Splittability of Permutation Classes

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Permutation Classes

Definition

- A permutation class is a set of permutations $\mathcal{C}$ such that for every $\pi \in \mathcal{C}$ and $\sigma \leq \pi$, we have $\sigma \in \mathcal{C}$.
- $\text{Av}(\sigma)$ is the class of all permutations avoiding σ.
- A principal permutation class is the class of the form $\text{Av}(\sigma)$ for some σ.

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Merging Permutations

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Permutation π is a merge of permutations σ and τ if the symbols of π can be colored red and blue, so that the red symbols are a copy of σ and the blue ones of τ.

Example

3175624 is a merge of 231 and 1342.

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For two sets P and Q of permutations, let $P \odot Q$ be the set of permutations obtained by merging a $\sigma \in P$ with a $\tau \in Q$.

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Example

Every permutation avoiding 123 is a union of two decreasing subsequences, i.e., $\mathrm{Av}(123) \subseteq \mathrm{Av}(12) \odot \mathrm{Av}(12)$. Therefore, Av(123) is splittable.

Example (Claesson–J.–Steingrímsson; PP'12)

$\mathrm{Av}(1324) \subseteq \mathrm{Av}(132) \odot \mathrm{Av}(213)$. In particular, Av(1324) is splittable.

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Figure out which (principal) classes are splittable.

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Simple Lemma

The following are equivalent:

- *$\mathcal{C}$ is splittable.*
- *There are classes $\mathcal{A}$ and $\mathcal{B}$, neither of which is a superset of $\mathcal{C}$, such that $\mathcal{C} \subseteq \mathcal{A} \odot \mathcal{B}$.*
- *There are two permutations $\sigma, \pi \in \mathcal{C}$ such that $\mathcal{C} \subseteq Av(\sigma) \odot Av(\pi)$.*
- *For some k, $\mathcal{C}$ has k proper subclasses $\mathcal{A}_1, \ldots, \mathcal{A}_k$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_k$.*

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Direct Sums

Definition

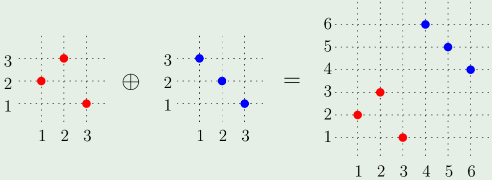
Given two permutations $\pi = \pi_1, \ldots, \pi_k$ and $\sigma = \sigma_1, \ldots, \sigma_m$, define the direct sum $\pi \oplus \sigma$ as

$$\pi \oplus \sigma = \pi_1, \ldots, \pi_k, \sigma_1 + k, \ldots, \sigma_m + k.$$

A permutation is $\oplus$-decomposable if it is a direct sum of two nonempty permutations.

Example

$231 \oplus 321 = 231654$



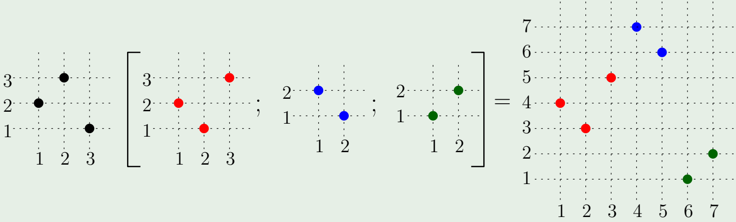
Inflations

Definition

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation and $\sigma_1,\dots,\sigma_n$ a sequence of n permutations. The inflation of π by $\sigma_1,\dots,\sigma_n$, denoted by $\pi[\sigma_1;\dots;\sigma_n]$ is the permutation obtained by replacing each π_i by a copy $\overline{\sigma_i}$ of σ_i, so that if $\pi_i < \pi_j$, then all elements of $\overline{\sigma_i}$ are smaller than those of $\overline{\sigma_j}$.

Example

231[213; 21; 12]= 4357612



Wreath Products and Simple Permutations

Definition

The wreath product of two sets A and B (denoted $A \wr B$) is the set

$$\{\alpha[\beta_1; \ldots; \beta_n];\ \alpha \in A, \beta_1, \ldots, \beta_n \in B\}$$

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A class of permutations $\mathcal{C}$ is wreath-closed if $\mathcal{C} \wr \mathcal{C} \subseteq \mathcal{C}$.

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A permutation is simple if it cannot be obtained from smaller permutations by inflation.

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The Main Results

These classes are unsplittable:

- Av(σ) for a simple permutation σ
- more generally, any wreath-closed class $\mathcal{C}$
- Av(213)
- the class Av($231, 312$) of layered permutations

These classes are splittable:

- Av(σ) for a $\oplus$-decomposable σ of size at least 4

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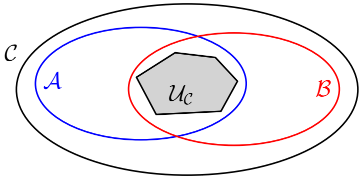
Proving Unsplittability

Definition

Let $\mathcal{C}$ be a permutation class. A permutation $\pi \in \mathcal{C}$ is unavoidable in $\mathcal{C}$ if whenever $\mathcal{C}$ is split into proper subclasses $\mathcal{A}$ and $\mathcal{B}$, then $\pi \in \mathcal{A}$ and $\pi \in \mathcal{B}$.

$\mathcal{U}_\mathcal{C}$ is the set of unavoidable permutations in $\mathcal{C}$.



