Coupling policy iteration with semi-definite relaxation to compute accurate numerical invariants in static analysis

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Computing numerical invariant is an important problem.

A lot of related works (non exhaustive list) in static analysis of numerical programs:

Bagnara	SAS 20005
Cousot et al	Astrée
Feret	NSAD 2005
Feron	arxiv:012.1986 2008
Goubault/Putot	SAS 2006, CAV 2009, CAV 2010
Manna/Sankaranarayanan	VMCAI 2005
Miné	VMCAI 2005
Monniaux	CAV 2005

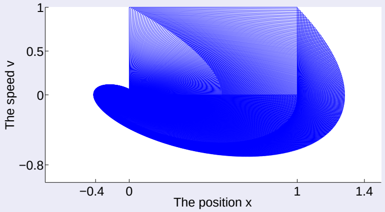
Technique used: Galois based abstract interpretation to compute overapproximation of invariants.

Running example

We consider an harmonic oscillator $\ddot{x} + c\dot{x} + x = 0$.

Figure: An harmonic oscillator, its Euler integration scheme and the loop invariant found at control point 2

```
x = [0,1];
v = [0,1];
h = 0.01;
c = 1;  [1]
while (true) {  [2]
  ov = v;
  v = v*(1-hc)-h*x;
  x = x+h*ov;  [3] }
```



Trajectories of the Euler scheme integration

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```

Convex polyhedra fail:

$x = \top, \; v = \top$

$h = 0.01, \; c = 1$

(e.g Apron-interproc)

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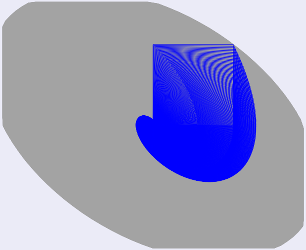
This talk:



$$\left\{ \begin{array}{c} -1.8708 \leq x \leq 1.8708 \\ -1.5275 \leq v \leq 1.5275 \\ 2x^2 + 3v^2 + 2xv \leq 7 \end{array} \right\}$$

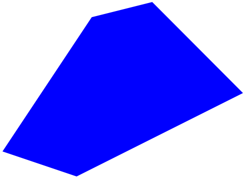
First ingredient:
Non linear templates

Linear templates

Domain of polyhedra (Cousot,Halbwachs 78):

Problem: number of extreme points and faces blows up!
An approach: Manna, Sankaranarayanan, and Sipma (VMCAI05).
Polyhedra with prescribed normals of facets so no exponential blow up.

The user provided linear templates: $p_1(x,y) = \frac{1}{2}x - y$, $p_2(x,y) = y - \frac{3}{2}x$, $p_3(x,y) = y - \frac{1}{4}x$, $p_4(x,y) = y + x$ and $p_5(x,y) = -y - \frac{1}{3}x$



$$\left\{ (x,y) \;\middle|\; \begin{array}{lcl} p_1(x,y) & \leq & 1 \\ p_2(x,y) & \leq & 5 \\ p_3(x,y) & \leq & \frac{5}{2} \\ p_4(x,y) & \leq & 2 \\ p_5(x,y) & \leq & 3 \end{array} \right\}$$

Figure: On the left, the geometric concretization of the abstract set on the right

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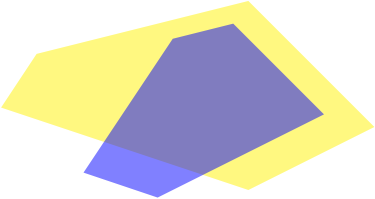

$$\left\{ (x,y) \;\middle|\; \begin{array}{rcr} p_1(x,y) & \leq & 2 \\ p_2(x,y) & \leq & 10 \\ p_3(x,y) & \leq & 3 \\ p_4(x,y) & \leq & 3 \\ p_5(x,y) & \leq & 2 \end{array} \right\}$$

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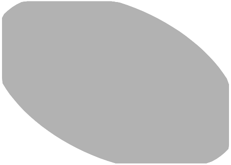
Special cases:
Linear templates are $\pm e_i$, we get intervals.
Linear templates are $\pm(e_i - e_j)$, we get zones.

