Consecutive patterns in permutations

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Consecutive patterns

$\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_m.$

Definition. π *contains* σ *as a consecutive pattern* if it has a subsequence of adjacent entries order-isomorphic to σ.

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Examples: 25134 avoids 132
42531 contains 132
15243 contains two occurrences of 132

In this talk, containment and avoidance will always refer to consecutive patterns.

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Consecutive patterns generalize basic combinatorial concepts:

- Occurrences of 21 are *descents*.
- Occurrences of 132 and 231 are *peaks*.
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Consecutive patterns arise naturally in dynamical systems, and play a role in distinguishing deterministic from random sequences.

Notation

For a fixed pattern σ, let

$$P_\sigma(u,z) = \sum_{n\ge 0} \sum_{\pi\in\mathcal{S}_n} u^{\#\{\text{occurrences of } \sigma \text{ in } \pi\}} \frac{z^n}{n!},$$

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where $\alpha_n(\sigma) = \#\{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}$.

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Let

$$\omega_\sigma(u,z)=\frac{1}{P_\sigma(u,z)}.$$

Some questions being studied

- Exact enumeration: find $P_\sigma(u,z)$ or $P_\sigma(0,z)$.

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- Classification of patterns according to *c-Wilf-equivalence*.
  We write $\sigma \sim \tau$ if $P_\sigma(u,z) = P_\tau(u,z)$.

  Example: $1342 \sim 1432$.

  In this talk: Classification of patterns of length up to 6.

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  In this talk: Classification of patterns of length up to 6.

- Comparison of $\alpha_n(\sigma)$ for different patterns.

  Example: $\alpha_n(132) < \alpha_n(123)$ for $n \geq 4$.

  In this talk: For which pattern $\sigma \in \mathcal{S}_m$ is $\alpha_n(\sigma)$ largest.

Patterns of small length

Length 3: **2** c-Wilf classes (compare: 1 Wilf class in classical case)

$123 \sim 321$

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Length 4: **7** c-Wilf classes (compare: 3 Wilf classes in classical case)

$1234 \sim 4321$

$2413 \sim 3142$

$2143 \sim 3412$

$1324 \sim 4231$

$1423 \sim 3241 \sim 4132 \sim 2314$

$1342 \sim 2431 \sim 4213 \sim 3124 \overset{*}{\sim} 1432 \sim 2341 \sim 4123 \sim 3214$

$1243 \sim 3421 \sim 4312 \sim 2134$

All $\sim$ follow from reversal and complementation except for $\overset{*}{\sim}$.

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$1234 \sim 4321$ — enumeration solved

$2413 \sim 3142$ — enumeration unsolved

$2143 \sim 3412$

$1324 \sim 4231$

$1423 \sim 3241 \sim 4132 \sim 2314$

$1342 \sim 2431 \sim 4213 \sim 3124 \overset{*}{\sim} 1432 \sim 2341 \sim 4123 \sim 3214$

$1243 \sim 3421 \sim 4312 \sim 2134$

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Clusters

We use an adaptation of the cluster method of Goulden and Jackson, based on inclusion-exclusion.

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Example: 142536879 is a 3-cluster w.r.t. 1324.

The cluster method

Let the EGF for clusters be

$$C_\sigma(u,z) = \sum_{n,k} c^\sigma_{n,k}\, u^k \frac{z^n}{n!},$$

where $c^\sigma_{n,k}$:= number of k-clusters of length n w.r.t. σ.

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Theorem (Goulden-Jackson '79, adapted)

$$P_\sigma(u,z)=\frac{1}{\omega_\sigma(u,z)}=\frac{1}{1-z-C_\sigma(u-1,z)}.$$

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$$P_\sigma(u,z) = \frac{1}{\omega_\sigma(u,z)} = \frac{1}{1-z-C_\sigma(u-1,z)}.$$

This reduces the computation of $P_\sigma(u,z)$ to the enumeration of clusters.

Clusters as linear extensions of posets

$\underline{\pi_1\pi_2\overline{\pi_3\pi_4\pi_5}}\overline{\pi_6\pi_7}\underline{\pi_7\pi_8\pi_9\pi_{10}\pi_{11}}$ is a cluster w.r.t. $\sigma = 14253$

$$\Updownarrow$$

$$\pi_1 < \pi_3 < \pi_5 < \pi_2 < \pi_4$$
$$\pi_3 < \pi_5 < \pi_7 < \pi_4 < \pi_6$$
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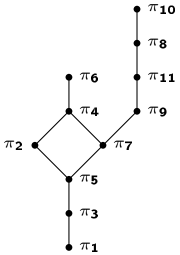

