Computing the smallest fixed point of monotone nonexpansive maps arising in static analysis and game theory

Assalé Adjé[1], Stéphane Gaubert[2] and Eric Goubault[3]

18th International Symposium on Mathematical Theory of Network Systems
Virginia Tech, Blacksburg

[1]CEA Saclay, Laboratoire d'Informatique de Polytechnique (LIX)
[2]INRIA Saclay, Centre de Mathématiques Appliquées de Polytechnique (C.M.A.P)
[3]CEA Saclay, Ecole Polytechnique

Static analysis by abstract interpretation see Cousot 77

A simple C program

```
int x,
x=0                //1
while(x<=99){      //2
      x=1+x;       //3
}                  //4
```

Pb: Find automatically the smallest overapproximation of the values of x at each breakpoint.

Static analysis by abstract interpretation see Cousot 77

A simple C program(on the left) and its abstract semantic equations (on the right)

```
int x,
x=0              //1
while(x<=99){    //2
      x=1+x;     //3
}                //4
```

$$\begin{aligned} x_1 &= \{0\} \\ x_2 &= (x_1 \cup x_3) \cap]-\infty, 99] \\ x_3 &= 1 + x_2 \\ x_4 &= (x_1 \cup x_3) \cap [100, +\infty[\end{aligned}$$

Find the smallest overapproximation of the values of x at each breakpoint.

$\longrightarrow$ Find the smallest vector x of intervals which satisfies the fixed point equation.

Static analysis by abstract interpretation see Cousot 77

A simple C program(on the left) and its abstract semantic equations (on the right)

```
int x,
x=0                 //1
while(x<=99){       //2
      x=1+x;        //3
}                   //4
```

$$\begin{aligned} x_1 &= \{0\} \\ x_2 &= (x_1 \cup x_3) \cap]-\infty, 99] \\ x_3 &= 1 + x_2 \\ x_4 &= (x_1 \cup x_3) \cap [100, +\infty[\end{aligned}$$

Pb: Find the smallest vector x of intervals which satisfies the fixed point equation.

By writing an interval $I = [-i^-, i^+]$ and after reductions:

Nonlinear fixed point equation given by line 2

$$\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0, (x_2^- - 1)) \\ \min(99, \max(0, (x_2^+ + 1)) \end{pmatrix}.$$

Static analysis by abstract interpretation see Cousot 77

A simple C program(on the left) and its abstract semantic equations (on the right)

```
int x,
x=0              //1
while(x<=99){    //2
      x=1+x;     //3
}                //4
```

$$\begin{aligned} x_1 &= \{0\} \\ x_2 &= (x_1 \cup x_3) \cap]-\infty, 99] \\ x_3 &= 1 + x_2 \\ x_4 &= (x_1 \cup x_3) \cap [100, +\infty[\end{aligned}$$

Pb: Find the smallest vector x of intervals which satisfies the fixed point equation.
By writing an interval $I = [-i^-, i^+]$ and after reductions:

Nonlinear fixed point equation given by line 2

$$\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0, (x_2^- - 1)) \\ \min(99, \max(0, (x_2^+ + 1)) \end{pmatrix}.$$

$\Longrightarrow (x_2^-, x_2^+) = (0, 99)$ (How to find it automatically?)

Game interpretation

Simple zero sum repeated game with stopping option:
One day game (date k): (Min) can decide to stop and pays 99 (the game ends) or to let (Max) play. Then (Max) can stop and wins 0 (the game ends) or continue to play and wins 1.

$$G = \begin{array}{|c|c|} \hline G+1 & 0 \\ \hline 99 & 99 \\ \hline \end{array} \qquad \left| \begin{array}{l} v^0 = 0 \\ v^k = \min(\max(v^{k-1}+1, 0), 99) \end{array} \right.$$

$v^\infty = 99$ (Value Iteration 99 steps!!!)

Game theory and abstract interpretation correspondences

Repeated games	Abstract interpretation
dynamical system	program
Dynamical Programming operator	functional
horizon n problem	n logical steps
limit of the value in horizon n	optimal invariant (bound)
value iteration	Kleene iteration

Method to compute the optimal invariant

- Classical Value Iteration (Kleene):
  - Theoretically guarantees the minimality of the fixed point found.
  - Slow (99 iterations for the previous simple case) and may not be convergent.
  - Often needs acceleration techniques: minimality no longer guaranteed.
- An alternative: Policy Iteration Algorithm: Howard (60) (stochastic control), extended by Hoffman and Karp (66) (stochastic games).
  Fast method (but complexity unknown).