Non linear templates

Taking $p_1(x,v) = x$, $p_2(x,v) = v$ and $p_3(x,v) = 2x^2 + 3v^2 + 2xv$.



$$\left\{ (x,y) \mid \begin{array}{rcr} p_1(x,y) & \leq & 1.8708 \\ -p_1(x,y) & \leq & 1.8708 \\ p_2(x,y) & \leq & 1.5275 \\ -p_2(x,y) & \leq & 1.5275 \\ p_3(x,y) & \leq & 7 \end{array} \right\}$$

Figure: On the left, the geometric concretization of the abstract set on the right

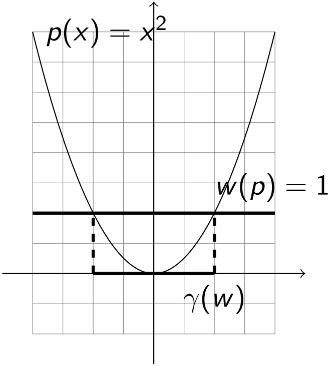
In all these sets, we fix a set of functions $P = \{p_1, p_2, \ldots, p_m\}$ and we give "bounds" on these functions.

Concretization map

P set of functions from $\mathbb{R}^d$ to $\mathbb{R}$ and $\mathcal{F}(P, \overline{\mathbb{R}})$ set of functions from P to $\overline{\mathbb{R}}$.

For $w \in \mathcal{F}(P, \overline{\mathbb{R}})$, we define the concretization map:

$$\boxed{\gamma(w) = \{x \in \mathbb{R}^d \mid p(x) \leq w(p),\ \forall\, p \in P\}}$$



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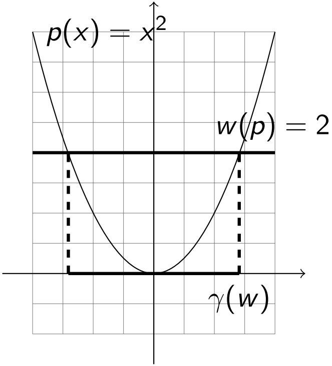

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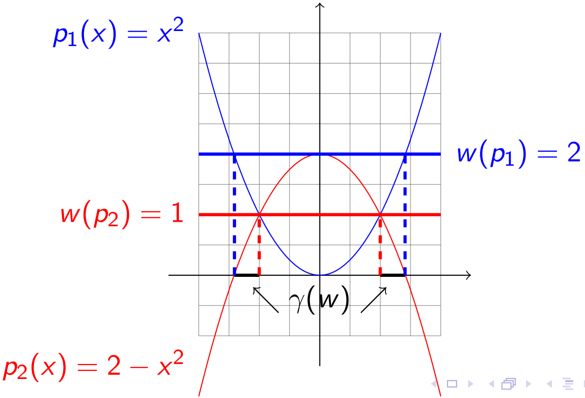

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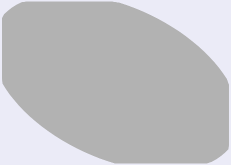
Example

$P = \{p_1, -p_1, p_2, -p_2, p_3\}$ with $p_1(x,v) = x$, $p_2(x,v) = v$ and $p_3(x,v) = 2x^2 + 3v^2 + 2xv$.
$w(p_1) = w(-p_1) = 1.8708$, $w(p_2) = w(-p_2) = 1.5275$ and $w(p_3) = 7$.

$\gamma(w) =$ 

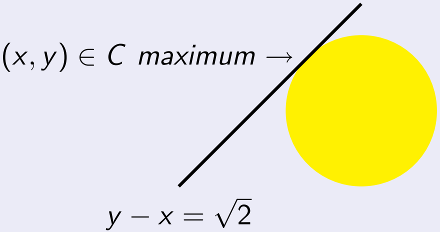
Abstraction map

For $C \subset \mathbb{R}^d$, we define the abstraction map:

$$\boxed{\begin{array}{rccl} \alpha(C): & P & \rightarrow & \overline{\mathbb{R}} \\ & p & \mapsto & \sup\{p(x) \mid x \in C\} \end{array}}$$

Example

$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ *and*
$P = \{p_1(x, y) = x, p_2(x, y) = y, p_3(x, y) = y - x\}$.



