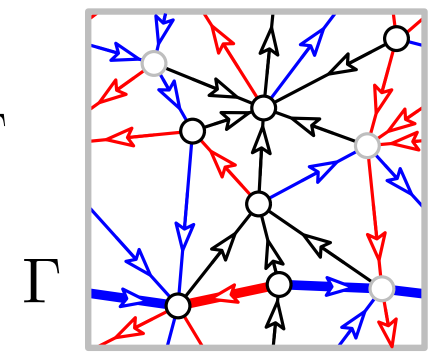
Structural properties of Schnyder woods computed by Algo 1

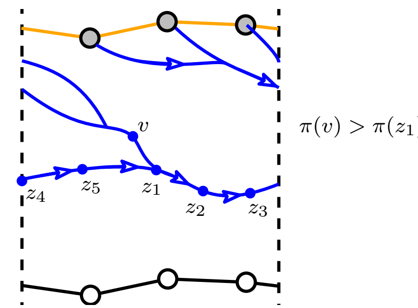


- edges of Γ are either 0 or 1
- 0 and 1-paths are oriented downward
- 2-paths are oriented upward
- 0, 1 and 2-paths cross the cycle Γ



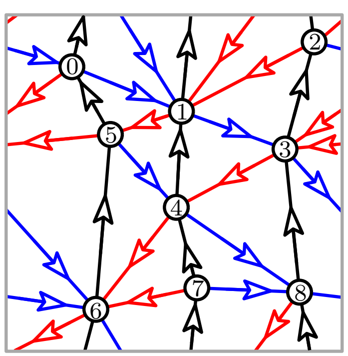
- 0, 1 and 2-cycles are never homotopic to Γ



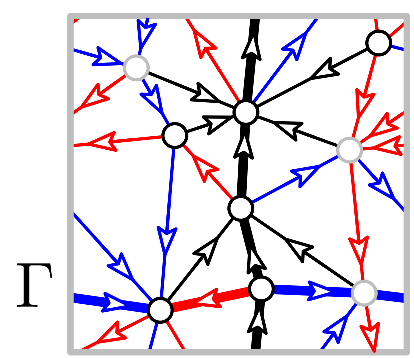


$\pi(z_1) > \pi(z_2) > \pi(z_3) > \dots \pi(z_5) > \pi(z_1)$

- If the Schnyder wood is (at least) half-crossing then the 0-cycles and 1-cycles are pairwise crossing



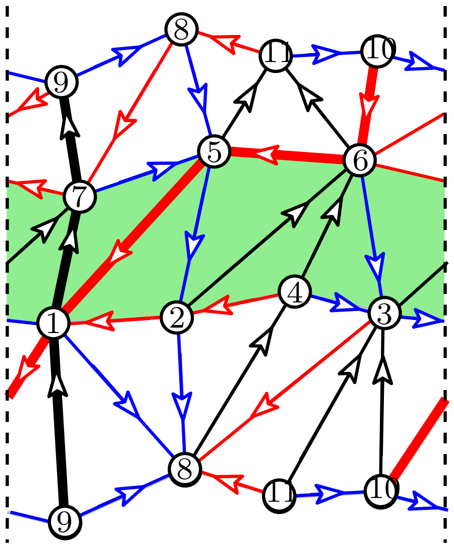
crossing Schnyder wood



edges on Γ are either blue or red

the Schnyder wood may be NOT crossing
but it is at least **balanced**

Toward half-crossing Schnyder woods
(with one connected mono-chromatic component)



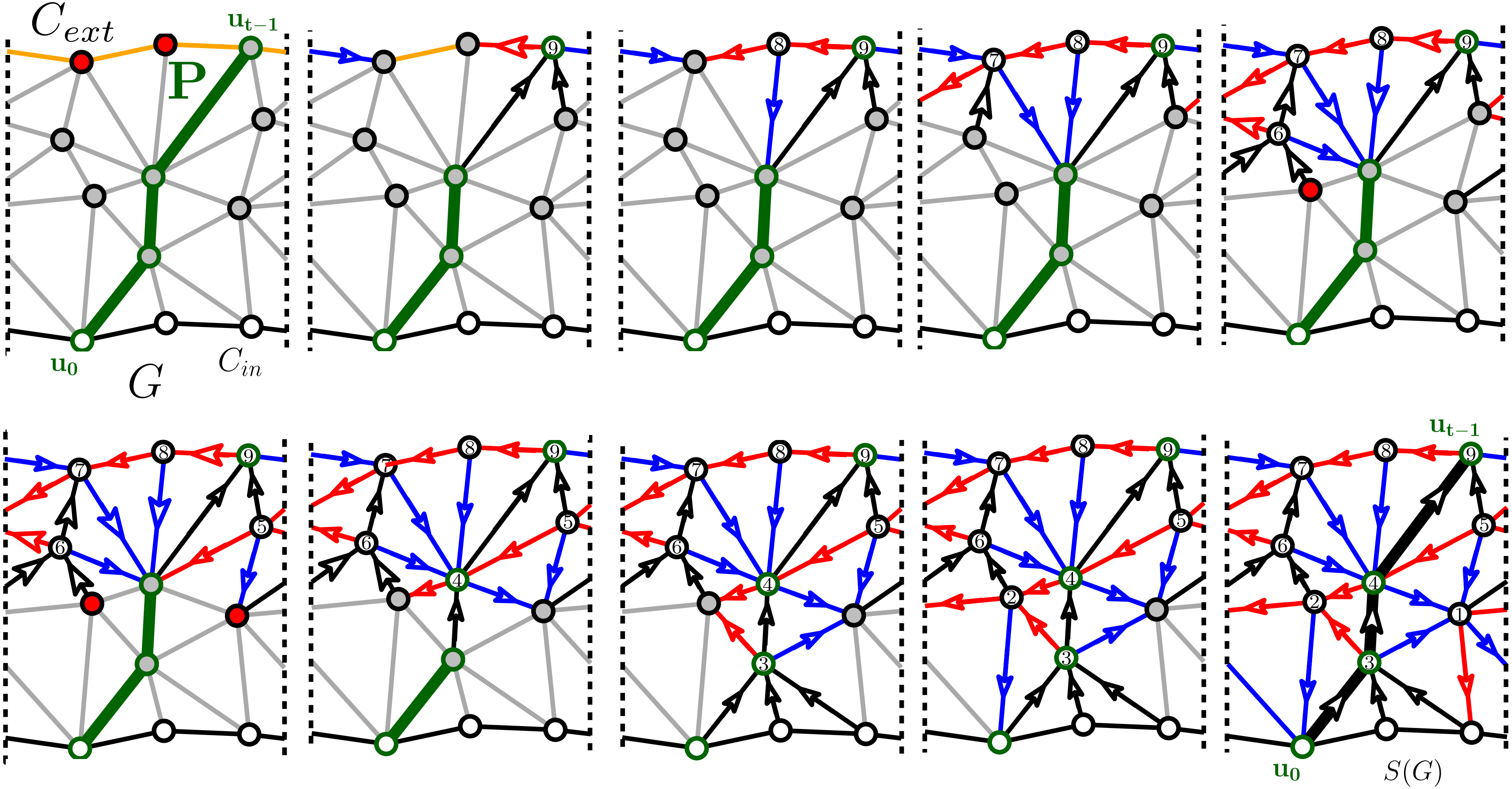
P-constrained (cylindric) Schnyder woods

