On symmetric quadrangulations

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Abstract

This note gathers observations on symmetric quadrangulations, with enumerative consequences. In the first part a new way of enumerating rooted simple quadrangulations is presented, based on two different quotient operations of symmetric simple quadrangulations. In the second part, based on results of Bouttier, Di Francesco and Guitter and on quotient and substitution operations, the series of three families of symmetric quadrangulations are computed, with control on the radius.

Keywords: planar maps, simple quadrangulations, orientations.

Introduction

A *planar map* is a connected graph embedded in the plane up to continuous deformation; the unique unbounded face of a planar map is called the *outer face*, the other ones are called *inner faces*. Vertices and edges are also called outer if they belong to the outer face and inner otherwise. A map is said to be *rooted* if an edge of the outer face is marked and oriented so as to have the outer face on its left. A quadrangulation is a map with all faces of degree 4. For $k \geq 1$, a *quadrangular dissection of a $2k$-gon* or *k-dissection* is a map whose outer face contour is a simple cycle of length $2k$, and with all inner faces of degree 4. A map is said to be *simple* if it has no multiple edges; a k-dissection is called *irreducible* if every 4-cycle delimits a face (possibly the outer one).

Enumeration of families of maps has received a lot of attention; several methods can be applied: the recursive method introduced by Tutte [12], the

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random matrix method introduced by Brézin et al [3], and the bijective method introduced by Cori and Vauquelin [6] and Schaeffer [11]. In the first part of this note, we show another method for the enumeration of rooted simple quadrangulations based on quotienting symmetric simple quadrangulations. Historically, the enumeration of symmetric maps of order k in a family $\mathcal{M}$ (*i.e.*, such that a rotation of order k fixes the map) was reduced to the enumeration of rooted maps in $\mathcal{M}$ via a quotient argument, a method used by Liskovets [9]. We proceed in the reverse way, namely we use two quotient operations of symmetric simple quadrangulations to build an algebraico-differential Equation (1) satisfied by the series of rooted simple quadrangulations which can be explicitly solved to obtain the formula for the number of rooted simple quadrangulations (due to Tutte [12] and bijectively proved by Schaeffer [11]).

One quotient operation is new and relies deeply on the existence and properties of α-orientations which we recall in Section 1.1. The other quotient operation is classical and we describe it now. For $k \geq 2$, a k-dissection D is called *k-symmetric* if the plane embedding (conveniently deformed) is invariant by a rotation of angle $2\pi/k$ centered at a vertex – called the *center* of the dissection. As observed by Liskovets [9], any two semi-infinite straight lines starting from the center and forming an angle of $2\pi/k$ delimit a sector of D. When keeping only this sector and pasting these two lines together, we obtain a planar map, called the *quotient-map* of D; note that in our case, this map has outer degree 2.

The results in the second part of the note are expressions of the series of several families of symmetric quadrangular dissections with control on their radius, defined as the distance from their central vertex to the outer face. Families of k-symmetric dissections have been counted according to the number of inner faces by Brown [4,5] using the recursive method (Liskovet's quotient method [9] can also be applied, reducing the enumeration to rooted quadrangular dissections). In Section 2, combining results by Bouttier *et al.* [1] with the quotient method and substitution operations, we count – for any $k \geq 2$ – k-symmetric general (resp. simple, irreducible) dissections according to the number of inner faces and the radius.

This is the first result on a distance parameter for irreducible quadrangulations; and it illustrates again the property that the series expression of a "well behaved" map family $\mathcal{M}$ refined by a distance parameter d is typically expressed in terms of the dth power of an algebraic series of singularity type $z^{1/4}$ (implying that asymptotically the distance parameter d on a random map of size n in $\mathcal{M}$ converges in the scale $n^{1/4}$ as a random variable).

1 Simple quadrangulations via symmetric ones

In this section, simple quadrangulation are required to have at least 2 faces (to avoid the degenerated case with two edges and one face) and we call symmetric simple quadrangulations the simple 2-symmetric dissections.