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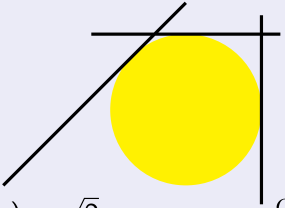

A Galois connection

We recall that,
for $w \in \mathcal{F}(P, \overline{\mathbb{R}})$,

$$\gamma(w) = \{x \in \mathbb{R}^d \mid p(x) \leq w(p),\ \forall\, p \in P\}$$

and for $C \subset \mathbb{R}^d$,

$$\begin{array}{rrcl} \alpha(C): & P & \rightarrow & \overline{\mathbb{R}} \\ & p & \mapsto & \sup\{p(x) \mid x \in C\} \end{array}$$

We get:

Proposition

(α, γ) *defines a Galois connection between* $\mathcal{F}(P, \overline{\mathbb{R}})$ *and* $\mathcal{P}(\mathbb{R}^d)$.

Second ingredient: Semidefinite positive relaxation

Quadratic templates

P is a finite set of functions:

Quadratic templates

$$p \mapsto x^T A_p x + x^T b_p + c_p$$

where A_p are $d \times d$ symmetric matrices, b_p are $\mathbb{R}^d$ vectors and c_p are scalars.

For instance:

$P = \{p_1, -p_1, p_2, -p_2, p_3\}$ with $p_1(x, v) = x$, $p_2(x, v) = v$ and $p_3(x, v) = 2x^2 + 3v^2 + 2xv$ is a set of quadratic templates.

Compute bounds in abstract interpretation

In abstract interpretation, we solve classically: $F^\sharp(w) = \alpha \circ F \circ \gamma(w) = w$. So,

$$(F^\sharp(w))(p) = \sup\{p(y) \mid y = F(x),\ x \in \gamma(w)\}$$

For a linear assignment $y := Mx$, assume that $q(x) \leq w(q),\ \forall q \in P$. In quadratic templates, we have:

$$\boxed{\begin{array}{rll} (F^\sharp(w))(p) = & \sup & x^T M^T A_p M x + x^T M^T b_p + c_p \\ & \text{s.t} & x^T A_q x + x^T b_q + c_q \leq w(q) \\ & & \forall q \in P \end{array}}$$

Can be solved in polynomial time when every templates p are both concave and convex i.e linear.

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In general this problem is NP-Hard [PV91]!

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$$\boxed{\begin{array}{rll} (F^\sharp(w))(p) = & \sup & x^T M^T A_p M x + x^T M^T b_p + c_p \\ & \text{s. t} & x^T A_q x + x^T b_q + c_q \leq w(q) \\ & & \forall q \in P \end{array}}$$

However we propose to compute an overapproximation in polynomial time (under technical hypothesis very often satisfied!) by the Shor relaxation scheme.

Concepts of Lagrange Duality (1)

$g, g_1, \ldots, g_n$ be functions from $\mathbb{R}^d$ to $\mathbb{R}$.
A constrained optimization problem:

$$\begin{array}{cc} \sup & g(x) \\ \mathrm{s.t} & g_1(x) \leq 0 \\ & \vdots \\ & g_n(x) \leq 0 \end{array}$$

We write: $L(\lambda, x) = g(x) - \sum_{i=1}^{n} \lambda_i g_i(x)$ and we remark that:

$$\inf_{\lambda \in \mathbb{R}^n_+} L(\lambda, x) = \begin{cases} g(x) & \text{if } \forall\, i \; g_i(x) \leq 0 \\ -\infty & \text{otherwise} \end{cases}$$

The entries of vectors $\lambda \in \mathbb{R}^n_+$ are called Lagrange multipliers.

Concepts of Lagrange Duality (2)

g concave, $g_1, \ldots, g_n$ convex and if there exists x s.t $g_i(x) < 0$ (Strong duality theorem):

$$\inf_{\lambda \in \mathbb{R}^n_+} \sup_{x \in \mathbb{R}^d} L(x, \lambda) = \sup\{g(x) \mid g_i(x) \leq 0 \ \forall i\}$$

In quadratic templates, to apply this theorem, the templates p should be linear!

However:

In general, we just have an overapproximation (Weak duality theorem):

$$\boxed{\inf_{\lambda \in \mathbb{R}^n_+} \sup_{x \in \mathbb{R}^d} \left(g(x) - \sum_{i=1}^{n} \lambda_i g_i(x) \right) \geq \sup\{g(x) \mid g_i(x) \leq 0 \ \forall i\}}$$

In our case: $F^{\mathcal{R}} = \inf F^{\lambda}$ with $F^{\lambda} = \sup_{x \in \mathbb{R}^d} \left(g(x) - \sum_{i=1}^{n} \lambda_i g_i(x) \right)$.

