Canonical triangulation of a graph, with a coding application

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Compression of polygonal meshes

- lossless compression of the connectivity information;
- 3-connected planar graphs: polygonal meshes topologically equivalent to a sphere (genus 0), with one connected component and no boundary.

Related works

- Turan ('84): general graphs;
- Chuang/He/Kao/Lu ('98): 3-connected simple planar graph;
- Bonichon/Gavoille/Hanusse et al. ('03): simple planar graph;
- Degree-Valence approach: Khodakovsky/Alliez/Desbrun/Schroder ('02), Isenburg ('02)

Polygonal meshes

Tutte entropy(1963)

- enumeration of the class of 3-connected planar graphs with m edges :

$$\Phi_m \approx \frac{9}{2}\sqrt{\frac{6}{\pi}}m^{-\frac{5}{2}}\,4^m \tag{1}$$

- theoretical bound for the entropy *bpe* (bits/edge):

$$e = \frac{1}{m}\log_2 \Phi_m \text{ bpe}$$

$$e = \log_2 4 = 2 \text{ bpe} \tag{2}$$

Encoding of 3-connected planar graphs

Algorithm in two phases, based on some properties of realizers and canonical orderings:

- we compute a triangulation T of G;
- we use only the connectivity of T and the sequence of the degrees of the faces to encode the original graph G;
- we compress T with an optimal coder for triangulation;

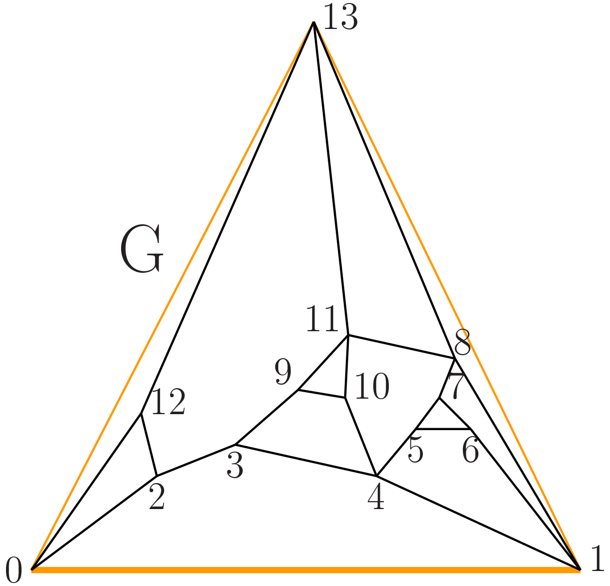
Main ideas of our algorithm:

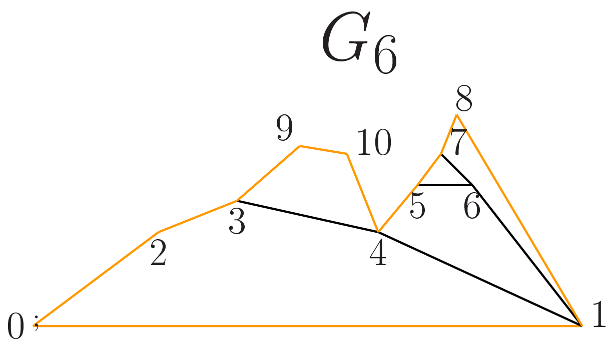
- we realize a traversing of the graph: without *splits*;
- near-optimality of the based *valence/degree* approach;
- optimal encoding of triangulations needs at most $3.24bpv$;
- upper bound guaranteed: 2.62 *bpe*;
- adaptive behaviour: short code for regular meshes.

Canonical ordering for 3-connected planar graphs

Definition (Kant'96). Let $\pi = \{V_1, \dots V_K\}$ be an ordered partition of the vertices V. Define G_k to be the subgraph of G induced by $V_1 \cup, \dots, \cup V_k$ and C_k the boundary of the exterior face of G_k (orange edges). We say that π is a ***canonical ordering*** of G if:

- $V_1 = \{v_0, v_1\}$, $V_K = \{v_{n-1}\}$ and each C_k is a cycle containing (v_0, v_1);
- each G_k $(k > 1)$ is 2-connected and internally 3-connected



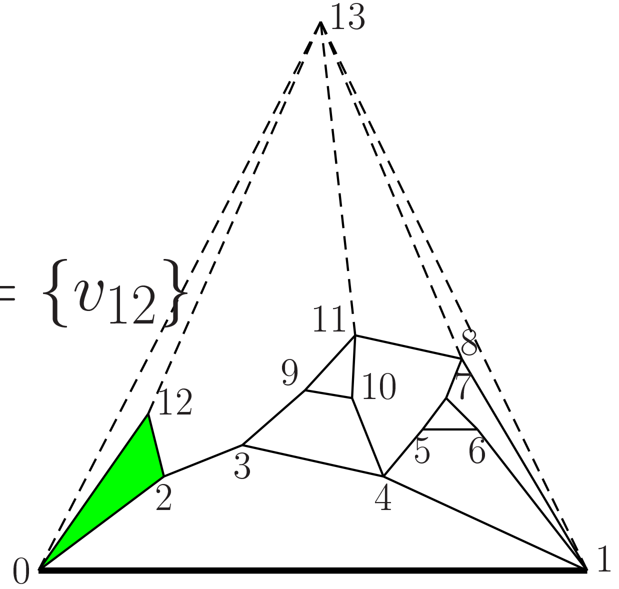


Canonical ordering for 3-connected planar graphs

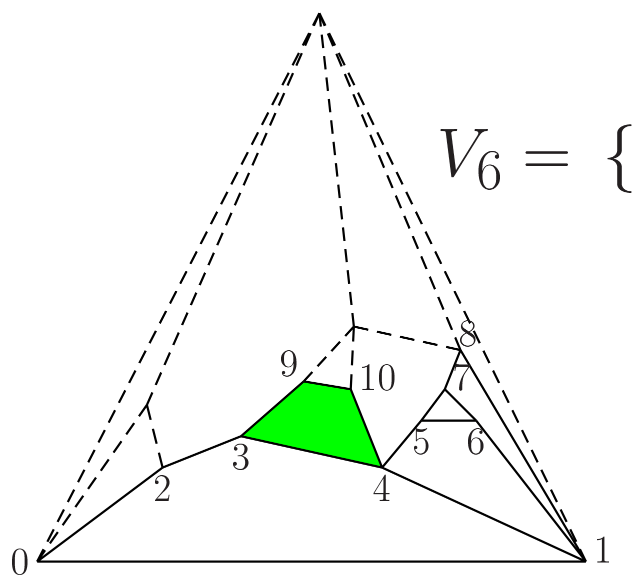
For each $k \in \{2, \ldots, K-1\}$ one of the following conditions hold:

1. $V_k = \{z\}$ t.q. $z \in C_k$ and z has at least one neighbor in $G - G_k$
2. $V_k = \{z_1, \ldots, z_l\}$, each z_i has at least one neighbor in $G - G_k$ and z_1, z_l have exactly one neighbor on C_{k-1}.

$V_8 = \{v_{12}\}$

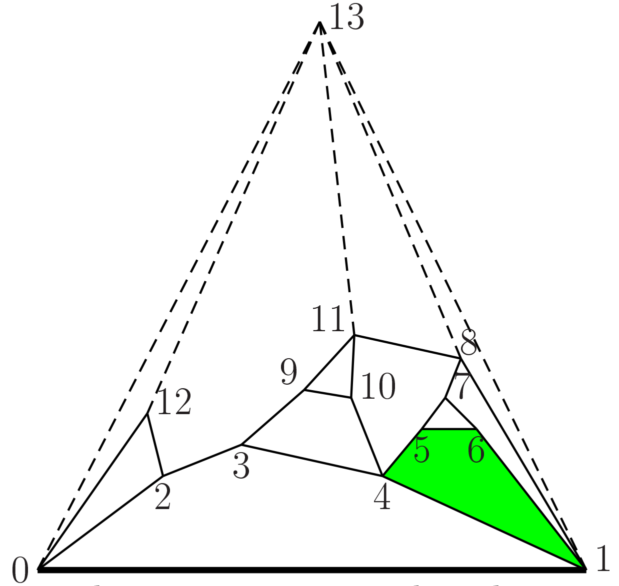


$V_6 = \{v_9, v_{10}\}$



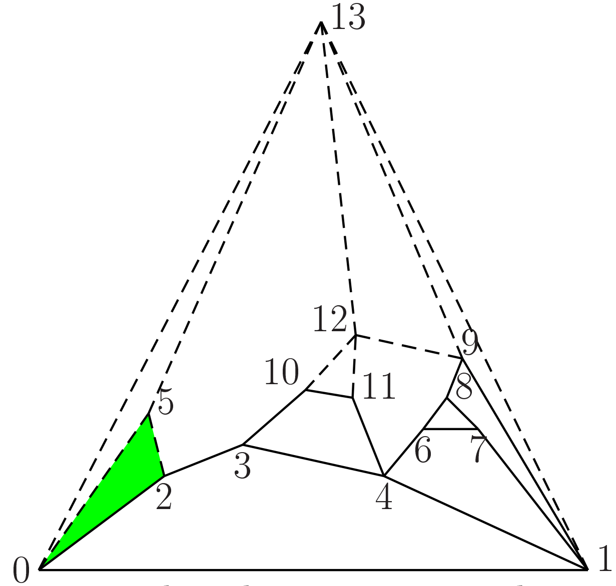
Rightmost canonical ordering

Definition. (roughly speaking) A canonical ordering is ***rightmost*** if when there are more than one vertex or chain V_k that ca be added to G_{k-1}, we always add the rightmost one.



rightmost canonical ordering

$V_1 = \{0, 1\}, V_2 = \{2, 3, 4\}, V_3 = \{5, 6\}$



canonical ordering: not rightmost

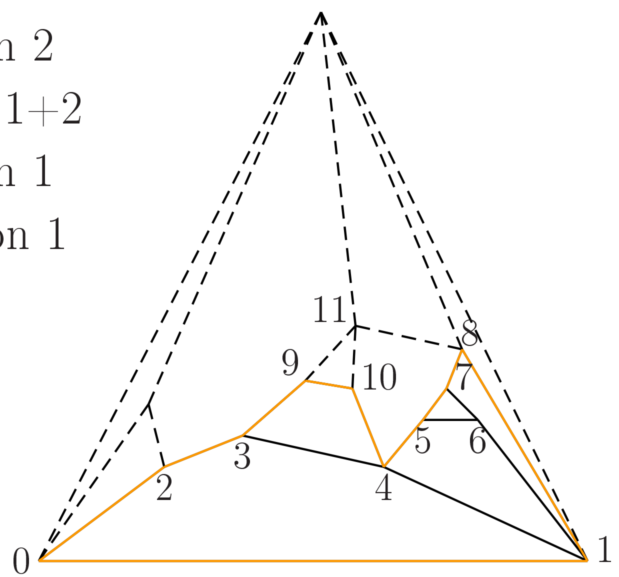
$V_1 = \{0, 1\}, V_2 = \{2, 3, 4\}, V_3 = \{5\}$

Some notations: free vertex

Definition. A vertex v ($v \neq v_0, v_1$)on the boundary C_k of the outer face of the subgraph G_k, is ***free*** if the following conditions are satisfied:

1. v has at least one neighbour in $G - G_k$ (incident to a dashed line);
2. v is not incident to a separation face (face whose deletion changes the 2-connexity of the subgraph G_k).

v_2 is not free: condition 2
v_3 is not free: condition 1+2
v_9 is not free: condition 1
v_{11}, v_8 are free: condition 1



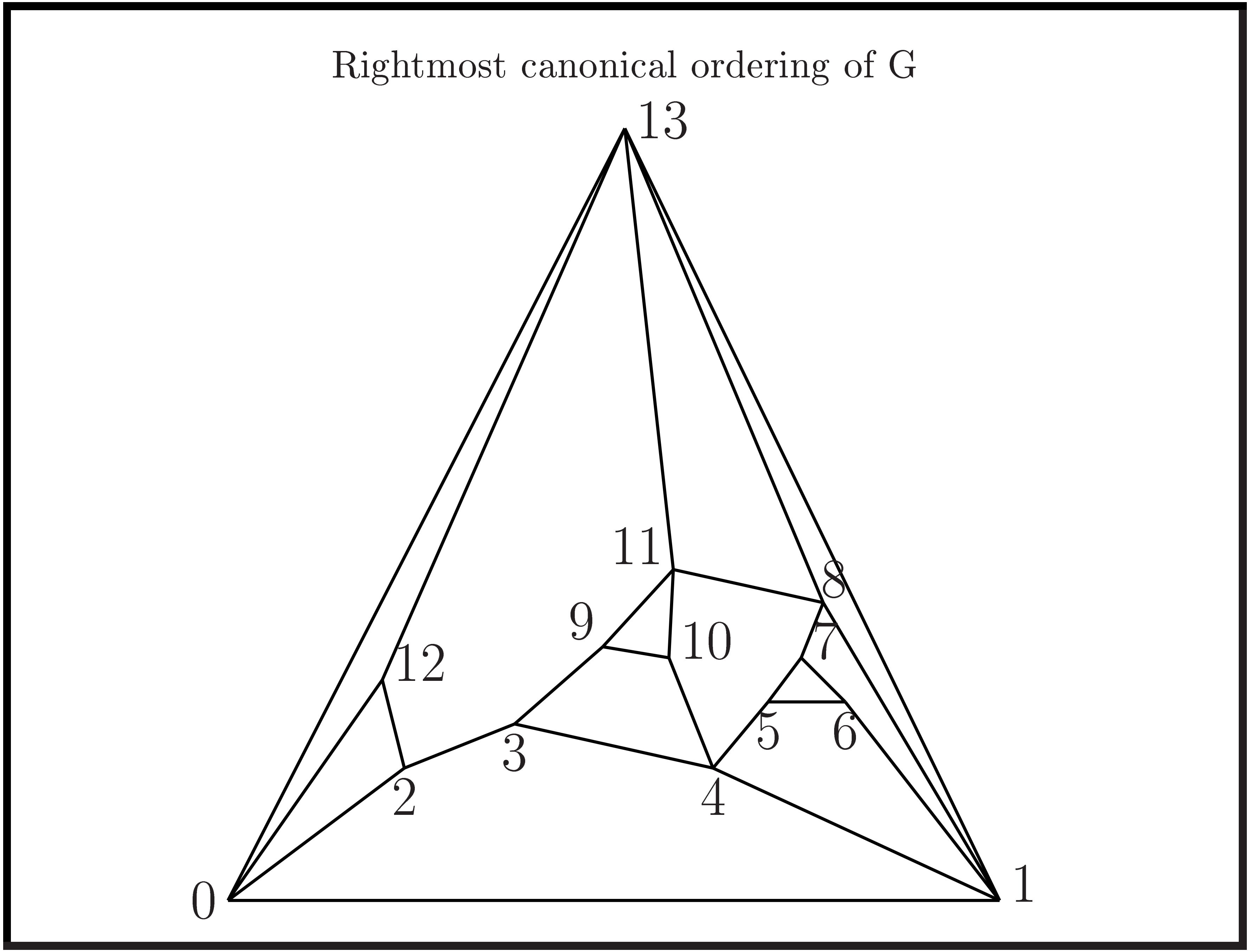
Encoding/decoding of 3-connected planar graphs

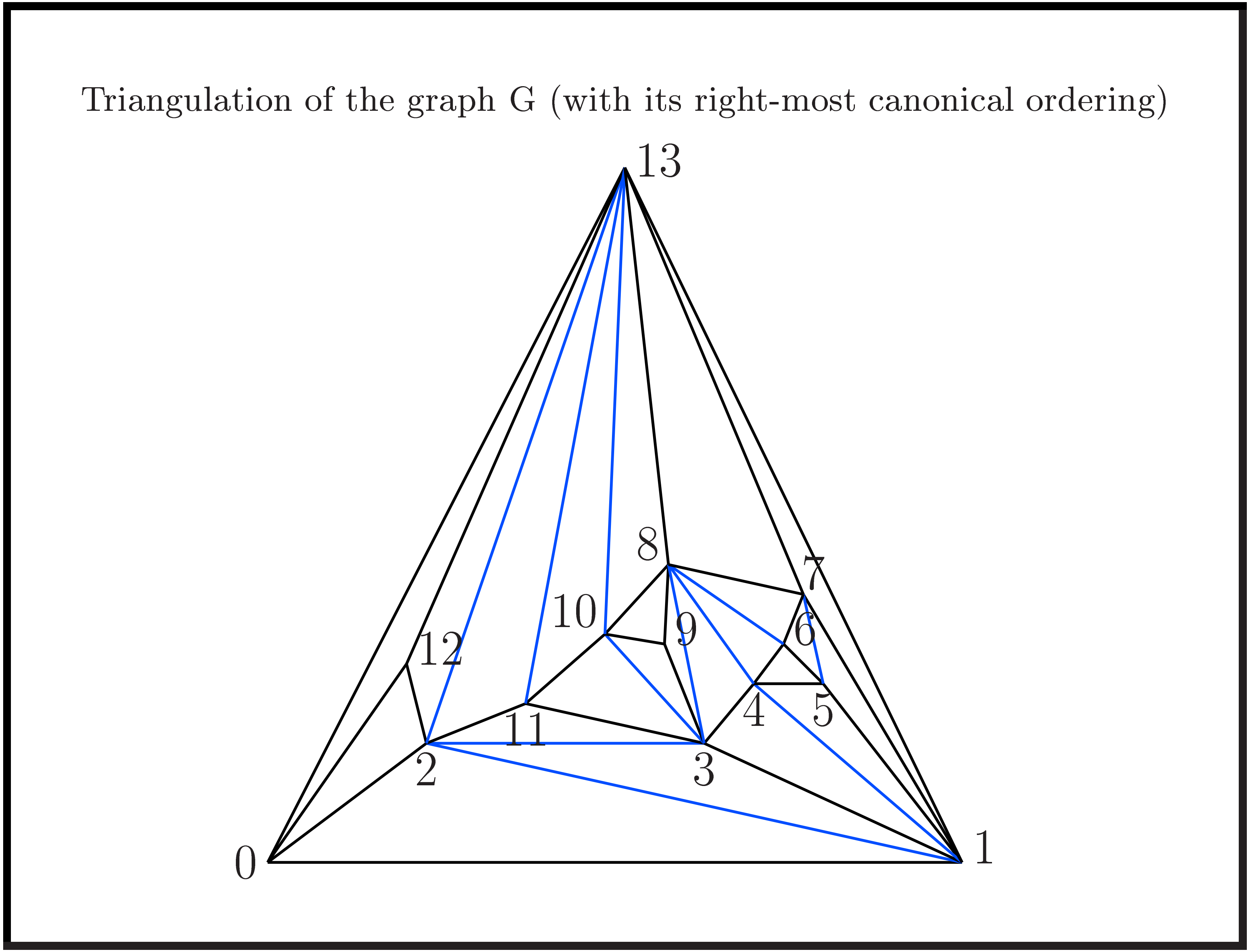
Main steps of our algorithm

- we compute a triangulation T of G;
- we compress T using an optimal coder for triangulations;
- we traverse T using a modified version of the right-most canonical ordering;
- writing/reading the degre of the faces visited during the traversing of G (visited the first time);

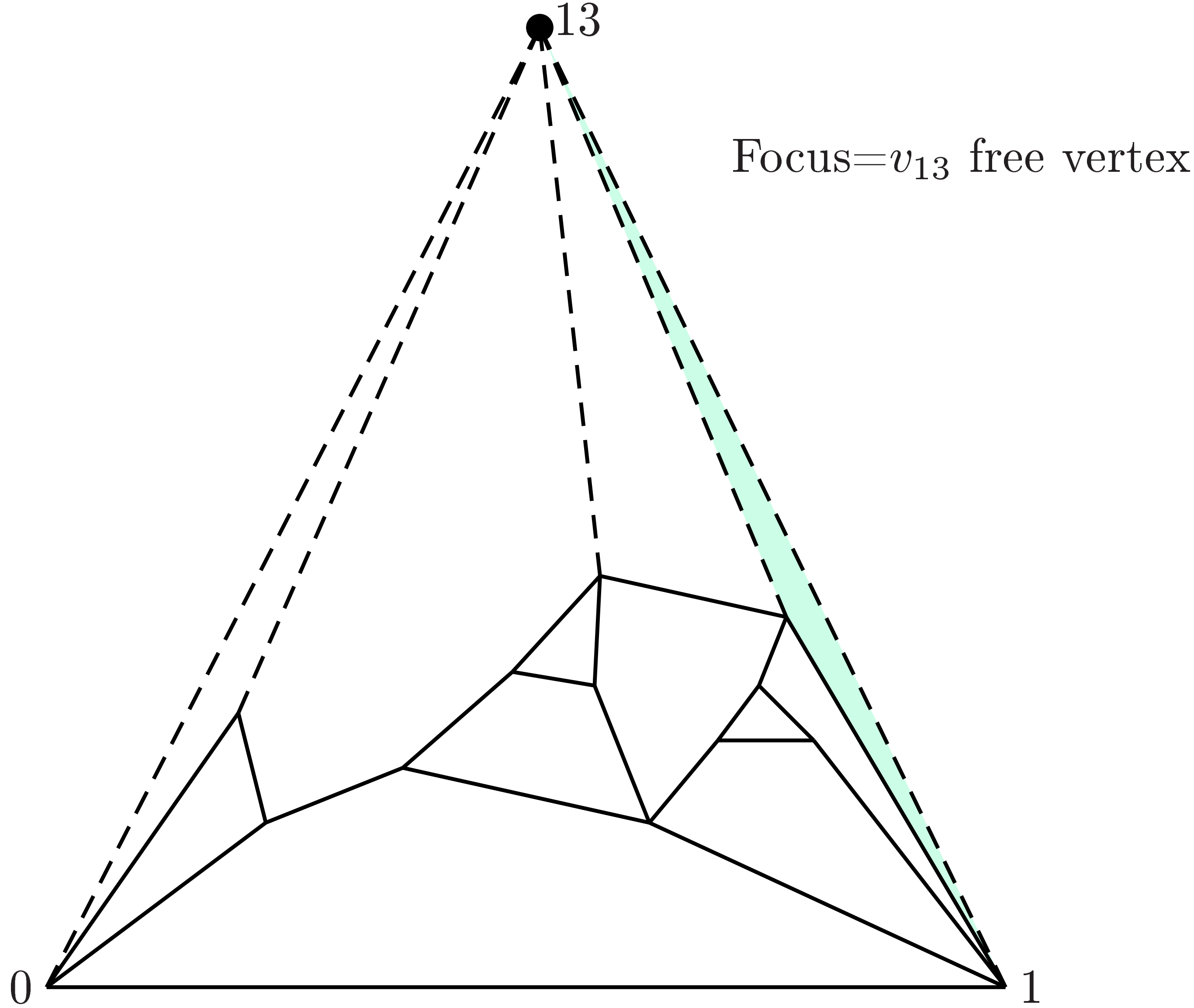
Features of our approach:

- it is a conquest of the graph by retraction;
- at each step we delete (left-most) free vertex on the boundary of the outer face (active list)
- we visit the faces around a vertex in direct order: the output code is the degree of the face visited.

