About half permutations

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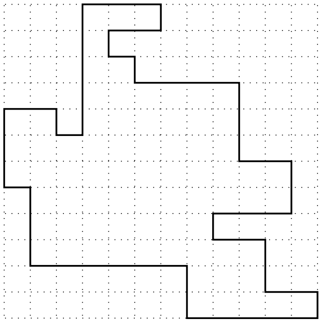
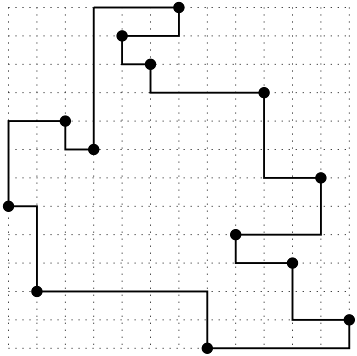
Basic definitions

A *permutomino* of size n is a polyomino (with no holes) having n rows and n columns, such that for each abscissa (ordinate) between 1 and $n+1$ there is exactly one vertical (horizontal) bond in the boundary of P with that coordinate.



Basic definitions

A permutomino P of size n is uniquely defined by a pair of permutations of length $n+1$, denoted by $\pi_1(P)$ and $\pi_2(P)$, called the *first* and the *second* components of P, respectively.



$$\pi_1 = (6, 3, 9, 8, 12, 11, 13, 1, 5, 10, 4, 7, 2)$$

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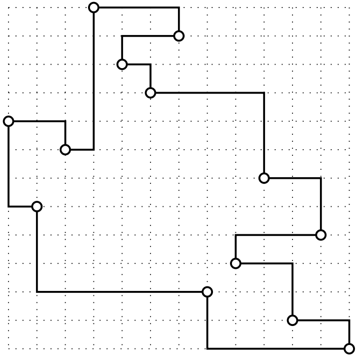


$\pi_2 = (9, 6, 8, 13, 11, 10, 12, 3, 4, 7, 2, 5, 1)$

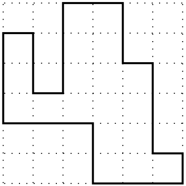
Directed column-convex permutominoes

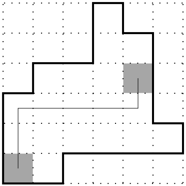
Definition

A permutomino P is said to be *column-convex* if all its columns are connected.

Definition

A permutomino P is said to be *directed column-convex* if it is a column-convex permutomino and all its cells can be reached from a distinguished cell – called *source* – by means of a path, internal to the permutomino, and using only north and east unit steps.





Directed column-convex permutominoes

Proposition (Beaton, Disanto, Guttman, Rinaldi, 2010)

The number of directed column-convex permutominoes of size n is $\frac{(n+1)!}{2}$.

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Remark

The authors prove this result analytically.

Enumeration of directed column-convex permutominoes

We present a bijective proof that the number of directed column-convex permutominoes of size n is $\frac{(n+1)!}{2}$.

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We prove that:

- every directed column-convex permutomino P is uniquely determined by its second component $\pi_2(P)$;
- the set

$$\{\pi_2(P) : P \text{ is a directed column-convex permutomino of size } n\}$$

is in bijective correspondence with its complement in S_{n+1}, where S_{n+1} denotes the set of permutations of length $n+1$.

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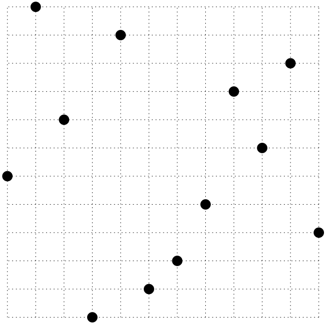
Proof.

Let $\pi = \pi_2(P)$ for some directed column-convex P.



