A bijection between fractional trees and d-angulations

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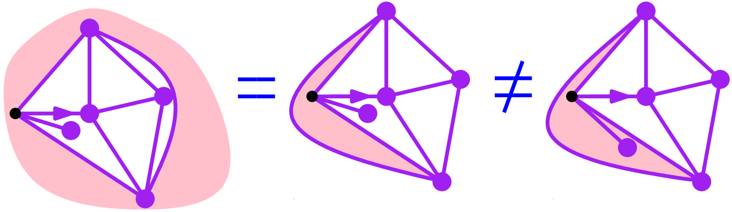
Definition of planar maps

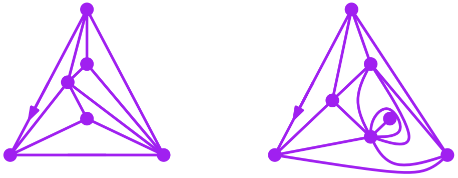
- Planar map = planar connected graph embedded properly in the sphere up to a direct homomorphism of the sphere
- Rooted planar map = an oriented edge is marked.
- with a planar embedding = the "outer face" is chosen.



Triangulations, quadrangualations, . . .

Faces = connected components of the plane without the edges of the map.

Triangulation, quadrangulation, pentagulation, d-angulation, . . . =
map whose faces are all of degree 3, 4, 5, d, . . .



Girth = length of the shortest cycle.

From now on, only d-angulations of girth d.

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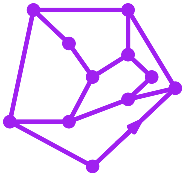

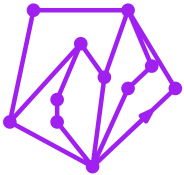

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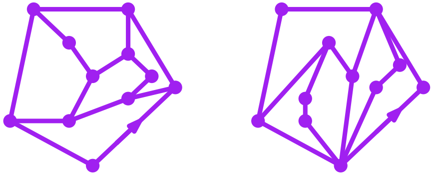

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Enumeration

One of the main question when studying some families of maps :

How many maps belong to this family ?

- Tutte '60s: recursive decomposition
- Matrix integrals: t'Hooft '74,Brézin, Itzykson, Parisi and Zuber '78 ,
- Representation of the symmetric group: Goulden and Jackson '87 ,
- Bijective approach with labeled trees: Cori-Vauquelin '81, Schaeffer '98, Bouttier, Di Francesco and Guitter '04, Bernardi, Chapuy, Fusy, Miermont, . . .
- Bijective approach with blossoming trees: Schaeffer '98, Schaeffer and Bousquet-Mélou '00, Poulalhon and Schaeffer '05, Fusy, Poulalhon and Schaeffer '06.

Rooted simple triangulations

The number of rooted simple triangulations with $2n$ faces, $3n$ edges and $n+2$ vertices is equal to:

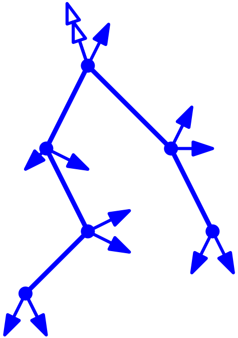
$$\frac{2(4n-3)!}{n!(3n-1)!} = \frac{1}{n} \cdot \underbrace{\frac{2}{(4n-2)}\binom{4n-2}{n-1}}_{\text{number of blossoming trees with } n \text{ nodes}} .$$

Blossoming tree = rooted plane tree where each node (= inner vertex) carries exactly two leaves.

Theorem (Poulalhon and Schaeffer '05)

There exists a one-to-one correspondence between the set of balanced plane trees with n nodes and two leaves adjacent to each node, and the set of rooted simple triangulations of size n.

Closure of a blossoming tree



Root of the tree is not involved in the local closure ⇒ the tree is balanced.

n trees correspond to the same rooted triangulation.