Input: a cylindric triangulation G and a chord-free path $P := \{u_0, \ldots, u_{t-1}\}$

the path P must intersect the two boundary cycles only at u_0 and u_{t-1}

Output: a Schnyder wood $S_P(G)$ such that the edges of P are of color 2

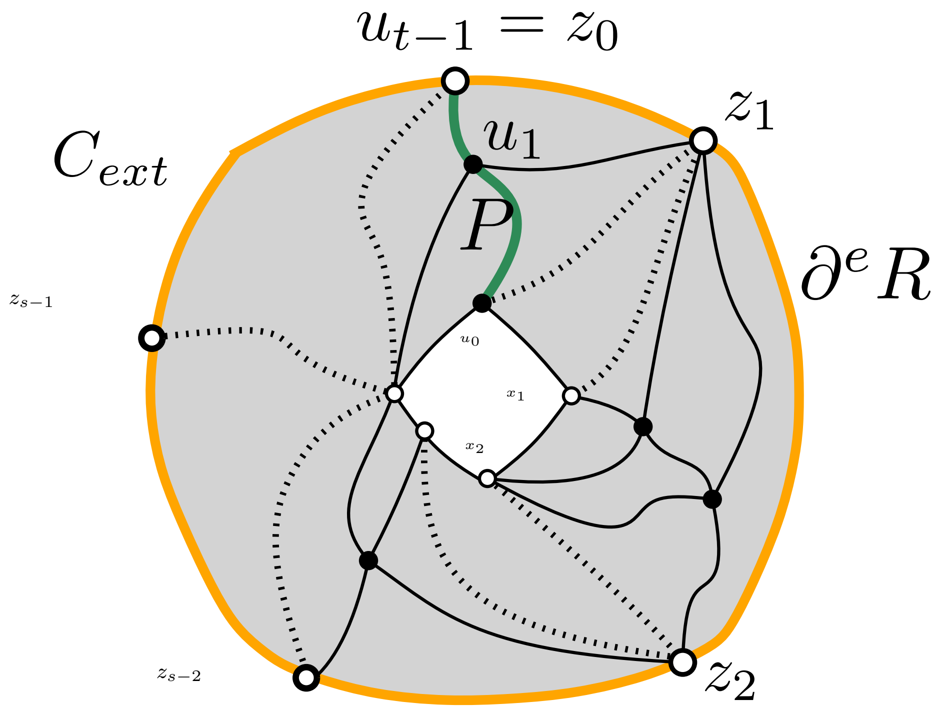
Solution: perform vertex shellings only for (boundary) vertices which are not adjacent to an inner vertex of P



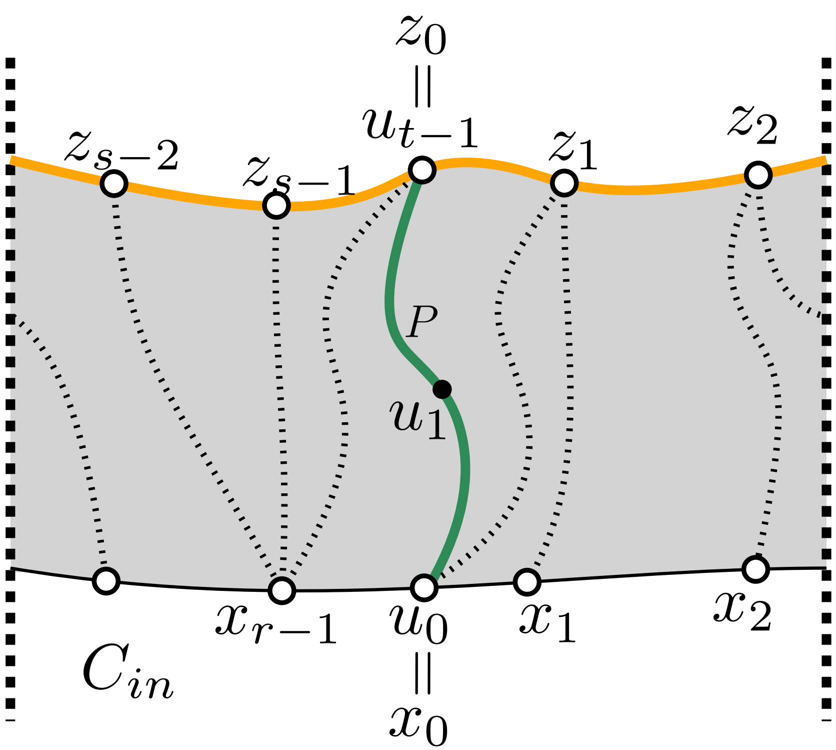
Definition of a river

Def: a **river** is a cylindric triangulation such that the two boundaries are disjoint and chordless and such every vertex is incident to a non-trivial chord (connecting the two boundaries)

