The Shape of Random Pattern-Avoiding Permutations

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Joint work with Igor Pak

Outline

I. Historical Context and Definitions

II. Results

III. Proof and Applications

IV. Extensions and Open Problems

Pattern avoidance

Definition

We say $\sigma \in S_n$ *contains* $\tau \in S_m$, if there are $x_1 < x_2 < \ldots < x_m$ s.t. $(\sigma(x_1), \ldots, \sigma(x_m))$ have the same relative order as τ. If σ does not contain τ, we say σ is τ-*avoiding*.

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Example:

$$(3,5,1,2,4) = (3,5,1,2,4) \text{ contains } (1,3,2).$$

However, $(4,2,3,1,5)$ is **132**-avoiding.

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However, $(4,2,3,1,5)$ is $\mathbf{132}$-avoiding.

Definition

We use $\mathcal{S}_n(\tau)$ to denote the set of τ-avoiding permutations of length n.

$$\mathcal{S}_n(\tau) =: \{\sigma \in S_n : \sigma \text{ is } \tau\text{-avoiding}\}.$$

Pattern avoidance

Examples: The permutation $\sigma = (3, 5, 1, 2, 4)$ contains $\tau = (1, 3, 2)$, since

$$\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \color{red}{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \color{red}{1} \\ 0 & \color{red}{1} & 0 & 0 & 0 \end{pmatrix} \text{ contains } \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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The permutation

$$\pi = (4,2,3,1,5) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ is } \mathbf{132}\text{-avoiding.}$$

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Also, if $w = (w_1, \ldots, w_n)$ is a list of n distinct integers, we use $red(w)$ to denote the permutation obtained from w by replacing the i-th smallest entry with i.

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Example:

$$red(2,6,9,3,7) = (1,3,5,2,4).$$

Pattern avoidance

Theorem (MacMahon '15, Knuth '65)

Let $\tau \in S_3$. Then

$$|\mathcal{S}_n(\tau)| = C_n, \quad \textit{for all } n.$$

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- $|\mathcal{S}_n(123)| = |\mathcal{S}_n(321)|$ and

$$|\mathcal{S}_n(132)| = |\mathcal{S}_n(213)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)|$$

  by symmetry.
- Many distinct bijections between

$$\mathcal{S}_n(123) \text{ and } \mathcal{S}_n(132).$$

Statistics on Catalan structures:

Definition

Let A be a finite set. A map $\alpha : A \to \mathbb{Z}$ is called a *statistic* on A. Let A and B be two finite sets with $|A| = |B|$, and let α and β be statistics on A and B, respectively. If $|\alpha^{-1}(z)| = |\beta^{-1}(z)|$ for all $z \in Z$, we say α and β are *equidistributed*.

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Many results are known on the asymptotic behavior of statistics on various Catalan structures (see Flajolet & Sedgewick, *Analytic Combinatorics*, 2009).

Statistics on Catalan structures

Theorem

Let h_n be the height of a random plane tree with $n+1$ vertices, and m_n the height of a random Dyck path of length $2n$. Then h_n and m_n are equidistributed, and

$$\mathbb{E}[h_n], \mathbb{E}[m_n] \sim \sqrt{\frac{\pi n}{2}}, \quad \textit{as} \quad n \to \infty.$$

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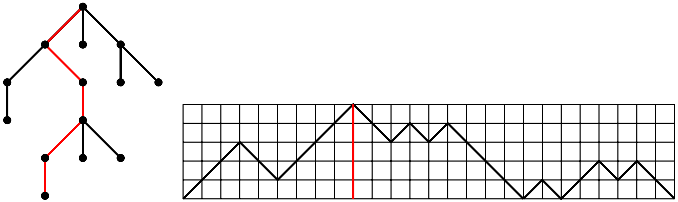

Shape of 123-avoiding permutations

Definition:

We define

$$P_n(j,k) := |\{\sigma \in \mathcal{S}_n(123) : \sigma(j) = k\}| = \sum_{\sigma \in \mathcal{S}_n(123)} M(\sigma)_{j,k},$$

where $M(\sigma)$ is the matrix representation of σ.

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Even more specifically, for fixed $c \in \mathbb{R}$ and $\alpha \in [0,1)$, how does $P_n(an - cn^{\alpha}, bn - cn^{\alpha})$ depend on a, b, c and α?

