Bijective maps based on mesh patterns

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Permutations

Definition

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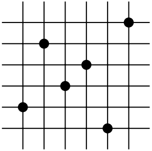

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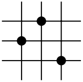
Patterns

Classical patterns are permutations

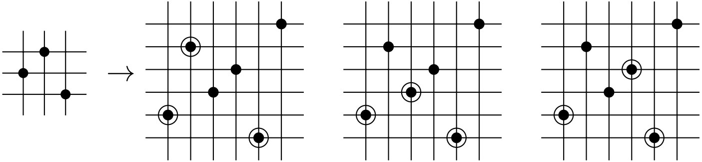
$231 =$ 

Definition

A permutation contains a pattern if we can find an occurrence of the pattern in the permutation. If a permutation does not contain a pattern it is said to avoid it.

Example

For example the permutation 253416 has three occurrences of the classical pattern 231.



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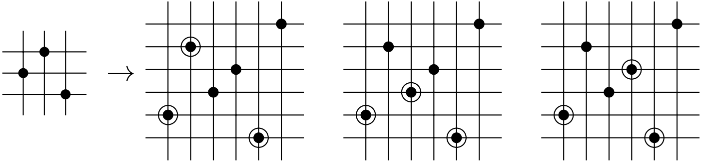

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Mesh patterns

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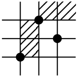
A mesh pattern is a pair (p, R), where p is a classical pattern and R is a set of coordinates of shaded squares.

Here we have the mesh pattern $(132, \{(1,1),(1,2),(2,3),(3,3)\})$



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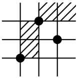

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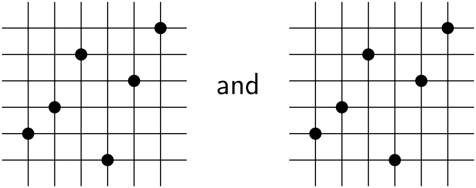

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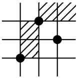

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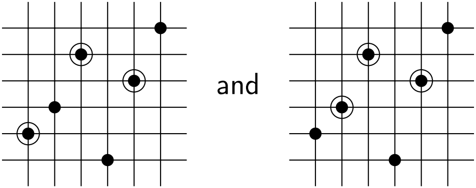

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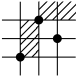

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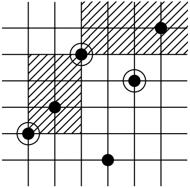

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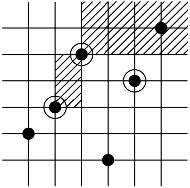

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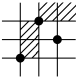

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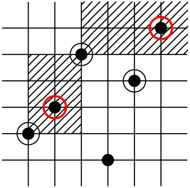

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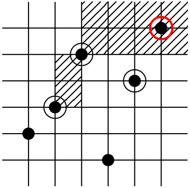


Wilf-equivalence

Definition

Two patterns p and q are Wilf-equivalent if equally many permutations of length n avoid p and q, for all n.

Definition

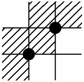
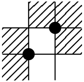
Wilf-classification is the sorting of patterns into classes by Wilf-equivalence.

Motivation

- In 2011 Hilmarsson, Jónsdóttir, Sigurðardóttir, Ulfarsson and Viðarsdóttir begun the Wilf-classification of mesh patterns of length 2
- These patterns have been divided into 65 pattern classes using automatic methods (basic symmetries, up-shift, toric-shift and the shading lemma)
- The number of pattern classes was decreased to 56
- We believe that further 18 pattern classes can be merged to 8
- In this talk I will show how two pattern classes can be merged to one

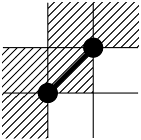
The two pattern classes

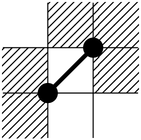
We have

$q_1 =$  $q_2 =$ 

These two patterns represent two different pattern classes. For small n the number of permutations of length n avoiding these two patterns respectively is the same.

To construct the map we have to be able to isolate a unique occurrence of the pattern in the permutation.
We are given a permutation containing at least one occurrence of q_1 to the left and q_2 to the right.





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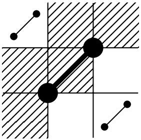

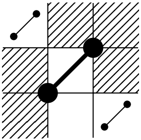

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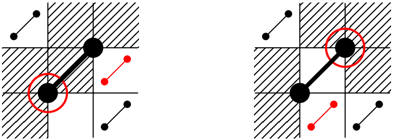


This figure shows how other occurrences of the patterns can lie inside the permutation in relation to each other.