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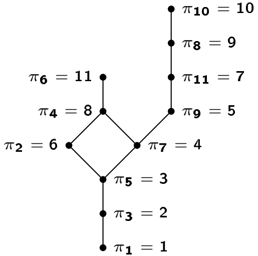

Ex: $\underline{1\ 6\ \overline{2\ 8\ 3}}\ \overline{11\ \underline{4}}\ \underline{9\ 5\ 10\ 7}$

The pattern $\sigma = 12\dots m$ and generalizations

Theorem (Goulden-Jackson '83, E.-Noy '01)

For $\sigma = 12\dots m$, $\omega_\sigma(u,z)$ *is the solution of*

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Example:

$$P_{1234}(0,z) = \frac{1}{\omega_{1234}(0,z)} = \frac{2}{\cos z - \sin z + e^{-z}}$$

The pattern $\sigma = 12\dots m$ and generalizations

More generally...

Theorem (E.-Noy '11)

Let $\sigma \in \mathcal{S}_m$ be such that all its cluster posets are chains. Then $\omega_\sigma(u,z)$ is the solution of

$$\omega^{(m-1)} + (1-u)\sum_{d\in O_\sigma} \omega^{(m-d-1)} = 0,$$

for a certain set O_σ easily defined from σ.

An example of such a pattern is

$$\sigma = 12\dots(s-1)(s+1)s(s+2)(s+3)\dots m.$$

Non-overlapping patterns

$\sigma \in \mathcal{S}_m$ is non-overlapping if two occurrences of σ can't overlap in more than one position.

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Proposition (Dotsenko-Khoroshkin, Remmel '10)

For $\sigma \in \mathcal{S}_m$ non-overlapping, $P_\sigma(u, z)$ depends only on σ_1 and σ_m.

Non-overlapping patterns

Theorem (E.-Noy '01)

Let $\sigma \in \mathcal{S}_m$ be non-overlapping with $\sigma_1 = 1$, $\sigma_m = b$. Then $\omega_\sigma(u,z)$ is the solution of

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Example:

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E.-Noy '11: Similar differential equations for $\omega_\sigma(u,z)$ for $\sigma = 12534$ and $\sigma = 13254$ (which aren't non-overlapping).

The pattern $134\dots(s+1)2(s+2)(s+3)\dots m$

Theorem (E.-Noy, Liese-Remmel, Dotsenko-Khoroshkin)

For $\sigma = 1324$, $\omega_\sigma(u,z)$ *is the solution of*

$$z\omega^{(5)} - ((u-1)z-3)\omega^{(4)} - 3(u-1)(2z+1)\omega^{(3)} + (u-1)((4u-5)z-6)\omega'' + (u-1)(8(u-1)z-3)\omega' + 4(u-1)^2 z\omega = 0$$

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The construction generalizes to patterns of the form

$$\sigma = 134\dots(s+1)2(s+2)(s+3)\dots m.$$

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"Conjecture" (Noonan-Zeilberger '96)

For every classical pattern σ (i.e., where occurrences are not constrained to consecutive positions), the generating function for σ-avoiding permutations is D-finite.

Consecutive Wilf-equivalence

One can classify patterns of length up to 6 into consecutive-Wilf-equivalence classes, proving four conjectures of Nakamura:

n	# of classes
3	2
4	7
5	25
6	92

Theorem (E.-Noy '11)

- $123546 \sim 124536 \rightarrow$ *solution of* $\omega^{(5)} + (1-u)(\omega' + \omega) = 0.$
- $123645 \sim 124635 \rightarrow$ *solution of* $\omega^{(5)} + (1-u)z(\omega'' + \omega') = 0.$
- $132465 \sim 142365 \rightarrow$ *solution of* $\omega^{(5)} + (1-u)(\omega'' + z\omega') = 0.$
- $154263 \sim 165243.$

Asymptotic behavior

Theorem (E. '05)

For every σ, the limit

$$\rho_\sigma := \lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} \qquad \textit{exists.}$$

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This limit is known only for some patterns.

Theorem (Ehrenborg-Kitaev-Perry '11)

For every σ,

$$\frac{\alpha_n(\sigma)}{n!} = \gamma_\sigma \rho_\sigma^n + O(\delta^n),$$

for some constants γ_σ and $\delta < \rho_\sigma$.

The proof uses methods from spectral theory.

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Theorem (E. '12)

For every $\sigma \in \mathcal{S}_m$ there exists n_0 such that

$$\alpha_n(\sigma) \leq \alpha_n(12 \dots m)$$

for all $n \geq n_0$.

Interestingly, the analogous result for classical patterns (i.e., without the adjacency requirement) is false.