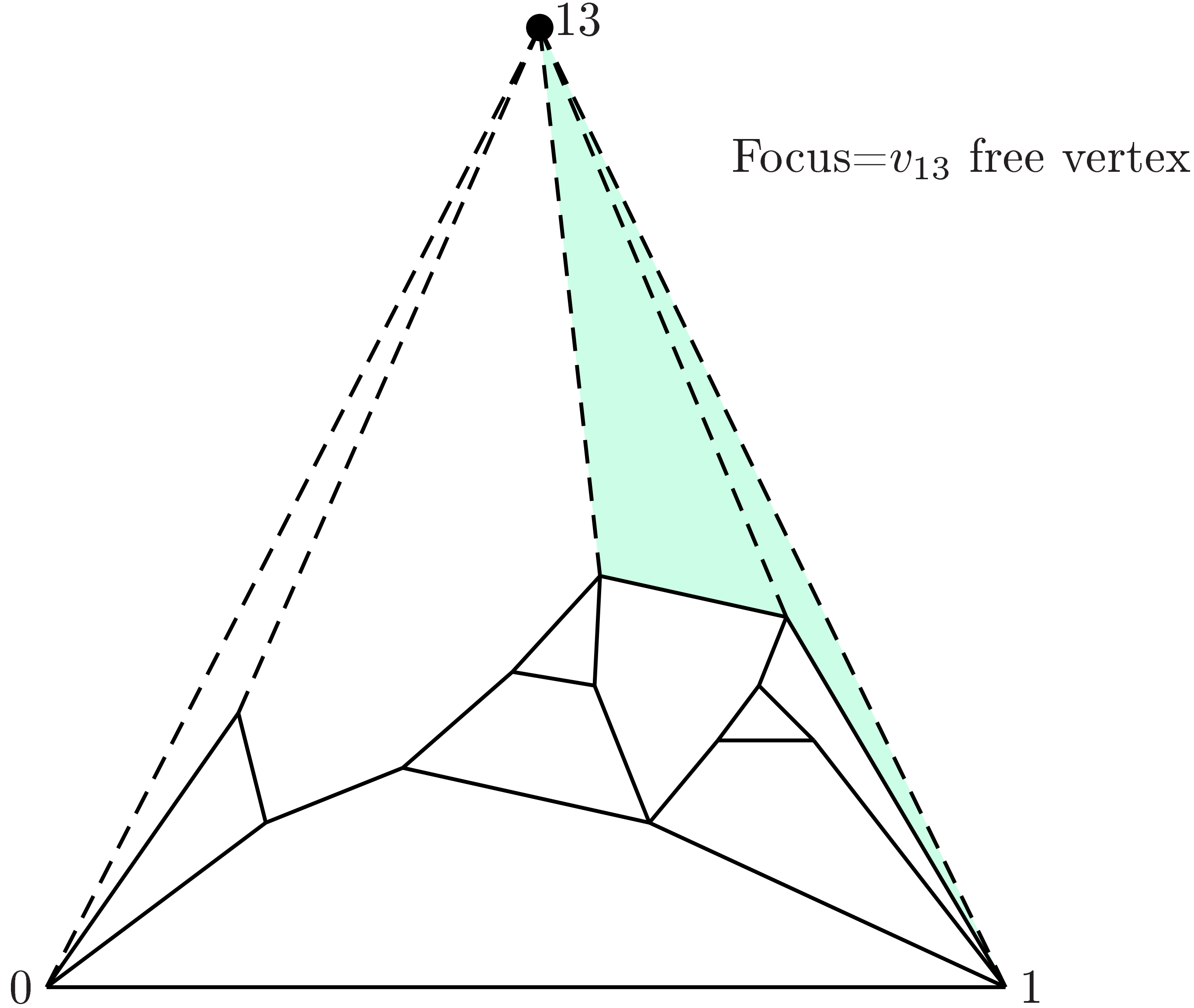

Rightmost canonical ordering of G
13
11
9
10
8
7
12
5
6
3
2
4
0
1


Triangulation of the graph G (with its right-most canonical ordering)
13
8
7
10
9
6
12
4
5
11
2
3
0
1