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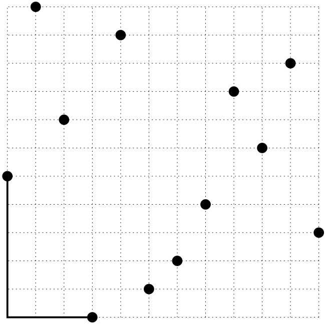
Proof.

Let $\pi = \pi_2(P)$ for some directed column-convex P.

- $\pi(1)$ is connected with $\pi(i) = 1$ (directed);



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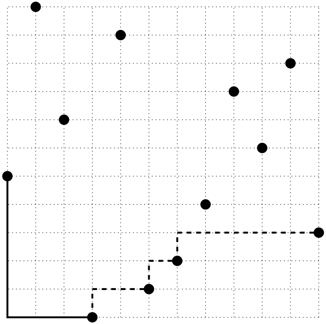
- the right-to-left minima of π have to be connected in sequence (directed);



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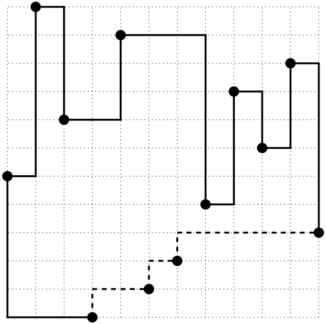
- $\pi(1)$ is connected with $\pi(i) = 1$ (directed);
- the right-to-left minima of π have to be connected in sequence (directed);
- the remaining entries of π have to be connected in sequence (column-convex).



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Enumeration of directed column-convex permutominoes

Definition

We define

$$\mathcal{P}''_n = \{\pi : \pi = \pi_2(P) \text{ for some } P \in \mathcal{D}_{n-1}\}.$$

The permutations of $\mathcal{P}''_n$ will be called *dcc-permutations*.

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We provide

- a characterization of dcc-permutations of size n;
- a bijective correspondence between dcc-permutations of length n and non dcc-permutations of length n.

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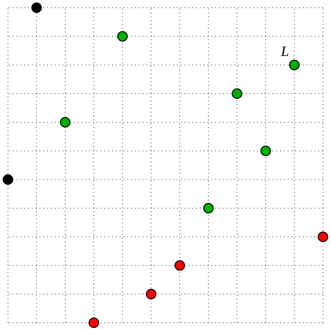
And so we prove in a bijective way that

$$|\mathcal{D}_{n-1}| = \frac{n!}{2}.$$

Characterization of dcc-permutations

Definition

- $\mathcal{R}(\pi)$: right-to-left minima of π;
- $\overline{\mathcal{R}}(\pi)$: $(\pi(j-1), \pi(j), \ldots, \pi(n))$ of π minus the points of $\mathcal{R}(\pi)$, where $\pi \in S_n$ $(n > 1)$ with $\pi(1) \neq 1$ and $\pi(j) = 1$;
- $L(\pi)$: the rightmost element of $\overline{\mathcal{R}}(\pi)$.



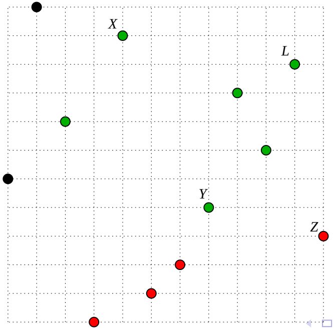
Characterization of dcc-permutations

Definition

Let $\pi \in S_n$ such that $\pi(1) \neq 1$, for each $X \in \overline{\mathcal{R}} - \{L\}$,

- Y: the leftmost point of $\overline{\mathcal{R}}$ on the right of X;
- Z: the leftmost point of $\mathcal{R}$ on the right of Y.

We set $C_X = (X, Y, Z)$.