Shape of **123**-avoiding permutations

Example: For $n = 3$, we have

$$\mathcal{S}_3(123) = S_3 - (123) = \{(132), (213), (231), (312), (321)\},$$

$$\text{so } P_3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

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$$\text{Also, } P_4 = \begin{pmatrix} 1 & 3 & 5 & 5 \\ 3 & 4 & 2 & 5 \\ 5 & 2 & 4 & 3 \\ 5 & 5 & 3 & 1 \end{pmatrix}.$$

Shape of **123**-avoiding permutations

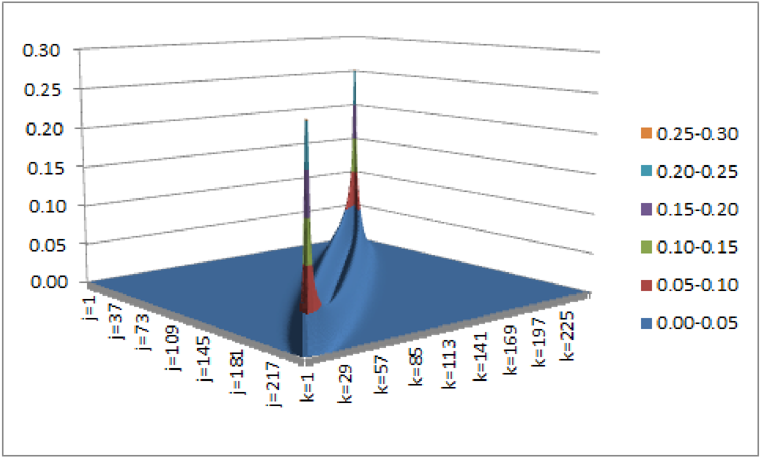

Figure: Surface of $P_{250}(j,k)/C_{250}$.

Shape of 123-avoiding permutations

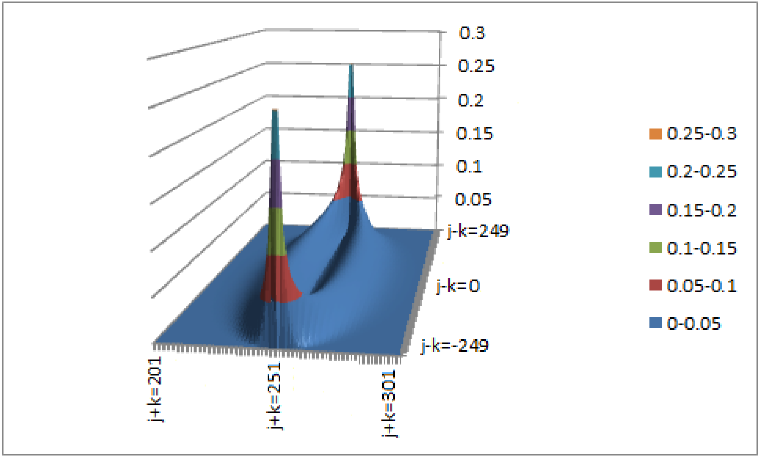

Figure: Closer look at $P_{250}(j,k)/C_{250}$, $201 \leq j+k \leq 301$.

Shape of **123**-avoiding permutations

Theorem (M.-Pak, 2013+)

Let $(a, b) \in [0, 1]^2$, $c \in \mathbb{R}$ *and* $\alpha \in [0, 1)$ *be fixed constants. Then there exists* $\delta = \delta(a, b) \in [0, 1)$ *such that*

$$\frac{P_n(an, bn)}{C_n} \begin{cases} \sim \delta^n & a + b \neq 1, \\ = \Theta\left(n^{-\frac{3}{2}}\right) & a + b = 1, \end{cases}$$

as $n \to \infty$.

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as $n \to \infty$. *Similarly, there exists* $\varepsilon = \varepsilon(a,c) \in (0,1)$ *such that*

$$\frac{P_n(an-cn^{\alpha},bn-cn^{\alpha})}{C_n} \begin{cases} \sim \delta^n & a+b \neq 1, \\ \sim \varepsilon^{n^{2\alpha-1}} & a+b=1, \alpha > \frac{1}{2}, \\ = \Theta\left(n^{-\frac{3}{2}+2\alpha}\right) & a+b=1, 0 \leq \alpha \leq \frac{1}{2}, \end{cases}$$

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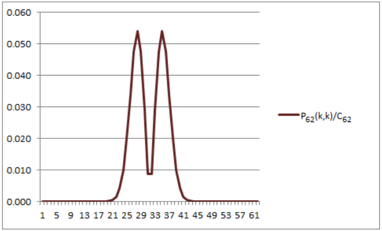

Figure: Values of $P_{62}(k,k)/C_{62}$.

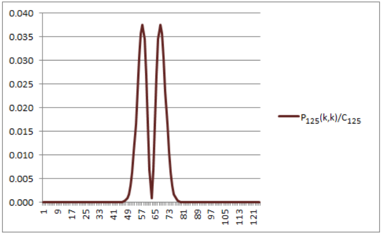

Figure: Values of $P_{125}(k,k)/C_{125}$.