$C_{ext} = \{z_0, z_1, z_2, \ldots, z_{s-1}\}$

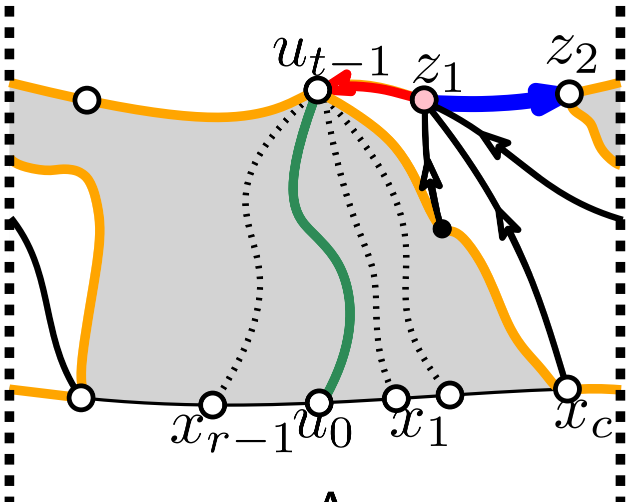


$C_{in} = \{x_0, x_1, x_2, \ldots, x_{r-1}\}$

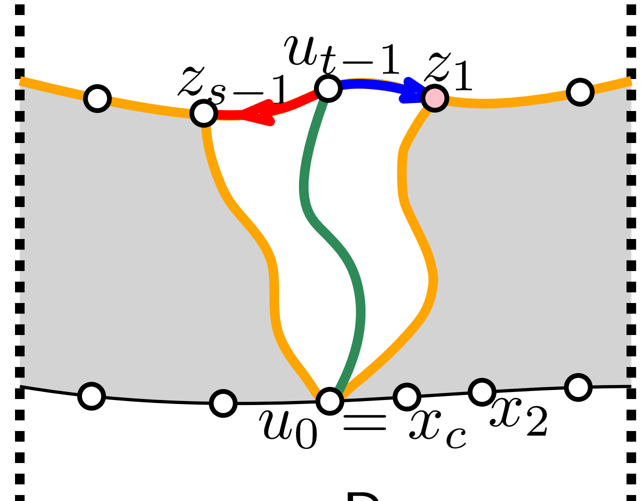


Right-most traversal of a river

u_{t-1} has chords both at the left and the right of P



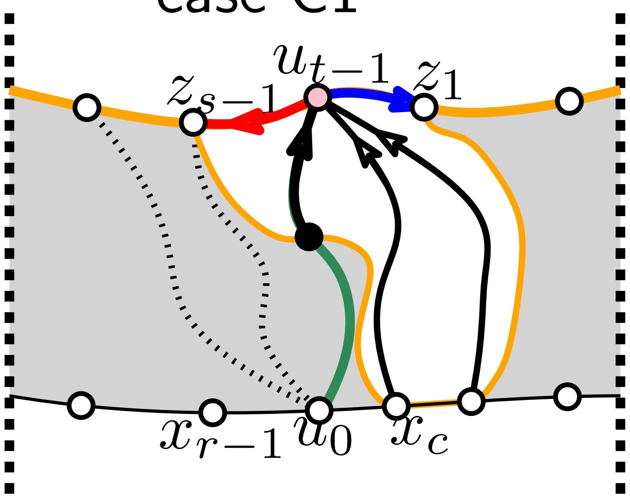
case A



case B

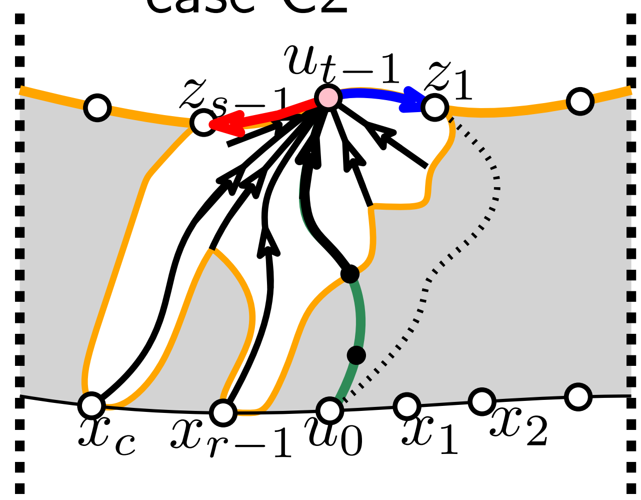
u_{t-1} has no chords other than (u_0, u_{t-1})