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The theorem is equivalent to ρ_σ being largest for $\sigma = 12\dots m$.

Proof idea — 1. Singularity analysis

Let $\sigma \in \mathcal{S}_m \setminus \{12\dots m, m\dots 21\}$. Want to show: $\rho_\sigma < \rho_{12\dots m}$.

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Recall: ρ_σ is the growth rate of the coefficients of

$$P_\sigma(0,z) = \frac{1}{\omega_\sigma(0,z)} = \sum_{n\geq 0} \alpha_n(\sigma)\frac{z^n}{n!},$$

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One can show that $\omega_\sigma(z) := \omega_\sigma(0,z)$ is analytic near the origin, so

- ρ_σ^{-1} is the smallest zero of $\omega_\sigma(z)$,
- $\rho_{12\dots m}^{-1}$ is the smallest zero of $\omega_{12\dots m}(z)$.

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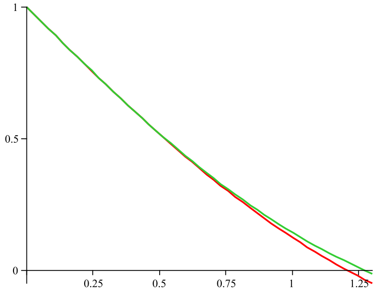
To show that $\rho_\sigma < \rho_{12\ldots m}$, it is enough to show that

$$\omega_{12\ldots m}(z) < \omega_\sigma(z)$$

for $0 < z < 1.276$.



Proof idea — 2. Comparing cluster numbers

We show that $\omega_{12\ldots m}(z) < \omega_\sigma(z)$ for $0 < z < 1.276$:

$$\omega_{12\ldots m}(z) = \sum_{j\geq 0}\left(\frac{z^{jm}}{(jm)!} - \frac{z^{jm+1}}{(jm+1)!}\right) < 1 - z + \frac{z^m}{m!} - \frac{z^{m+1}}{(m+1)!} + \frac{z^{2m}}{(2m)!},$$

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Key fact #1: The sequence $\{s^\sigma_k(z)\}_{k\geq 1}$ is decreasing.

Proof idea — 2. Comparing cluster numbers

We show that $\omega_{12\dots m}(z) < \omega_\sigma(z)$ for $0 < z < 1.276$:

$$\omega_{12\dots m}(z) = \sum_{j\geq 0}\left(\frac{z^{jm}}{(jm)!} - \frac{z^{jm+1}}{(jm+1)!}\right) < 1-z+\frac{z^m}{m!}-\frac{z^{m+1}}{(m+1)!}+\frac{z^{2m}}{(2m)!},$$

$$\wedge$$

$$\omega_\sigma(z) = 1 - z - \sum_{k\geq 1}(-1)^k \underbrace{\sum_n r^\sigma_{n,k}\frac{z^n}{n!}}_{s^\sigma_k(z)} > 1 - z + \frac{z^m}{m!} - s^\sigma_2(z).$$

Key fact #1: The sequence $\{s^\sigma_k(z)\}_{k\geq 1}$ is decreasing.

Key fact #2: $s^\sigma_2(z) < \frac{z^{m+1}}{(m+1)!} - \frac{z^{2m}}{(2m)!}$.

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Proposition (E. 12)

For every non-overlapping $\sigma \in \mathcal{S}_m$ there exists n_0 s.t.

$$\alpha_n(123\dots(m-2)m(m-1)) \leq \alpha_n(\sigma) \leq \alpha_n(134\dots m2)$$

for all $n \geq n_0$.

Consecutive patterns in dynamical systems

Deterministic or random?

Two sequences of numbers in $[0, 1]$:

.6416, .9198, .2951, .8320, .5590, .9861, .0550, .2078, .6584, .8996, .3612, .9230, .2844, .8141, .6054, . . .

.9129, .5257, .4475, .9815, .4134, .9930, .1576, .8825, .3391, .0659, .1195, .5742, .1507, .5534, .0828, . . .

Which one is random? Which one is deterministic?

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Which one is random? Which one is deterministic?

The first one is deterministic: taking $f(x) = 4x(1 - x)$, we have

$f(.6146) = .9198,$
$f(.9198) = .2951,$
$f(.2951) = .8320,$
. . .

Allowed patterns of a map

Let X be a linearly ordered set, $f : X \to X$. For each $x \in X$ and $n \geq 1$, consider the sequence

$$x, f(x), f^2(x), \dots, f^{n-1}(x).$$

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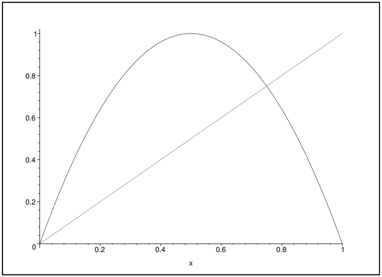
If there are no repetitions, the relative order of the entries determines a permutation, called an allowed pattern of f.