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- *$\mathcal{U}_\mathcal{C}$ is a permutation class.*
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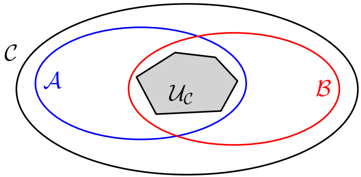

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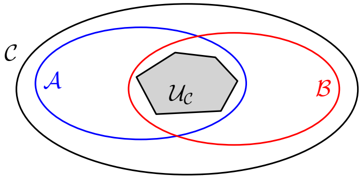

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Example of an Unavoidable Pattern

Example

If $\mathcal{C}$ is a nonempty $\oplus$-closed class, then $12\cdots k \in \mathcal{U}_{\mathcal{C}}$ for each k.

Proof:

- For contradiction, suppose $\mathcal{C}$ is split into are proper subclasses $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{A}$ does not contain $12\cdots k$.
- Pick any $\pi \in \mathcal{C} \setminus \mathcal{B}$. Then the k-fold sum $\pi^{\oplus k} = \pi \oplus \pi \oplus \cdots \oplus \pi$ belongs to $\mathcal{C}$.
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A Generalization

Proposition

- *If $\mathcal{C}$ is $\oplus$-closed, then $\mathcal{U}_{\mathcal{C}}$ is also $\oplus$-closed.*
- *If $X \wr \mathcal{C} \subseteq \mathcal{C}$ for some X, then $X \wr \mathcal{U}_{\mathcal{C}} \subseteq \mathcal{U}_{\mathcal{C}}$.*
- *If $\mathcal{C} \wr X \subseteq \mathcal{C}$ for some X, then $\mathcal{U}_{\mathcal{C}} \wr X \subseteq \mathcal{U}_{\mathcal{C}}$.*

Corollary

If $\mathcal{C}$ is wreath-closed then $\mathcal{C}$ is unsplittable. (Because $\mathcal{C} \wr \mathcal{C} \subseteq \mathcal{C}$ implies $\mathcal{U}_{\mathcal{C}} \wr \mathcal{C} \subseteq \mathcal{U}_{\mathcal{C}}$, hence $\mathcal{U}_{\mathcal{C}} = \mathcal{C}$.)

Corollary

The class $\mathcal{L} = Av(231, 312)$ of layered permutations is unsplittable. (Because it is $\oplus$-closed and also $\mathcal{L} \wr \{21\} \subseteq \mathcal{L}$, and it is the smallest nonempty class with these properties.)

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Theorem

If σ is $\oplus$-decomposable of size at least 4, then $Av(\sigma)$ is splittable.

- Av(132) and Av(12) are unsplittable, even though 132 and 12 are $\oplus$-decomposable.
- Proof of the theorem is difficult, splittings have a complicated structure.
- Some special cases are easier (see next slide).

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For nonempty α, β, γ, the class Av$(\alpha \oplus \beta \oplus \gamma)$ is splittable, and Av$(\alpha \oplus \beta \oplus \gamma) \subseteq$ Av$(\alpha \oplus \beta) \odot$ Av$(\beta \oplus \gamma)$.

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Corollary

For α and γ of size at least two, the class Av$(\alpha \oplus \gamma)$ is splittable, and Av$(\alpha \oplus \gamma) \subseteq$ Av$(\alpha \oplus 1) \odot$ Av$(1 \oplus \gamma)$.
(Because Av$(\alpha \oplus \gamma) \subseteq$ Av$(\alpha \oplus 1 \oplus \gamma)$.)

The hard part is to show that Av$(1 \oplus \gamma)$ is splittable for every $\oplus$-indecomposable γ of size at least 3.

Some 'Harder' Splittable Cases

Notation: $\mathcal{C}^{*k} = \underbrace{\mathcal{C} \odot \mathcal{C} \odot \cdots \odot \mathcal{C}}_{k \text{ times}}$

Theorem

If $Av(\sigma)$ is splittable, then $Av(1 \oplus \sigma)$ is also splittable.

In fact (under some technical assumptions), if $Av(\sigma) \subseteq Av(\pi_1) \odot Av(\pi_2) \odot \cdots \odot Av(\pi_m)$ then for a suitable K,

$$Av(1 \oplus \sigma) \subseteq Av(1 \oplus \pi_1)^{*K} \odot Av(1 \oplus \pi_2)^{*K} \odot \cdots \odot Av(1 \oplus \pi_m)^{*K}.$$

Example

Since $\text{Av}(321) \subseteq \text{Av}(21) \odot \text{Av}(21)$, we have $\text{Av}(1 \oplus 321) \subseteq \text{Av}(1 \oplus 21)^{*K}$, i.e. $\text{Av}(1432) \subseteq \text{Av}(132)^{*K}$.
(In this example, $K = 5$ is optimal.)

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Open Problems

- For which σ is the class Av(σ) splittable? What should we expect for 'typical' σ?
- Is there a 'nice' way to split Av(1342)? Can you split it into Av(1423), Av(3142) and Av(2413), possibly with repetitions? Which permutations are unavoidable in Av(1342)?
- Is there an example of a class $\mathcal{C}$ for which $\mathcal{U}_{\mathcal{C}}$ is splittable?
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Thank you for your attention!

See “Splittings and Ramsey Properties of Permutation Classes” (arXiv:1307.0027) for more results and full proofs