case C1



no chord $(u_{t-1}, \bar{x})$ at the left of P

case C2



no chord $(u_{t-1}, \bar{x})$ at the right of P

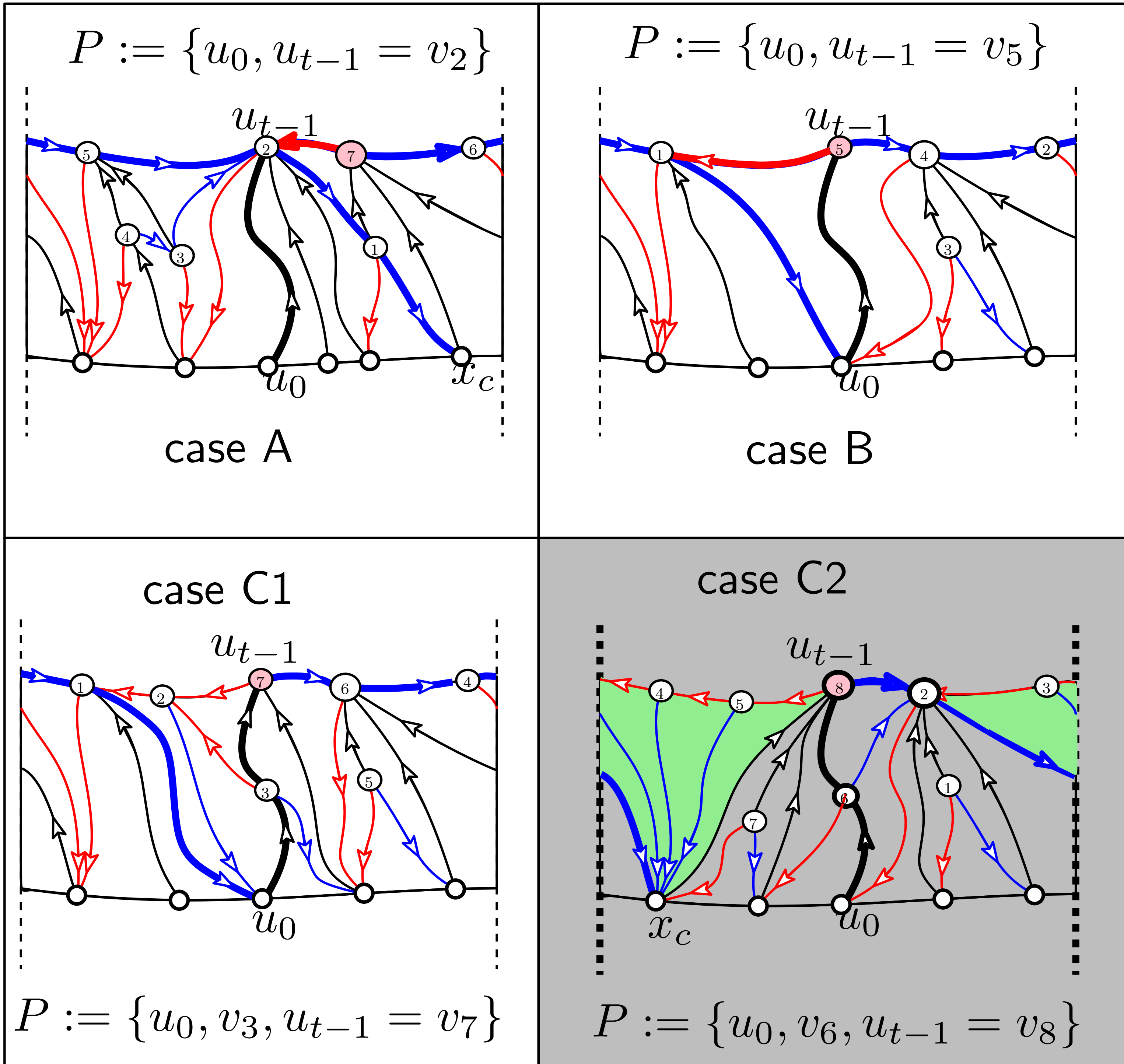
Right-most traversal of a river

Right-most traversal: remove at each step the left-most vertex without chords

Lemma

In cases (A), (B) and (C_1), the blue path P_1 visits all vertices on the top boundary and crosses P either at u_0 or at u_{t-1}

In case (C_2), the blue path P_1 may not cover all top boundary vertices (not crossing P), but then there exists a ccw-oriented (contractible) cycle (green region)

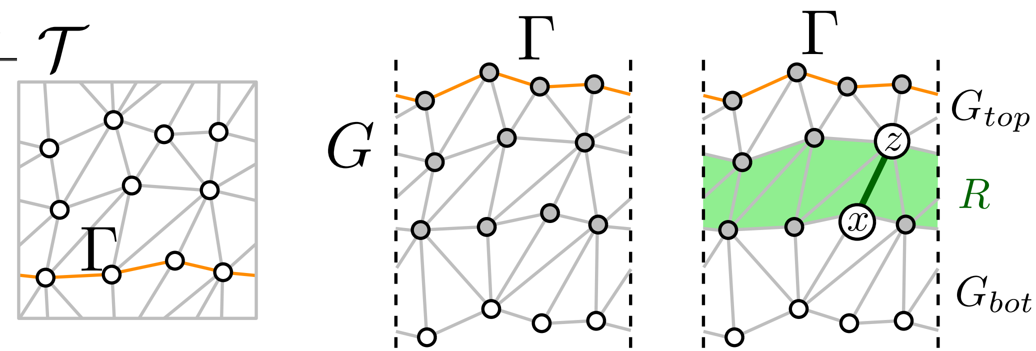


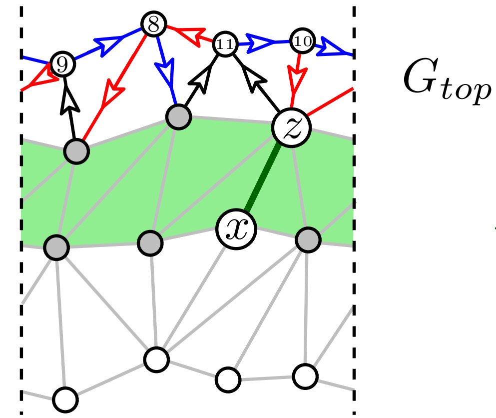
An algorithm for half-crossing Schnyder woods

Algo 2 Half-crossing Schnyder woods (with a connected mono-chromatic component)

Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood

// Pre-processing step
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
// First pass
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has **type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
// Run a second pass on R
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; // Remark: the r-cycle and 2-cycles are parallel
$P_2 \leftarrow \gamma_2 \cap R$; // restriction of γ_2 to the river R
$u \leftarrow \partial^e R \cap P_2$;
if *u has **type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
// Remark: $S(G_{bottom})$ and $S(G_{top})$ are P_2-constrained
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;





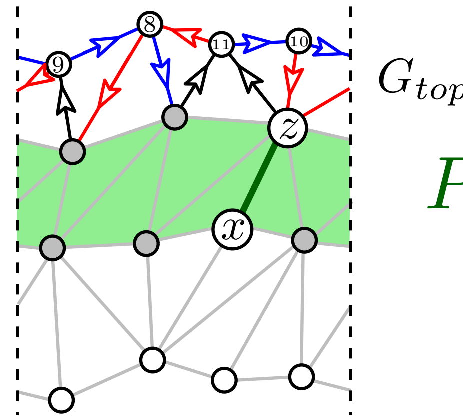
$P = \{x, z\}$

compute $S(G_{top})$

An algorithm for half-crossing Schnyder woods

Algo 2 Half-crossing Schnyder woods (with a connected mono-chromatic component)

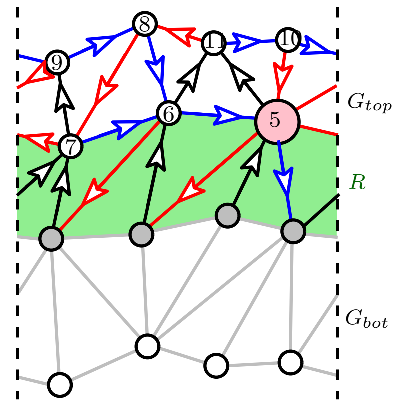
Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood
`// Pre-processing step`
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
`// First pass`
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
`// Run a second pass on` R
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; `// Remark: the` r`-cycle and 2-cycles are parallel`
$P_2 \leftarrow \gamma_2 \cap R$; `// restriction of` γ_2 `to the river` R
$u \leftarrow \partial^e R \cap P_2$;
if *u has* ***type (A), (B) or (C1)*** **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
`// Remark:` $S(G_{bottom})$ `and` $S(G_{top})$ `are` P_2`-constrained`
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;