  Extended by Costan, Gaubert, Goubault, Martel and Putot (CAV'05) to fixed point problems in static analysis.

Control Approach

$f : \mathbb{R}^d \mapsto \mathbb{R}^d$,

$$f_i = \inf_{a \in A(i)} \sup_{b \in B(i,a)} r_i^{a,b} + M_i^{a,b},$$

A **strategy** (policy) π is a map which associates to each state i an available action: $\pi(i) \in A(i)$.
Consider the one player dynamic operator:

$$f_i^{\pi} = \sup_{b \in B(i,\pi(i))} r_i^{\pi(i),b} + M_i^{\pi(i),b}.$$

The set $\{ f^{\pi} \mid \pi \text{ strategy } \}$ has the **lower selection property**:
$\forall\, x \in \mathbb{R}^d,\ \exists\, \pi$ s.t $f(x) = f^{\pi}(x)$.

Idea: To solve $f(x) = x$, we solve a sequence of one player problems i.e $f^{\pi}(x) = x$.

Policy Iteration Algorithm

$f : \mathcal{L} \mapsto \mathcal{L}$, $\mathcal{L}$ is a complete lattice E.g:
$\mathcal{L} = \bar{\mathbb{R}}^d = \{\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}\}^d$.
Assume $f = \inf_{\pi \in \Pi} f^\pi$ (f^π are minimum free).
The set $\{f^\pi \mid \pi \in \Pi\}$ has the lower selection property.

1. *Init*: Select a strategy π^0, $k = 0$.
2. *Value Determination* $(\mathbf{D_k})$: Compute a fixed point u^k of f^{π^k}.
3. Compute $f(u^k)$.
4. If $f(u^k) = u^k$, return u^k
5. *Policy Improvement* $(\mathbf{I_k})$: If $f(u^k) < u^k$ define π^{k+1} s.t $f(u^k) = f^{\pi^{k+1}}(u^k)$ and go to Step (D_{k+1}).

If we compute, at $(\mathbf{D_k})$ the smallest u^k and if $\{f^\pi \mid \pi \text{ strategy}\}$ is finite and f^π are order preserving, the algorithm stops.

Example 1

```
int x,
x=0              //1
while(x<=99){    //2
      x=1+x;     //3
}                //4
```

$$f\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0,(x_2^- - 1)) \\ \min(99,\max(0,(x_2^+ + 1))) \end{pmatrix}.$$

We start by:

$$f^{\pi^0}\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0,(x_2^- - 1)) \\ 99 \end{pmatrix}.$$

The smallest fixed point of this policy is $(0, 99)$ (Linear Programming...).

Example 1

```
int x,
x=0                 //1
while(x<=99){       //2
        x=1+x;      //3
}                   //4
```

$$f\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0,(x_2^- - 1)) \\ \min(99,\max(0,(x_2^+ + 1))) \end{pmatrix}.$$

We start by:

$$f^{\pi^0}\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0,(x_2^- - 1)) \\ 99 \end{pmatrix}.$$

The smallest fixed point of this policy is $(0, 99)$ (Linear Programming...).
This fixed point is also a fixed point for f so PI Algorithm stops. (0 iteration)

Example 2 ($x = 1+x \rightsquigarrow x = 1-x$)

```
int x,
x=0                  //1
while(x<=99){        //2
      x=1-x;         //3
}                    //4
```

The fixed point equation becomes:

$$f\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0,(x_2^+ - 1)) \\ \min(99,\max(0,(x_2^- + 1)) \end{pmatrix}.$$

We start by

$$f^{\pi^0}\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0,(x_2^+ - 1)) \\ 99 \end{pmatrix},$$

PI algorithm returns $(98, 99)$, i.e $x_2 = [-98, 99]$,

Example 2 ($x = 1+x \rightsquigarrow x = 1-x$)

```
int x,
x=0                //1
while(x<=99){      //2
        x=1-x;     //3
}                  //4
```

The fixed point equation becomes:

$$f\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0, (x_2^+ - 1)) \\ \min(99, \max(0, (x_2^- + 1)) \end{pmatrix}.$$

We start by

$$f^{\pi^0}\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0, (x_2^+ - 1)) \\ 99 \end{pmatrix},$$

PI algorithm returns $(98, 99)$, i.e $x_2 = [-98, 99]$, but $(0, 1)$ is the smallest fixed point i.e $x_2 = [0, 1]$.