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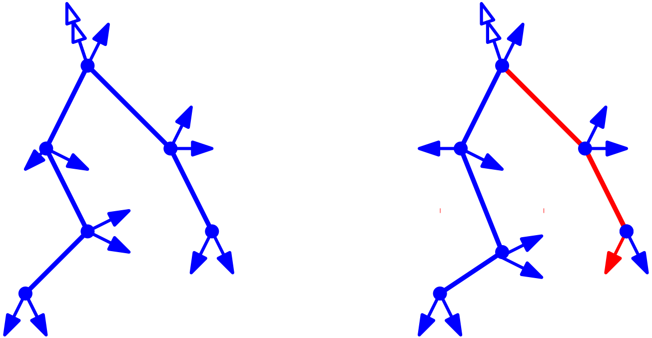

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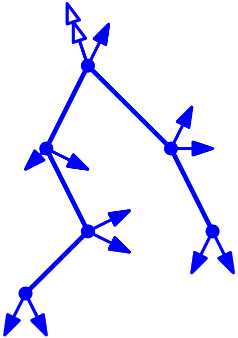

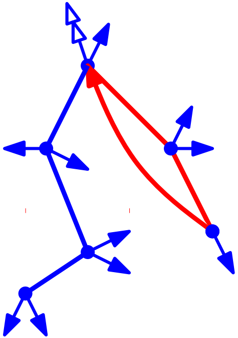

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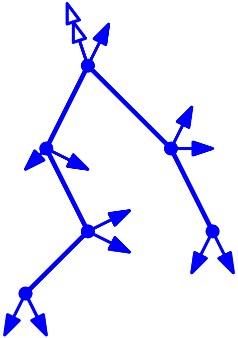

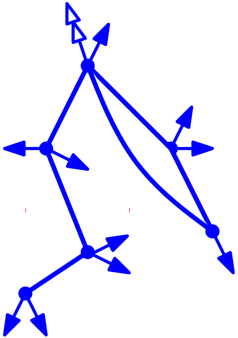

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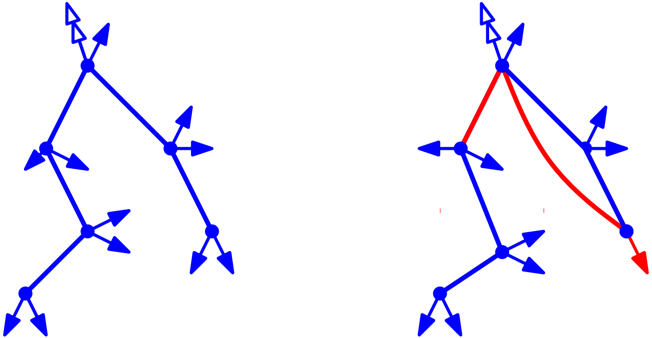

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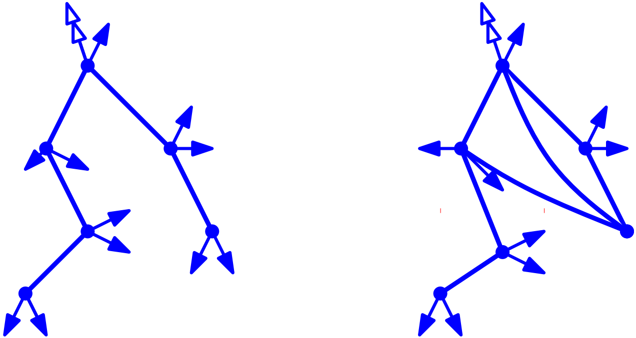

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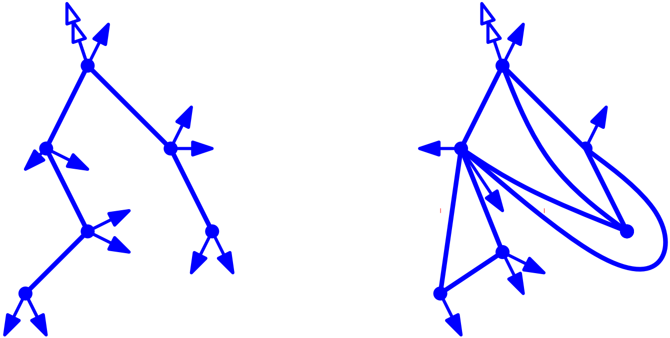

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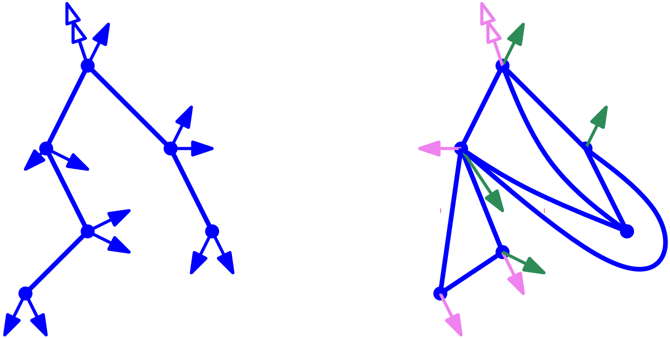