1.1 A new way of quotienting 2-symmetric quadrangulations

From [7], a quadrangulation is simple iff it admits an orientation of its inner edges so that inner vertices have outdegree 2 and outer ones have outdegree 0; such an orientation is called a *2-orientation*. By a general result of Felsner [8] on orientations with prescribed outdegrees, any simple quadrangulation Q admits a unique 2-orientation with no counterclockwise circuit, called the *minimal* 2*-orientation* of Q, see Figure 1(a). We now fix a 2-symmetric simple quadrangulation Q and describe a new way of quotienting Q relying on its minimal 2-orientation O, which is itself necessarily symmetric.

For each inner edge e of Q, call *leftmost path starting at* e the maximal oriented path P starting at e such that for any triple v, v', v'' of successive vertices along P, (v', v'') is the first outgoing edge after (v, v') in clockwise order around v'. It can be shown that it is a simple path ending at one outer vertex of Q.

Let u be the central vertex, e_1, e_2 its two outgoing edges, and $P_1 = (u = v_0, v_1, \ldots, v_p)$, $P_2 = (u = w_0, w_1, \ldots, w_p)$ the leftmost paths of e_1 and e_2 respectively. Clearly P_1 and P_2 map to one another by the rotation. Hence P_1 and P_2 cannot meet except at their starting point u. Let us cut Q along $P_1 \cup P_2$ to split Q into two isomorphic dissections, see Figure 1(c), and define Q_1 as the one with clockwise contour $u, v_1, v_2, \ldots$ If Q_1 is a quadrangulation, we set $\Phi(Q) := Q_1$ and mark the edge (u, v_1). Otherwise, for any $i \leq p-2$, we identify in Q_1 vertices v_{i+2} with w_i, and merge corresponding edges; this defines the map $\Phi(Q)$, in which we then mark the edge (v_1, v_2).

Concerning orientations, the identification of v_{i+2} with w_i creates an orientation conflict only when merging (u, v_1) with (v_1, v_2). We choose to orient the merged edge from v_1 to v_2. With this convention, $\Phi(Q)$ is naturally endowed with its minimal 2-orientation and the leftmost path of the marked edge (v_1, v_2) is $(v_1, v_2, \ldots, v_p)$. It is then easy to describe the inverse mapping and to obtain:

Theorem 1.1 *The mapping Φ is a one-to-one correspondence between symmetric simple quadrangulations with $2n$ inner faces and simple quadrangulations with n inner faces and a marked edge.*