Shor relaxation

For quadratic templates: $(F^{\mathcal{R}}(w))(p)$ can be rewritten as [Sho87, TN01]:

Minimize	a linear function
s. t	- a linear combination of matrices is SDP - the coefficient of combination are Lagrange multipliers

($SDP =$ Semi-definite positive i.e all eigenvalues are nonnegative.)

This minimization problem can be solved in polynomial time [NN94].

Shor relaxation

For quadratic templates: $(F^{\mathcal{R}}(w))(p)$ can be rewritten as [Sho87, TN01]:

$$\begin{array}{ll} \text{Minimize} & \eta \\ \text{s.t} & N(p \circ M) + \eta R(-1) + \sum_{q \in P} \lambda(q)[R(w(q)) - N(q)] \text{ is SDP} \\ & \lambda(q) \geq 0,\ \forall\, q \in P \\ & \eta \in \mathbb{R} \end{array}$$

Where $N(p) = \begin{pmatrix} c_p & \frac{1}{2} b_p^T \\ \frac{1}{2} b_p & A_p \end{pmatrix}$, and for $y \in \mathbb{R}$, $R_{1,1}(y) = y$ and 0 otherwise.

This minimization problem can be solved in polynomial time [NN94].

Third ingredient: Policy iteration

Fixpoint computation

Our semantic, for:

- Linear assignment and intersection(tests): we compute an overapproximation of abstract semantic $F^\sharp$ using Shor relaxation.
- Union: interpreted as a sup (no relaxation).

Fixpoint computation

We write $\underline{x}(x,v) = x$, $\underline{v}(x,v) = v$, $\underline{E}(x,v) = 2x^2 + 3v^2 + 2xv$ and $P = \{\underline{x}, -\underline{x}, \underline{v}, -\underline{v}, \underline{E}\}$.
For harmonic oscillator, the abstract semantic $F^\sharp$, given by Galois connection:

$$\boxed{\begin{array}{rcl} F_1^\sharp(w)(p) & = & \{0 \leq \underline{x}(x,v) \leq 1,\, 0 \leq \underline{v}(x,v) \leq 1,\, \underline{E}(x,v) \leq 7\} \\ F_2^\sharp(w)(p) & = & \sup\{w_1(p), w_3(p)\} \\ F_3^\sharp(w)(p) & = & \sup\limits_{(x,v)\in\gamma(w_2)} p(M(x,v)) \end{array}}$$

with $\begin{pmatrix} 1 & h \\ -h & 1-hc \end{pmatrix} = \begin{pmatrix} 1 & 0.01 \\ -0.01 & 0.99 \end{pmatrix}$

Fixpoint computation

We write $\underline{x}(x,v)=x$, $\underline{v}(x,v)=v$, $\underline{E}(x,v)=2x^2+3v^2+2xv$ and $P=\{\underline{x},-\underline{x},\underline{v},-\underline{v},\underline{E}\}$.
For harmonic oscillator, the relaxed semantic $F^{\mathcal{R}}$:

$$
\begin{array}{rcl}
F_1^{\mathcal{R}}(w)(p) & = & \{0\leq \underline{x}(x,v)\leq 1,\, 0\leq \underline{v}(x,v)\leq 1,\, \underline{E}(x,v)\leq 7\} \\
F_2^{\mathcal{R}}(w)(p) & = & \sup\{w_1(p),w_3(p)\} \\
F_3^{\mathcal{R}}(w)(p) & = & \inf\limits_{\lambda\in\mathcal{F}(P,\mathbb{R}_+)} \sup\limits_{(x,v)} \sum\limits_{q\in P}\lambda(q)(w_2(q)-q(x))
\end{array}
$$

To compute the least fixpoint of $F^{\mathcal{R}}$, we use classically Kleene iteration.
With widening, Kleene iteration returns:
$\{-2.45\leq x\leq 2.45,\ -2\leq v\leq 2,\ 2x^2+3v^2+2xv\leq 10\}$.

Policy iteration

We remark that:

- $F_3^{\mathcal{R}}(w)(p) = \inf_{\lambda \in \mathcal{F}(P,\mathbb{R}_+)} \sup_{(x,v)} \sum_{q \in P} \lambda(q)(w_2(q) - q(x))$
- And for technical reasons, when we fix w^*, there exists a Lagrange multiplier λ^* s.t $F_3^{\mathcal{R}}(w) = \sup_{(x,v)} \sum_{q \in P} \lambda^*(q)(w_2^*(q) - q(x))$.