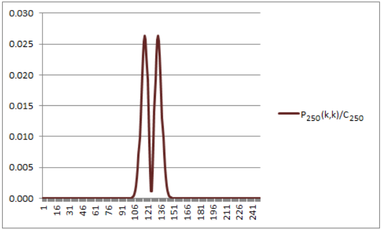

Figure: Values of $P_{250}(k,k)/C_{250}$.

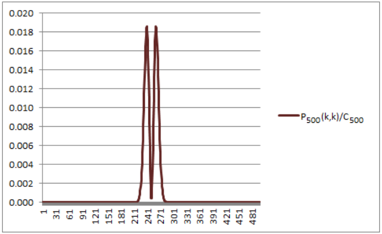

Figure: Values of $P_{500}(k,k)/C_{500}$.

Shape of **132**-avoiding permutations

Definition:

We define

$$Q_n(j,k) := |\{\sigma \in \mathcal{S}_n(132) : \sigma(j) = k\}| = \sum_{\sigma \in \mathcal{S}_n(132)} M(\sigma)_{j,k},$$

where $M(\sigma)$ is the matrix representation of σ.

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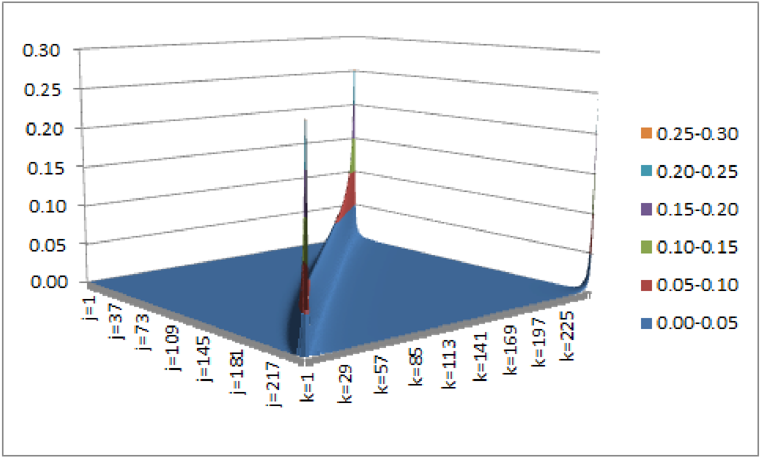
Example: $Q_3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$, $Q_4 = \begin{pmatrix} 1 & 3 & 5 & 5 \\ 3 & 5 & 4 & 2 \\ 5 & 4 & 3 & 2 \\ 5 & 2 & 2 & 5 \end{pmatrix}$.

Shape of **132**-avoiding permutations



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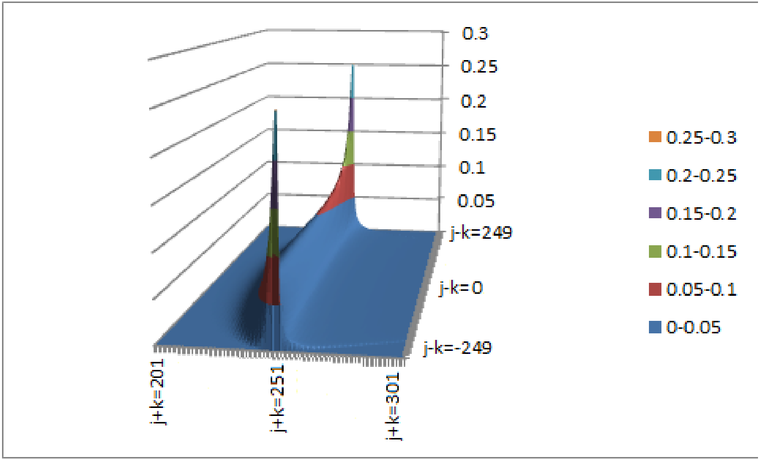
Shape of **132**-avoiding permutations



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Theorem (M.-Pak, 2013+)

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$$\frac{Q_n(an, bn)}{C_n} \begin{cases} \sim \delta^n & a+b<1, \\ = \Theta\left(n^{-\frac{3}{4}}\right) & a+b=1, \\ = \Theta\left(n^{-\frac{3}{2}}\right) & 1<a+b<2, \\ \sim \frac{1}{4} & a+b=2, \end{cases}$$

as $n \to \infty$.