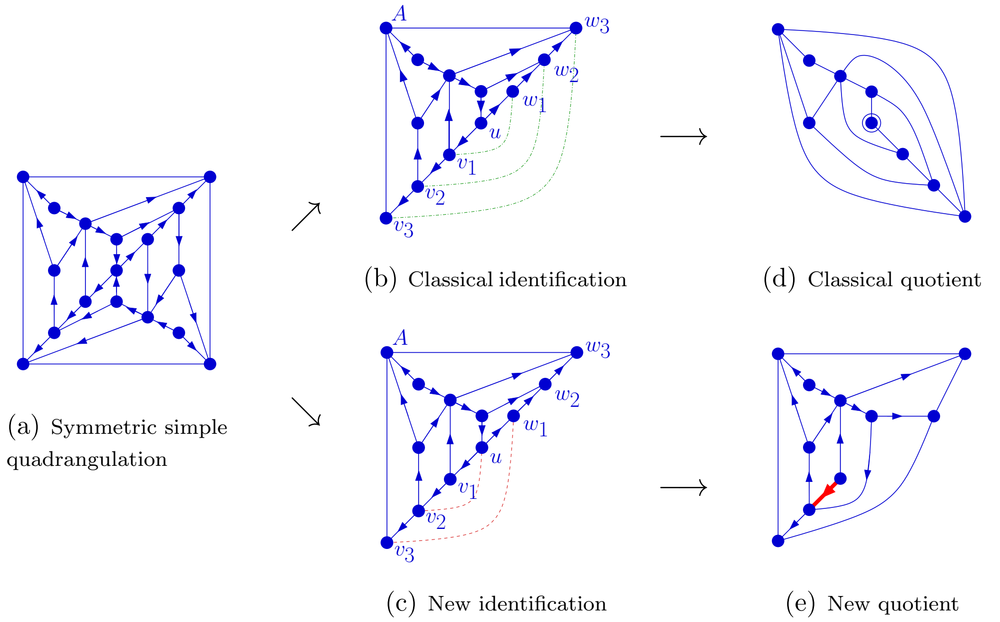


Figure 1. Classical quotient (b), (d) and the new quotient (c), (e) of a symmetric simple quadrangulation endowed with its minimal 2-orientation (a).

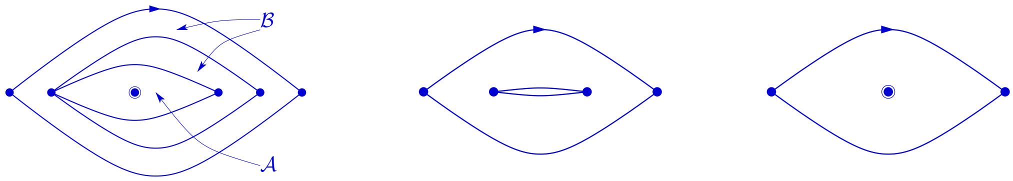
1.2 Classical quotient vs. new quotient, and getting a functional equation

A quadrangular dissection is said to be *pointed* if it has a marked vertex. As any k-symmetric dissection is implicitly pointed (at the center), its k-quotient is a pointed 1-dissection (*i.e.* with outer face of degree 2). A pointed dissection is called *quasi-simple* if the marked vertex lies strictly in the interior of any 2-cycle. Then the following can be shown (see Lemma 2.1 in Section 2 for a more general statement):

Proposition 1.2 *The 2-quotient is a one-to-one correspondence between symmetric simple quadrangulations with* $2n$ *inner faces and quasi-simple pointed 1-dissections with* n *inner faces.*

Any edge of a simple quadrangulation has an implicit orientation given by its minimal 2-orientation. Hence a simple quadrangulation with a marked edge corresponds to two distincts quadrangulations with a marked oriented edge. Similarly, because 1-dissections are bipartite, each pointed 1-dissection corresponds to two different rooted pointed 1-dissections. We obtain:

Corollary 1.3 *Simple quadrangulations with* n *inner faces and a marked oriented edge are in one-to-one correspondence with rooted quasi-simple pointed 1-dissections with* n *inner faces.*



(a) Nested 2-cycles (b) Element of $\mathcal{B}$ (c) Element of $\mathcal{A}$

Figure 2. The decomposition of a quasi-simple quadrangular 1-dissection

This correspondence allows us to get a functional equation for the generating series of the family $\mathcal{Q}$ of non-degenerated rooted simple quadrangulations. Let $q(x) = \sum_{n\geq 2} q_n x^n$ be the series of $\mathcal{Q}$ according to the number of faces (including the outer one). We will decompose both families involved in Corollary 1.3 in terms of $\mathcal{Q}$.

First any simple quadrangulation with a marked oriented edge can be seen as a rooted simple quadrangulation with a marked face (possibly the outer one), hence their generating series according to inner faces is equal to $q'(x)$.

Let now $\mathcal{D}$ denote the family of rooted quasi-simple 1-dissections. Observe that for any $D \in \mathcal{D}$, its separating 2-cycles are nested and therefore ordered from innermost to outermost. This yields a decomposition of D as a sequence of components. Let $\mathcal{A}$ and $\mathcal{B}$ be the families of rooted 1-dissections where the unique 2-cycle is the outer face with respectively a marked inner vertex and a marked inner edge. We can cut D along the nested 2-cycles to obtain a pointed 1-dissection in $\mathcal{A}$ and a sequence of maps in $\mathcal{B}$ (see Figure 2). Denoting respectively by $d(x)$, $a(x)$, $b(x)$ the series of $\mathcal{D}$, $\mathcal{A}$, and $\mathcal{B}$ according to the number of quadrangular faces. It gives $d(x) = a(x)(1-b(x))^{-1}$.