Example

$$\begin{array}{rccc} f: & [0,1] & \to & [0,1] \\ & x & \mapsto & 4x(1-x). \end{array}$$



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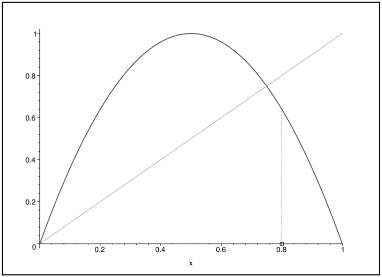

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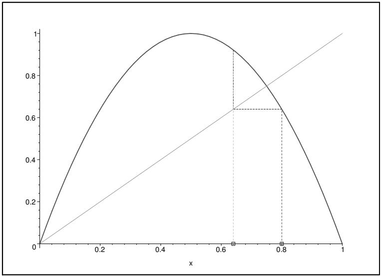

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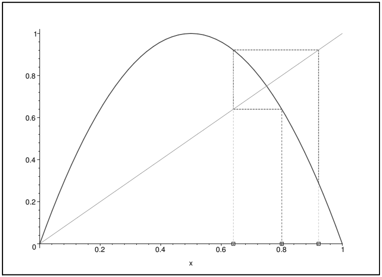

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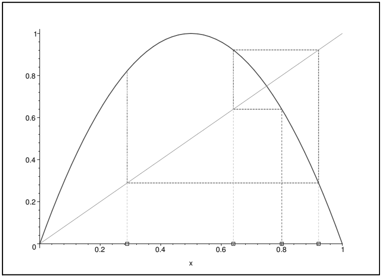

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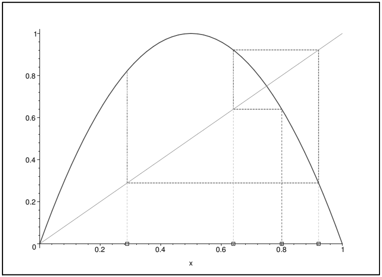

For $x = 0.8$ and $n = 4$, the sequence

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determines the permutation 3241, so it is an allowed pattern.

Allowed and forbidden patterns

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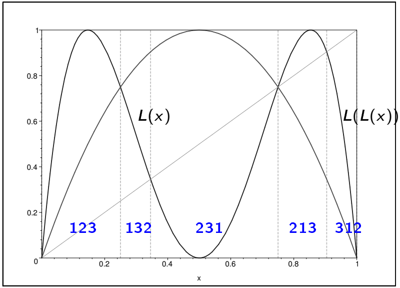
Thus, Allow(f) can be characterized by avoidance of a (possibly infinite) set of consecutive patterns.

The permutations not in Allow(f) are called forbidden patterns of f.

Example: $L(x) = 4x(1-x)$

Taking different $x \in [0,1]$, the patterns $123, 132, 231, 213, 312$ are realized. However, 321 is a forbidden pattern of L.



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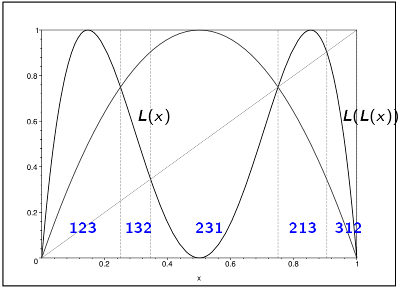

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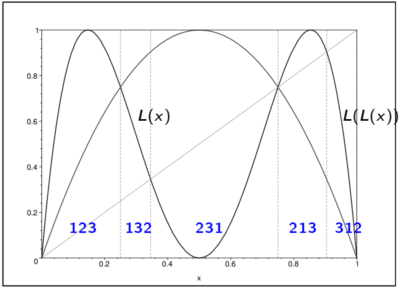

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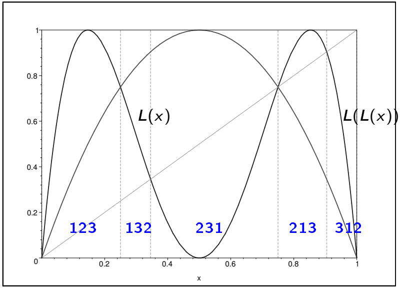

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Theorem (E.-Liu): L has infinitely many basic forbidden patterns.

Forbidden patterns

Let $I \subset \mathbb{R}$ be a closed interval.