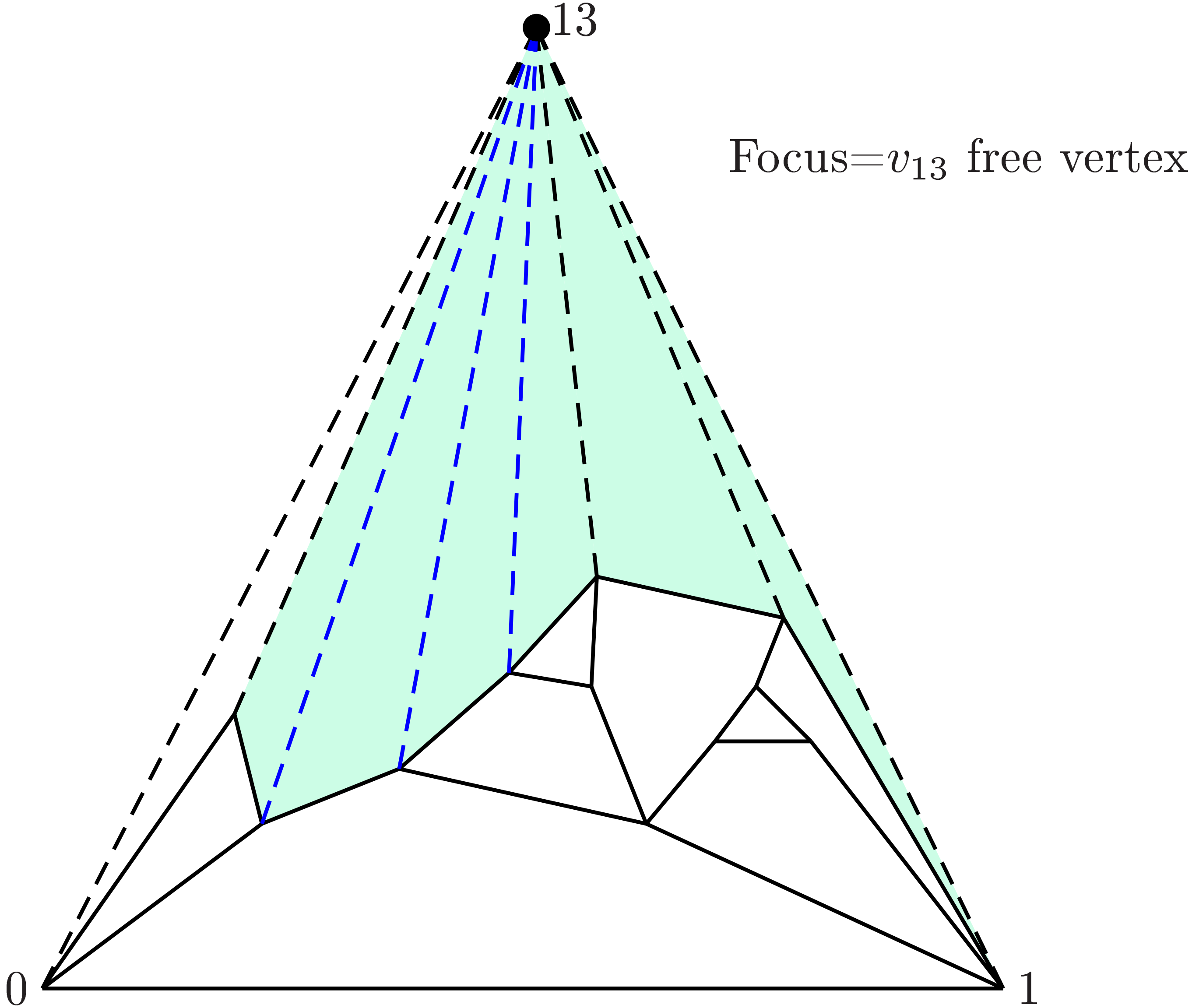
Starting the triangulation of G



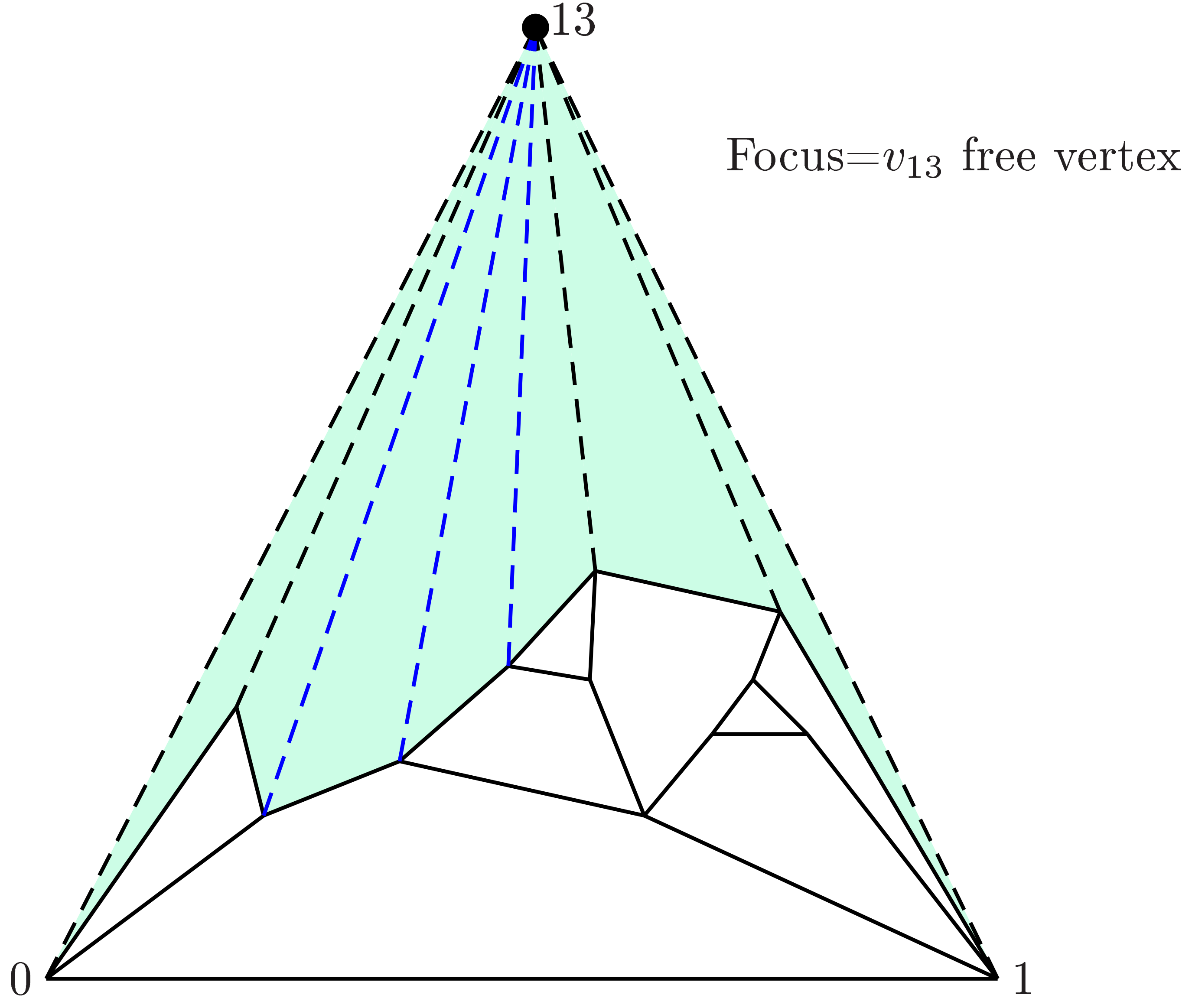
Computing the triangulation of G



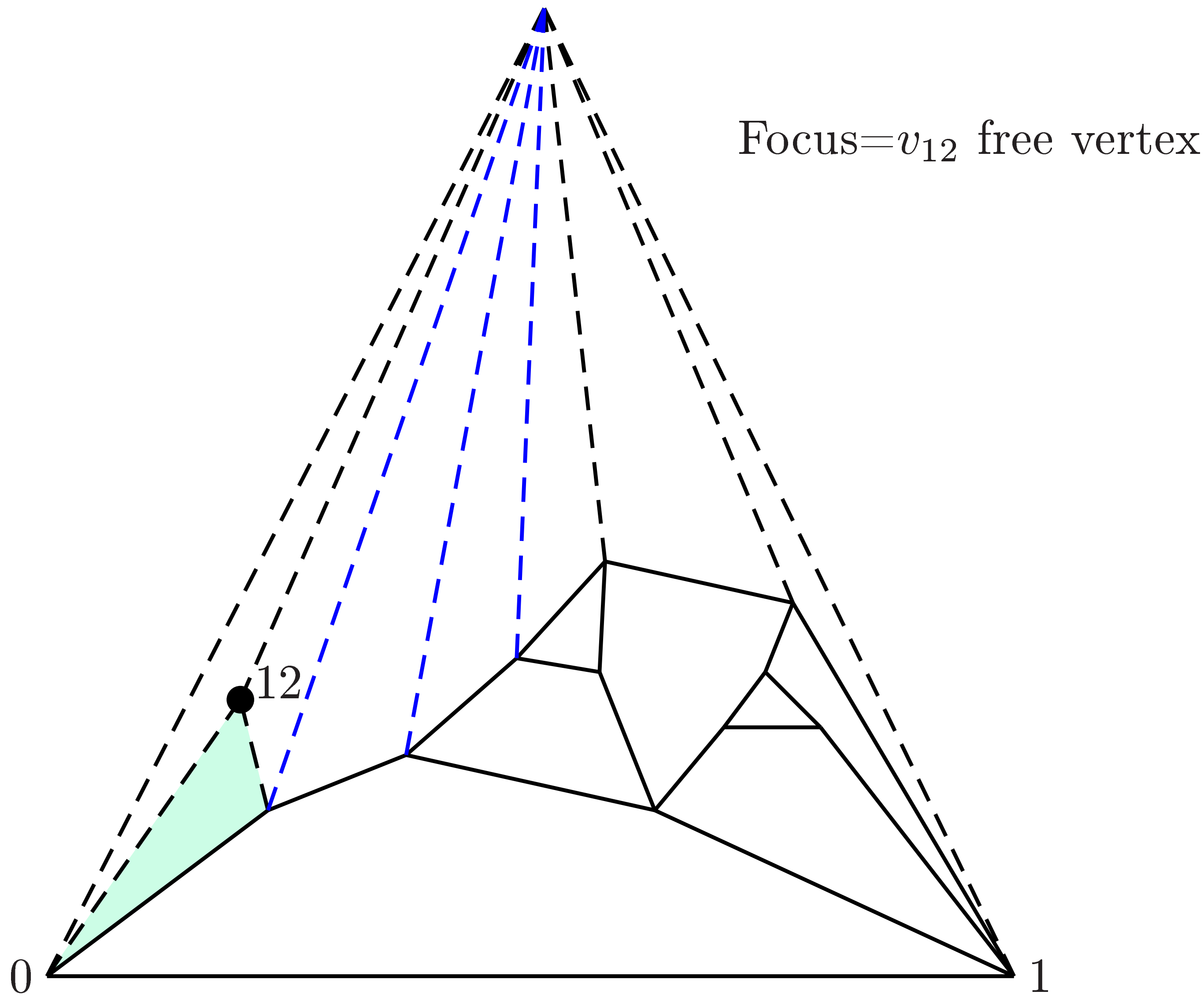
Computing the triangulation of G



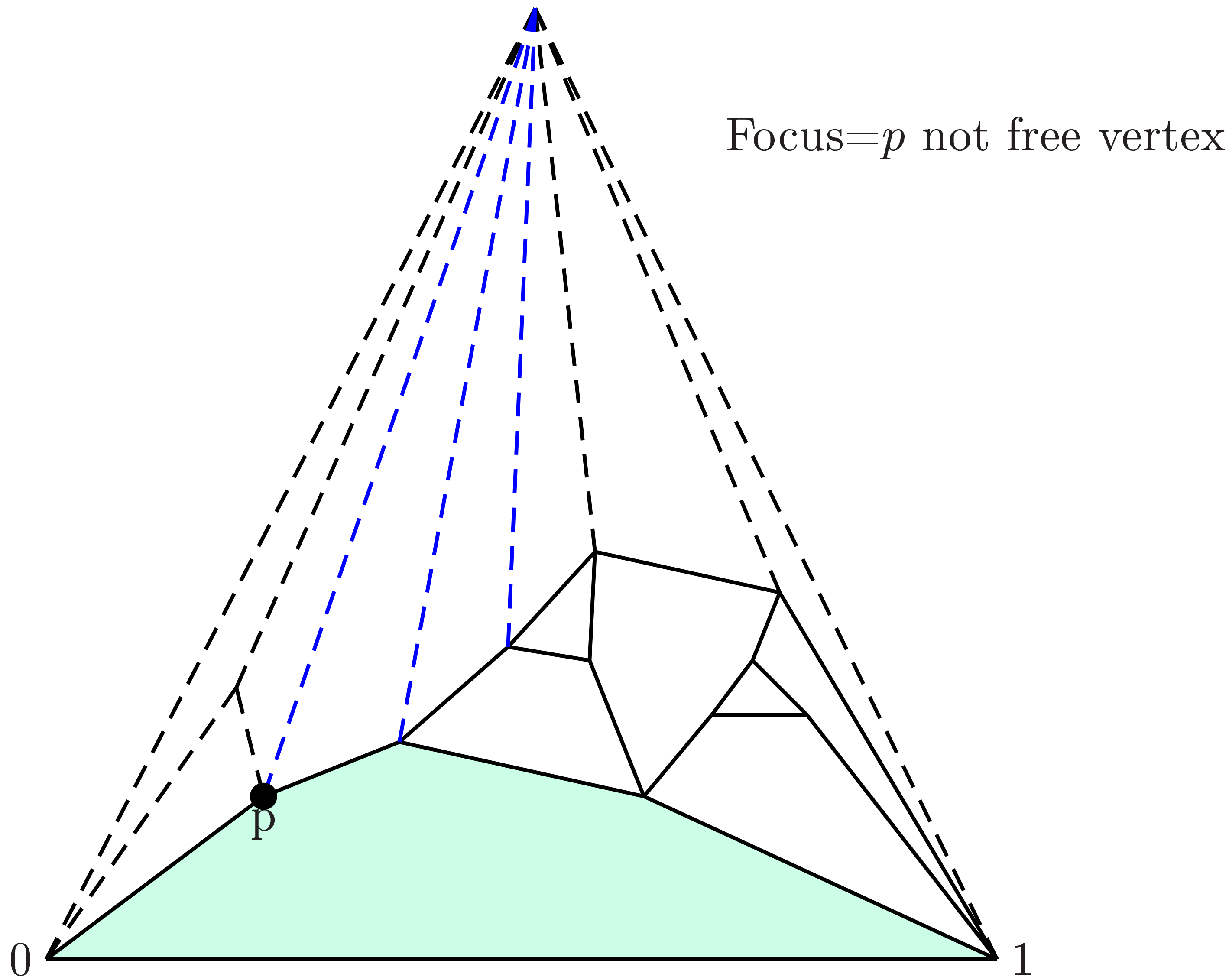
Starting the triangulation of G



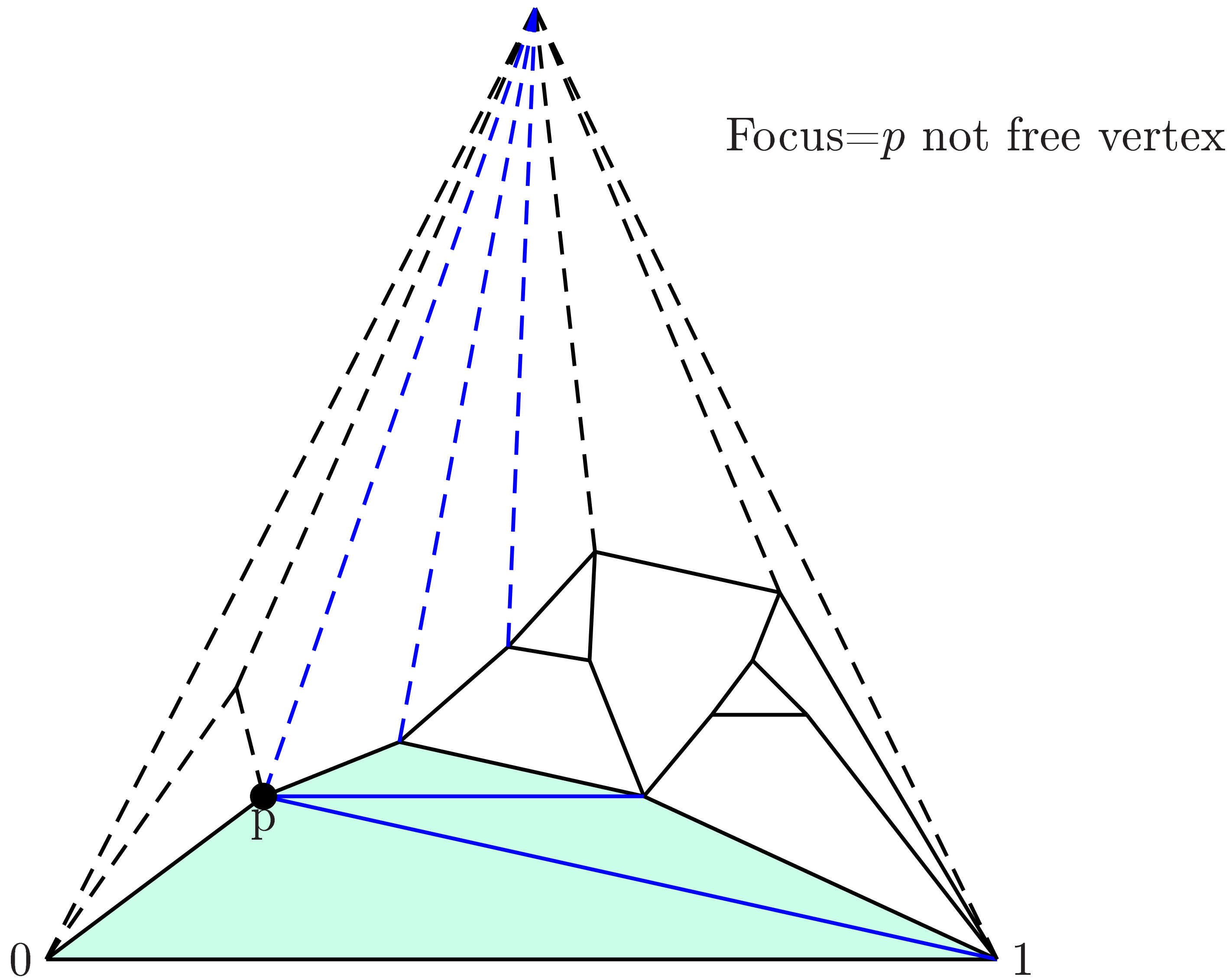
Computing the triangulation of G



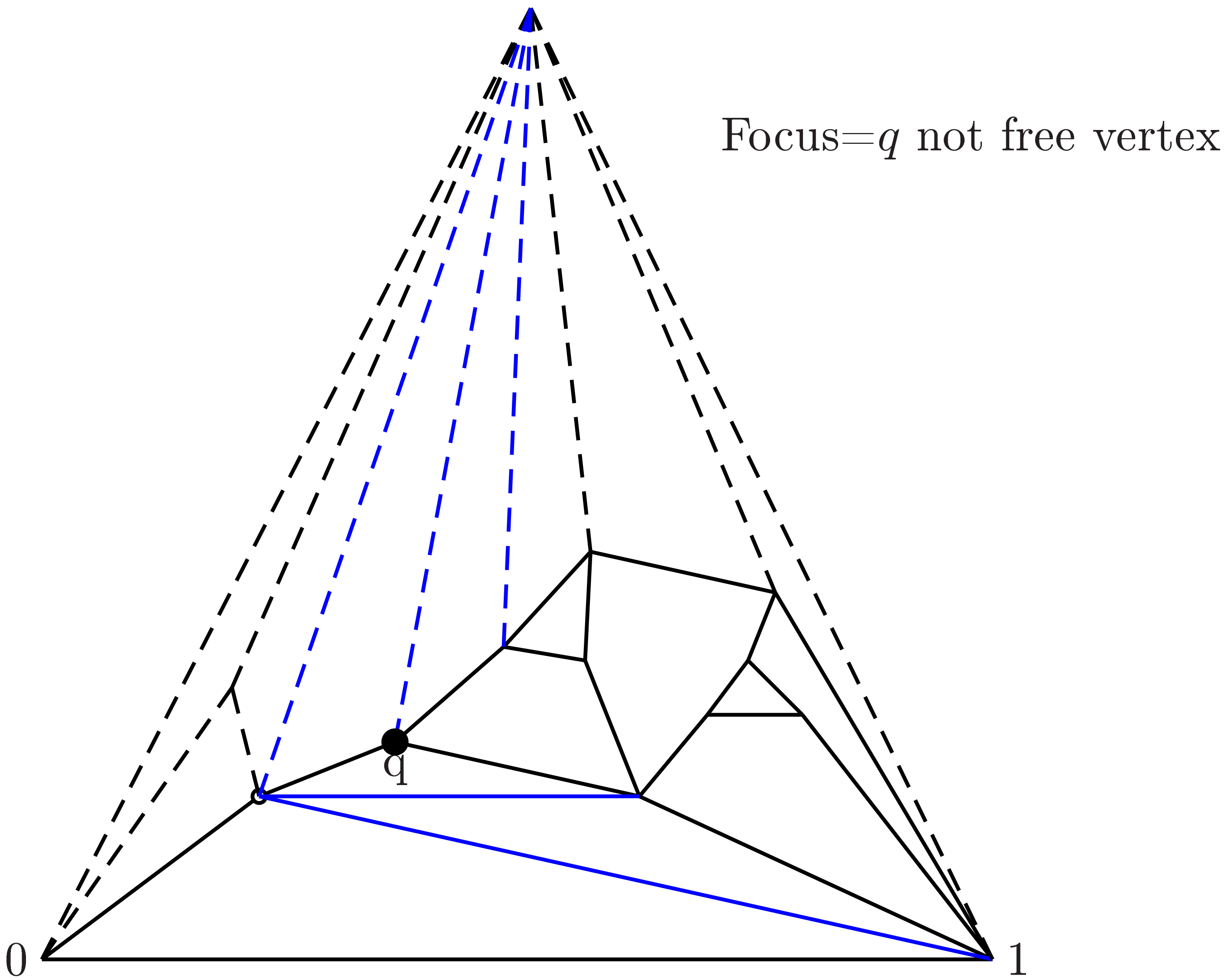
Computing the triangulation of G



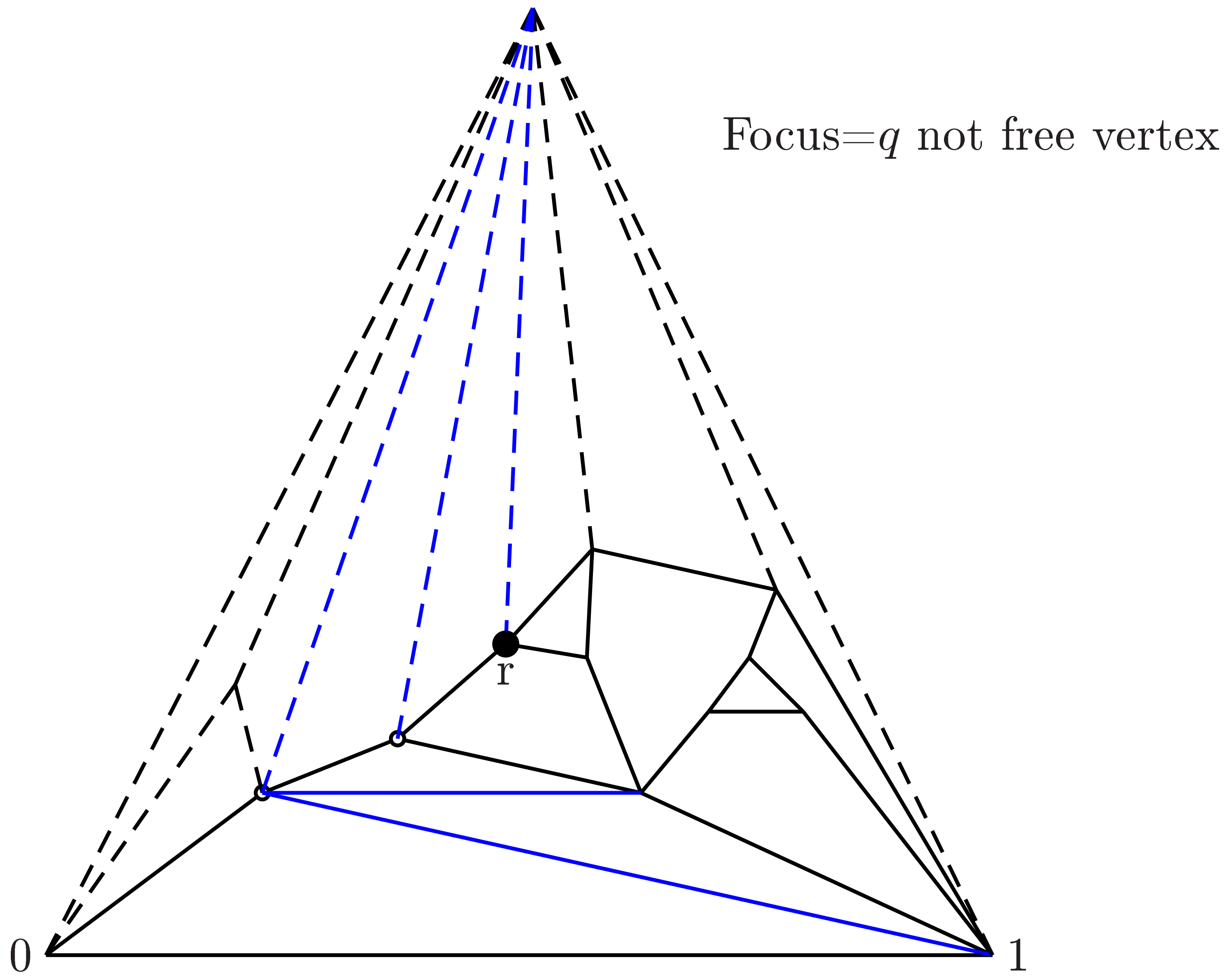
Computing the triangulation of G



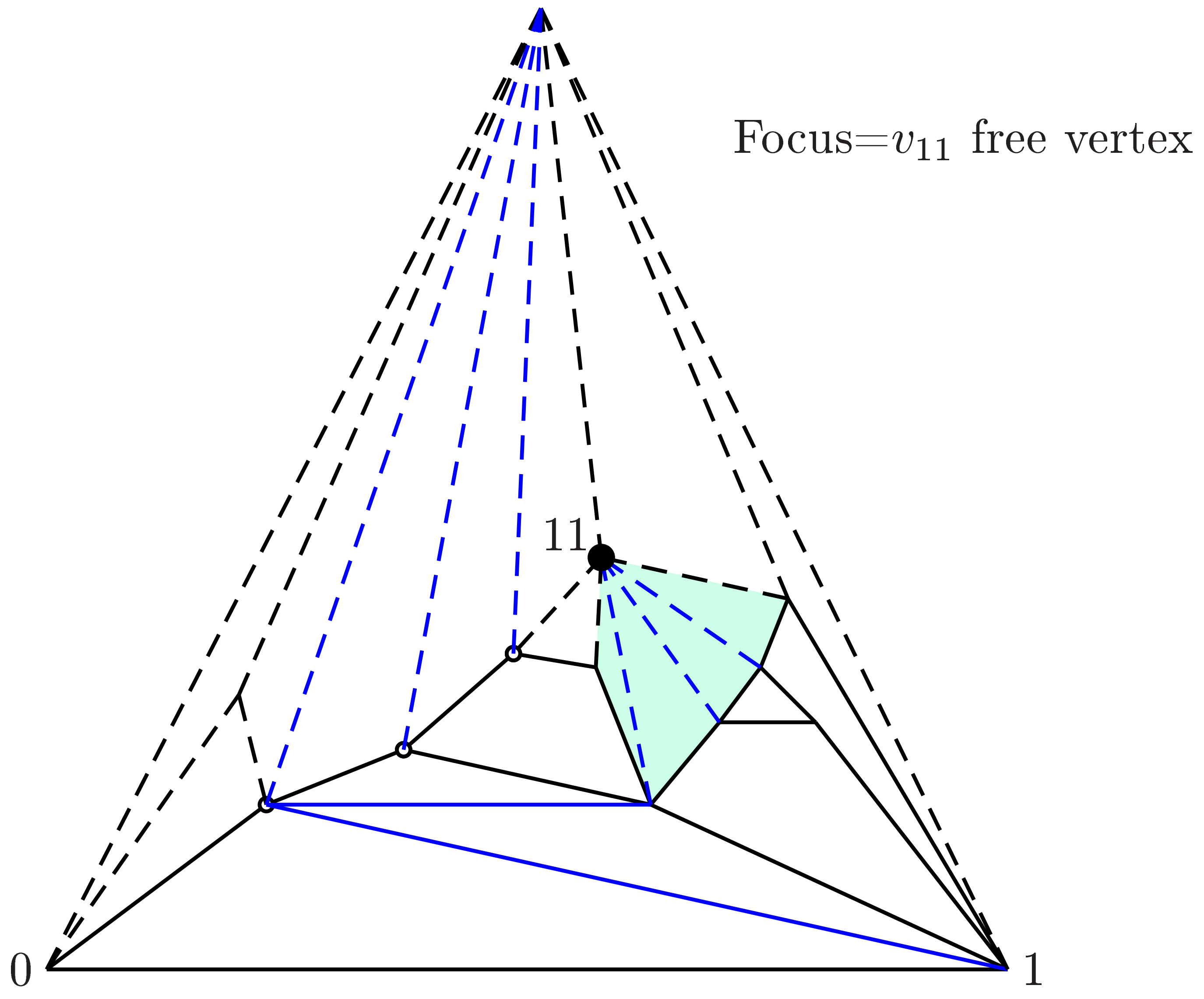
Computing the triangulation of G



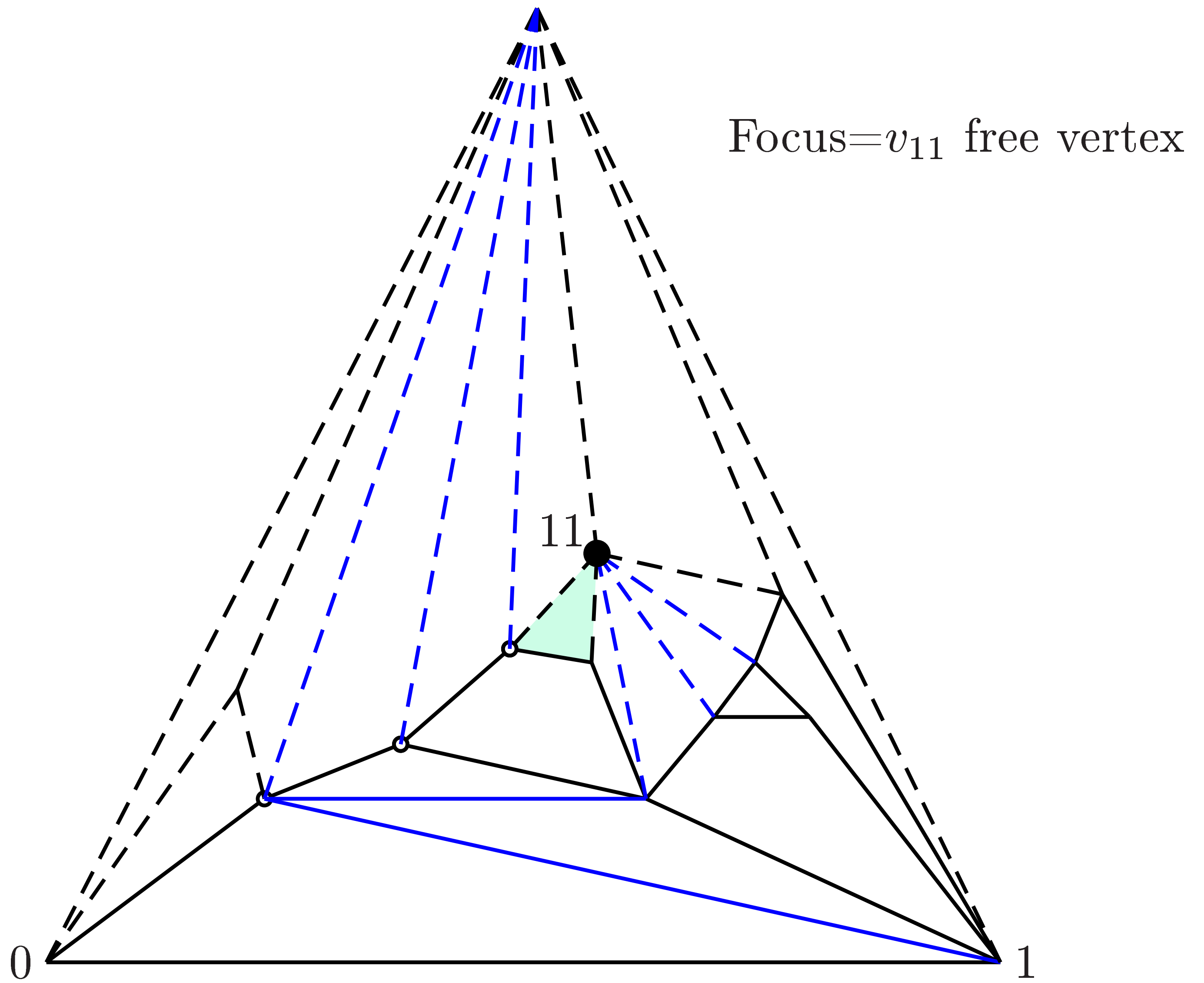