Characterization of dcc-permutations

Theorem

A permutation $\pi \in S_n$ is a dcc-permutation if and only if the following properties hold:

i) $\pi(1) \neq 1$;

ii) $\forall\, X \in \overline{\mathcal{R}}(\pi) - \{L\},\ C_X = (X, Y, Z)$, *we have* $X > Z$;

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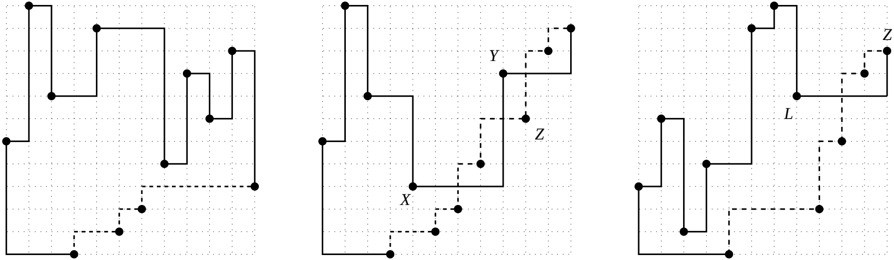

A bijection for dcc-permutations

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The number of dcc-permutations of length n is $\frac{n!}{2}$.

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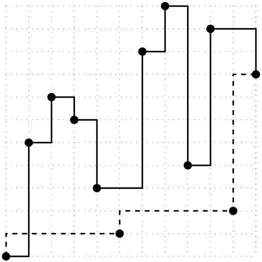
Proof.

We determine a bijective correspondence $\phi : S_n \setminus \mathcal{P}''_n \to \mathcal{P}''_n$.

case 1) $\pi(1) = 1$



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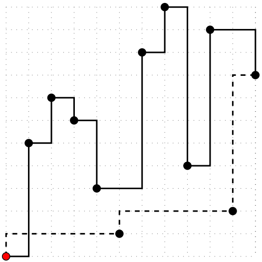

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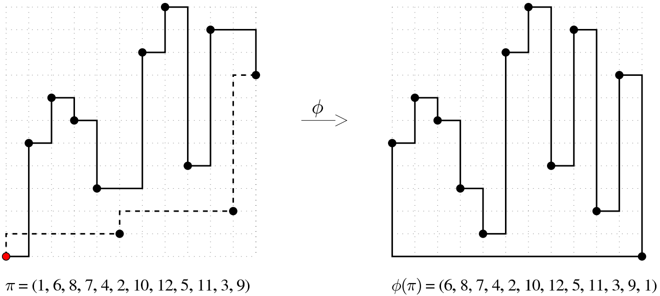

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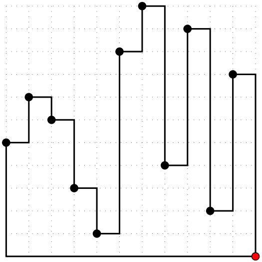

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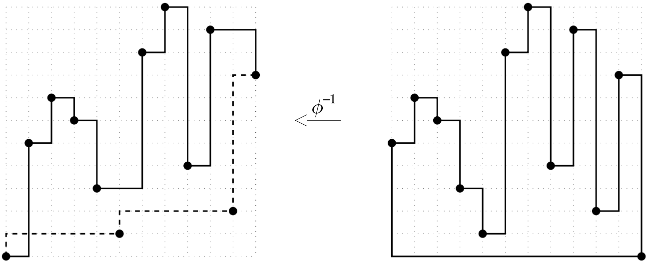

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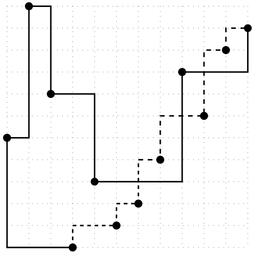

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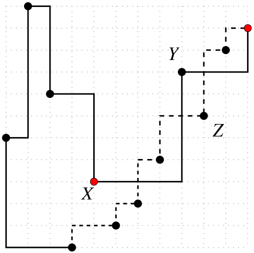

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Let X be the leftmost of the elements which do not satisfy ii).

We exchange X with $\pi(n)$.