(sometimes) one pass suffices

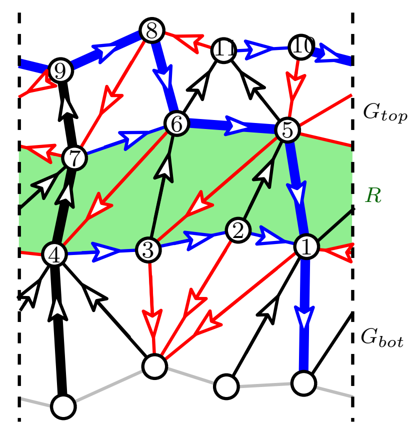
$$P = \{x, z\}$$

compute a constrained rightmost traversal of R



z has type A

compute $S(G_{bot})$



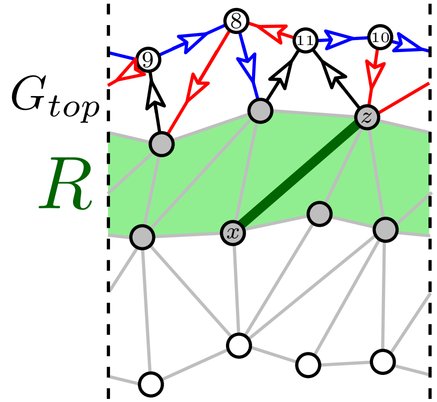
There is one connected (blue) 1-cycle, crossing the 2-cycles

return the Schnyder wood

An algorithm for half-crossing Schnyder woods

Algo 2 Half-crossing Schnyder woods (with a connected mono-chromatic component)

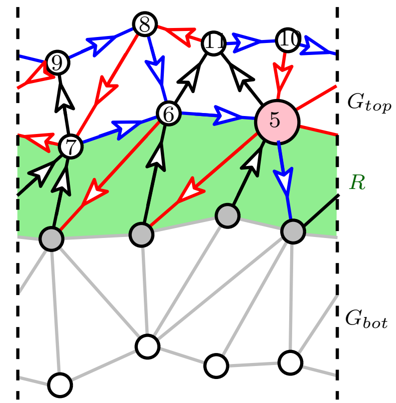
Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood
`// Pre-processing step`
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
`// First pass`
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has **type** (A), (B) or (C1)* **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
`// Run a second pass on` R
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; `// Remark: the` r`-cycle and 2-cycles are parallel`
$P_2 \leftarrow \gamma_2 \cap R$; `// restriction of` γ_2 `to the river` R
$u \leftarrow \partial^e R \cap P_2$;
if *u has **type** (A), (B) or (C1)* **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
`// Remark:` $S(G_{bottom})$ `and` $S(G_{top})$ `are` P_2`-constrained`
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;



(sometimes) two passes are required

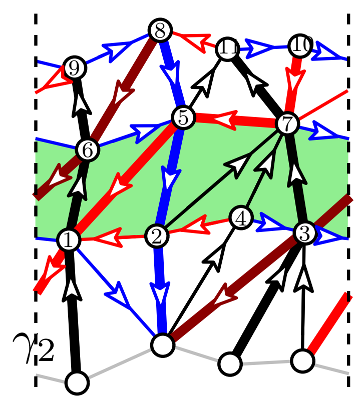
$P = \{x, z\}$

compute a constrained rightmost traversal of R



z has type C_1

compute $S(G_{bot})$



there is one connected 1-cycle

the (blue) 1-cycle and (black) 2-cycles are NOT crossing
(red) 0-cycles cross (black) 2-cycles but have 2 components

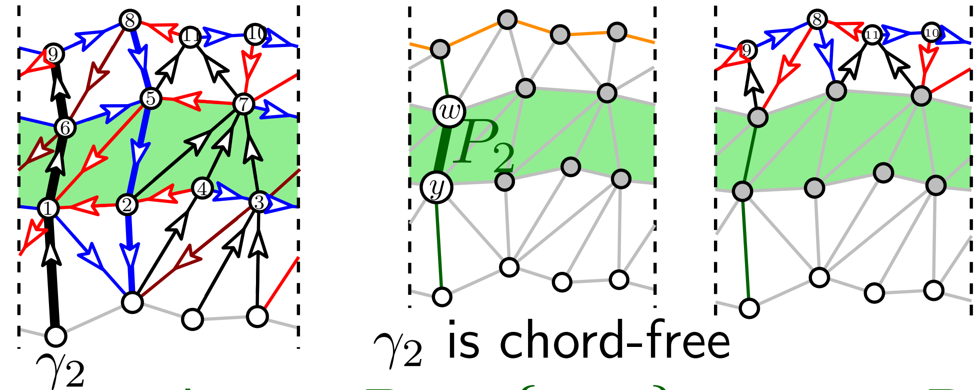
we need a second pass
(black) 2-cycles are chord-free

An algorithm for half-crossing Schnyder woods

Algo 2 Half-crossing Schnyder woods (with a connected mono-chromatic component)

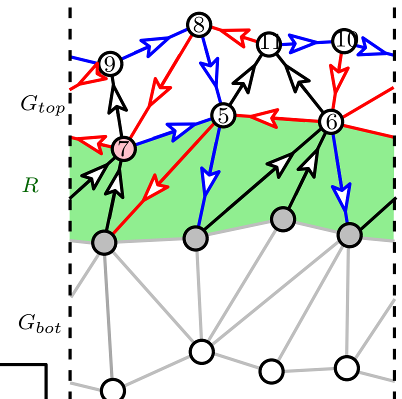
Data: a simple toroidal triangulation $\mathcal{T}$, a non-contractible chordless cycle Γ
Result: a half-crossing Schnyder wood
`// Pre-processing step`
cut $\mathcal{T}$ along Γ: let G be the resulting cylindric triangulation ;
compute a river R and the partition $G = G_{top} \cup R \cup G_{bottom}$;
`// First pass`
compute a Schnyder wood $S(G_{top})$ of G_{top} ;
choose an arbitrary non trivial chord $e = (x, z)$ of R ;
$P \leftarrow \{x, z\}$;
if *z has **type** (A), (B) or (C1)* **then**
  run the right-most P-constrained traversal of (R, P);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P) ;
  $r \leftarrow 0$;
end
compute a Schnyder wood $S(G_{bottom})$ of G_{bottom} ;
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
if *the r-cycle and 2-cycles are crossing in $S(\mathcal{T})$* **then**
  return $S(\mathcal{T})$;
end
`// Run a second pass on R`
$\gamma_2 \leftarrow$ any 2-cycle of $S(\mathcal{T})$; `// Remark: the r-cycle and 2-cycles are parallel`
$P_2 \leftarrow \gamma_2 \cap R$; `// restriction of γ2 to the river R`
$u \leftarrow \partial^e R \cap P_2$;
if *u has **type** (A), (B) or (C1)* **then**
  run the right-most P-constrained traversal of (R, P_2);
  $r \leftarrow 1$;
else
  run the left-most P-constrained traversal of (R, P_2) ;
  $r \leftarrow 0$;
end
`// Remark: S(Gbottom) and S(Gtop) are P2-constrained`
glue boundary cycles together and let $S(\mathcal{T}) = S(G_{bottom}) \cup S(R) \cup S(G_{top})$;
return $S(\mathcal{T})$;