Deleting the non-root outer edge of a rooted 1-dissection gives a rooted quadrangulation (possibly degenerated). Taking into account the marked inner vertex or edge, we get:

$$\begin{cases} a(x) = 2x + \sum_{n\geq 2} n q_n x^n = 2x + xq'(x) \\ b(x) = 2x + \sum_{n\geq 2} (2n-1) q_n x^n = 2x + 2xq'(x) - q(x). \end{cases}$$

Proposition 1.4 *The series $q(x)$ satisfies the following equation:*

$$x(2 + 2q'(x)^2 + 3q'(x)) = q'(x) \cdot [1 + q(x)]. \tag{1}$$

This equation can be solved leading to a new proof that $q_{n+1} = \dfrac{4(3n)!}{n!(2n+2)!}$.

2 Distance from the central vertex to the boundary

We define the *radius* $r(P)$ of a pointed dissection P as the distance between its marked vertex and the outer face (for instance, the example of Figure 1(a) has radius 3). We also denote by $o(P)$ its outer degree, $\ell(P)$ the length of a shortest cycle not equal to the outer face boundary and strictly enclosing u and $m(P)$ (resp. $\widetilde{m}$ (P)) the length of a shortest (resp. non-facial) cycle not strictly enclosing u. The following lemma relates distances in k-symmetric dissections and their k-quotients:

Lemma 2.1 *For $k \geq 2$ let D be a k-symmetric dissection, and E the k-quotient of D, we have:*

$$o(D) = ko(E),\ \ell(D) = k\ell(E),\ m(D) = m(E),\ \widetilde{m}(D) = \widetilde{m}(E),\ r(D) = r(E).$$

For $k \geq 2$, $i > 0$, and $\mathcal{D}^{(k)}$ a family of k-symmetric dissections, let $\mathcal{D}_i^{(k)}$ be the family of dissections in $\mathcal{D}^{(k)}$ of radius i. Let $\mathbf{D}_i^{(k)}$ be the image of $\mathcal{D}_i^{(k)}$ by the k-quotient operation. By Lemma 2, the counting series $D_i^{(k)}(x)$ of $\mathbf{D}_i^{(k)}$ according to the number of inner faces is also the series of $\mathcal{D}_i^{(k)}$ according to the number of orbites of inner faces.

Previous work by Bouttier *et al.* [1] yields the following expression for the series of the family $\mathcal{F}_i^{(k)}$ of all k-symmetric 1-dissections, which is the series of the family $\mathbf{F}_i$ of pointed 1-dissections:

$$F_i^{(k)}(x) = X_i - X_{i-1} \text{ and } X_i = X_\infty \frac{(1-X^i)(1-X^{i+3})}{(1-X^{i+1})(1-X^{i+2})},$$

where $X + X^{-1} + 1 = (xX_\infty^2)^{-1}$ and $X_\infty = 1 + 3xX_\infty^2$. We want to extend this result to other families, namely those of simple and irreducible k-symmetric dissections, respectively denoted by $\mathcal{G}^{(k)}$ and $\mathcal{H}^{(k)}$.

To compute $G_i^{(k)}$ (which is the series of $\mathbf{G}_i$, the family of quasi-simple pointed 1-dissections with radius i), we proceed by substitution (an equivalent approach formulated on some labelled trees is discussed in [2]). We map each element of $\mathbf{F}_i$ to an element of $\mathbf{G}_i$, by collapsing each 2-cycle not strictly enclosing its pointed vertex into a single edge. Radius does not change because there is no way of shortening distances by travelling inside a 2-cycle not enclosing u. Conversely each element of $\mathbf{F}_i$ is uniquely obtained from an element of $\mathbf{G}_i$ – with n inner faces – by substituting some of its $2n+1$ edges by rooted 1-dissections. We obtain