How can we refine PI to find the smallest fixed point?

Optimization point of view

Smallest fixed point problem

↓

Minimization problem

↓

First order characterization

Maps are not C^1 neither convex $\dashrightarrow$ weaker differential notion: semidifferential.

Implementable characterization $\dashrightarrow$ nonlinear spectral radius.

Mathematical tools

Definition (Semidifferential see Rockafellar and Wets (98))

The semidifferential f'_u of f at u is the homogeneous continuous map s.t: $f(u+h) = f(u) + f'_u(h) + o(\|h\|)$.

E.g,
$f(x,y) = \min(1, x, \max(y,0))$, $f'_{(0,0)}(h_1,h_2) = \min(h_1, \max(h_2,0))$.

Definition (Spectral radius)

The spectral radius $\rho_{\mathbf{C}}(g)$ of a homogeneous continuous map g on a closed convex pointed cone $\mathbf{C}$ is the nonnegative number:

$$\rho_{\mathbf{C}}(g) = \sup\{\lambda \geq 0 \mid \exists\ x \in \mathbf{C}\backslash\{0\}, g(x) = \lambda x\}$$

A characterization of local minimality

u is a locally minimal fixed point if there exists a neighborhood which does not contain fixed point smaller than u.

Theorem (Characterization of a locally minimal fixed point)

Let $u \in \mathbf{Fix}(f)$. Consider the following statements:

1. *u is a locally minimal fixed point.*
2. $\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$.
3. $\rho_{\mathbb{R}^d_-}(f'_u) < 1$.

Then $3 \implies 2 \implies 1$.

Sketch of Proof: $2 \implies 1$, $y_n := \|h_n\|^{-1} h_n$,
$\|h_n\|^{-1}[f(u + \|h_n\| y_n) - f(u)] - f'_u(y_n) \to 0$
$\implies y_n - f'_u(y_n) \to 0 \implies y = f'_u(y)$ and $\|y\| = 1$ and y is negative.

Nonexpansiveness: local to global

f is (sup-norm) nonexpansive if: $\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty$, $x, y \in \mathbb{R}^d$.

Theorem (Local is Global)

If f is an order preserving nonexpansive map then u is locally minimal $\iff$ u is the smallest fixed point.

Lemma (Retraction of $\mathbb{R}^d$)

There exists a nonexpansive and order preserving map P s.t $P(\mathbb{R}^d) = \mathbf{Fix}(f)$ and $\mathbf{Fix}(P) = \mathbf{Fix}(f)$.

Proposition

If f is nonexpansive then $\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$ if and only if $\rho_{\mathbb{R}^d_-}(f'_u) < 1$.

Piecewise affine maps

see C.D. Aliprantis and R. Tourby (07)
$f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is piecewise affine if

$$f_i = \min_{a \in A_i} \max_{b \in B_a} g_{a,b}$$

where $g_{a,b}$ is affine and A_i and all B_a are finite.

Proposition (Semidifferential of piecewise affine maps)

A piecewise affine map f is semidifferentiable for all $u \in \mathbb{R}^d$ and:

1. *Let* $\bar{A}_j = \{a \in A_j \mid f_j(u) = \max_{b \in B_a} g_{a,b}(u)\}$ *and* $\bar{B}_a = \{\bar{b} \in B_a \mid g_{a,\bar{b}}(u) = \max_{b \in B_a} g_{a,b}(u)\}$, *then*

$$(f'_u)_j = \min_{a \in \bar{A}_j} \max_{b \in \bar{B}_a} \nabla g_{a,b}.$$

2. *For h small enough, $f(u+h) = f(u) + f'_u(h)$.*

...So, if f is piecewise affine then u is a locally minimal fixed point $\iff$ $\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$.

Smallest fixed point equivalent characterization

Theorem (Characterization of the smallest fixed point)

$f : \mathbb{R}^d \mapsto \mathbb{R}^d$ be an order preserving nonexpansive piecewise affine map. The following assertions are equivalent:

1. *u is the smallest fixed point of f.*
2. $\mathbf{Fix}_{|\mathbb{R}^d_-}(f'_u) = \{0\}$.
3. $\rho_{\mathbb{R}^d_-}(f'_u) < 1$.