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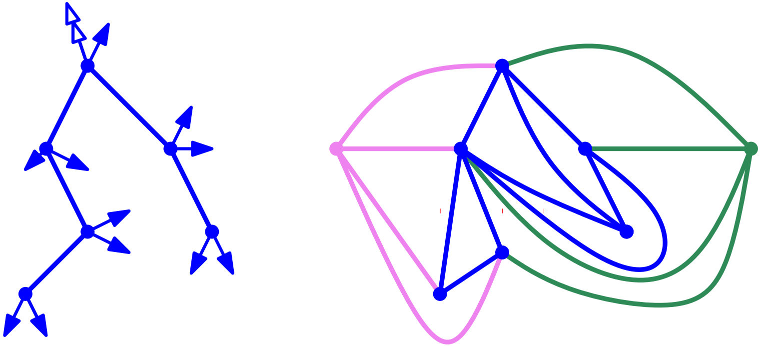

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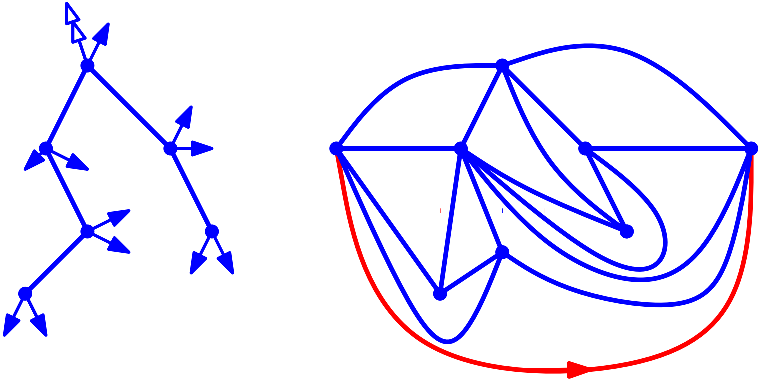

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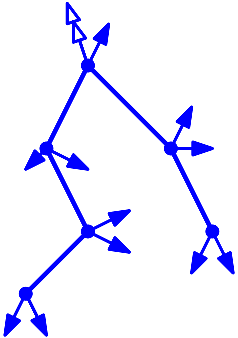

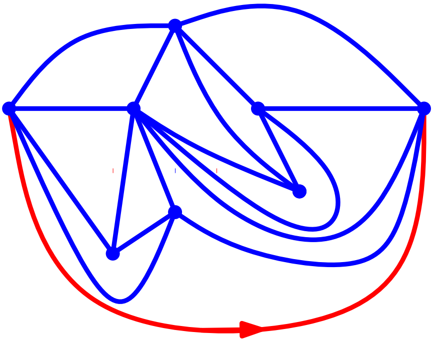

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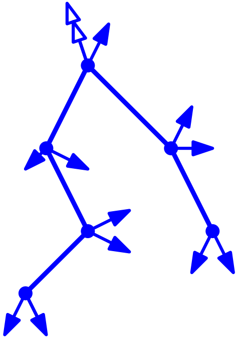

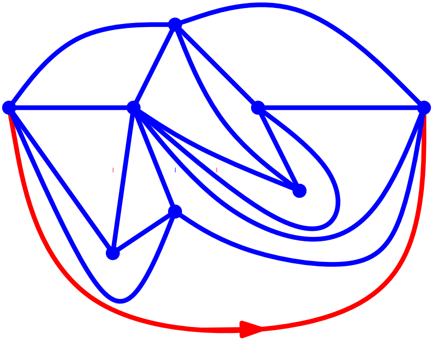

How to describe the inverse construction ? with orientations.

Orientations

Orientation of a planar map = an orientation is given to each edge

We want to consider orientations where the outdegree of each vertex is prescribed
$\rightarrow$ general theory of α-orientation (Felsner).

For triangulations:

$$\text{3-orientation} = \begin{cases} \text{out}(v) = 3 & \text{for each } v \text{ not in the root face} \\ \text{out}(v) = 0 & \text{otherwise.} \end{cases}$$

Theorem (Schnyder '89, Felsner '04)

Each rooted triangulation of girth 3 admits a unique minimal 3-orientation, ie. a 3-orientation without counterclockwise cycle.
Moreover there exists a directed path from any vertices to the root face : the orientation is accessible.