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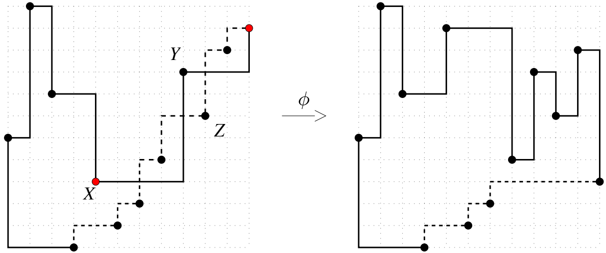

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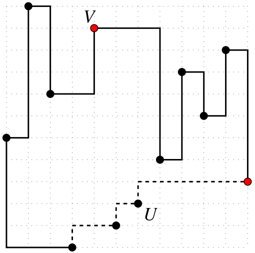

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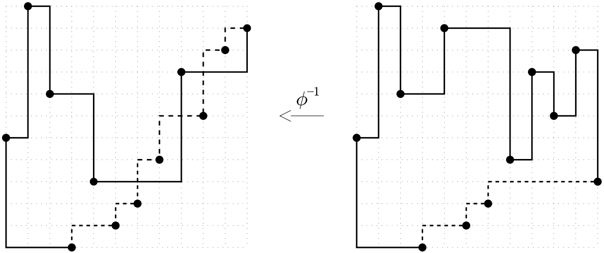

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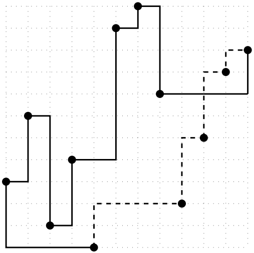

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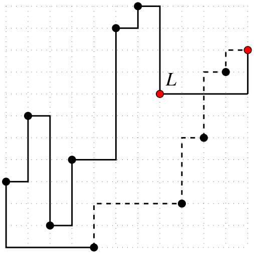

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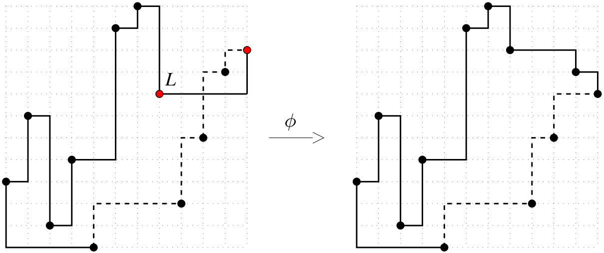

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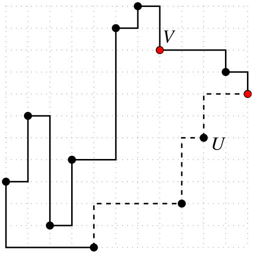

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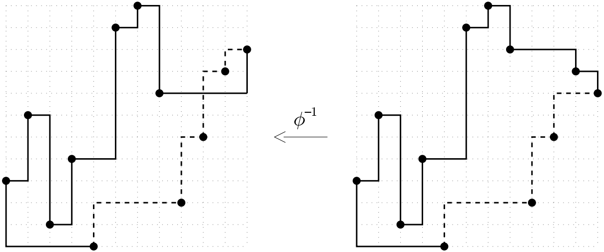

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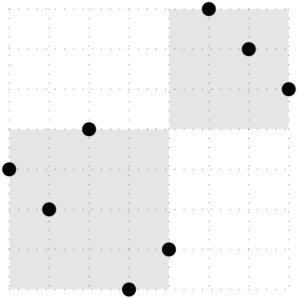
Combinatorial characterizations of dcc-permutations

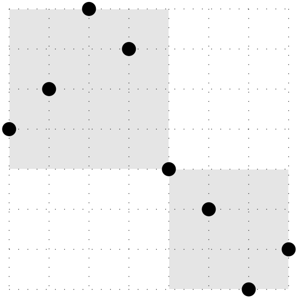
π *decomposable*: there is an index $i < n$ s.t. $(\pi(1), \ldots, \pi(i))$ is a permutation.