Compute the smallest fixed point by Policy Iteration

$$f_i = \min_{a \in A(i)} \max_{b \in B_a} r_i^{a,b} + M_i^{a,b} \qquad (\Box)$$

where $A(i)$ and all B_a are finite and $M_i^{a,b}$ are substochastic vectors.

1. Compute a fixed point u_k of f by Policy Iteration.
2. *Policy Improvement* $(\mathbf{I_k^2})$:Compute $\alpha_k := \rho_{\mathbb{R}_-^d}(f'_{u^k})$.
   - If $\alpha_k < 1$, returns u^k.
   - If $\alpha_k = 1$, take $h \in \mathbb{R}_-^d \setminus \{0\}$ s.t $f'_{u^k}(h) = h$.
     Take $\pi^{k+1}(j)$ which attains the min in
     $(f'_{u^k})_j(h) = \min_{a \in \bar{A}_j} (f'_a)_{u^k}(h)$ where $\bar{A}_j = \{a \mid f_a(u^k) = f(u^k)\}$.
     Initialize a new Policy Iteration with $f^{\pi^{k+1}}$.

This algorithm stops.

Computational details

In policy Iteration

Proposition (Smallest fixed point of a policy)

The smallest (real) fixed point of f^π is the unique optimal solution of the linear program: $\min\{\sum_{1\leq i\leq d} x_i \mid x\in\mathbb{R}^d,\ f^\pi(x)\leq x\}$.

In refinement

$\rho_{\mathbb{R}^d_-}(f'_u) < 1$ can be checked by a Power Algorithm type... Using equivalent spectral radius definition... (see Nussbaum (86) and Mallet-Paret and Nussbaum (02)) this latter algorithm may not stop when $\rho_{\mathbb{R}^d_-}(f'_u) = 1$.

Computational details

To compute spectral radius...an extension of min-max function (Olsder (91), Gunawardena (94)):

A homogeneous min-max function of the variables $h_1, \ldots, h_d$ is a term in the grammar: $X \mapsto \min(X, X), \max(X, X), h_1, \cdots, h_d, 0$.
E.g, $f(h_1, h_2, h_3) = \min(\max(0, h_2), h_3)$.

Proposition (Spectral radius of homogeneous min-max maps)

Let g be a homogeneous min-max map on $\mathbb{R}^d$, $e \equiv -1$, then

1. $\rho_{\mathbb{R}^d_-}(g) \in \{0, 1\}$.
2. $\rho_{\mathbb{R}^d_-}(g) = 0 \iff \lim_{k \to +\infty} g^k(e) = 0$.
3. *This latter limit is reached in at most d steps.*

Come back to Example 2

$$f\begin{pmatrix} x_2^- \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \max(0,(x_2^+ - 1)) \\ \min(99,\max(0,(x_2^- + 1))) \end{pmatrix}.$$

We found $(98, 99)$, the semidifferential at $(98, 99)$:

$$f'_{(98,99)}(h_1, h_2) = (h_2, \min(0, h_1)).$$

$\rho_{\mathbb{R}^2_-}(f'_{(98,99)}) = 1$ and $f'_{(98,99)}(e) = e$.

This fixed point leads to a new policy:
$f^{\pi^1}(x_2^-, x_2^+) := (\max(0,(x_2^+ - 1)), \max(0,(x_2^- + 1)))$

(Policy Iteration) returns $(0, 1)$, $f'_{(0,1)}(h_1, h_2) = (\max(0, h_2), h_1)$.
$\rho_{\mathbb{R}^2_-}(f'_{(0,1)}) = 0$ so $(0, 1)$ is the smallest fixed point.

Example 3

A more complicated example in the paper

```
int x,int y,
x=[0,2];y=[10,15]    //1
while (x<=y) {       //2
  x=x+1;             //3
  while (5<=y) {     //4
    y=y-1;           //5
  }                  //6
}                    //7
```

Conclusion

Conclusion

- Computional method to compute the smallest fixed point of monotone piecewise affine nonexpansive maps .
- Prototype implementation in C.
- Fast method.

Future Work

- Big “jumps“ problem (the map is no longer nonexpansive).
- Improvement of calculus of the spectral radius and the negative fixed point.