Computing the triangulation of G



Computing the triangulation of G

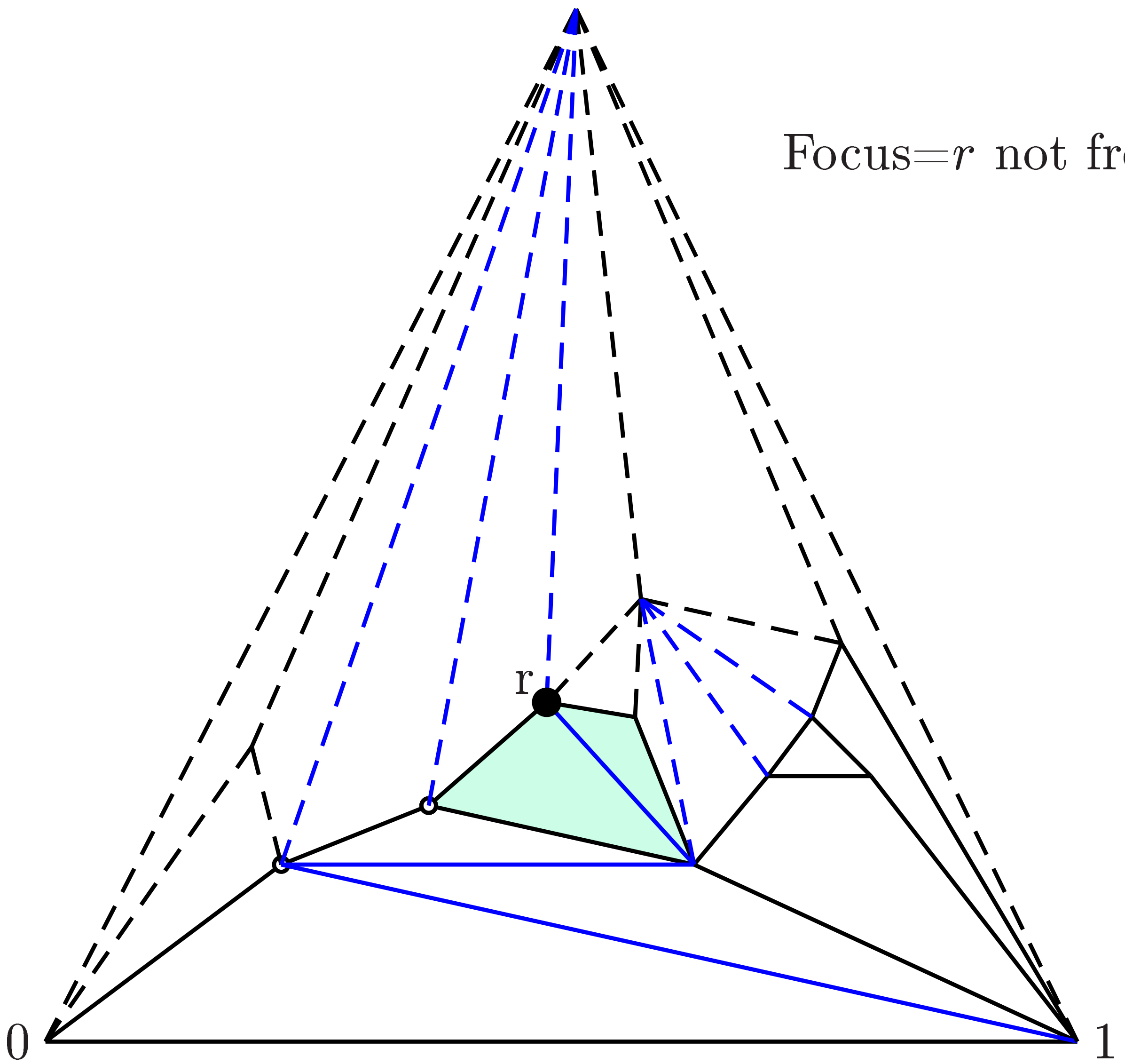


Computing the triangulation of G

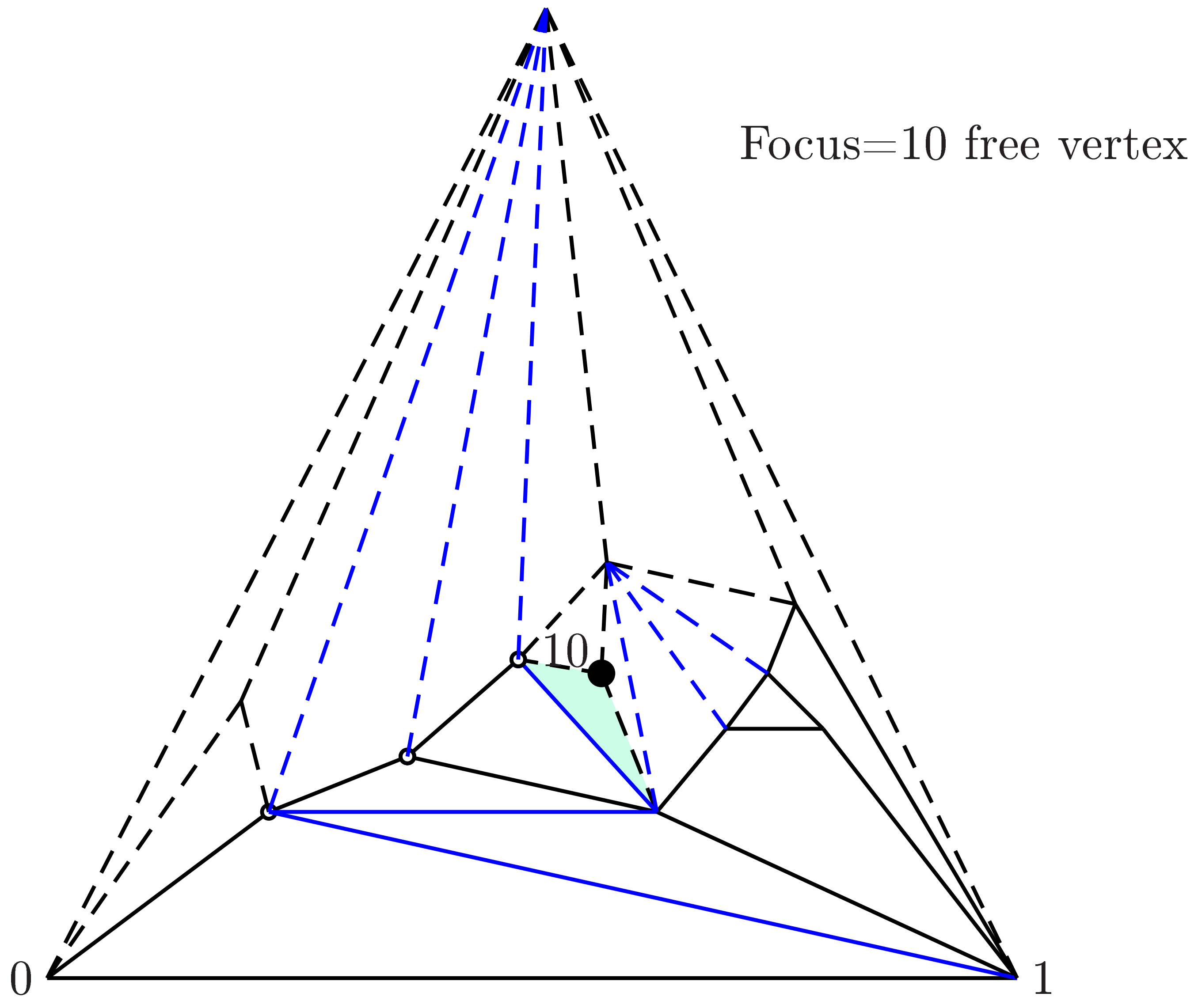


Computing the triangulation of G

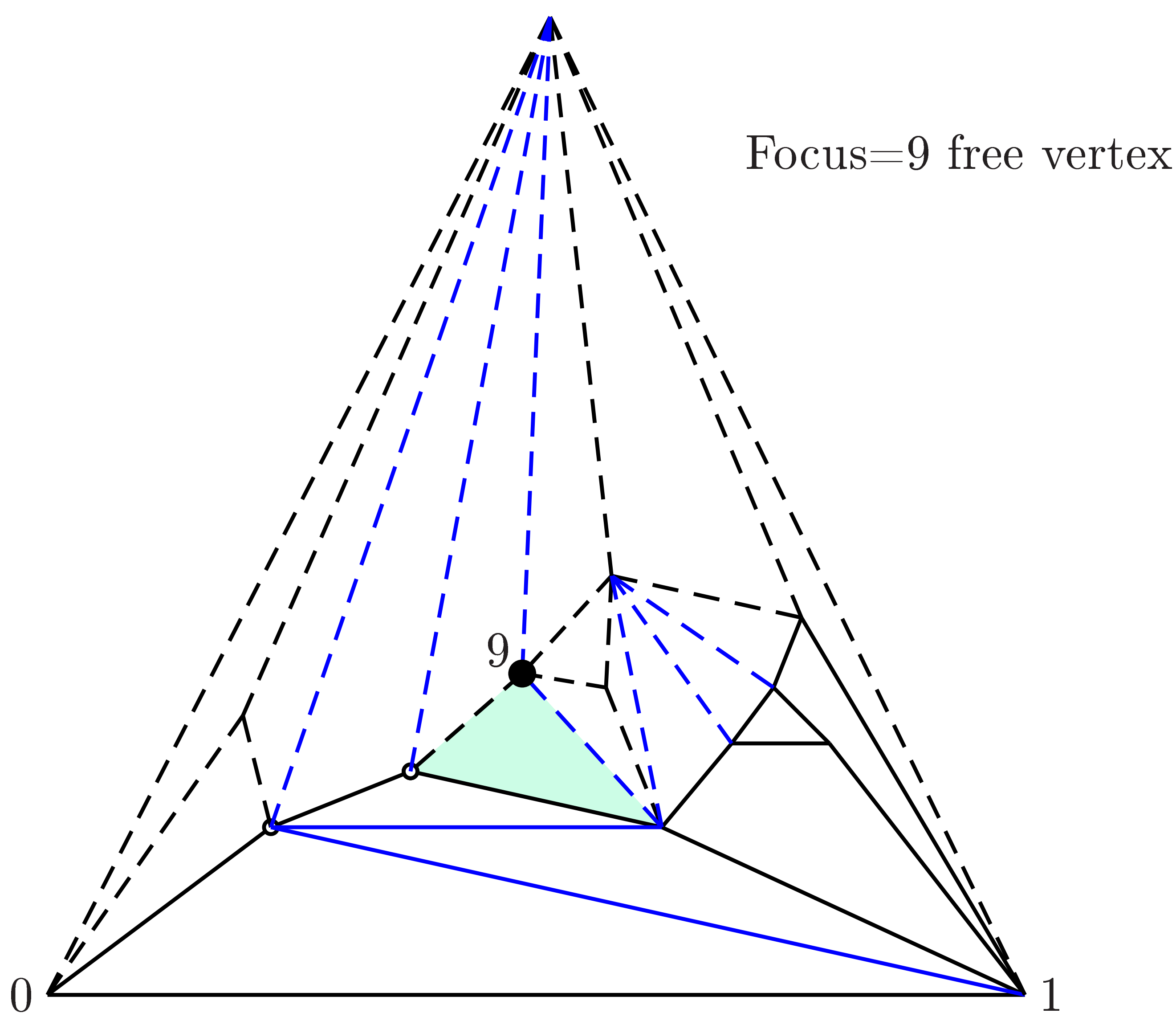
Focus=r not free vertex



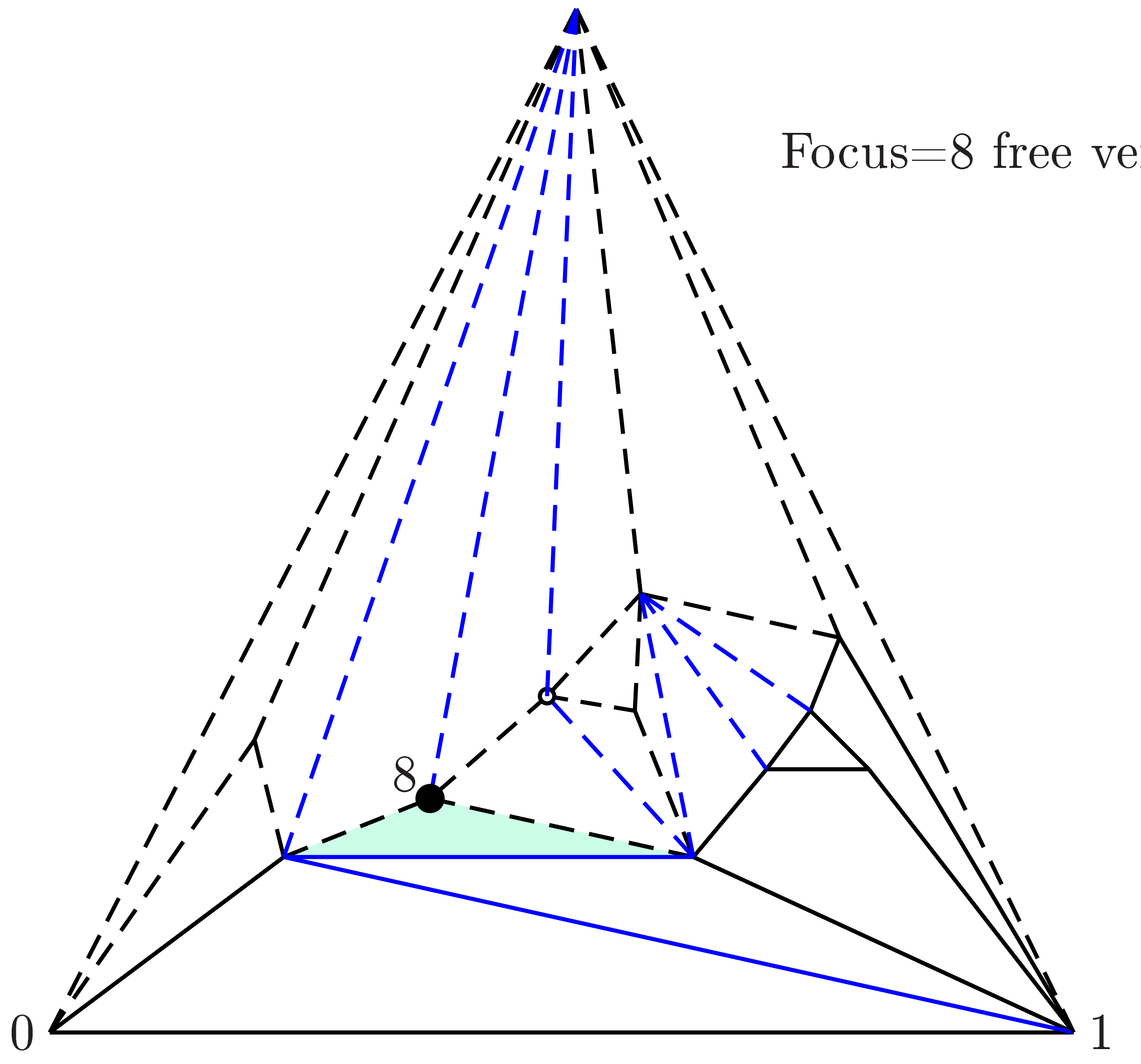
Computing the triangulation of G



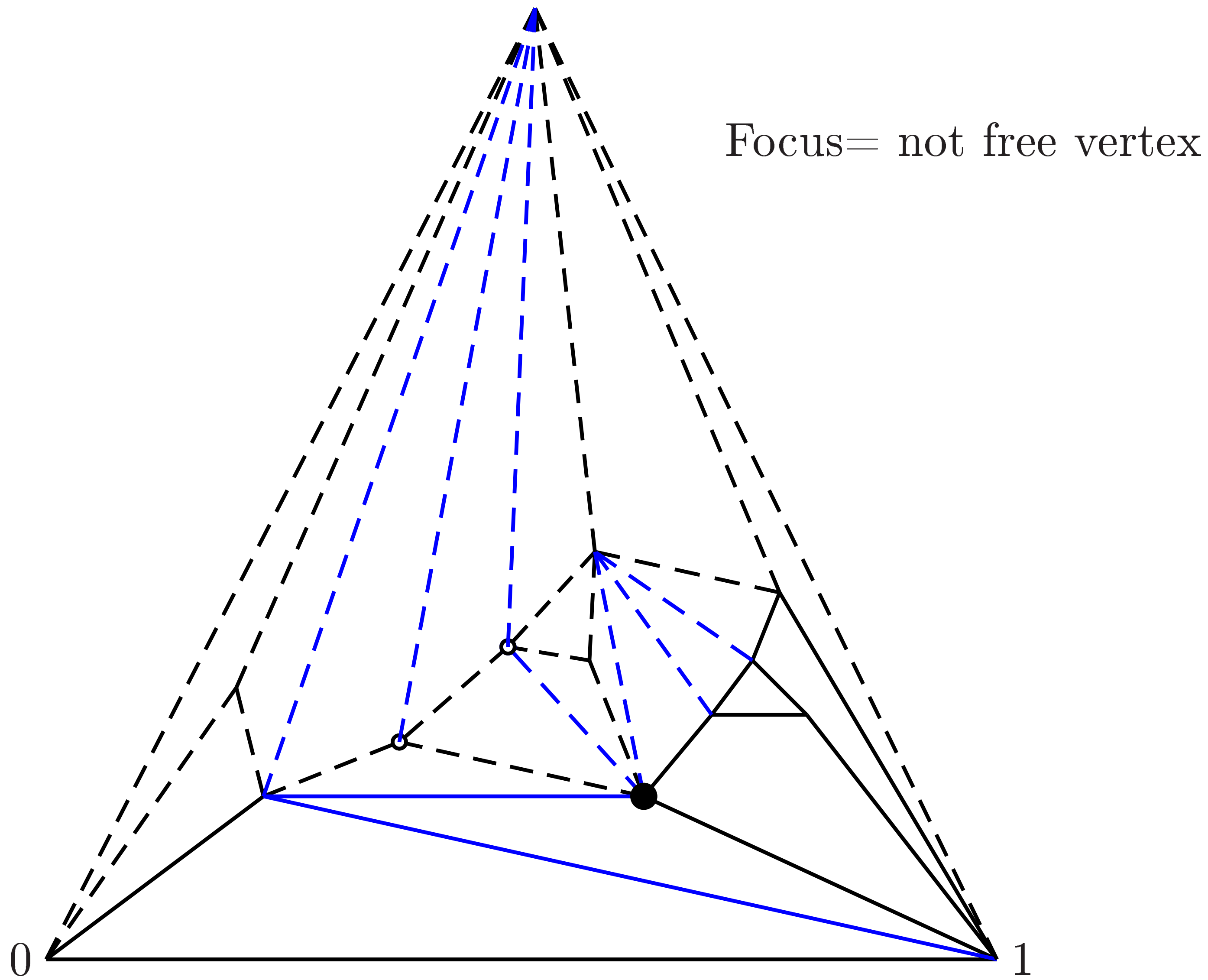
Computing the triangulation of G



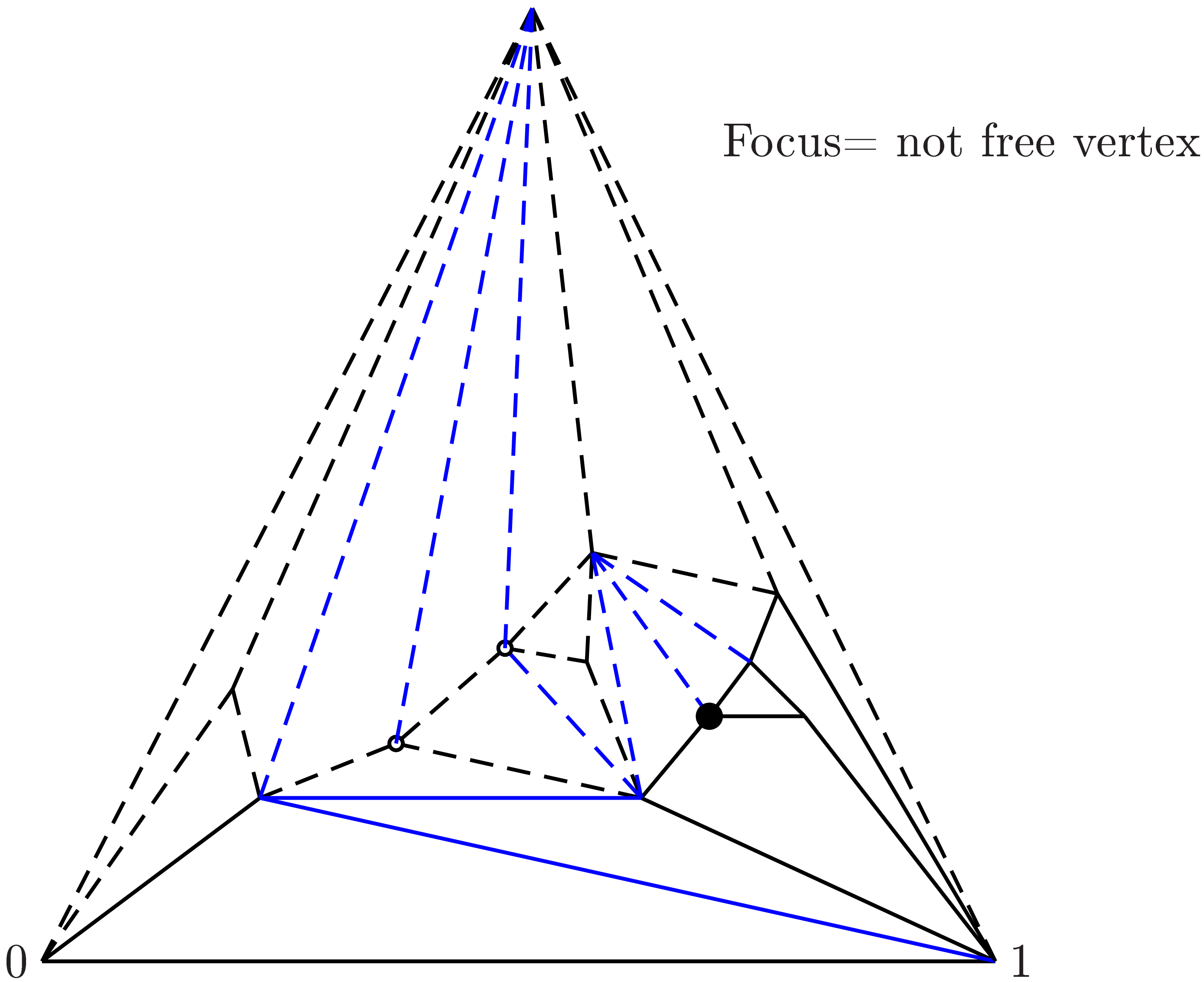
Computing the triangulation of G



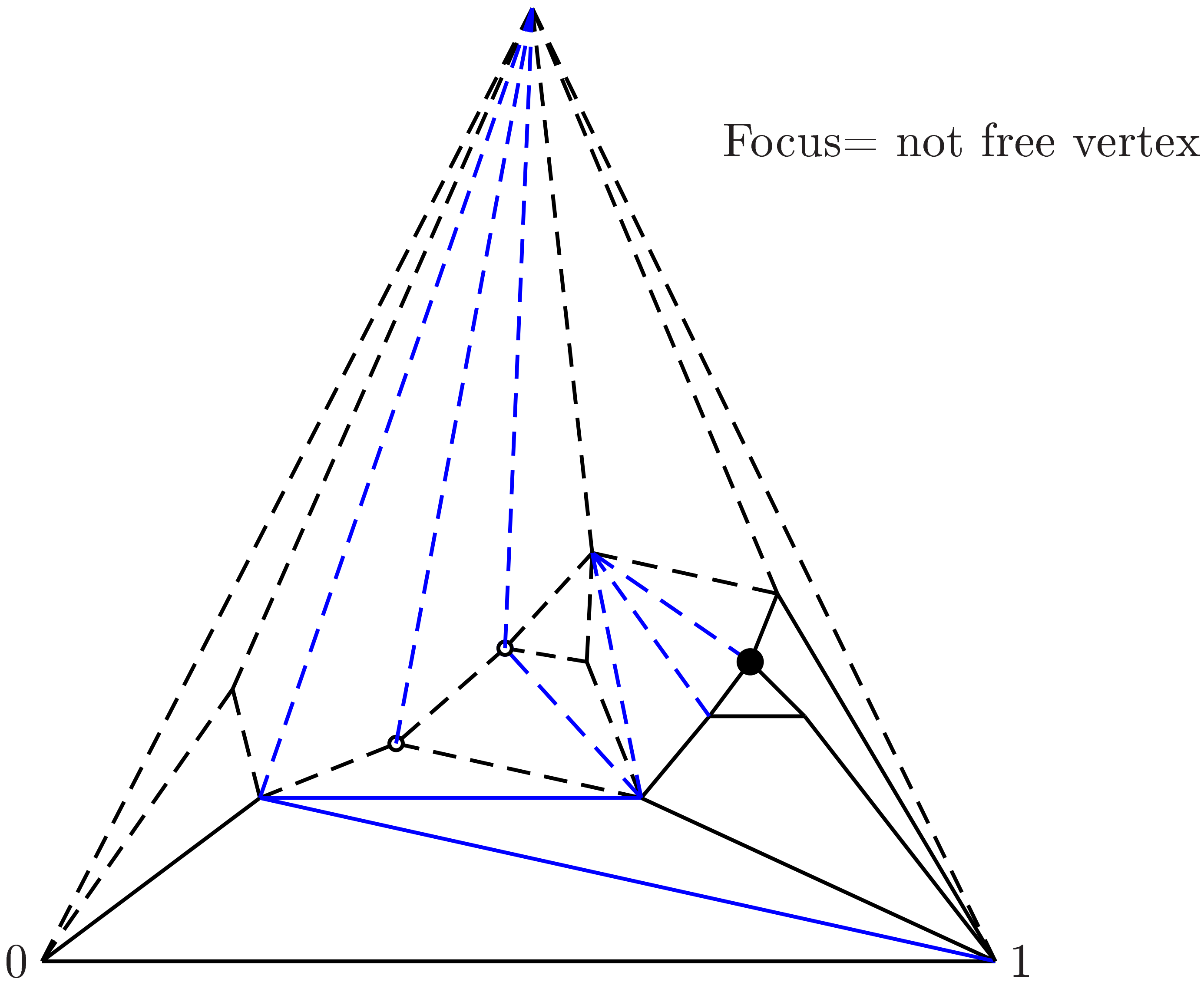
Computing the triangulation of G