Theorem (Bandt-Keller-Pompe '02)

Let $f : I \to I$ be a piecewise monotone map. Then

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Provides a combinatorial way to compute the topological entropy, which is a measure of the complexity of the dynamical system.

Deterministic vs. random sequences

Back to the original sequence:

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If it was a random sequence, any pattern would eventually appear.

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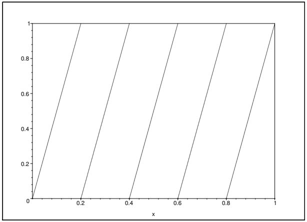
Shift maps

$$\begin{array}{rccc} M_k : & [0,1) & \rightarrow & [0,1) \\ & x & \mapsto & \{kx\} \end{array}$$

(fractional part)



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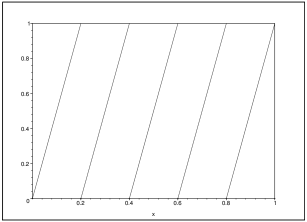

Considering the expansions in base k of $x \in [0,1)$, this map is "equivalent" to the shift map on the set $\mathcal{W}_k = \{0, 1, \dots, k-1\}^{\mathbb{N}}$ of infinite words on a k-letter alphabet, ordered lexicographically:

$$\begin{array}{rccc} \Sigma_k : & \mathcal{W}_k & \longrightarrow & \mathcal{W}_k \\ & w_1 w_2 w_3 \dots & \mapsto & w_2 w_3 w_4 \dots \end{array}$$

Example

The permutation 4217536 is realized (i.e., allowed) by Σ_3, because taking $w = 2102212210\ldots \in \mathcal{W}_3$, we have

$$\left.\begin{array}{rcl} w = 2102212210\ldots & & 4 \\ \Sigma_3(w) = 102212210\ldots & & 2 \\ {\Sigma_3}^2(w) = 02212210\ldots & & 1 \\ {\Sigma_3}^3(w) = 2212210\ldots & & 7 \\ {\Sigma_3}^4(w) = 212210\ldots & & 5 \\ {\Sigma_3}^5(w) = 12210\ldots & & 3 \\ {\Sigma_3}^6(w) = 2210\ldots & & 6 \end{array}\right\} \text{lexicographic order of the shifted words}$$

Forbidden patterns of shifts

Theorem (Amigó-E.-Kennel)

Σ_k has no forbidden patterns of length $n \leq k+1$, but it has basic forbidden patterns of each length $n \geq k+2$.

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The shortest forbidden patterns of Σ_4 are

$$615243, 324156, 342516, 162534, 453621, 435261.$$

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This characterizes permutations realized by Σ_k, and can be used to deduce a (complicated) formula for $|\text{Allow}_n(\Sigma_k)|$, for given n and k.

Signed shifts

For fixed $\sigma = \sigma_0\sigma_1\ldots\sigma_{k-1} \in \{+,-\}^k$, the signed shift with signature σ is

$$\begin{array}{cccc} \Sigma_\sigma: & \mathcal{W}_k & \longrightarrow & \mathcal{W}_k \\ & w_1w_2w_3\ldots & \mapsto & \begin{cases} w_2w_3w_4\ldots & \text{if } \sigma_{w_1} = +, \\ \bar{w}_2\bar{w}_3\bar{w}_4\ldots & \text{if } \sigma_{w_1} = -, \end{cases} \end{array}$$

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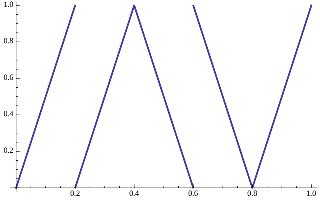
Thinking of words as expansions in base k of numbers in $[0,1)$, Σ_σ is "equivalent" to a piecewise linear map.



Σ_{++--+}

Signed shifts

Archer '13:

- Characterization of permutations realized by Σ_σ, for any σ (fixing and simplifying a result of Amigó).
- Upper and lower bounds on $|\mathrm{Allow}(\Sigma_\sigma)|$.

Periodic orbits

Let $\mathcal{P}_n(\Sigma_\sigma)$ be the set of permutations realized by the *periodic orbits* of Σ_σ of size n.

Theorem (Archer-E. '12)

Assuming $\sigma \neq -^k$ *or* $n \neq 2 \bmod 4$,
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Examples:	$\pi \in \mathcal{P}_n(\Sigma_{+-})$	$\Leftrightarrow$	$\hat{\pi}$ is unimodal.
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Corollary (Archer-E. '12)

Enumeration formulas for cyclic permutations avoiding some sets of patterns (in the classical sense).

Thank you