Run the second pass



γ_2 is chord-free

choose $P_2 = \{y, w\} := \gamma_2 \cap R$

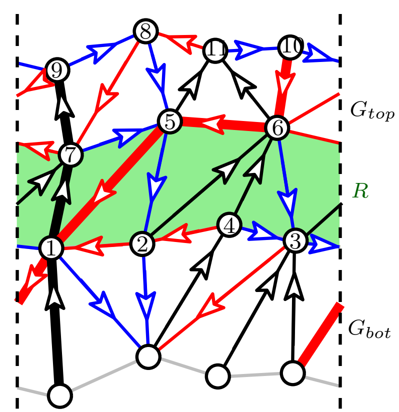


w has type C_2

compute a constrained leftmost traversal of R

compute $S(G_{bot})$

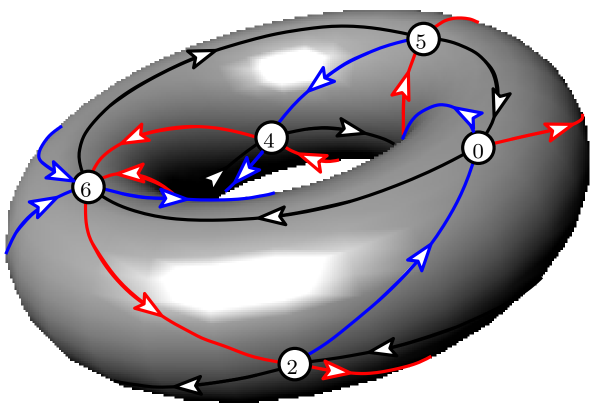
we need a second pass

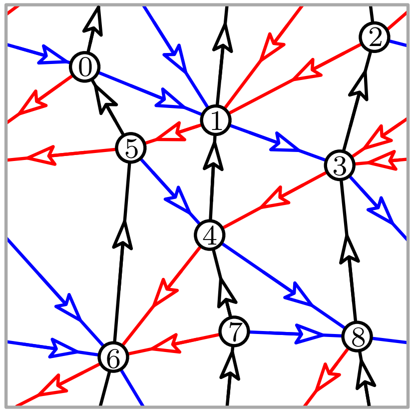


there is only one connected 0-cycle

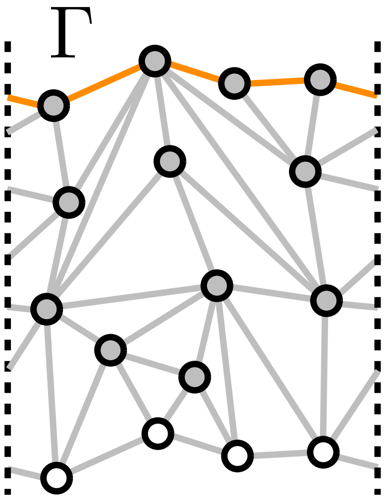
the 0-cycle(s) and the 2-cycle are crossing

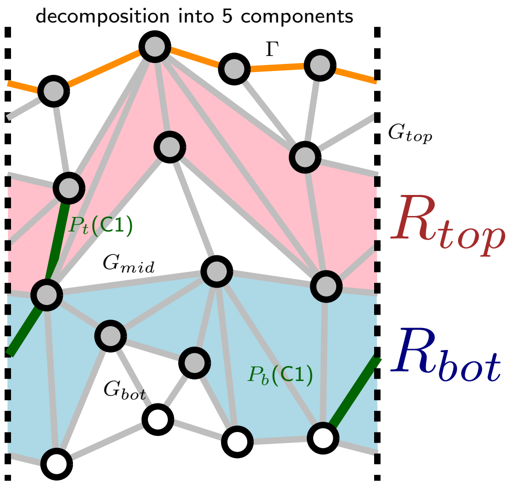
Toward crossing Schnyder woods
(with two connected mono-chromatic components)





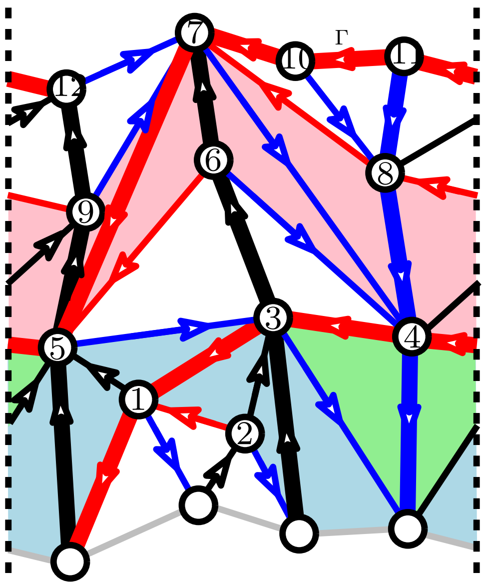
An algorithm for crossing Schnyder woods