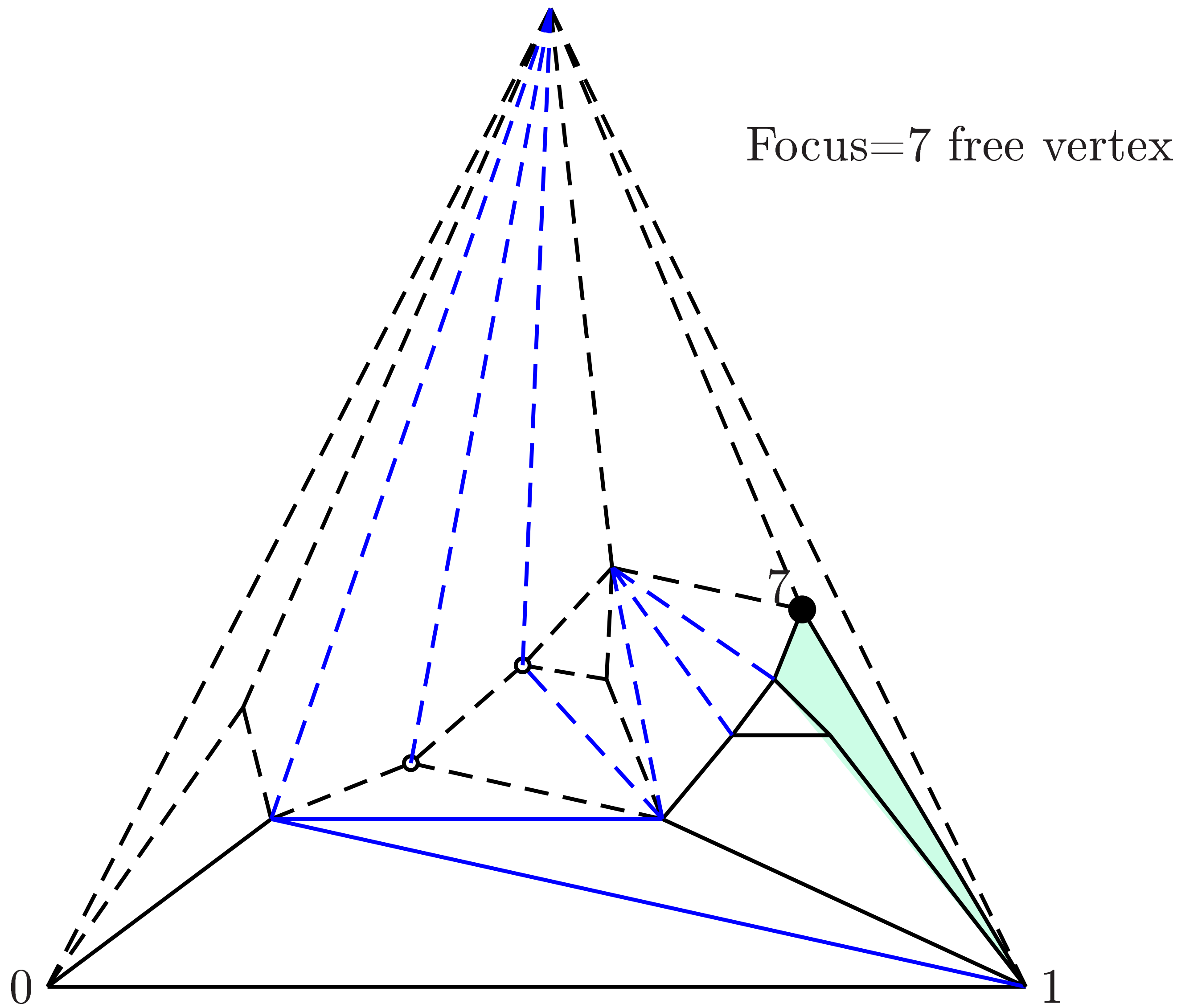
Computing the triangulation of G



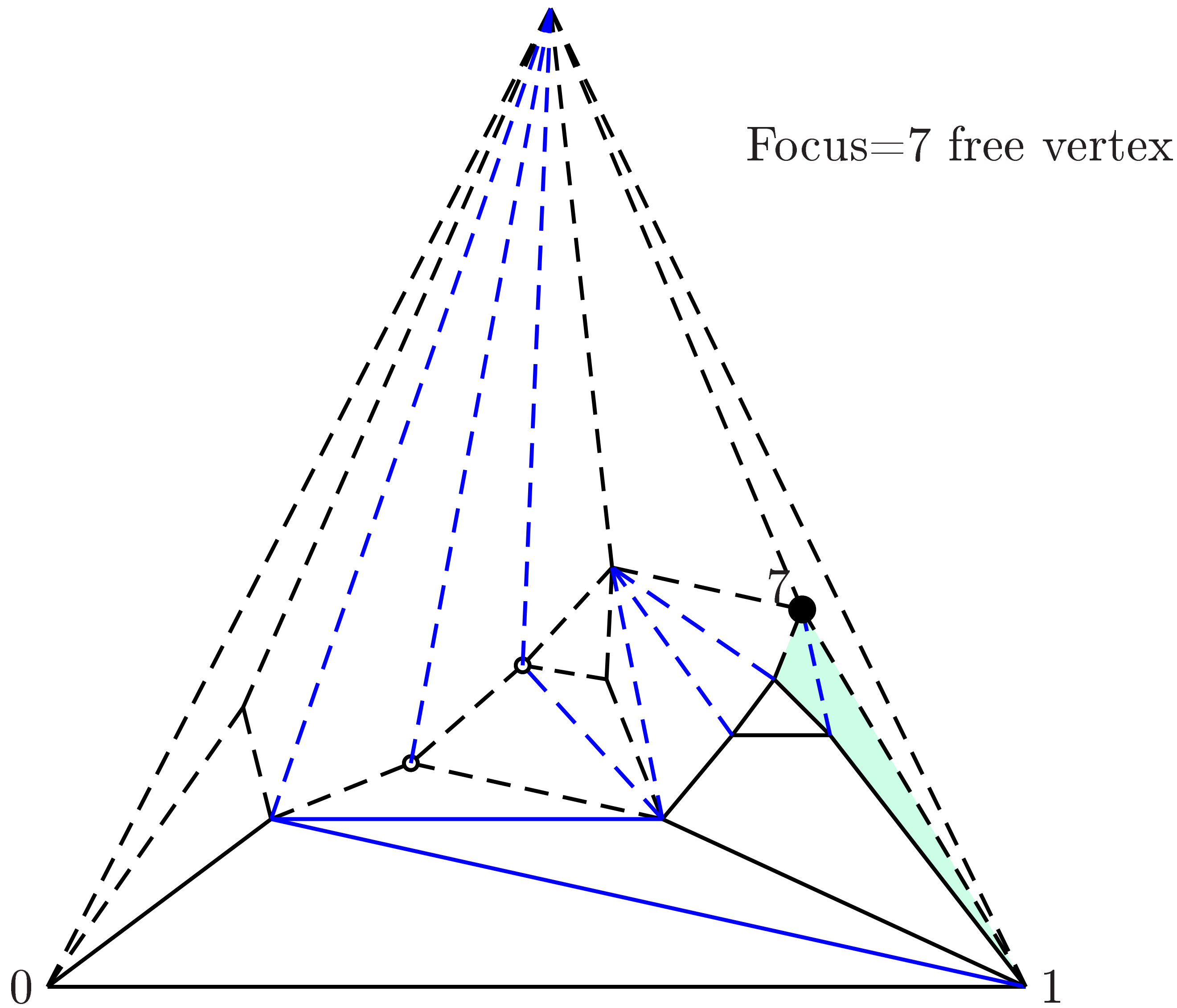
Computing the triangulation of G



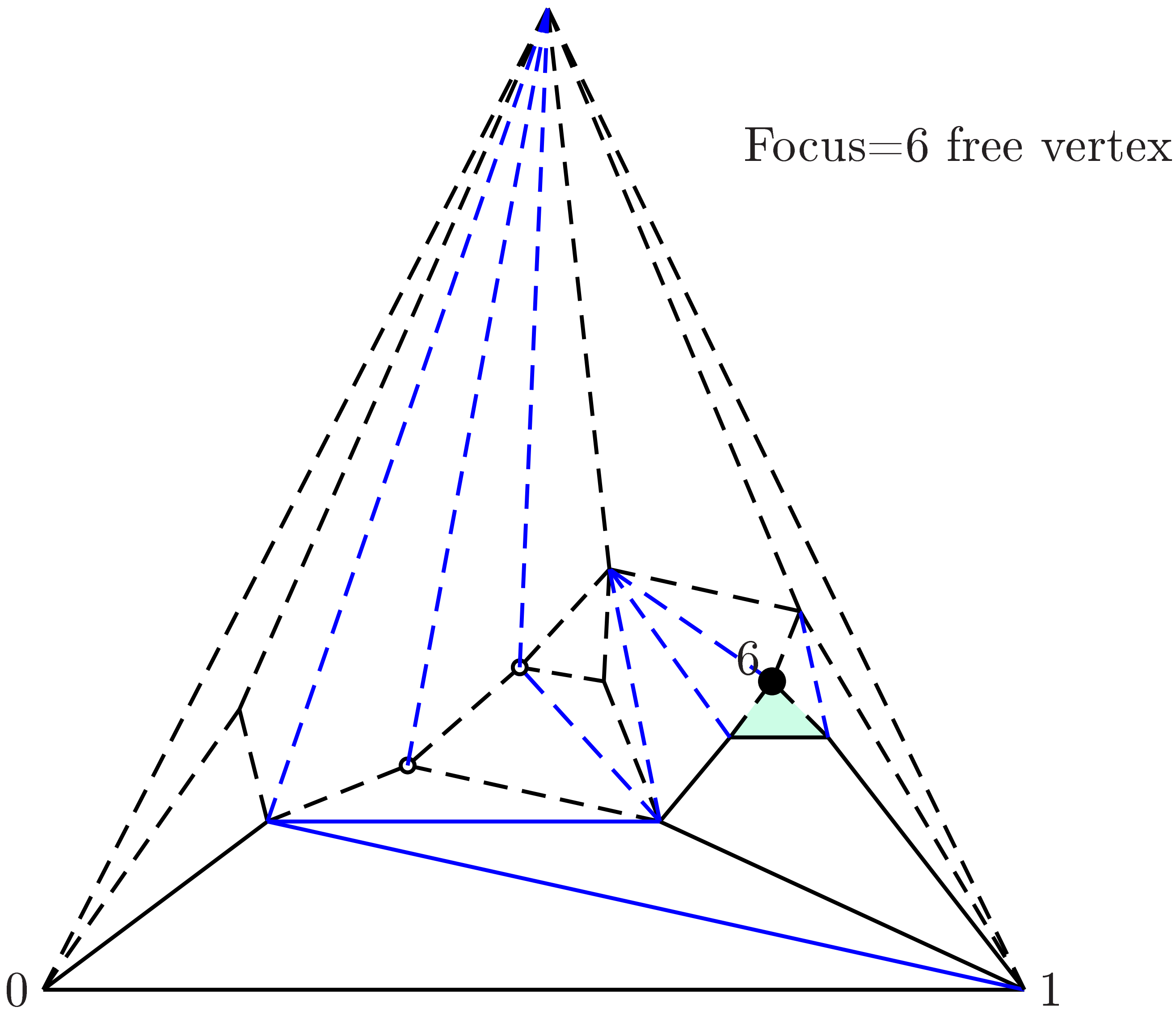
Computing the triangulation of G



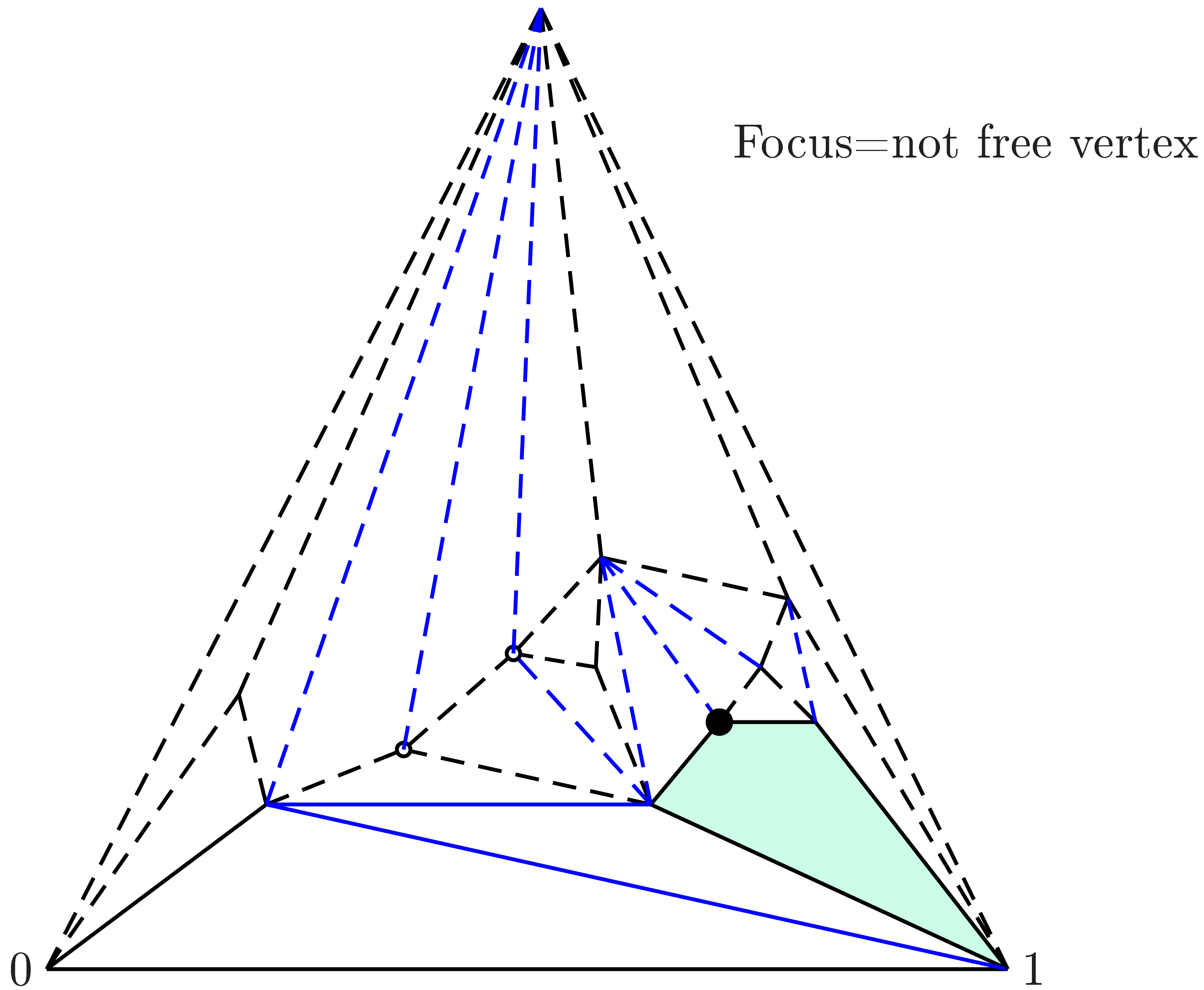
Computing the triangulation of G



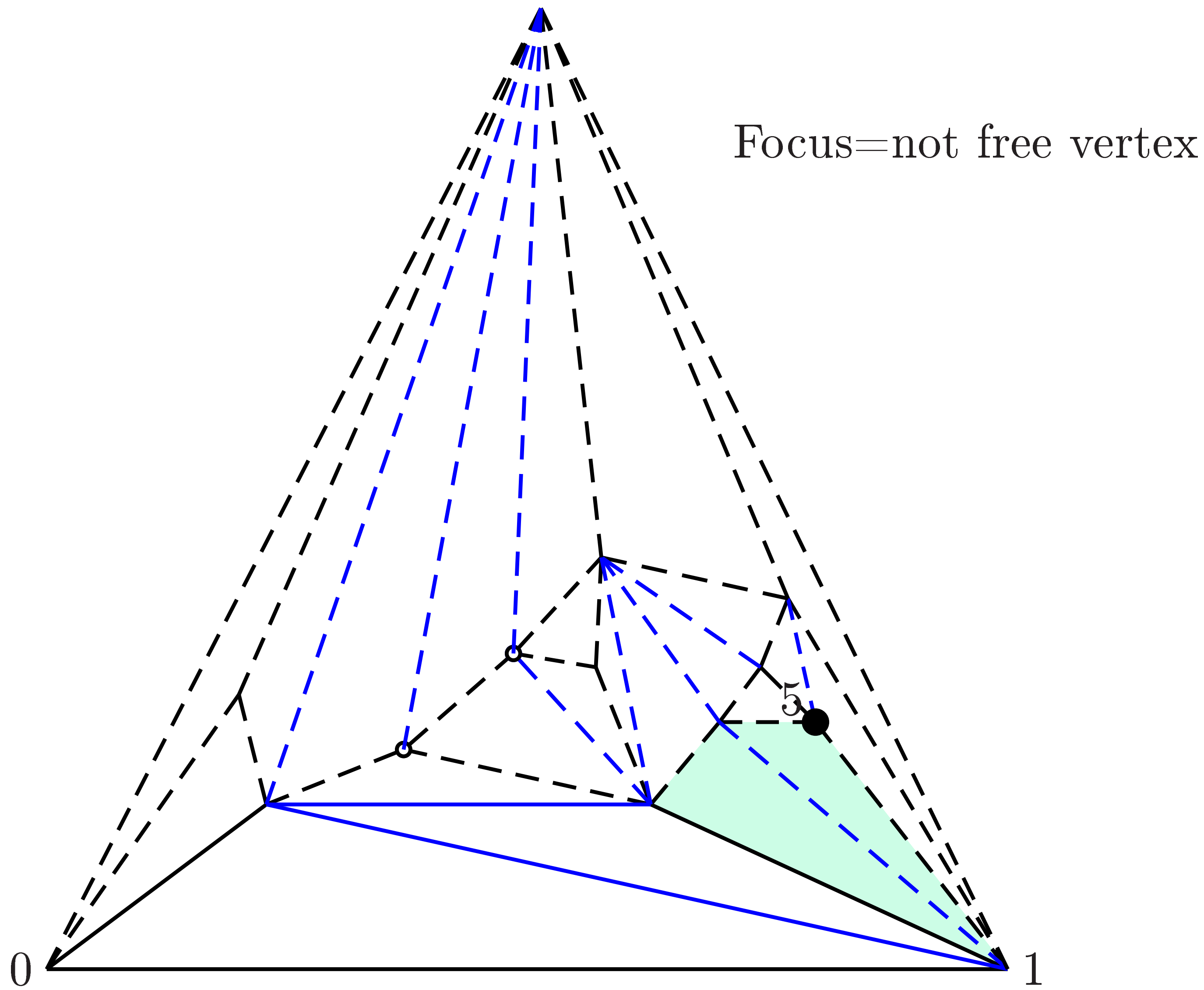
Computing the triangulation of G



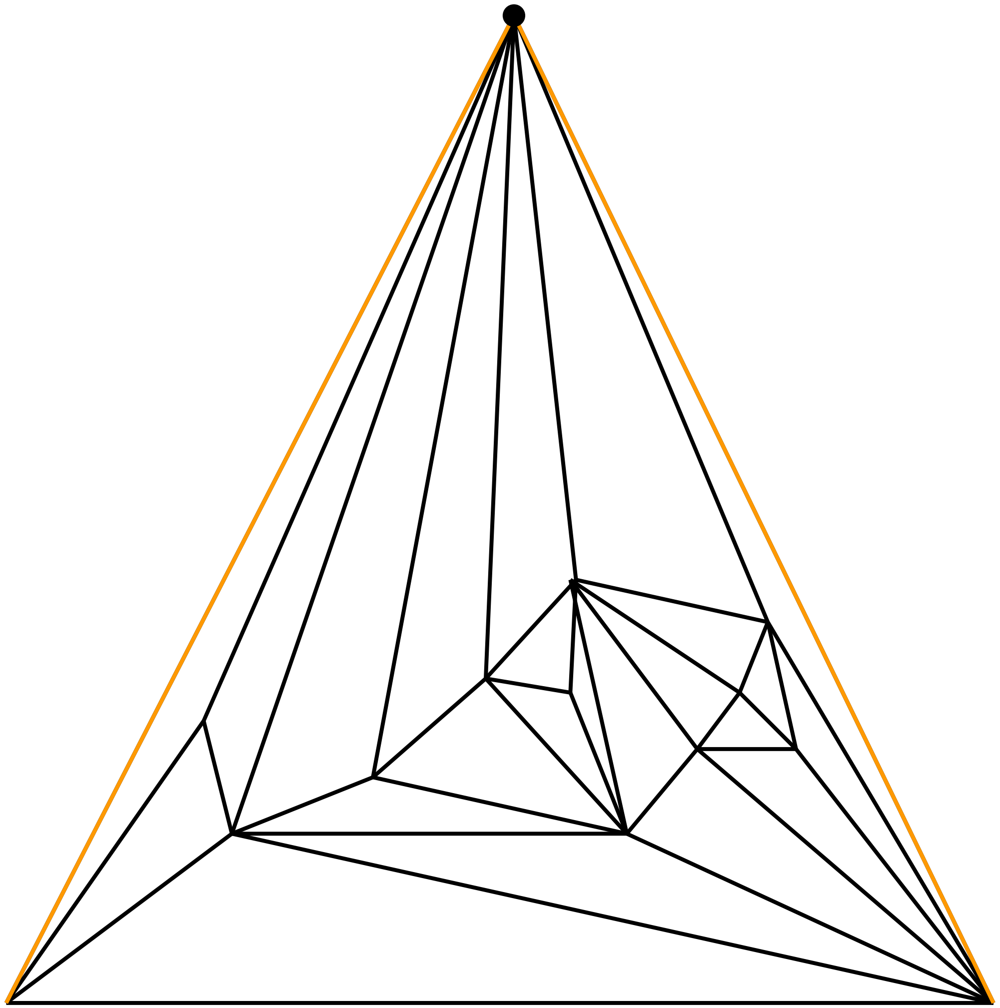
Computing the triangulation of G



G is now triangulated

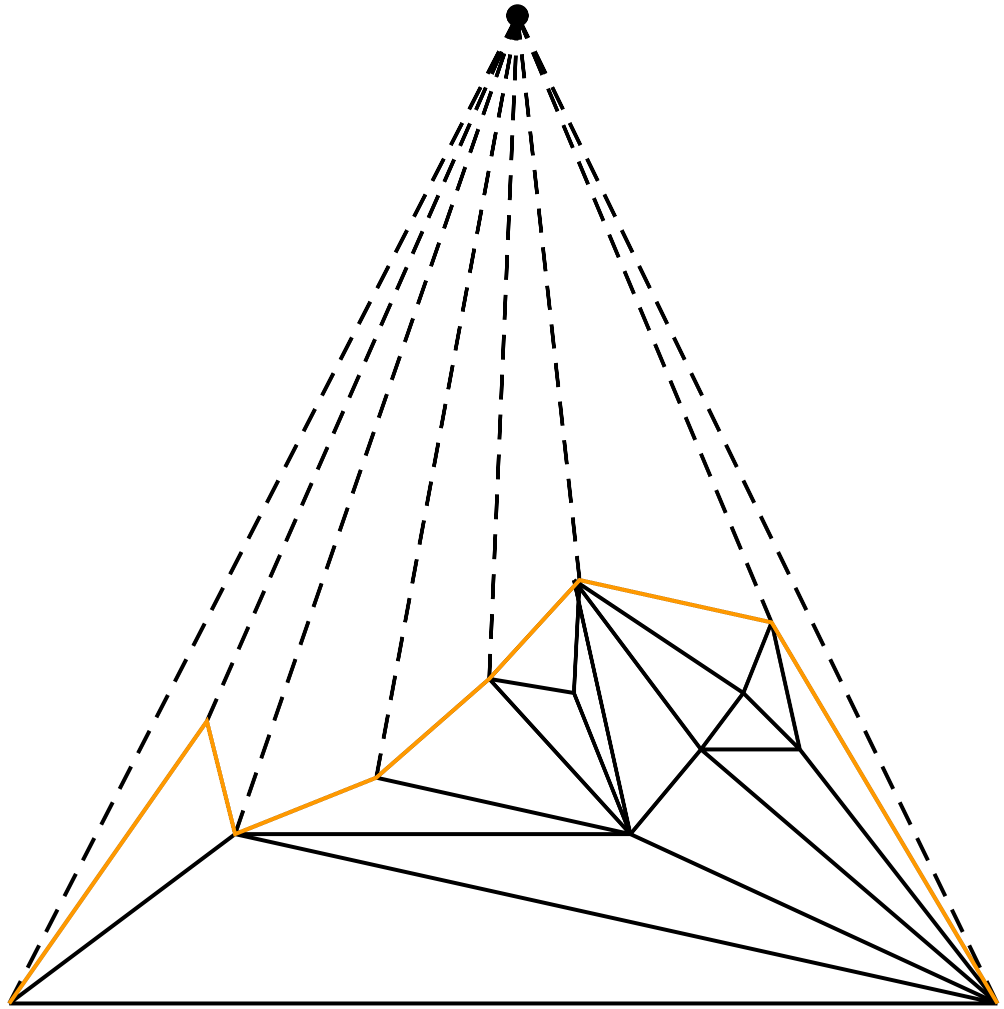


Start decoding



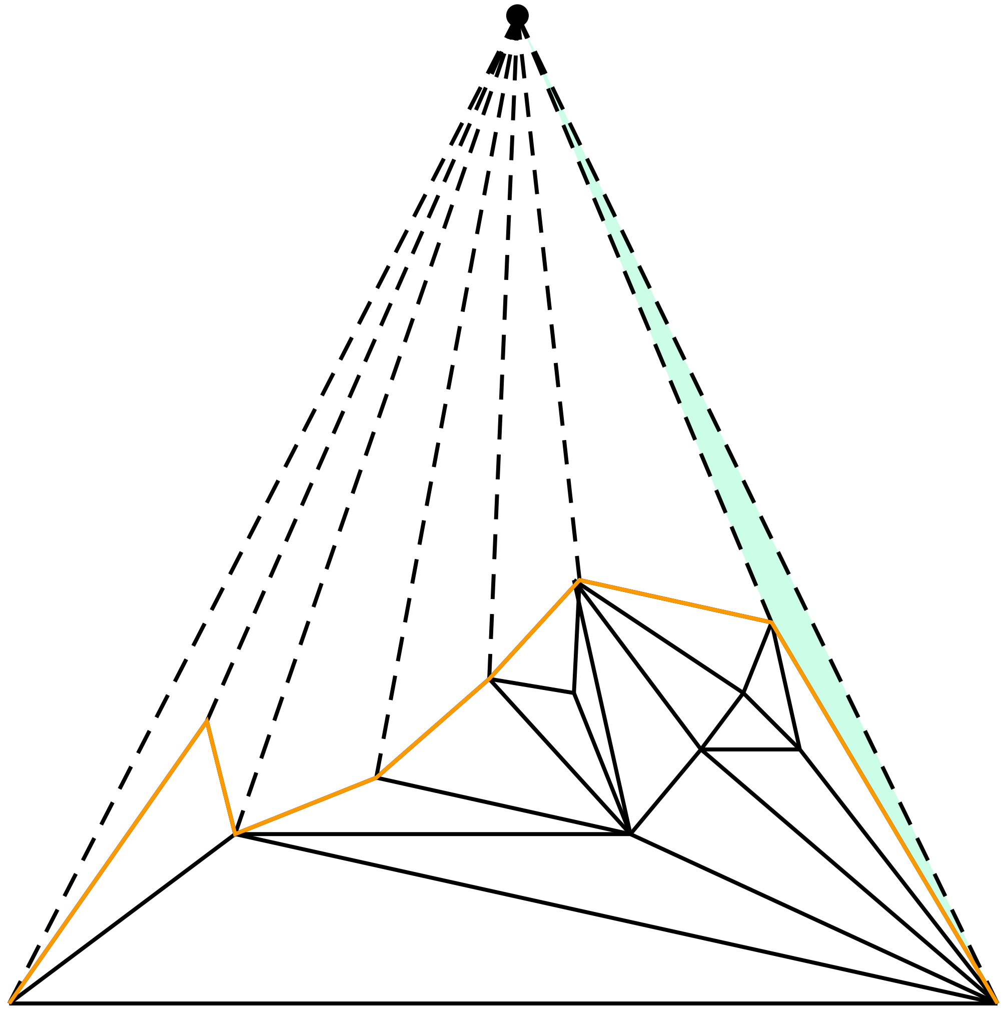
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4, 5, 4, 3, 4$