π *m-decomposable*: if its mirror image π^M is decomposable.



$\pi = (4, 3, 5, 1, 2, 8, 7, 6)$

decomposable



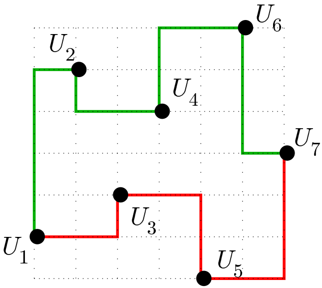
$\pi = (5, 6, 8, 7, 4, 3, 1, 2)$

m - decomposable

Combinatorial characterizations of dcc-permutations

Let P be a *column-convex* permutomino of size n, let π_1 be the first component of P, and let $U_i = (i, \pi_1(i))$, $1 \leq i \leq n+1$, be the points of the graphical representation of π_1.

We call *upper* (resp. *lower*) path of P the part of the boundary of P running from U_1 to U_{n+1} and starting with a north step (resp. east step).



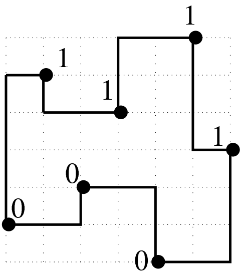
Combinatorial characterizations of dcc-permutations

We define a *valuation* v on the points of a permutation $\pi = \pi_1(P)$ for some column-convex permutomino P of size n in this way:

- $v(U_i) = 1$ iff U_i belongs to the upper path or $i = n+1$;
- $v(U_i) = 0$ iff U_i belongs to the lower path or $i = 1$;

Remark

A column-convex permutomino P of size n is uniquely determined by $\pi_1(P)$, and by the array $v(\pi_1) = (v(U_1), \dots, v(U_{n+1}))$.



$\pi_1 = (2, 6, 3, 5, 1, 7, 4)$

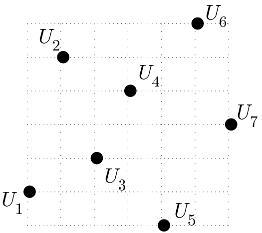
$v(\pi_1) = (0, 1, 0, 1, 0, 1, 1)$

Combinatorial characterizations of dcc-permutations

Definition

The pair (U_i, U_j) forms an *inversion* if and only $i < j$ and $\pi(i) > \pi(j)$.

The array $[U_i, U_j] = (U_i, U_{i+1}, \ldots, U_j)$ is a *locally decomposable* (*m-decomposable*) permutation if the normalization of $(\pi(i), \pi(i+1), \ldots, \pi(j))$ is a decomposable (m-decomposable) permutation.



Combinatorial characterizations of dcc-permutations

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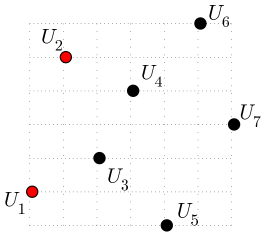

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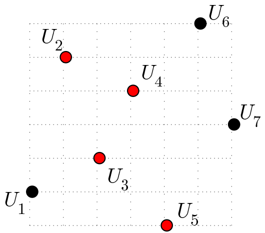

[U1, U2] locally decomposable permutation

[U2, U5] locally m-decomposable permutation

Combinatorial characterizations of dcc-permutations

Given $\pi \in S_n$ we define a set of logic implication formulas $\mathcal{F}(\pi)$ on the variables $\mathcal{U} = \{U_1, \ldots, U_n\}$ in this way:

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For any pair $U_i, U_j \in \mathcal{U}$ we have that $U_j \to U_i \in \mathcal{F}(\pi)$ if and only if

- (U_i, U_j) is an inversion;
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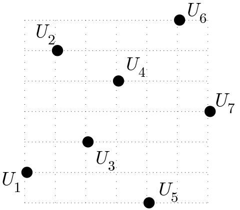

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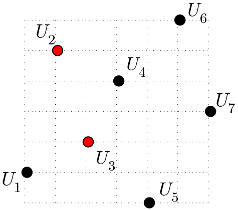

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Remark

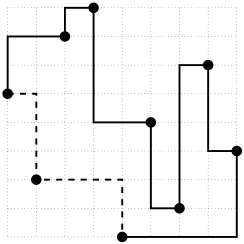
Each valuation v that satisfies $\mathcal{F}(\pi)$ corresponds to a column-convex permutomino P of size $n-1$ such that $\pi = \pi_1(P)$.