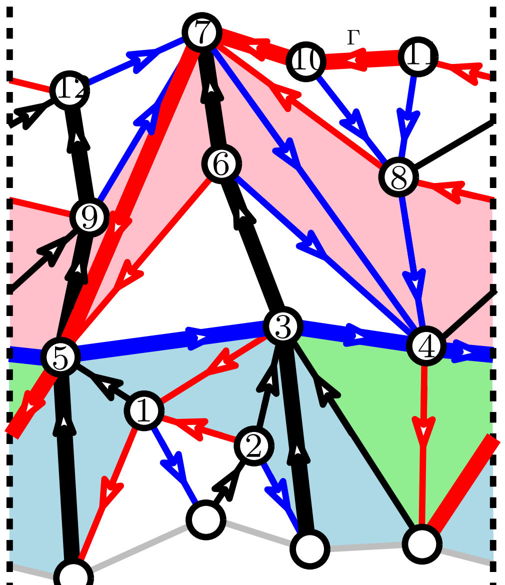
compute two non overlapping rivers

the 2-cycles and the 1-cycle are NOT crossing



half-crossing before reversing

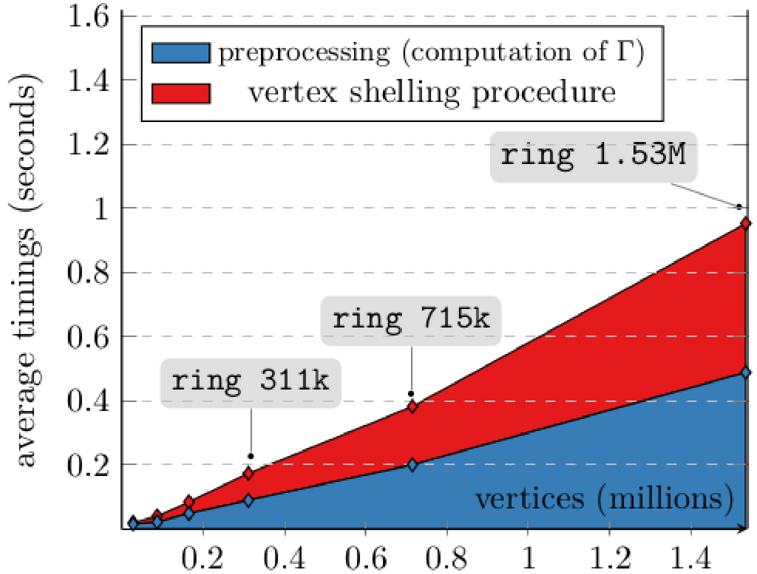
an oriented cycle in R_{bot} to be reversed



crossing after reversing

Experimental results

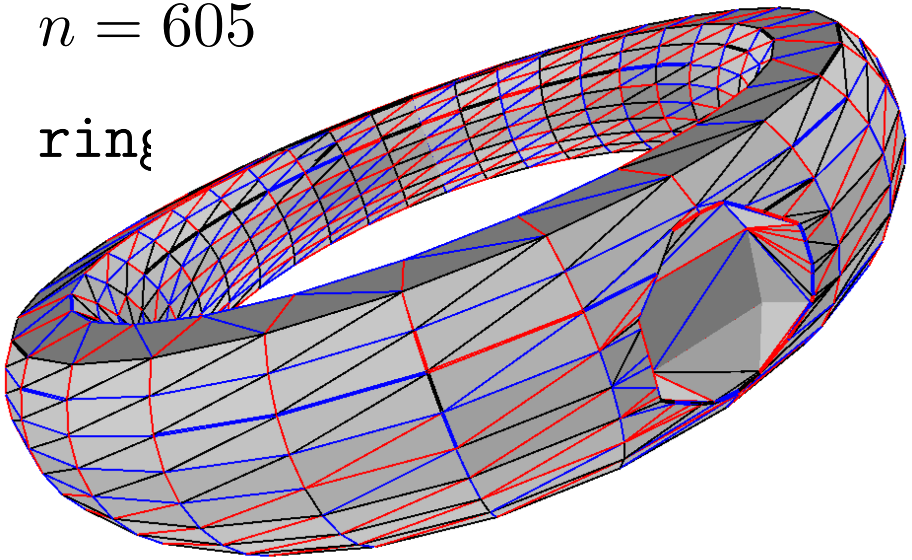
Fast linear-time implementation

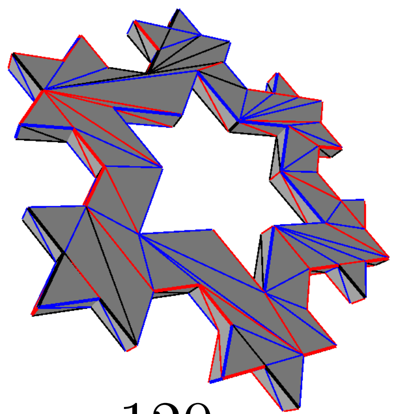


(with Java 1.8, on a Dell Laptop, Intel core i7 2.6GHz, 8GB RAM)