Start decoding



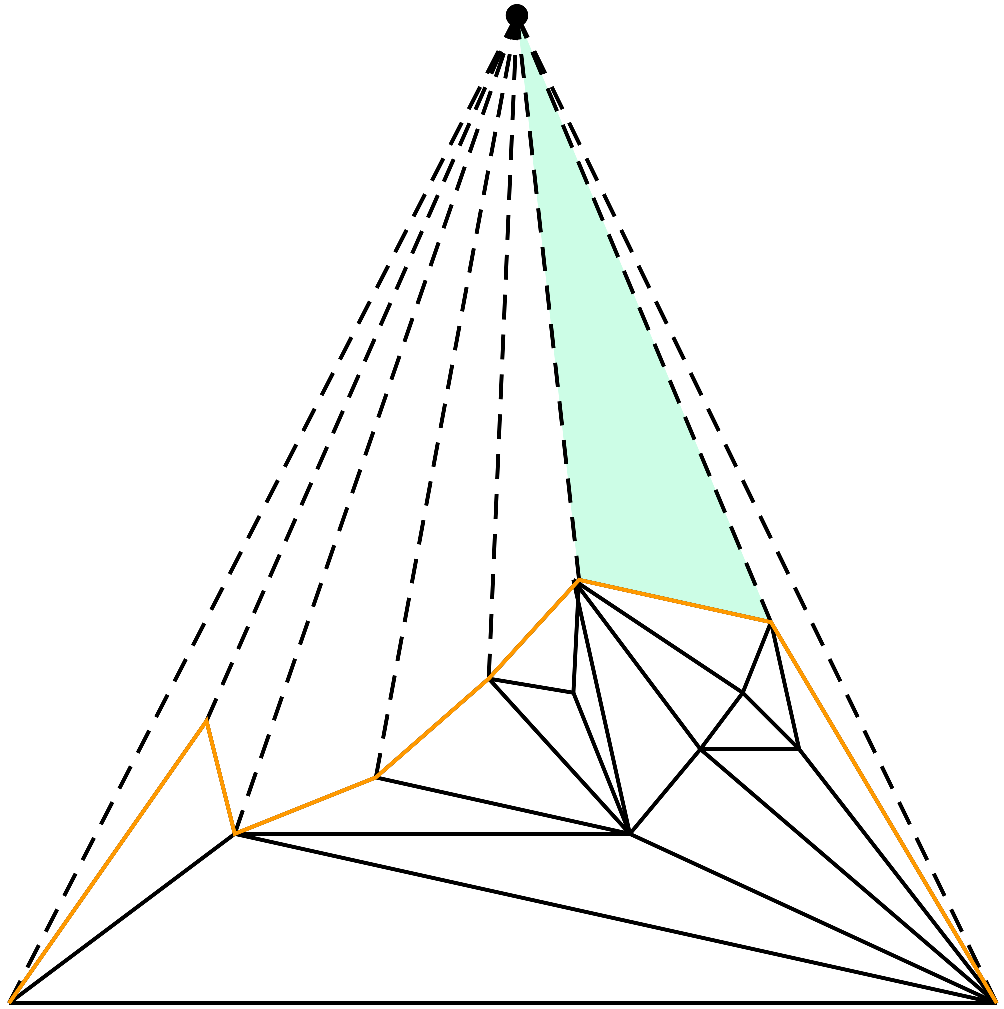
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4, 5, 4, 3, 4$

Start decoding



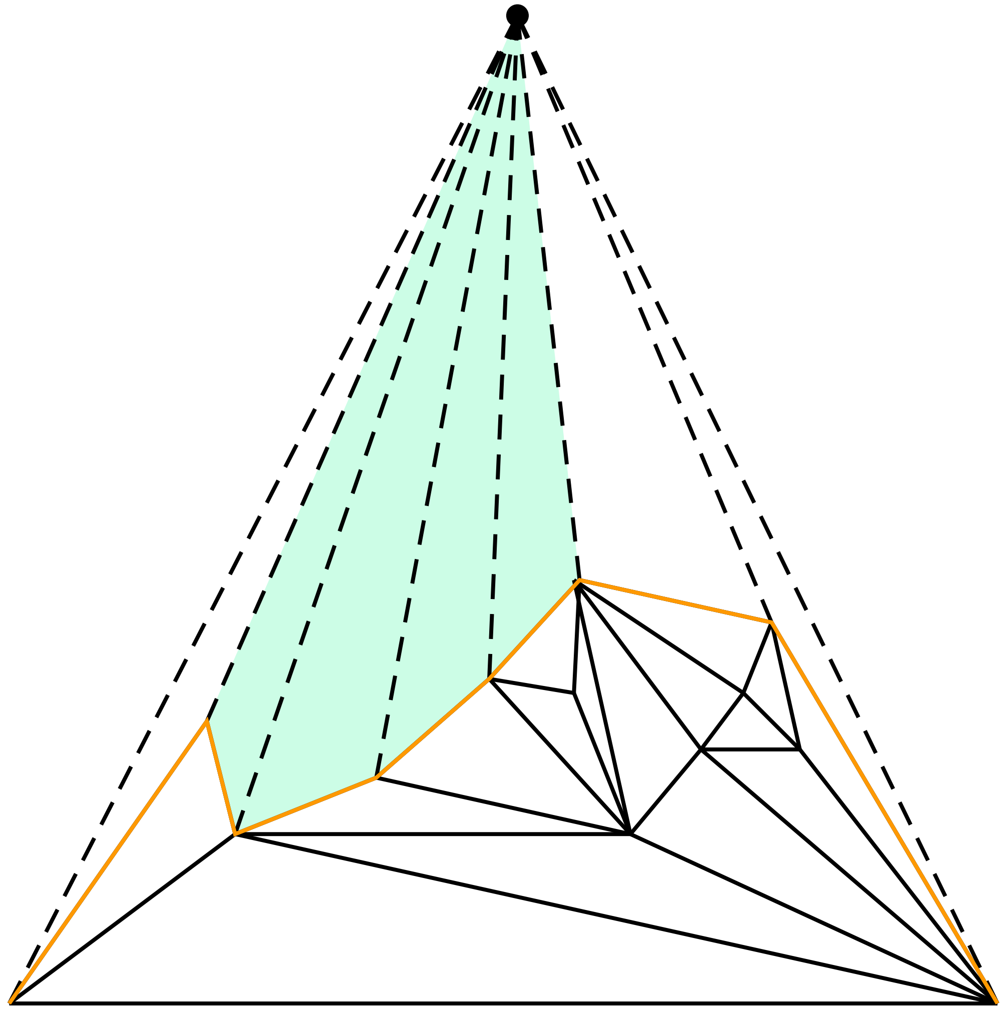
degrees of the faces: $\boxed{\underline{\mathbf{3}}}$ $, 3, 6, 3, 3, 6, 3, 4, 5, 4, 3, 4$

Start decoding



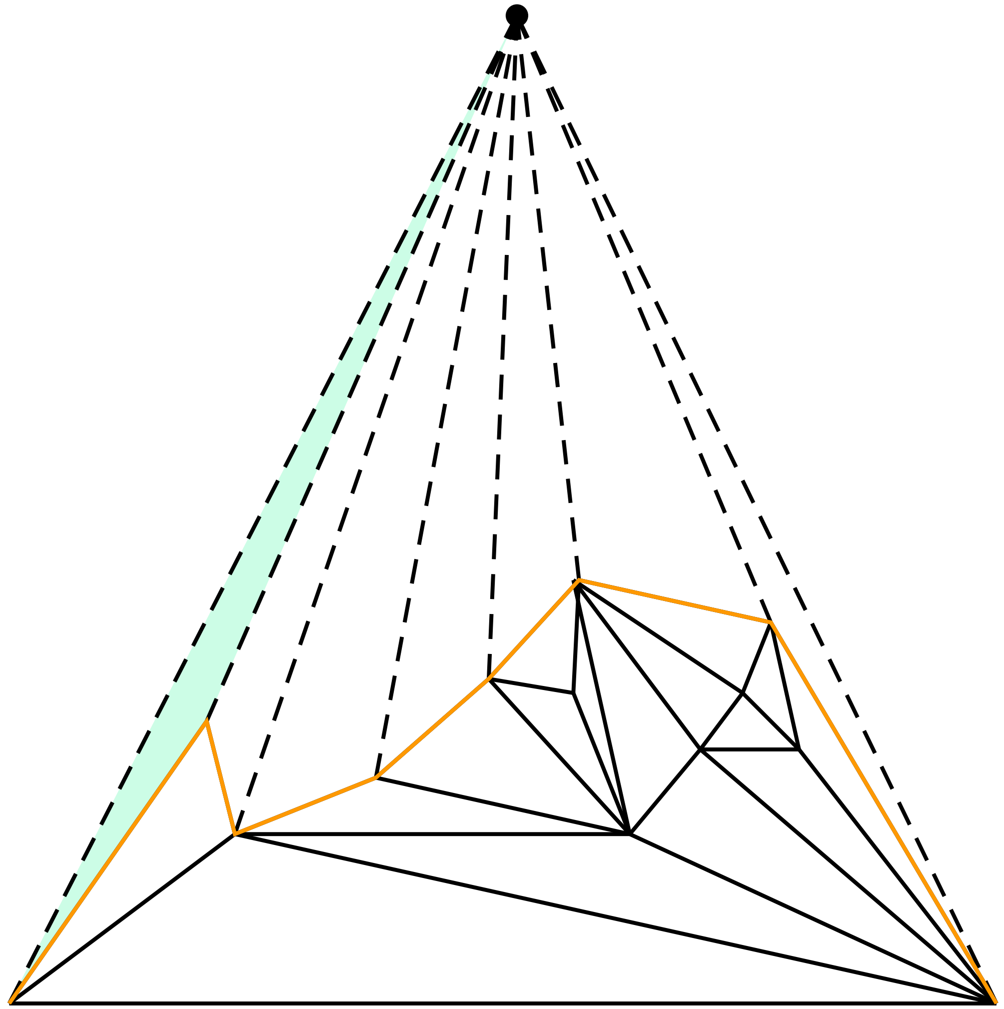
degrees of the faces: 3, $\boxed{\underline{\mathbf{3}}}$, 6, 3, 3, 6, 3, 4, 5, 4, 3, 4

Start decoding



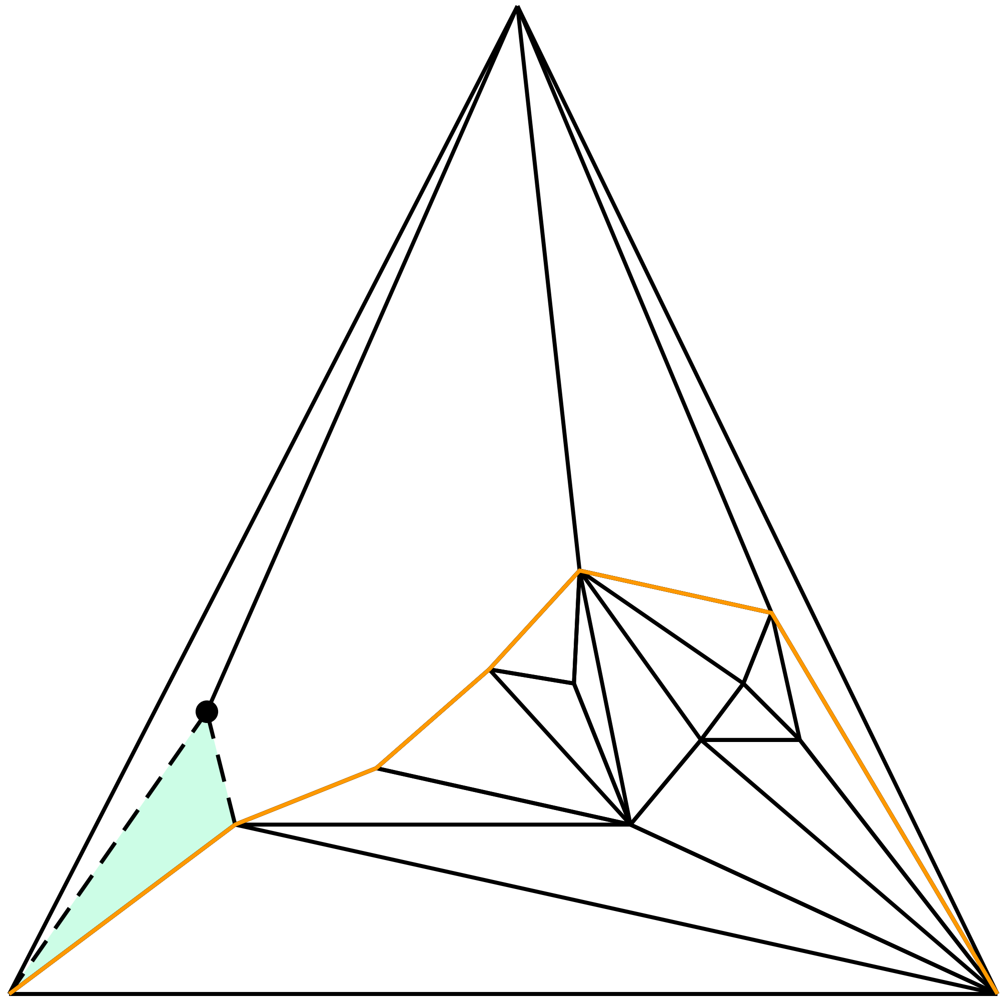
degrees of the faces: $3, 3\ \boxed{\underline{\mathbf{6}}}\ , 3, 3, 6, 3, 4, 5, 4, 3, 4$

Start decoding



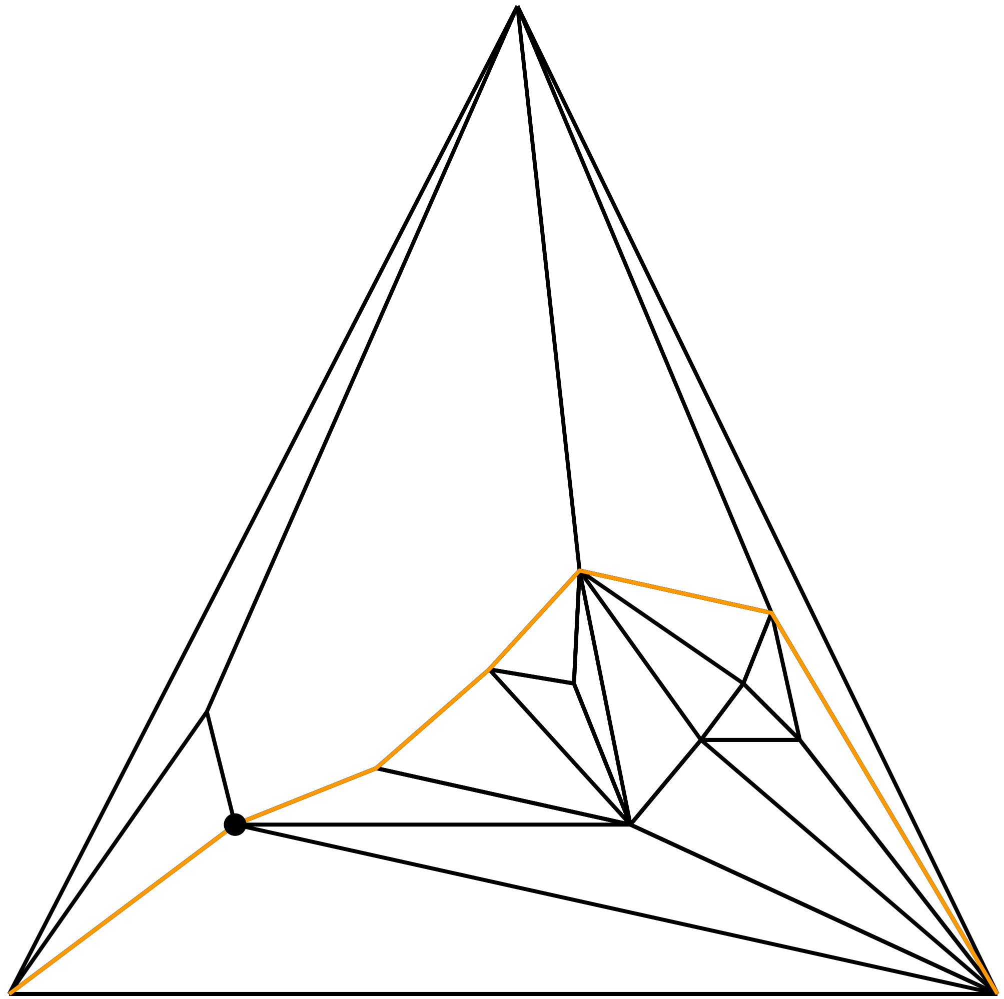
degrees of the faces: $3, 3, 6\ \boxed{\underline{\mathbf{3}}}\ , 3, 6, 3, 4, 5, 4, 3, 4$

decoding



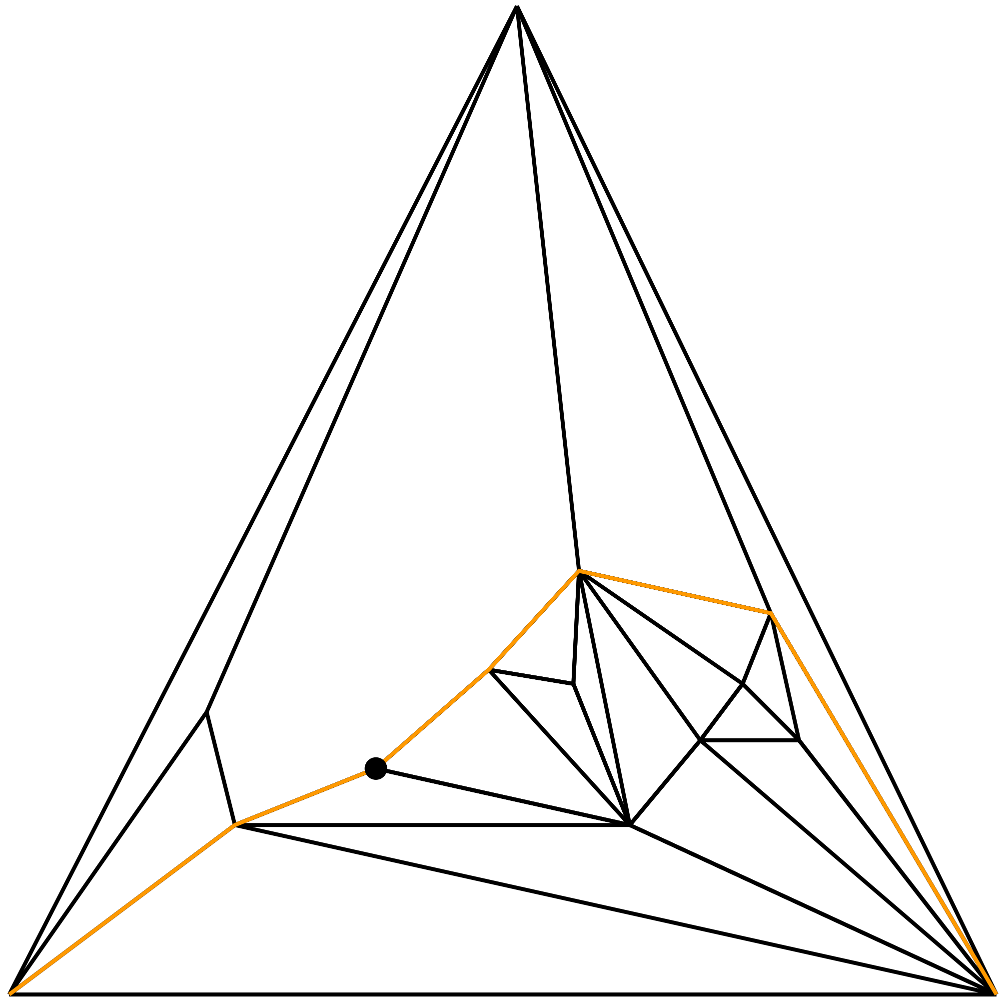
degrees of the faces: $3, 3, 6, 3, \boxed{\underline{\mathbf{3}}}, 6, 3, 4, 5, 4, 3, 4$