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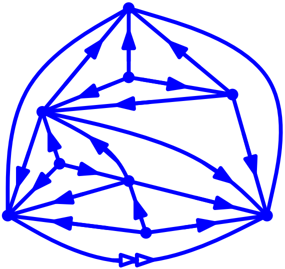

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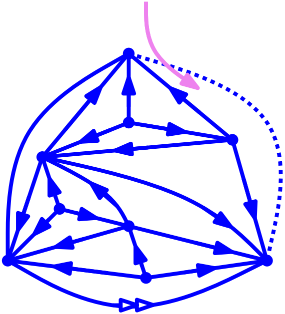

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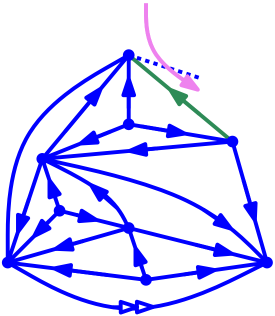

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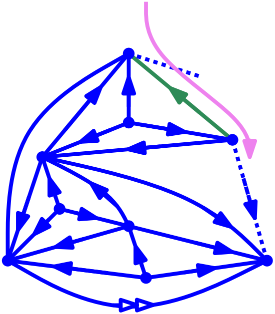

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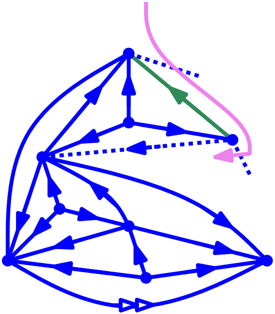

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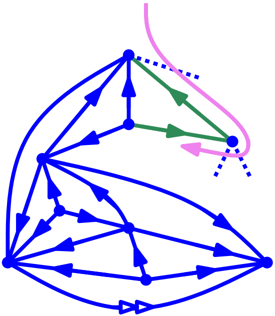

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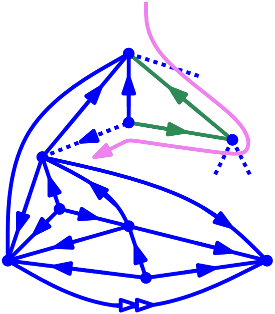

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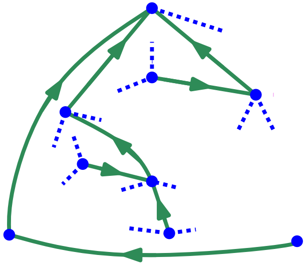

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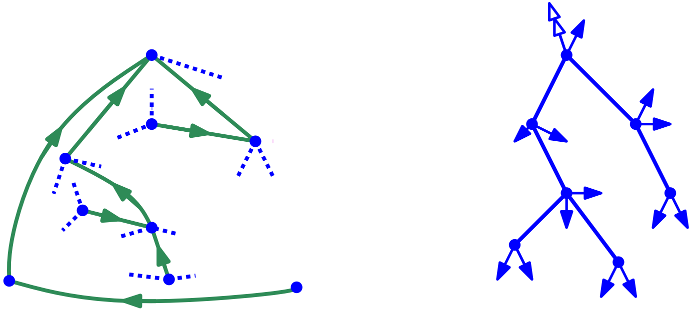

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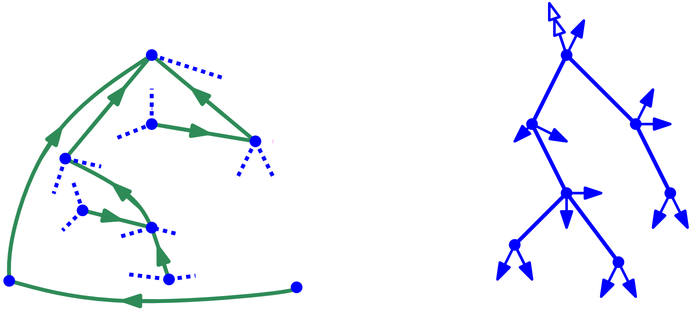

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And for d-angulations ?

k-fractional orientation = orientation of the expended map where each edge is replaced by k copies.

$$j/k\text{-orientation} = \begin{cases} \text{out}(v) = j & \text{for each } v \text{ not in the root face} \\ \text{out}(v) = k & \text{otherwise.} \end{cases}$$

Theorem (Bernardi and Fusy '11)

Any rooted d-angulation of girth d admits a unique minimal $\frac{d}{d-2}$-orientation such that the root face is a clockwise cycle.
Moreover this orientation is accessible.

d-fractional trees

d-fractional tree = rooted plane tree where each edge carries a flow (possibly in two directions) such that:

- sum of the flows in the edge $= d - 2$,
- for each node u, $\text{out}(u) = d$,
- for each leaf l, $\text{out}(l) = 0$,
- there exists a directed path from each node to the root.