The bijection

- The bijection $\varphi : \mathfrak{S}_n \longrightarrow \mathfrak{S}_n$ is inductively defined
- The bijection restricts to a bijection from permutations containing the pattern q_1 to permutations containing the pattern q_2 and from permutations avoiding the pattern q_1 to permutations avoiding the pattern q_2.
- For permutations of lengths 0 and 1, that is for the permutations ϵ and 1, we let

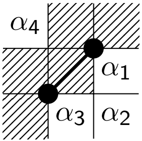
$$\varphi(\epsilon) = \epsilon \quad \text{and} \quad \varphi(1) = 1.$$

- Take a permutation $\pi \in \mathrm{Co}_n(q_1)$ and construct a permutation $\pi' \in \mathrm{Co}_n(q_2)$ and define $\varphi(\pi) = \pi'$.

- Find the left- and topmost occurrence of the pattern q_1 in π.

- Split up into subwords

$$\pi = \alpha_4 \; a \; \alpha_3 \; b \; (\alpha_1 \cup \alpha_2).$$



Since we chose the left -and topmost occurrence of the pattern q_1 α_4 must be an avoider of the pattern.

Definition

A flattening of a word $w = w_1 w_2 \cdots w_n$ of distinct integers, denoted $\mathrm{fl}(w)$, is the permutation obtained by replacing the ith smallest letter in w with i.

Example

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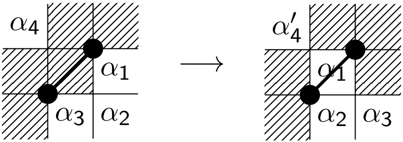
We construct π' as follows

$$\pi = \alpha_4 \ a \ \alpha_3 \ b \ (\alpha_1 \cup \alpha_2) \longrightarrow \pi' = \varphi(\mathrm{fl}(\alpha_4))^{\uparrow b} a(\alpha_1 \cup \alpha_2) b \alpha_3$$



- The letters a and b hold their values
- The words in α_1, α_2 and α_3 are moved as shown on the figure
- The word $\alpha'_4 = \varphi(\mathrm{fl}(\alpha_4))$ is an avoider of the pattern q_2 by induction hypothesis since $\mathrm{fl}(\alpha_4)$ is an avoider of the pattern q_1 of length less than n
- The image permutation, π', has the same construction as a permutation containing at least one occurrence of q_2 where the top- and leftmost occurrence is ab

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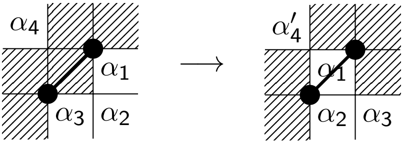

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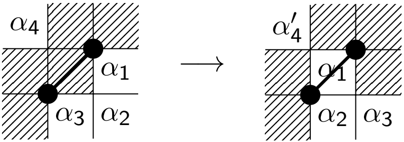

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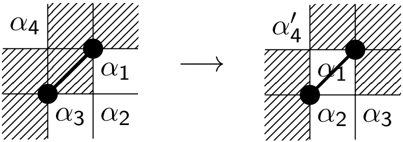

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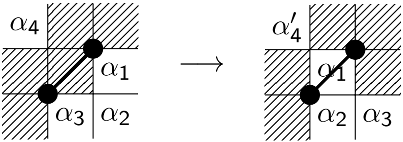

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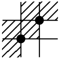
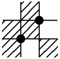
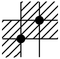
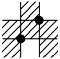
- This map, $\mathrm{Co}(q_1) \longrightarrow \mathrm{Co}(q_2)$ is a bijection because the inverse map can easily be defined.
- The containers of q_1 and the containers of q_2 are equally many so the same applies for the avoiders of the two patterns. Therefore we can construct a bijection from the avoiders of q_1 to the avoiders of q_2 by alphabetical order.
- Now we have shown by induction that the map φ exists for all $n \geq 0$ and it is a bijection.
- The existence of a bijection between the set of avoiders of q_1 and the avoiders of q_2, gives that the avoiders must be equally many so the two patterns are Wilf-equivalent.

Conclusions and Future work

With similar arguments we have shown that

$q_1 =$  and $q_3 =$ 

are Wilf-equivalent and also

$p_1 =$  and $p_2 =$ 

Our method allows us to bring the number of classes down to 53, and we believe it can be used to complete the classification of mesh patterns of length 2, showing that the total number of Wilf-classes is 46.

Definition

Two patterns p and q are said to be strongly Wilf-equivalent if the number of permutations of length n with k occurrences of the pattern p is the same as the number of permutations of length n with k occurrences of q for all $n \geq 0$ and $k \geq 0$.

Conjecture

The patterns q_1 and q_2 are strongly Wilf-equivalent.

If this turns out to be true, these patterns will be the shortest known (non-trivially) strongly Wilf-equivalent patterns, as currently the shortest strongly Wilf-equivalent patterns now known are vincular patterns of length four.

Thank you!

Questions?