decoding



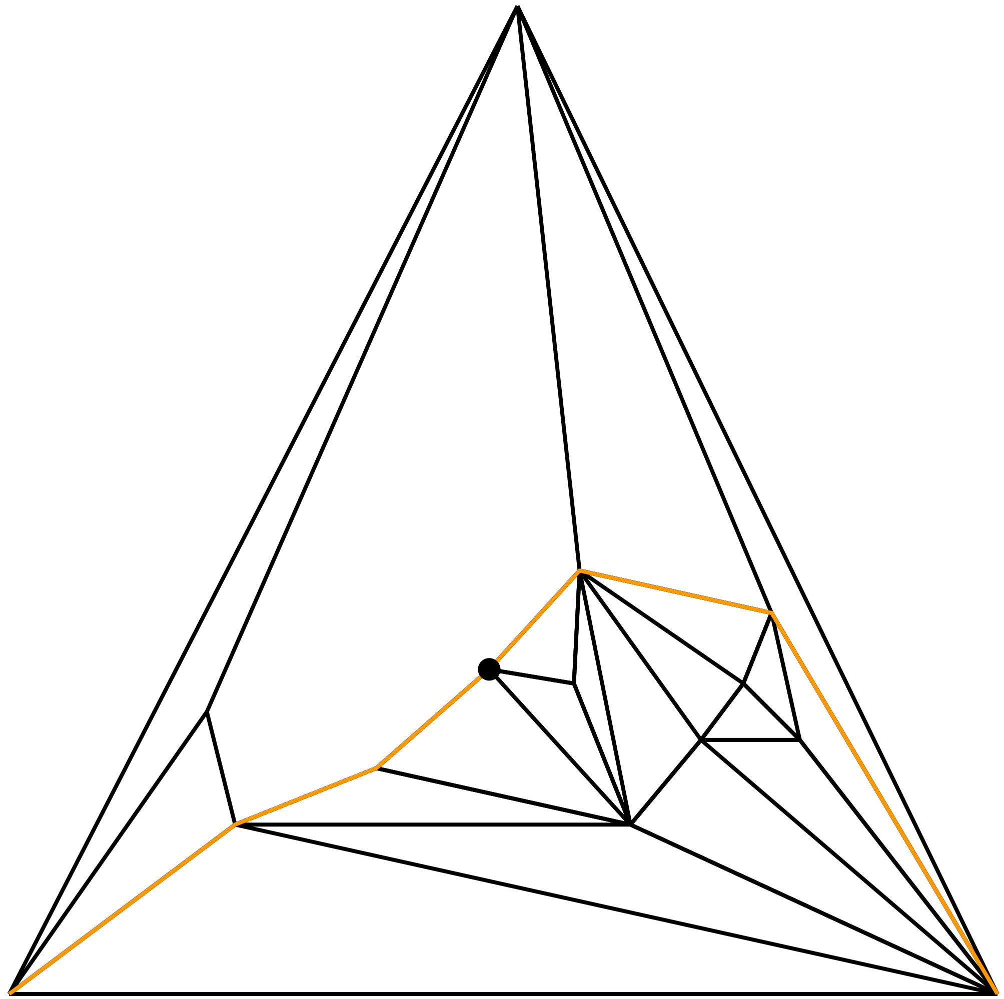
degrees of the faces: $3, 3, 6, 3, \boxed{\underline{\mathbf{3}}}, 6, 3, 4, 5, 4, 3, 4$

decoding



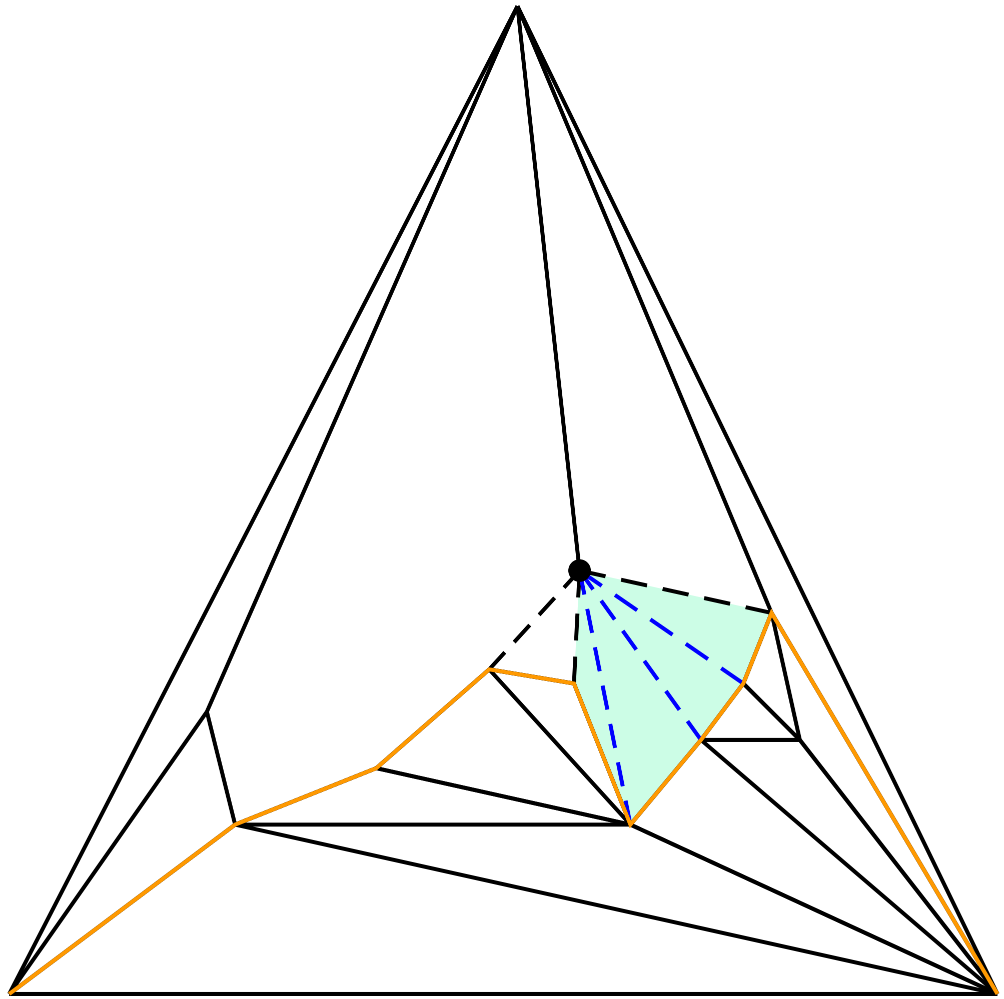
degrees of the faces: $3, 3, 6, 3, \boxed{\underline{\mathbf{3}}}, 6, 3, 4, 5, 4, 3, 4$

decoding



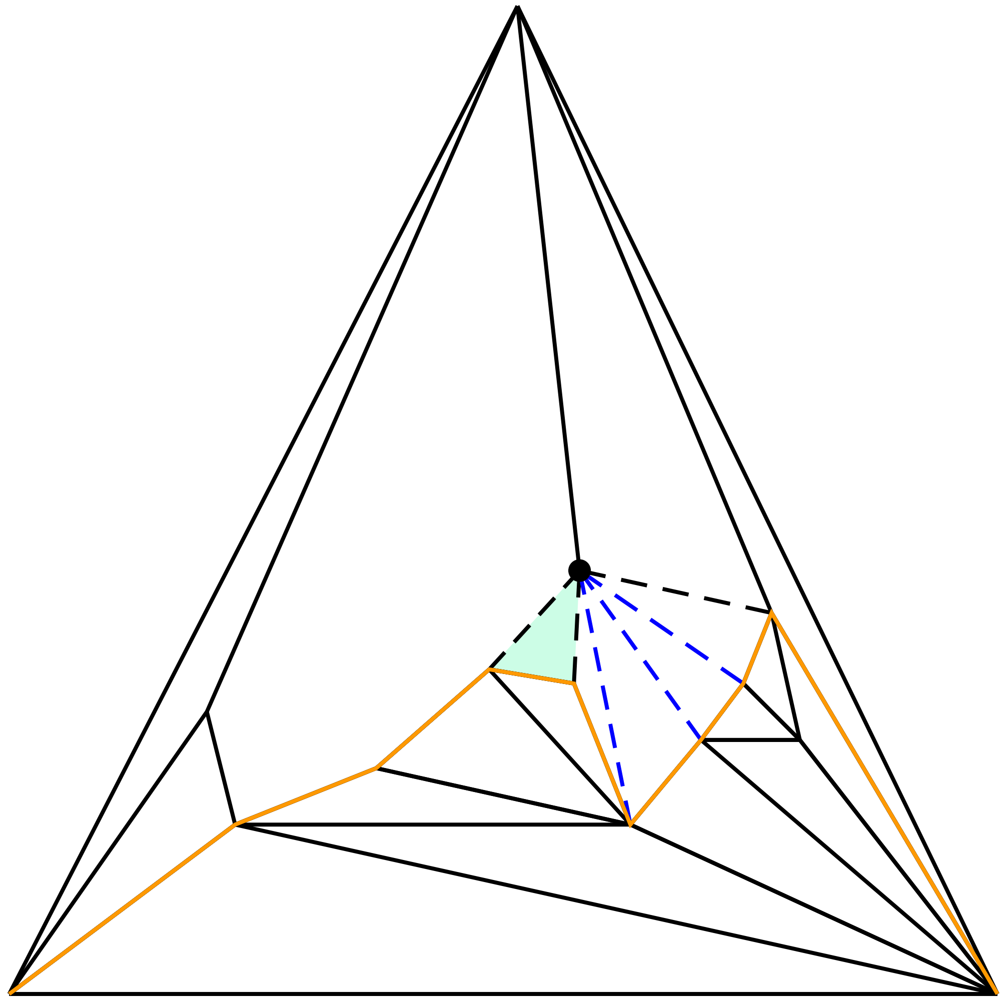
degrees of the faces: $3, 3, 6, 3, \boxed{\underline{\mathbf{3}}}, 6, 3, 4, 5, 4, 3, 4$

decoding



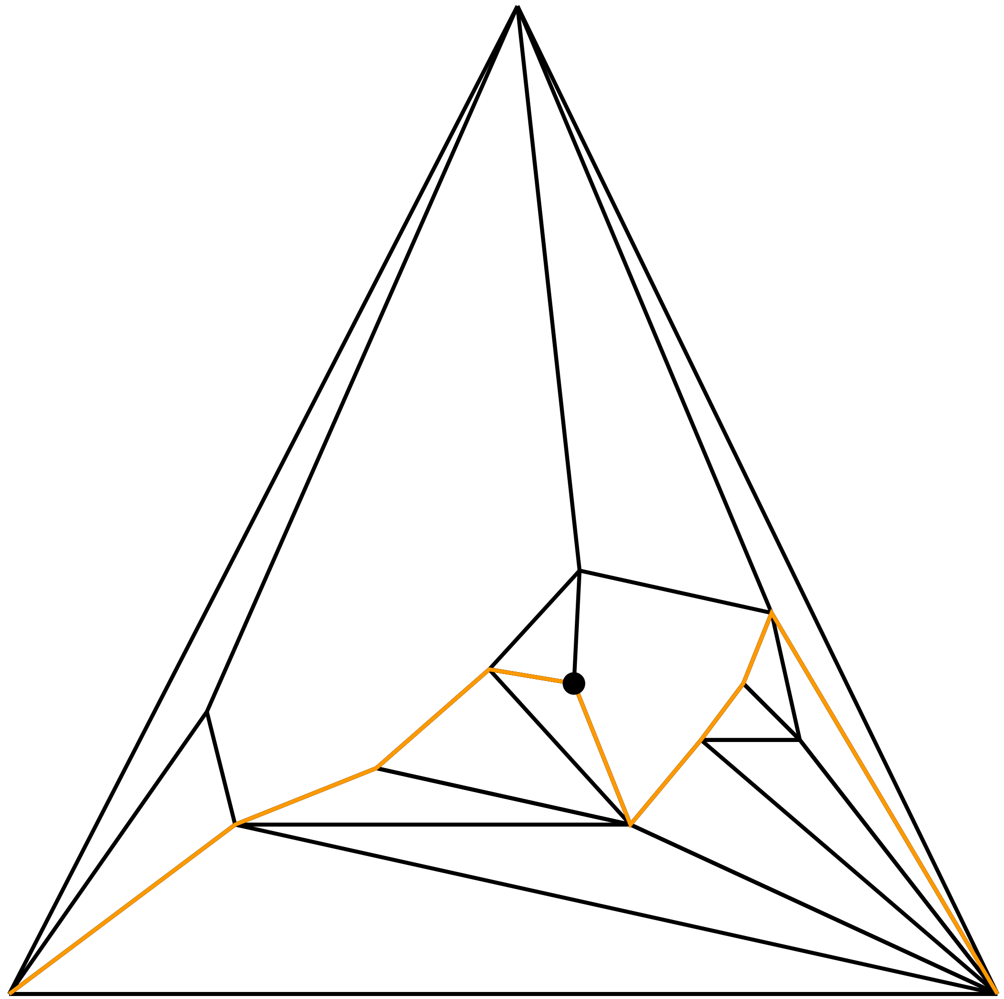
degrees of the faces: $3, 3, 6, 3, 3, \boxed{\underline{\mathbf{6}}}, 3, 4, 5, 4, 3, 4$

decoding



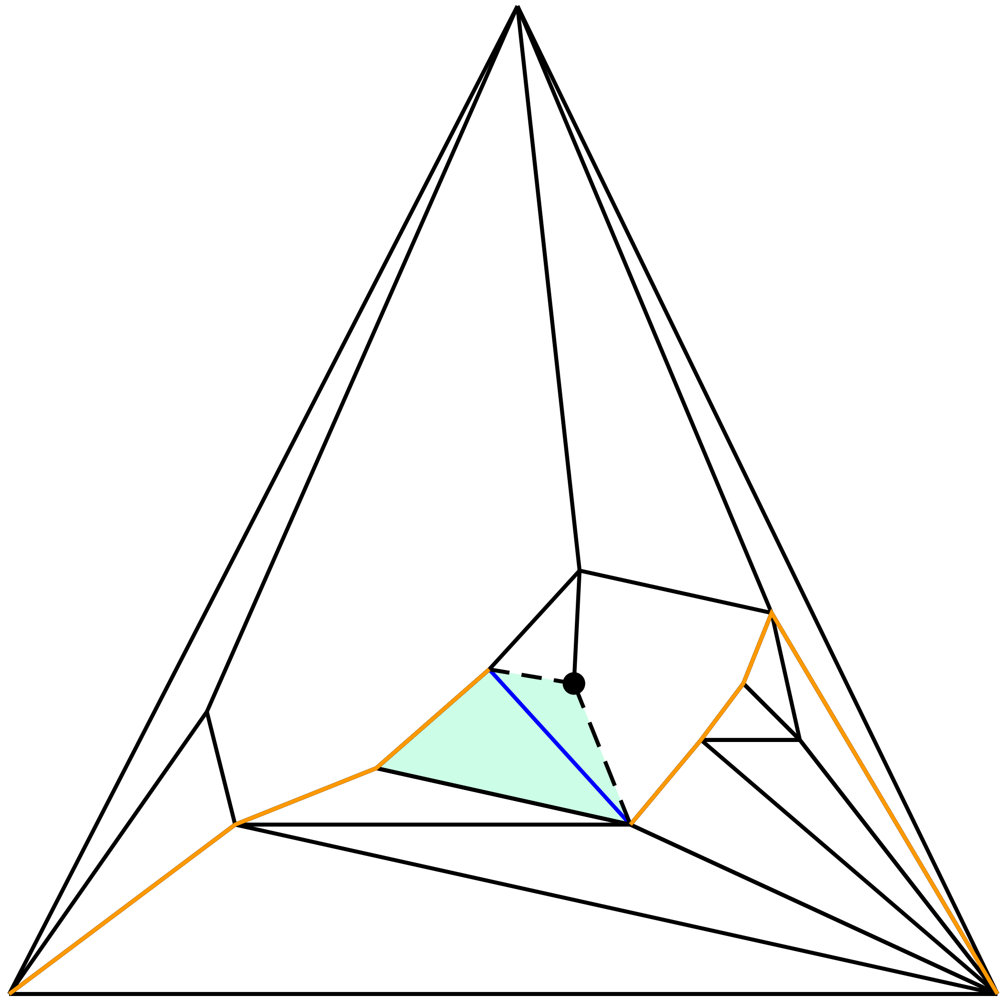
degrees of the faces: $3, 3, 6, 3, 3, 6, \boxed{\underline{\mathbf{3}}}\ , 4, 5, 4, 3, 4$

decoding



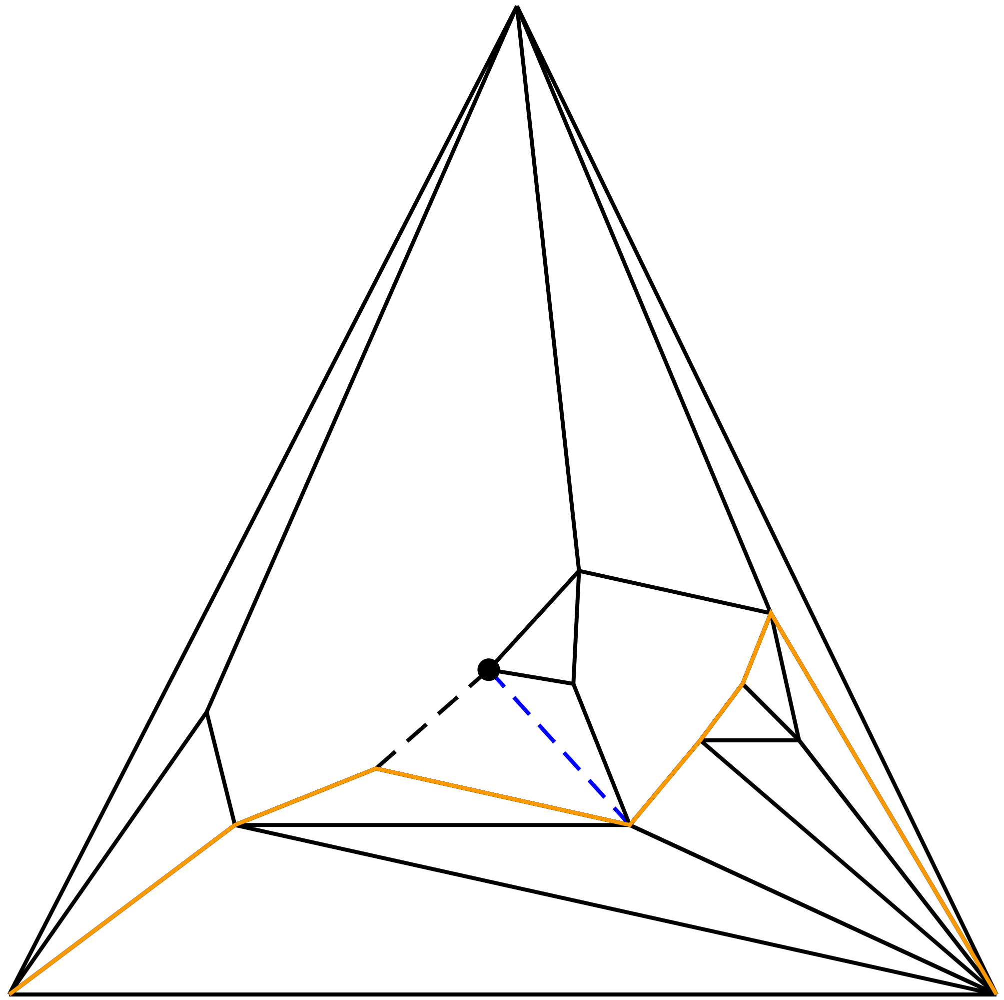
degrees of the faces: $3, 3, 6, 3, 3, 6, \boxed{\underline{\mathbf{3}}}, 4, 5, 4, 3, 4$

decoding



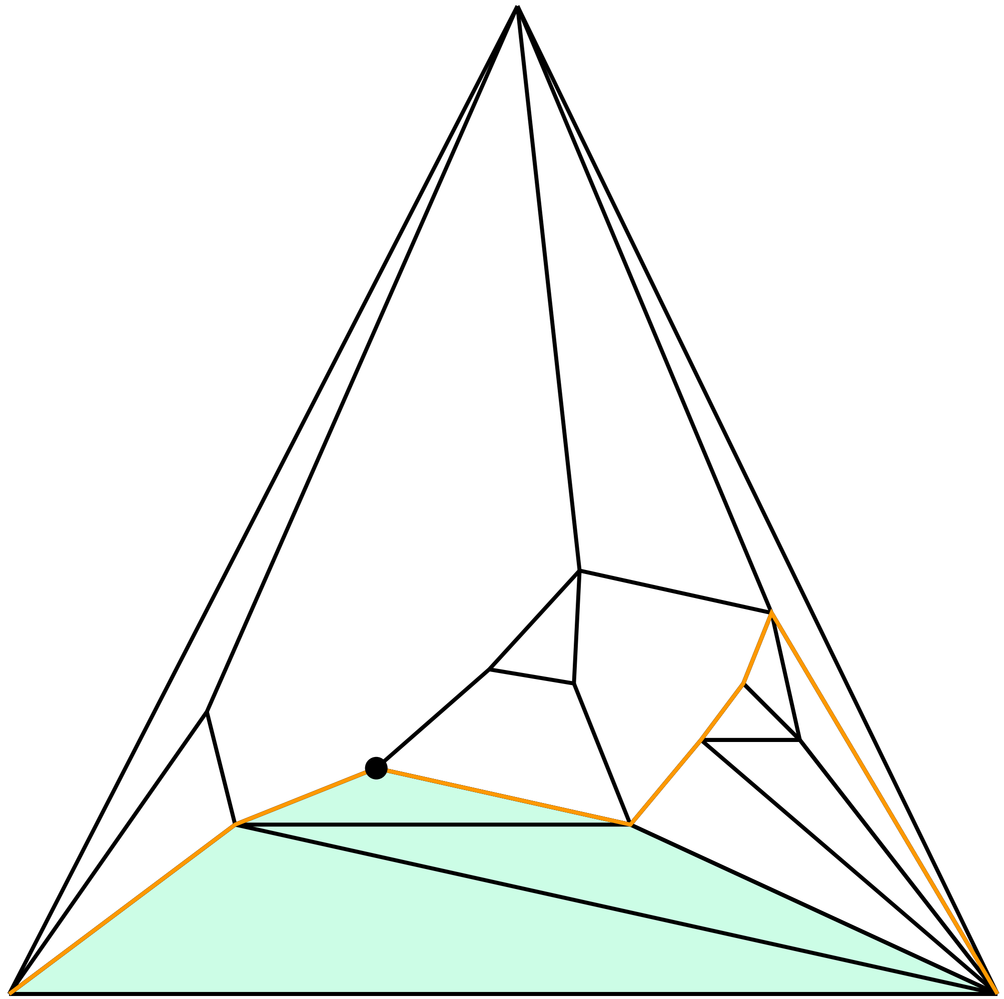
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, \boxed{\underline{\mathbf{4}}}, 5, 4, 3, 4$