This two remarks allows us to use policy iteration [CGG+05, GGTZ07].

The policies are the set on which we minimize so, here, policies are the vectors of Lagrange multipliers.

Policy iteration

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The policies are the set on which we minimize so, here, policies are the vectors of Lagrange multipliers.

1. First, PI consists in fixing a minimum i.e we fix a policy λ.
   For instance, $p = \underline{E}$, we can choose
   $\lambda(\underline{x}) = \lambda(-\underline{x}) = \lambda(\underline{v}) = \lambda(-\underline{v}) = 0$ and $\lambda(\underline{E}) = 1$. We get
   $(F_3^{\lambda}(w))(p) = w_2(\underline{E}) + \sup_{(x,v)} p(M(x,v)) - \underline{E}(x,v)$.
2. Then, we compute the LFP, w^{λ} of F^{λ} by solving a linear program.
3. Finally, we test if w^{λ} is a fixpoint of $F^{\mathcal{R}}$, if not, we change the policy.

Policy iteration

Howard (1-player game 60) Hoffman and Karp (Stochastic Games 66).

Costan,Gaubert,Goubault,Martel,Putot (2005) for Static analysis.

For $f := \inf_{\pi\in\Pi} f^{\pi}$ over a lattice L.

Assume that $\forall\, x \in L,\ \exists\, \pi$ s.t $f(x) = f^{\pi}(x)$. e.g: the set Π is finite.

Policy iteration (PI):

1. Select an initial policy π^0, $k = 0$
2. Compute the least fixpoint x^k of f^{π^k}
3. If $f(x^k) = x^k$ STOP else select π^{k+1} s.t $f(x^k) = f^{\pi^{k+1}}(x^k)$ go to 2.

Here

- The lattice L is the set $\mathcal{F}(P, \overline{\mathbb{R}})$.
- The function f is $F^{\mathcal{R}}$.
- Our policies are Lagrange multipliers.

Step 1

For all p, we take $\lambda(\underline{x}) = \lambda(-\underline{x}) = \lambda(\underline{v}) = \lambda(-\underline{v}) = 0$ and $\lambda(\underline{E}) = 1$ as initial policy.

We have to compute the least fixpoint of the "policy":

$$\begin{array}{rcl}(F_1^\lambda(w))(p) & = & \{0 \leq \underline{x}(x,v) \leq 1,\, 0 \leq \underline{v}(x,v) \leq 1,\, \underline{E}(x,v) \leq 7\} \\ (F_2^\lambda(w))(p) & = & \sup\{w_1(p), w_3(p)\} \\ (F_3^\lambda(w))(p) & = & w_2(\underline{E}) + \sup_{(x,v)} p(Mx) - \underline{E}(x,v)\end{array}$$

Step 2

we compute it by solving the linear program:

$$\text{Minimize } \sum_{i=1}^{3} \sum_{p\in P} w_i(p)$$

$$\begin{array}{c}(F_1^\lambda(w)) \leq w_1 \\ (F_2^\lambda(w)) \leq w_2 \\ (F_3^\lambda(w)) \leq w_3\end{array}$$

For instance, after a cloture operation, at the third component:

$$\begin{array}{rl} w_3^0(\underline{x}) = & 2.0493 \\ w_3^0(-\underline{x}) = & 2.0493 \\ w_3^0(\underline{v}) = & 1.6733 \\ w_3^0(-\underline{v}) = & 1.6733 \\ w_3^0(\underline{E}) = & 7.0000 \end{array}$$

Step 3

We evaluate $F^{\mathcal{R}}(w)$ using Shor relaxation.
We find $F^{\mathcal{R}}(w) \neq w$ but
Shor relaxation provides us optimal Lagrange multipliers which we use as policies.

At each $p \in P$, the policy $\pi_3^1(p)$ is the vector of Lagrange multipliers: $(\lambda(\underline{x}), \lambda(-\underline{x}), \lambda(\underline{v}), \lambda(-\underline{v}), \lambda(\underline{E}))$.

For instance: $\pi_3^1(\underline{x}) = (0.9035, 0, 0, 0, 0.0134)$.