$\rightarrow$ Trees not stable by rerooting, do not lead to nice combinatorial equalities.

$\Rightarrow$ Cyclic closure operation

d-fractional forest = simple rooted cycle of length d, on which are grafted d-fractional trees.

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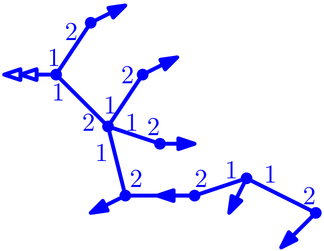

Theorem

There exists a one-to-one constructive correspondence between d-fractional forests with n nodes and rooted d-angulations of girth d with n vertices.

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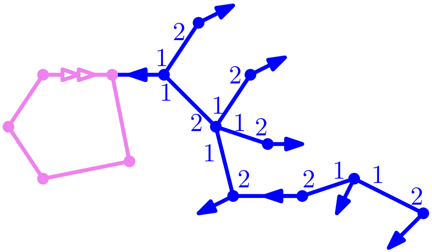

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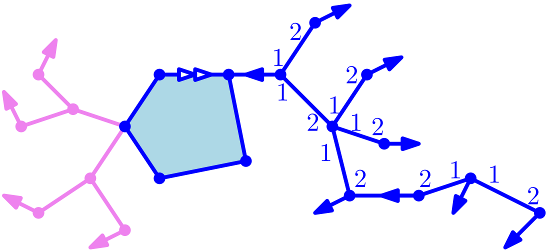

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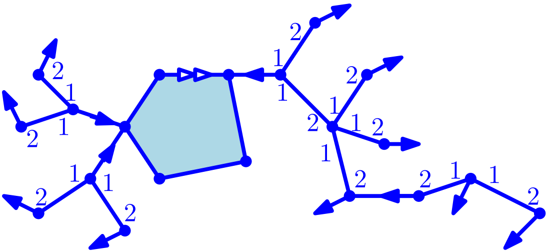

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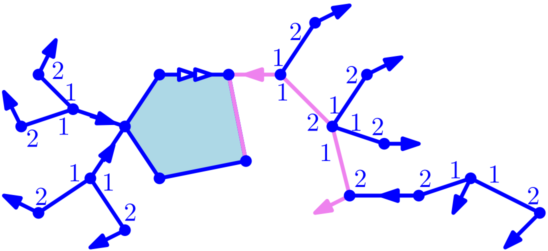

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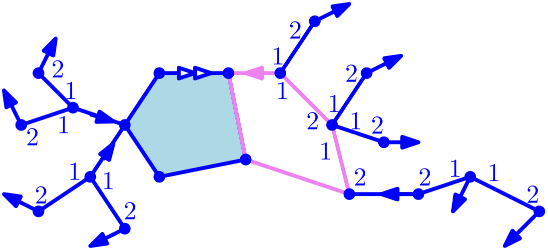

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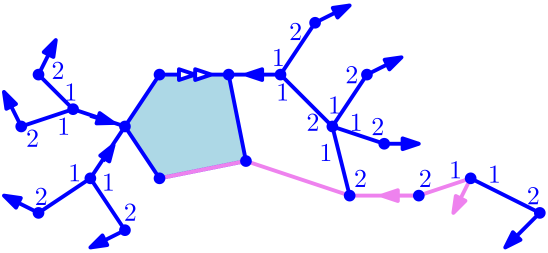

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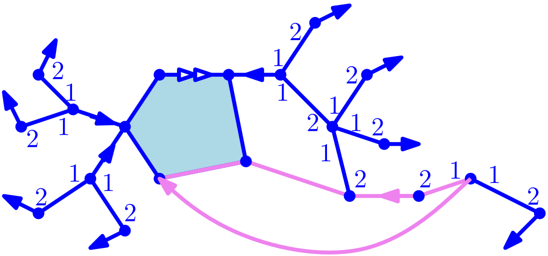