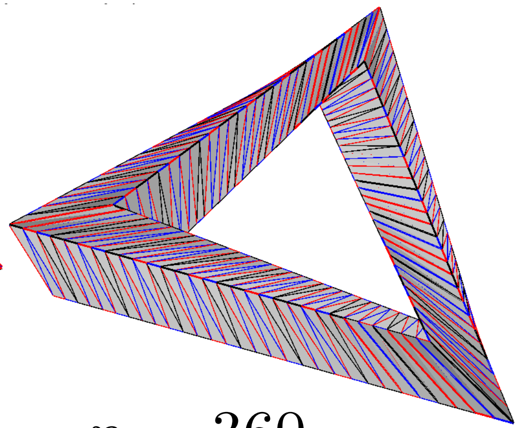
$n = 605$

ring



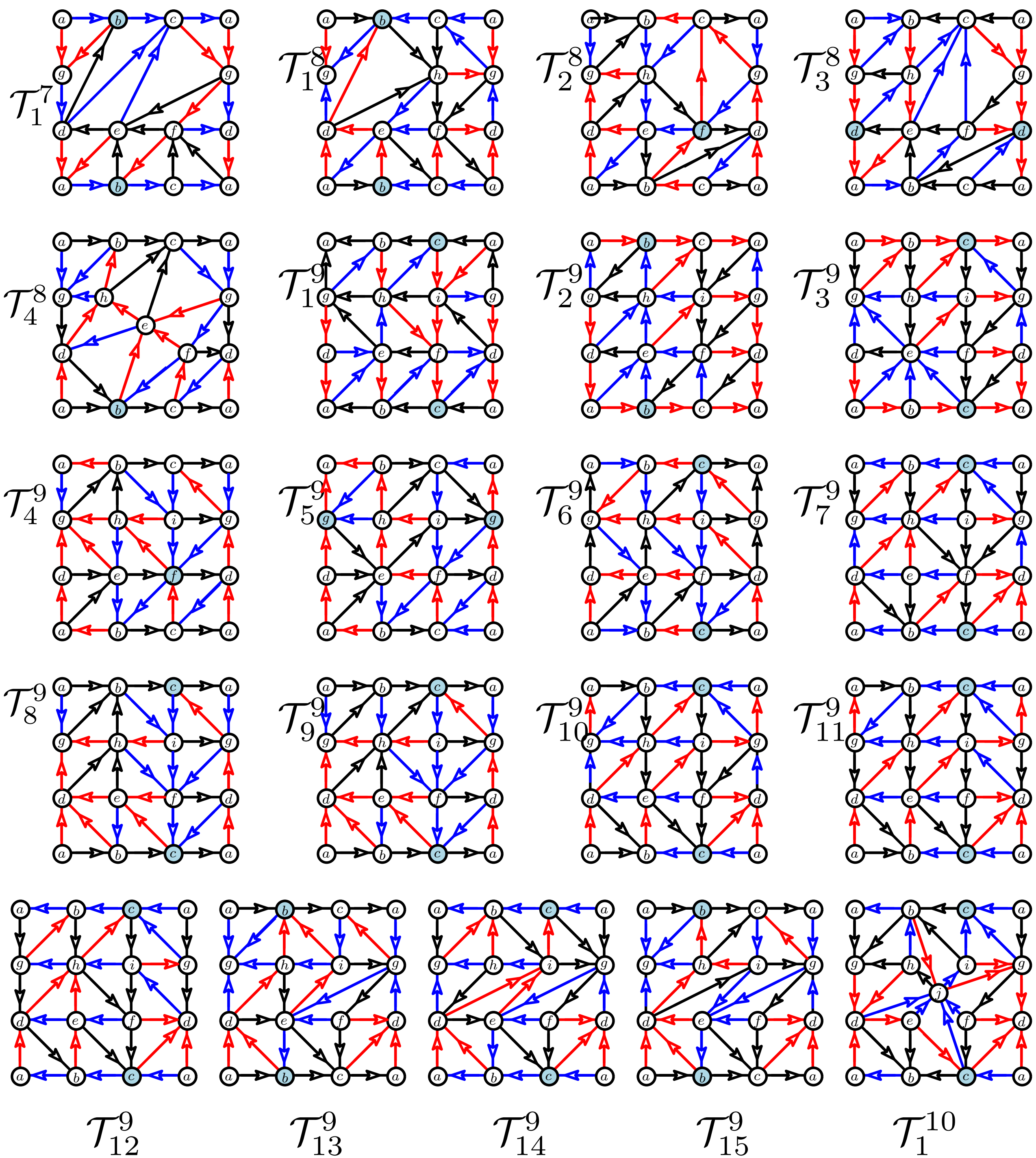


$n = 120$
Koch
snowfla



$n = 360$
Penrose
triangl

Conjectures on toroidal Schyder woods: experimental confirmation



Open problem: is it possible to find (at least) one toroidal Schnyder wood with connected mono-chromatic components and such the intersection of the three cycles is a single vertex?

(true for all triangulations of size at most $n = 11$)

n	# irreducible triangulations	#triangulations (g = 1)
7	1	1
8	4	7
9	15	112
10	1	2109
11	–	37867

triangulations are generated with **surftri** software [Sulanke, 2006]

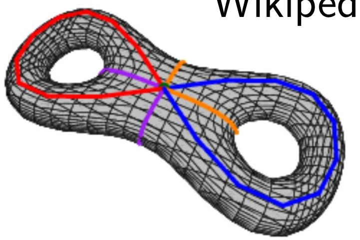
Concluding remarks

Schyder woods for $g \geq 2$

Thm (3-orientations for graphs on surfaces, of arbitrary genus)
[Albar Goncalves Knauer, 2014]

Any triangulation of a surface (the sphere and the projective plane) admits a '3-orientation': orientation without sinks s.t. every vertex has outdegree divisible by three

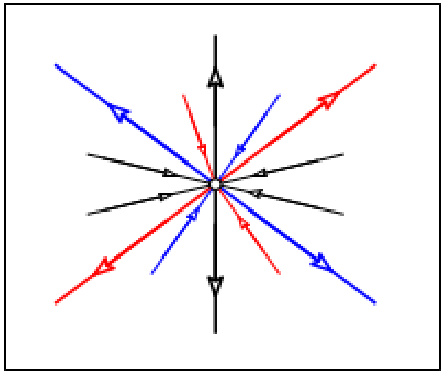
Wikipedia picture



Open problem [Goncalves Knauer Lévêque, 2016]
Existence of Schnyder woods for higher genus triangulations

Multiple local Schnyder condition: the outdegree of every vertex is a **positive** multiple of 3.

(there are no **sinks**)



Thm [Suagee, 2021]

Simple triangulations of genus $g \geq 1$ having "large" **edgewidth** do admit Schnyder woods

edgewidth $\geq 40(2^g - 1)$

(size of the smallest non contractible cycle)

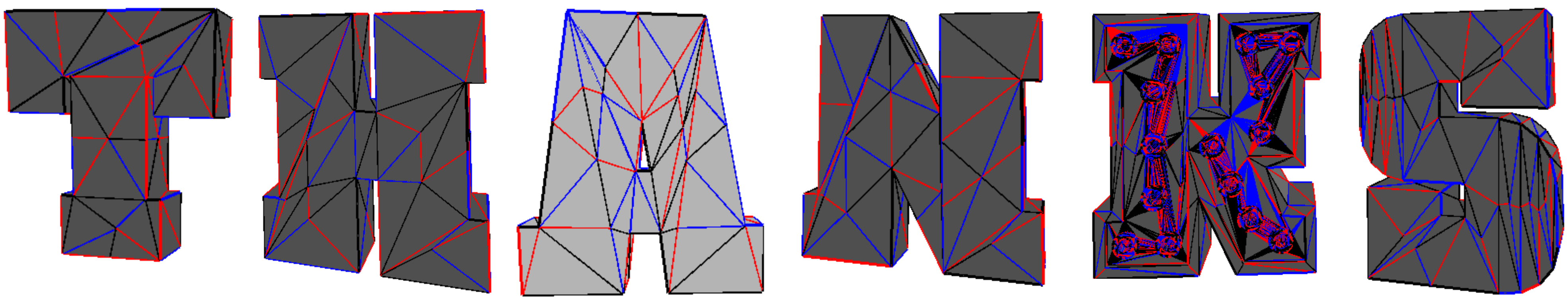
Experimental confirmation $(g = 2)$

exaustive generation of all 3-orientations for all triangulations with $g = 2$, $n \leq 11$

All simple triangulations of genus $g = 2$ and size ≤ 11 admit Schnyder woods

n	# irreducible triangulations	#triangulations ($g = 2$)
7	–	–
8	–	–
9	–	–
10	865	865
11	26276	113506

surftri software [Sulanke, 2006]


THANKS