Go back to **Step 2**.

Finally, after 5 iterations, the invariant of the loop i.e. $\gamma(w_2)$ at control point 2 is:

$$\{-1.8708 \leq x \leq 1.8708,\ -1.5275 \leq v \leq 1.5275,\ 2x^2 + 3v^2 + 2xv \leq 7\}.$$

Programs	M	#P	#lin	#var	#lps	#Ite.	Qual	Tim
Rotation2	P	2	2	2	0	0	FP	0.72
Rotation2	K	2	2	2	0	1	FP	1.07
Rotation10	P	2	2	10	0	0	FP	1.17
Rotation10	K	2	2	10	0	1	FP	1.82
Filter	P	5	3	2	1	2	FP	9.35
Filter	K	5	3	2	1	2	FP	19.7
Oscillator	P	5	3	2	1	5	FP	12
Oscillator	K	5	3	2	1	15	FP	18.8
Symplectic	P	5	3	2	1	0	FP	3
Symplectic	K	5	3	2	1	15	FP	18.3
SymplecticSeu	P	5	5	2	1	30	PFP	125.3
SymplecticSeu	K	5	5	2	1	30	PFP	78.9
Arrow-Hurwicz	P	2	14	4	3	10	PFP	44.6
Arrow-Hurwicz	K	2	14	4	3	26	PFP	81.7

Quality of invariant is very good with PI.

Programs	M	#P	#lin	#var	#lps	#Ite.	Qual	Tim
Rotation2	P	2	2	2	0	0	FP	0.72
Rotation2	K	2	2	2	0	1	FP	1.07
Rotation10	P	2	2	10	0	0	FP	1.17
Rotation10	K	2	2	10	0	1	FP	1.82
Filter	P	5	3	2	1	2	FP	9.35
Filter	K	5	3	2	1	2	FP	19.7
Oscillator	P	5	3	2	1	5	FP	12
Oscillator	K	5	3	2	1	15	FP	18.8
Symplectic	P	5	3	2	1	0	FP	3
Symplectic	K	5	3	2	1	15	FP	18.3
SymplecticSeu	P	5	5	2	1	30	PFP	125.3
SymplecticSeu	K	5	5	2	1	30	PFP	78.9
Arrow-Hurwicz	P	2	14	4	3	10	PFP	44.6
Arrow-Hurwicz	K	2	14	4	3	26	PFP	81.7

Time is quite big because the prototype is Matlab code (interpreted code). But a small time is passed in solvers (0.2 seconds).

Programs	M	#P	#lin	#var	#lps	#Ite.	Qual	Tim
Rotation2	P	2	2	2	0	0	FP	0.72
Rotation2	K	2	2	2	0	1	FP	1.07
Rotation10	P	2	2	10	0	0	FP	1.17
Rotation10	K	2	2	10	0	1	FP	1.82
Filter	P	5	3	2	1	2	FP	9.35
Filter	K	5	3	2	1	2	FP	19.7
Oscillator	P	5	3	2	1	5	FP	12
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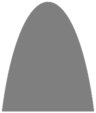
SymplecticSeu	P	5	5	2	1	30	PFP	125.3
SymplecticSeu	K	5	5	2	1	30	PFP	78.9
Arrow-Hurwicz	P	2	14	4	3	10	PFP	44.6
Arrow-Hurwicz	K	2	14	4	3	26	PFP	81.7

```
x = [0,1];
v = [0,1];
tau = 0.1 [1]
while [2] ((v>=1/2) [3]) { [4]
  x = (1-tau/2)*x+(tau-(tau^3)/4)*v;
  v = -tau*x+(1-tau/2)*v; [5]};
```



$$\left\{\begin{array}{c} -1.049 \leq x \leq 1.049 \\ -0.5 \leq v \leq 1.163 \\ x^2 + 0.9975 v^2 \leq 1.35 \end{array}\right\}$$

Figure: The symplectic implementation code with a threshold

Contributions and future works

Contributions

- New Domain for abstract interpretation.
- Generalization of linear templates Manna et al [SSM05, SCSM06]: To find tigher bounds on variables, we use nonlinear templates.
- We can compute an overapproximation of our abstract semantic in polynomial time.
- Policy iteration algorithm [CGG$^+$05, GGTZ07].

Future work

Consider programs with polynomial arithmetic and deal with sum-of-squares (SOS) relaxations.

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