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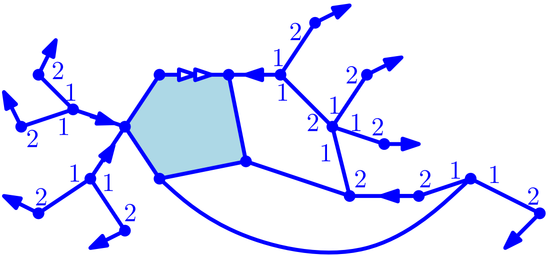

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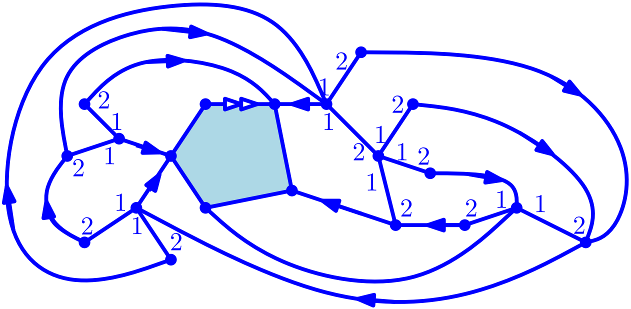

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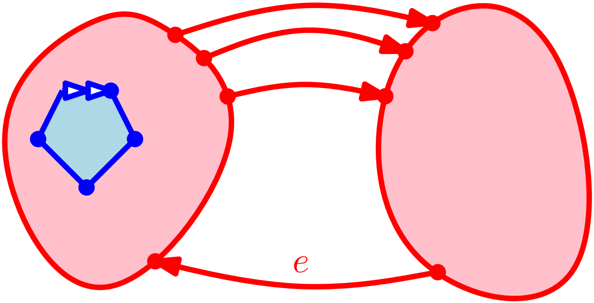
Proof of the theorem

- Induction on the number of faces of M.
- There exists a saturated clockwise edge e on the outer face:
  1. if $M \backslash e$ is still accessible: delete e.
  2. otherwise, there exists such a partition:



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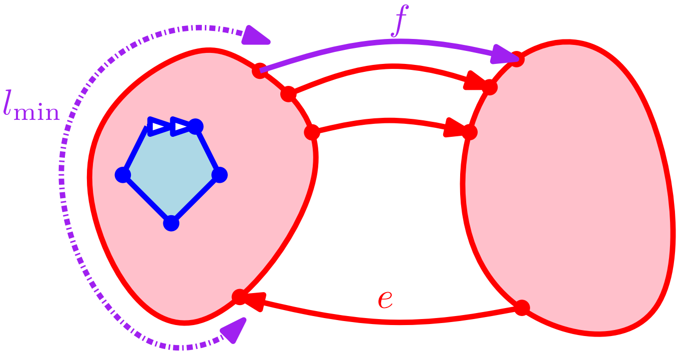

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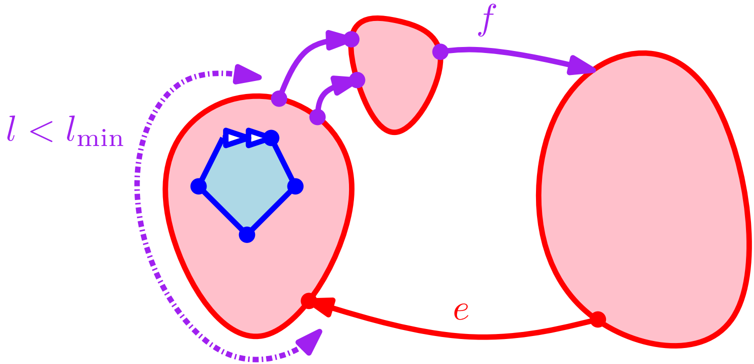

Generalization

"Theoretical proof" in quadratic time: relying on it, we can give a direct method to identify the closure edges.

$\Rightarrow$ Opening algorithm in linear time.

- Method generalizes directly to p-gonal d-angulations (ie. map with faces of degree d but root face of degree p).
- Enumerative consequences: recursive decomposition of the d-fractional trees $\Rightarrow$ Equations for the generating series of d-angulations.

General framework to obtain a bijection between maps endowed with a minimal accessible orientation and blossoming trees.

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That's all ... Thank you !