$$F_i^{(k)}(x) = \sum_{n\geq 1}[y^n]G_i^{(k)}(y)\cdot x^n(1+f(x))^{2n+1} = (1+f(x))\cdot G_i^{(k)}(x(1+f(x))^2),$$

where $f(x)$ is the series of rooted quadrangulations according to the number of faces, which is well known to be algebraic and satisfy $1+f(x) = X_\infty - xX_\infty^3$. Define $y(x) = x(1+f(x))^2$ and

$$Y_\infty(y) := \frac{X_\infty(x)}{1+f(x)}, \quad Y(y) := X(x) \quad \text{and} \quad Y_i(y) := \frac{X_i(x)}{1+f(x)}.$$

Then it is easily checked that $Y_\infty = 1 + yY_\infty^3$ (see [10,2]), $xX_\infty^2 = yY_\infty^2$,

$$Y + \frac{1}{Y} + 1 = \frac{1}{yY_\infty^2} \quad , \quad Y_i = Y_\infty\frac{(1-Y^i)(1-Y^{i+3})}{(1-Y^{i+1})(1-Y^{i+2})},$$

and that $G_i^{(k)}(y) = Y_i - Y_{i-1}$, which does not depend on k.

We now use a substitution approach at faces (instead of edges) to get an expression for $H_i^{(k)}(z)$ for $k \geq 2$ and $i > 0$. This time cases $k = 2$ and $k > 2$ differ; we start $k > 2$ which is simpler. By Lemma 2.1, $\mathbf{H}_i^{(k)}$ is the family of pointed 1-dissections where the 2-cycles and non-facial 4-cycles strictly enclose the pointed vertex u; we call such pointed dissections *quasi-irreducible*. The *core* of $\gamma \in \mathbf{G}_i$ is obtained by emptying each maximal (for the enclosed area) 4-cycle of γ not strictly enclosing u; this yields a quasi-irreducible 1-dissection with radius i (because there is no way of shortening this distance by travelling inside a 4-cycle not enclosing u). Conversely each element of $\mathbf{G}_i$ is uniquely obtained from an element of $\mathbf{H}_i^{(k)}$ where at each face a rooted simple quadrangulation is patched. For $k \geq 3$ and $i > 0$, we obtain

$$G_i^{(k)}(y) = H_i^{(k)}(g(y)),$$

where $g(y)$ is the series of rooted quadrangulations according to the number of inner faces, which is algebraic and satisfies $g(y) = y\cdot(-Y_\infty^2 + 3Y_\infty - 2)$. Under the change of variables $z = g(y)$, we define $Z_\infty(z) := Y_\infty(y)$, $Z(z) := Y(y)$ and $Z_i(z) := Y_i(y)$. It is easily checked that $Z_\infty(z) = 1 + z + (Z_\infty - 1)^2$ (see [10]), $1/(yY_\infty^2) = 1 + 1/(Z_\infty - 1)$,

$$Z + \frac{1}{Z} = \frac{1}{Z_\infty - 1}, \quad Z_i = Z_\infty\frac{(1-Z^i)(1-Z^{i+3})}{(1-Z^{i+1})(1-Z^{i+2})} \quad \text{and} \quad H_i^{(k)}(z) = Z_i - Z_{i-1}.$$

For $i > 0$, by Lemma 2.1, $\mathbf{H}_i^{(2)}$ is the family of pointed 1-dissections where the unique 2-cycle is the outer boundary and where non-facial 4-cycles strictly enclose the pointed vertex u. These dissections are the cores of pointed 1-dissections with radius i and where the unique 2-cycle is the outer boundary. From Bouttier and Guitter [2], we get ($\delta_{i,j}$ is the Kronecker symbol):

$$H_i^{(2)}(z) = \delta_{i,1} - [u^{i-1}]\frac{1}{\sum_{i>0}(Z_{i+1}Z_i - Z_iZ_{i-1})u^{i-1}}.$$

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