decoding



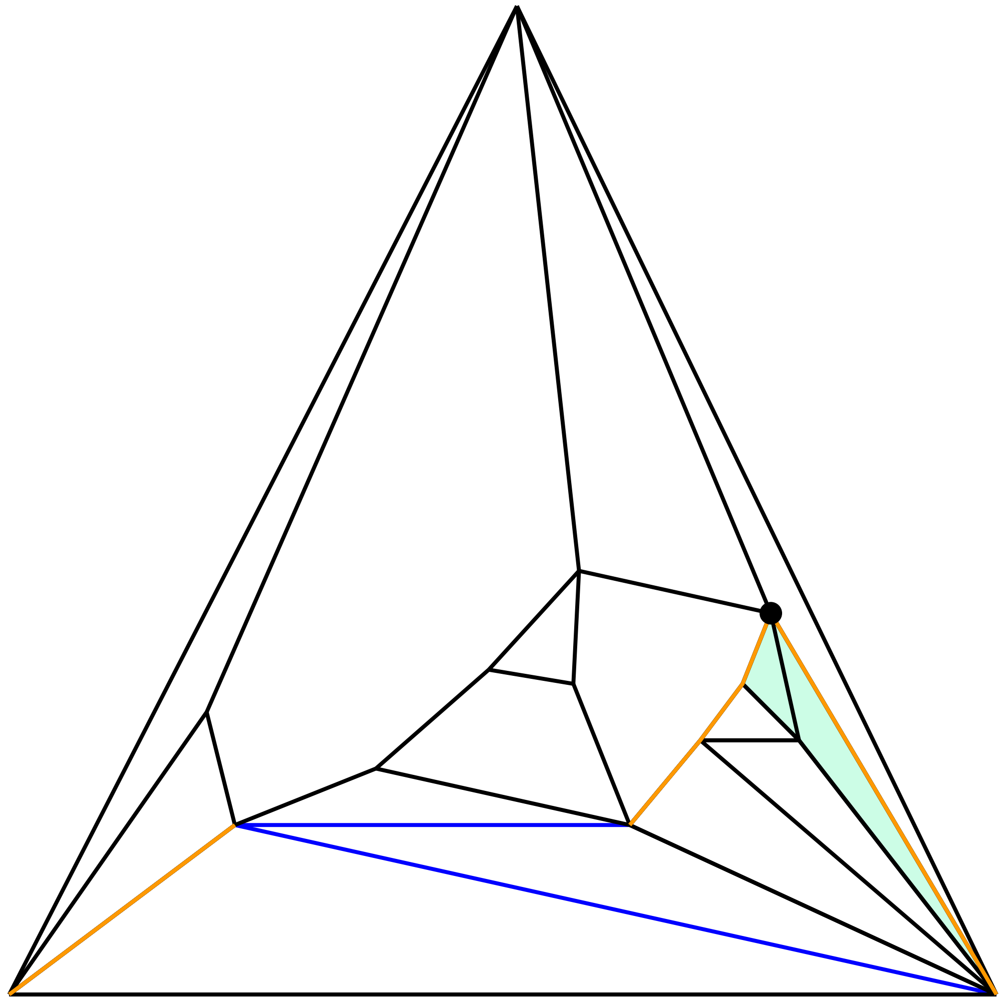
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, \boxed{\underline{\mathbf{4}}}, 5, 4, 3, 4$

decoding



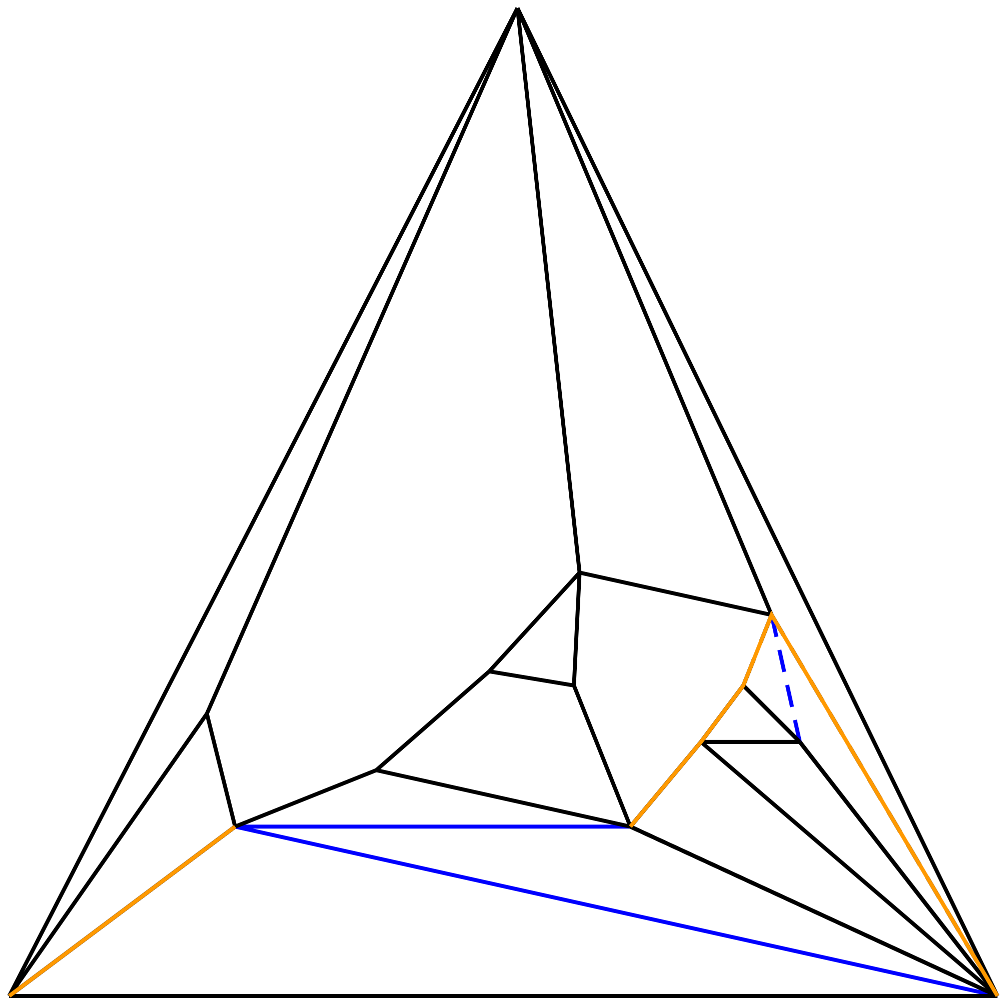
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4$ $\boxed{\underline{\mathbf{5}}}$ $, 4, 3, 4$

decoding



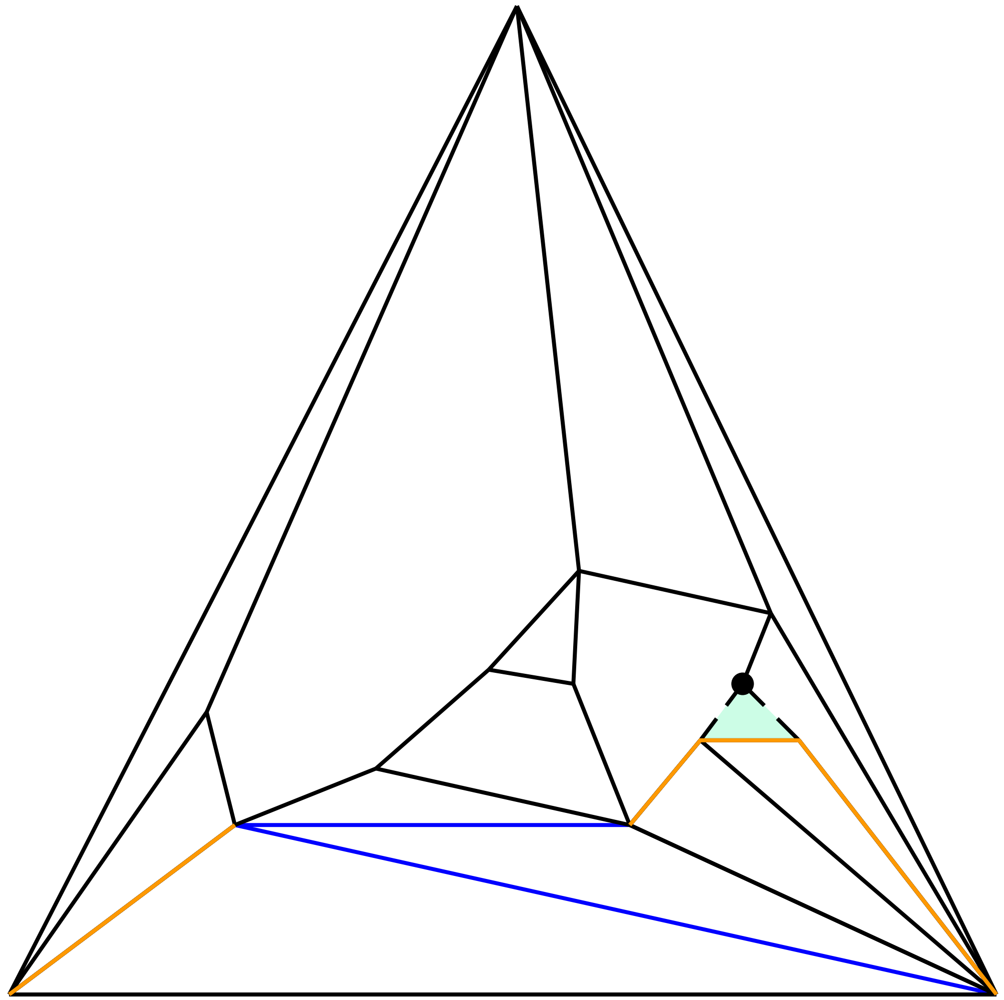
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4, 5\ \boxed{\underline{\mathbf{4}}}\ , 3, 4$

decoding



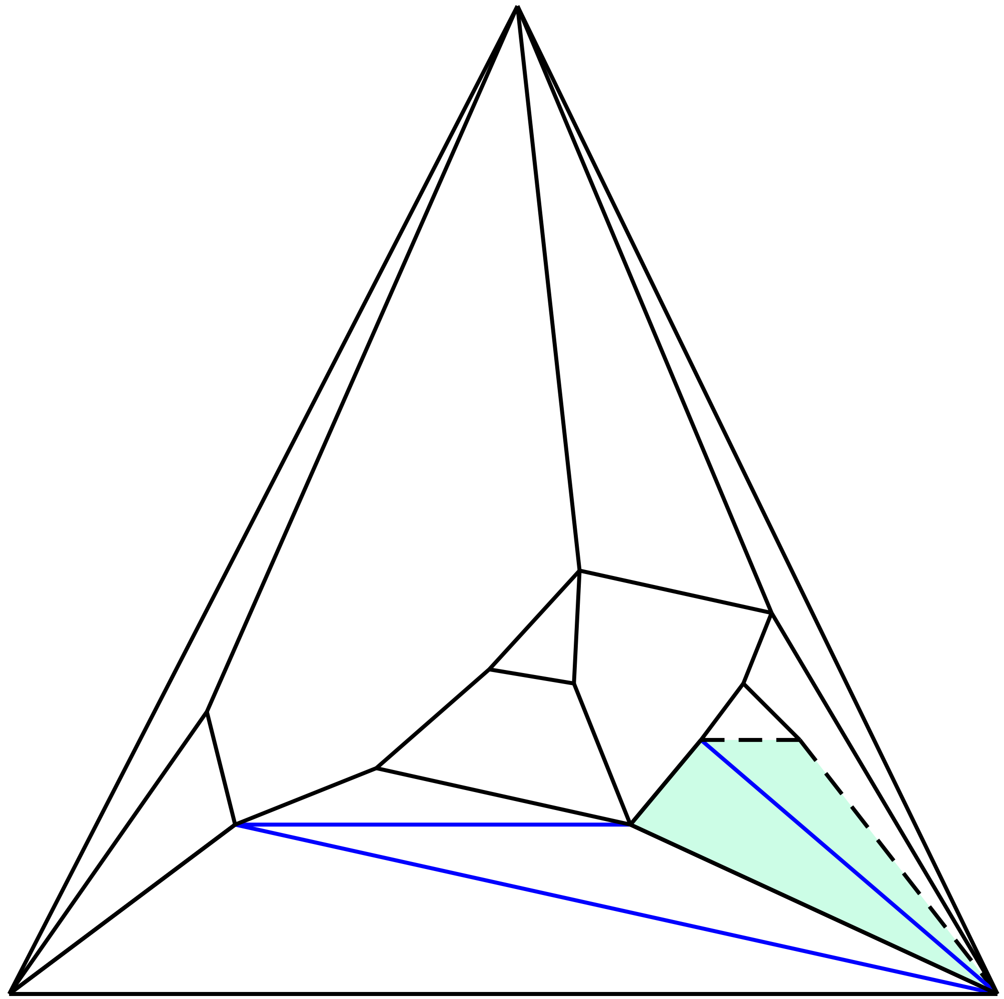
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4, 5\ \boxed{\underline{\mathbf{4}}}\ , 3, 4$

decoding



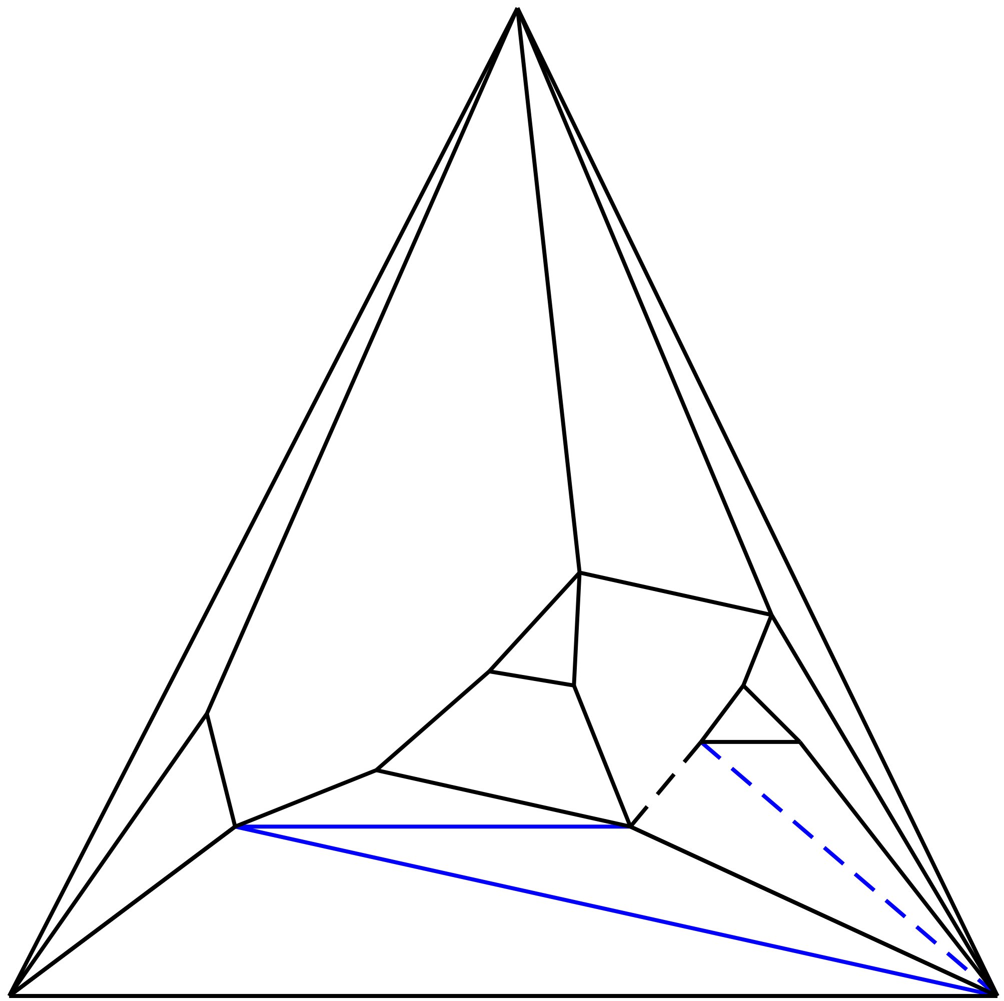
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4, 5, 4\ \boxed{\underline{\mathbf{3}}}\ , 4$

decoding



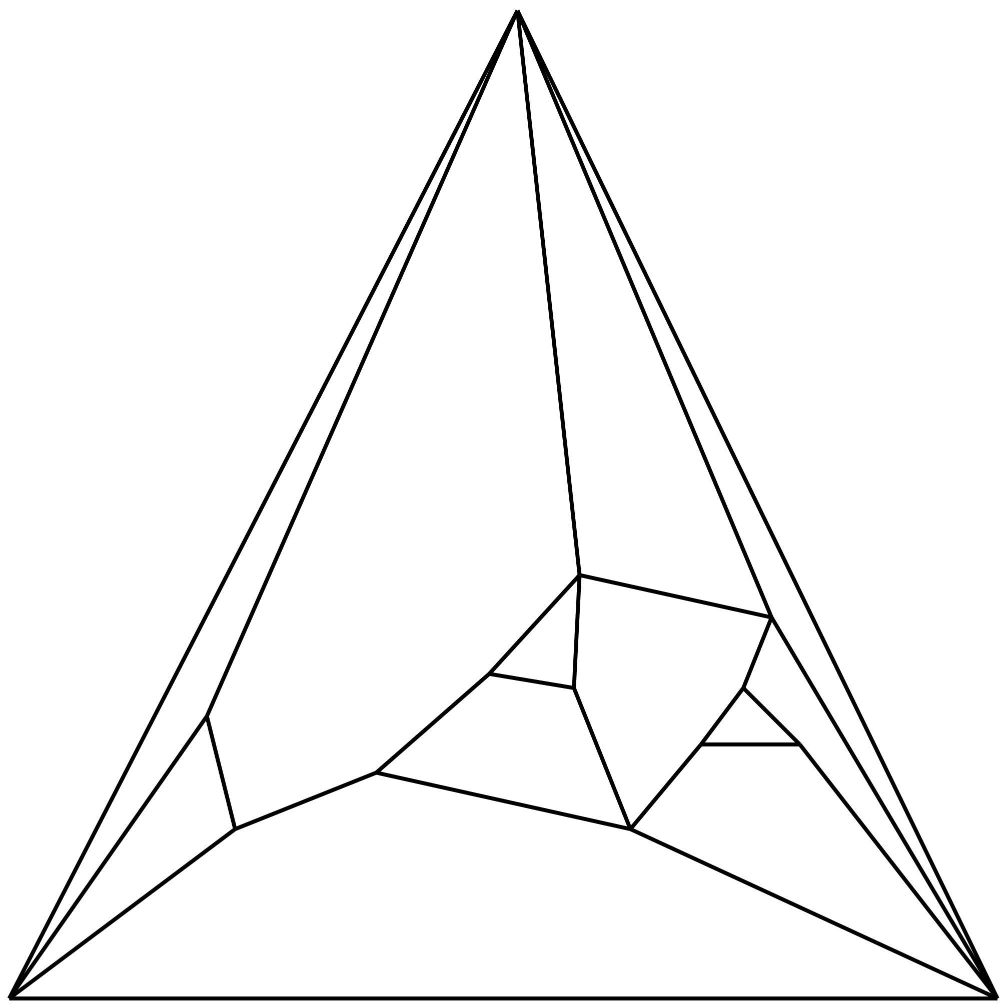
degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4, 5, 4, 3$ $\boxed{\underline{\mathbf{4}}}$

decoding



degrees of the faces: $3, 3, 6, 3, 3, 6, 3, 4, 5, 4, 3, 4$

decoding



Bit rate upper bound

Lemma 1. *Let G be a 3-connected planar graph with n vertices. If we suppose to apply an optimal coder to compress the triangulation T_G, it is possible to encode G with at most*

$$(3.2451 + 1.9988)\, n \ \textit{bits}$$

- For random polygon meshes, the valence frequencies p_i are asymptotically for large i

$$p_i \approx 18\sqrt{\frac{3}{2}}(\pi i)^{-\frac{1}{2}}\,\frac{1}{2^i}$$

- evaluation of the entropy (polygon mesh degrees) (Gotsman '03)

$$\sum_{i\geq 3} p_i \log_2 \frac{1}{p_i} = 1,998832...bpv$$

Bit rate upper bound

Theorem. *Any simple 3-connected plane graph with n vertices, m edges and f faces can be encoded in linear time using at most*

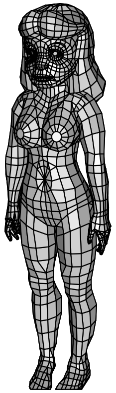
$$(3.2451 + 1.9988) \min\{n, f\} + 1 = 2.62e + o(e)$$

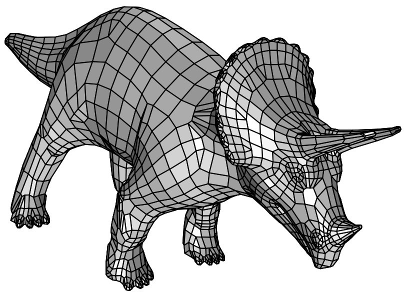
- If G is 3-connected then so is its dual G^*;
- if $n > 3$ ($f > 3$) then G^* is simple;
- $\|code(G, G^*)\| \leq (3.2451 + 1.9988) \min\{n, f\}$ bits.

$$\frac{(3.2451 + 1.9988)}{2}(e+1) + 1 = 2.62e + o(e)$$

Experimental results

modèle 3D	V	F	E	G bpe	G^* bpe
triceratops	2832	2834	5664	2.14	**1.65**
cupie	2984	3032	6004	**2.07**	2.15
teapot	1189	1290	2479	**1.94**	2.10





Comparison of compression results (upper bounds)

Algorithm	planar	triangulation	3-connected
Turan ('84)	$4m$		
Keeler ('95)	$3.58m$	$1.53m$	$3m$
Chuang ('98)(*)		$\frac{4}{3}m$	$2.377m$
He ('99)		$\frac{4}{3}m$	$2.835m$
Bonichon ('03)	$2.90m$	$3.37n$, $1.123m$	$2.90m$
Poulalhon/Schaeffer ('03)		$3.24n$, $1.08m$	
our al. + optimal coder		$3.24n$, $1.08m$	$\mathbf{2.62m}$