Combinatorial characterizations of dcc-permutations

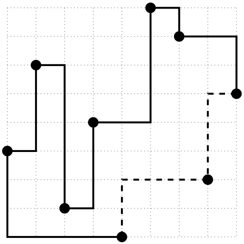
Remark

Given a permutomino P, the first component of P is just the mirror image of the second component of the polyomino P^M obtained by reflecting P with respect to the y-axis. Namely,

$$\pi_1(P) = (\pi_2(P^M))^M.$$



$\pi_1(P) = (6, 3, 8, 9, 1, 5, 2, 7, 4)$



$\pi_2(P^M) = (4, 7, 2, 5, 1, 9, 8, 3, 6)$

Combinatorial characterizations of dcc-permutations

The valuation $\hat{v}$ of π is defined as follows:

$\hat{v}(U_i) = 0$ if and only if U_i is a left-to-right minimum.

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Proposition

A permutation π is a dcc-permutation if and only if the valuation $\hat{v}$ satisfies $\mathcal{F}(\pi^M)$.

Combinatorial characterizations of dcc-permutations

Theorem

A permutation π of length n is a dcc-permutation if and only if:

- $\pi(1) \neq 1$,
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Corollary

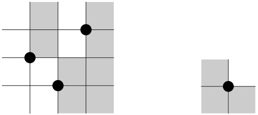
A permutation π of length n is a dcc-permutation if and only if $\pi(1) \neq 1$ and there is no point U_i of π such that $[U_i, U_n]$ is a locally decomposable permutation and U_i is not a right-to-left minimum.

Combinatorial characterizations of dcc-permutations

The previous result can be used to provide a characterization of the class of dcc-permutations in terms of *mesh patterns*.

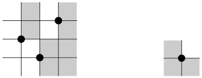
Theorem

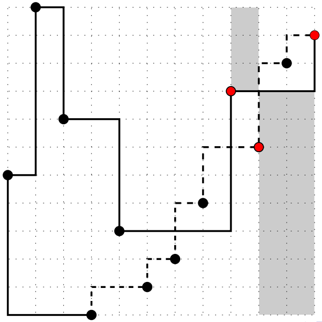
A permutation π is a dcc-permutation if and only if π avoids the mesh patterns represented below



Theorem

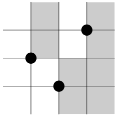
A permutation $\pi \in S_n$ is a dcc-permutation if and only if π avoids the mesh patterns represented below





The class $\mathcal{B}_n$ and its enumeration

Let $\mathcal{B}_n$ be the class of permutations avoiding the mesh pattern



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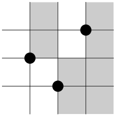


Proposition

We have that:

$$|\mathcal{B}_n| = 1 + \sum_{i=2}^{n} \frac{i!}{2}.$$

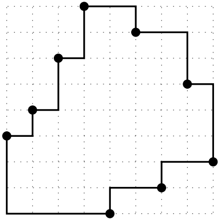
Further works

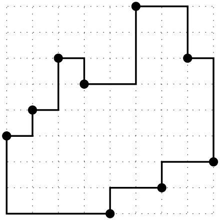
Enumeration of directed column-convex permutominoes according to the semi-perimeter

Let P be directed column-convex permutomino of size n.

$$\deg(P) = sp(P) - 2n.$$

$\mathcal{D}_{n,k}$: directed column-convex permutominoes of size n and degree k.





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Thank you!!