Pattern Avoidance in Latin Squares

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Outline

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Latin Squares

A *Latin Square* is an $n \times n$ grid in which each number $1, \ldots, n$ is used exactly once in each row and in each column.

2	1	3	4
1	3	4	2
4	2	1	3
3	4	2	1

Pattern Containment and Avoidance

- A permutation p is said to avoid a pattern π if p does not contain π

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- A permutation p is said to avoid a pattern π if p does not contain π
- A Latin Square is said to avoid a pattern π if no row or column, read respectively from left to right and top to bottom, contains π

123 Avoidance in Latin Squares

Question

How many Latin Squares avoid 123*?*

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New Question

How many Latin Squares avoid 123 *in just the columns?*

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How many Latin Squares avoid 123 in just the columns?

Theorem

The number of n by n Latin Squares avoiding the pattern 123 in the columns is $n!$.

123 Avoidance in Columns

Proof.

1. Fix the first row

1	3	2	4

123 Avoidance in Columns

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1. Fix the first row
2. There is then exactly one Latin Square with that first row avoiding 123 in the columns

1	3	2	4
4			
3			
2			

123 Avoidance in Columns

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1	3	2	4
4		1	
3		4	
2		3	

123 Avoidance in Columns

Proof.

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1	3	2	4
4	2	1	3
3	1	4	2
2	4	3	1

Corollaries

- There are $n!$ Latin Squares avoiding 123 in just the rows

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- There are $n!$ Latin Squares avoiding 123 in just the rows
- You will avoid 123 in every column if and only if each column has the decreasing structure

$$i, i-1, i-2, ..., 2, 1, n, n-1, ..., i+1.$$

1	3	2	4
4	2	1	3
3	1	4	2
2	4	3	1

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Theorem

The number of n by n Latin Squares avoiding the pattern 123 *in both the rows and the columns is n.*

123 Avoidance

Proof.

1. Fix a 1 in the first column

1			

123 Avoidance

Proof.

1. Fix a 1 in the first column
2. Complete that row to avoid 123

1	4	3	2

123 Avoidance

Proof.

1. Fix a 1 in the first column
2. Complete that row to avoid 123
3. By the previous corollary, there is exactly one Latin Square avoiding 123 in the columns with this row

3			
2			
1	4	3	2
4			

123 Avoidance

Proof.

1. Fix a 1 in the first column
2. Complete that row to avoid 123
3. By the previous corollary, there is exactly one Latin Square avoiding 123 in the columns with this row

3	2	1	4
2	1	4	3
1	4	3	2
4	3	2	1

123 Avoidance: Corollary

The Latin Squares avoiding 123 look like the following:

i	$i-1$			1	n			$i+2$	$i+1$
$i-1$	$\ddots$		1	n			$i+2$	$i+1$	i
		1	n			$i+2$	$i+1$	i	
	1	n	$\ddots$		$i+2$	$i+1$	i		
1	n			$i+2$	$i+1$	i			
n			$i+2$	$i+1$	i				1
		$i+2$	$i+1$	i		$\ddots$		1	n
	$i+2$	$i+1$	i				1	n	
$i+2$	$i+1$	i				1	n	$\ddots$	
$i+1$	i				1	n			$i+2$

Avoiding Longer Patterns

Theorem

Let π_n and π'_n be patterns of size n. Then $L_n(\pi_n) = L_n(\pi'_n)$, where $L_n(\pi)$ denotes the number of n by n Latin Squares avoiding π.

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Example

- $L_3(\pi_3) = 3$

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Example

- $L_3(\pi_3) = 3$
- $L_4(\pi_4) = 400$

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Example

- $L_3(\pi_3) = 3$
- $L_4(\pi_4) = 400$
- $L_5(\pi_5) = 148120$

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Theorem

$L_n(\pi) = L_n(\pi^{rev}) = L_n(\pi^c)$.

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Do other equivalence classes exist?

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Do other equivalence classes exist?

Example

Let $S_n(\pi)$ denote the number of permutations avoiding the pattern π. Then:

- $S_n(\pi) = S_n(\pi^{-1})$.
- $S_n(1342) = S_n(2413)$

Do Other Equivalence Classes Exist?

Example

- $S_n(\pi) = S_n(\pi^{-1})$.
- $S_n(1342) = S_n(2413)$

Answer (No)

- $L_5(1423) = 26492$, *but* $L_5((1423)^{-1}) = L_5(1342) = 26616$.
- $L_5(1342) = 26616$, *but* $L_5(2413) = 27797$

Monotone Subsequences

Theorem

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- *In particular, a permutation of length at least* $(m-1)^2+1$ *contains a monotone subsequence of length* m.

Monotone Subsequences

Theorem

- *Every permutation of length $pq + 1$ contains either an increasing subsequence of length $p + 1$ or a decreasing subsequence of length $q + 1$.*
- *In particular, a permutation of length at least $(m - 1)^2 + 1$ contains a monotone subsequence of length m.*

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Can we say something stronger for Latin Squares?

Monotone Subsequences: A (Slightly) Improved Bound

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One row must begin with a 1.

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Theorem

Every Latin Square of at least dimension $(m-1)(m-2)+2$ *has a monotone subsequence of length* m.

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Proof.

- Look at the row beginning with a 1

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Proof.

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- Need an increasing subsequence of length $m-1$ or a decreasing subsequence of length m in remaining $n-1$ entries
- Previous theorem guarantees this with $(m-1)(m-2)+1$ additional entries

Monotone Subsequences: An Optimal Bound

Consider the $n^2 \times n^2$ Latin Square obtained by permuting the following row, which has longest monotone sequence n:

$$n \quad 2n \ldots n^2 \qquad n-1 \quad 2n-1 \ldots n^2-1 \quad \ldots \quad 1 \quad n+1 \ldots n^2-n+1.$$

Example

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This Latin Square has longest monotone subsequence $n+1$

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- For a fixed pattern, say $\pi = 123...m$, can anything be said about the growth rate of $L_n(\pi)$?

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- For a fixed pattern, say $\pi = 123...m$, can anything be said about the growth rate of $L_n(\pi)$?
- Which patterns are the easiest to avoid in Latin Squares? The hardest? Do non-trivial equivalences exist?
- How many Latin Squares contain 123