Shape of **132**-avoiding permutations

Theorem (M.-Pak, 2013+ (continued))

Furthermore, let $c \in \mathbb{R}$ *and* $\alpha \in [0, 1)$ *be fixed constants. Then there exists* $\varepsilon = \varepsilon(a, c) \in (0, 1)$ *such that* $\frac{Q_n(an-cn^\alpha, bn-cn^\alpha)}{C_n}$ *is asymptotically equivalent to*

$$\begin{cases} \delta^n & a+b<1, \\ \varepsilon^{n^{2\alpha-1}} & a+b=1, c>0, \alpha>\frac{1}{2}, \\ \Theta\left(n^{-\frac{3}{4}}\right) & a+b=1, 0 \le \alpha \le \frac{3}{8}, \\ \Theta\left(n^{-\frac{3}{2}+2\alpha}\right) & a+b=1, c>0, \frac{3}{8} \le \alpha \le \frac{1}{2}, \\ \Theta\left(n^{-\frac{3}{2}\alpha}\right) & a+b=1, c<0, \frac{1}{2}<\alpha<1, \\ \Theta\left(n^{-\frac{3}{2}}\right) & 1<a+b<2, \\ \Theta\left(n^{-\frac{3}{2}\alpha}\right) & a+b=2, \end{cases}$$

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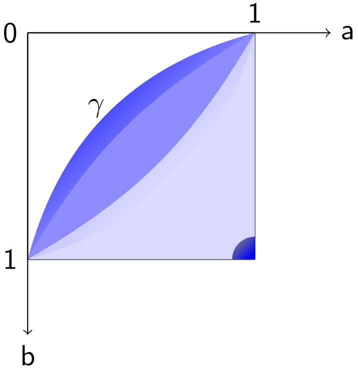
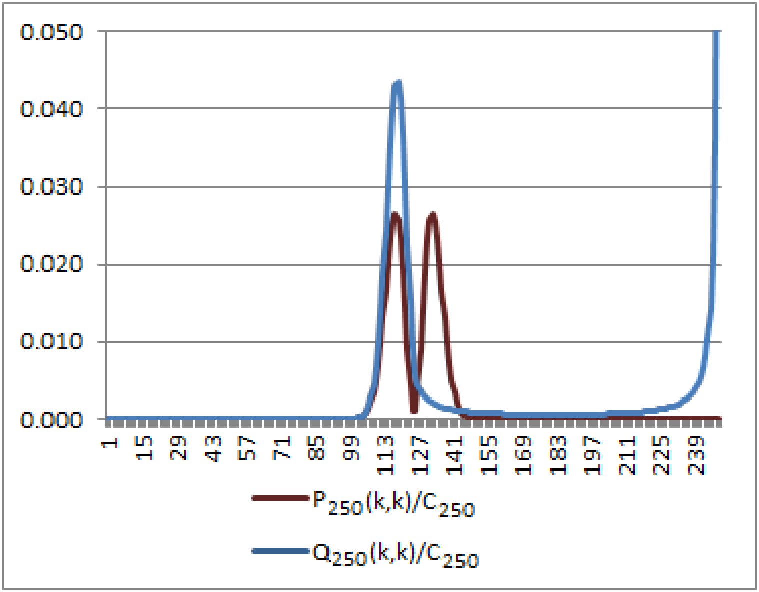


$$\gamma : \{a + b = 1 - \tfrac{2c}{\sqrt{n}}\}$$

Figure: Region where $Q_n(an - cn^\alpha, bn - cn^\alpha) \sim C_n/n^d$ for some d.

Comparison of $P_{250}(k,k)$ and $Q_{250}(k,k)$.



Sketch of proofs for $P_n(j,k)$ and $Q_n(j,k)$

DEFINITION:

Let $0 \leq k \leq n-1$. The *ballot number*, $b(n,k) = \frac{n-k}{n+k}\binom{n+k}{n}$, is the number of partial Dyck paths from $(0,0)$ to $(n+k, n-k)$ that finish with an upstep.

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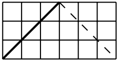
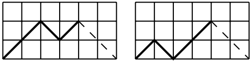
Example: $n = 3$

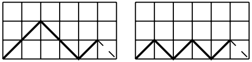
b(3,0)=1



b(3,1)=2



b(3,2)=2



Sketch of proofs for $P_n(j,k)$ and $Q_n(j,k)$

Lemma (Ballot numbers)

For every $1 \leq k \leq n$, we have $P_n(1,k) = Q_n(1,k) = b(n,k-1)$.

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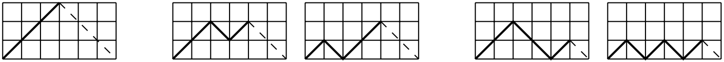

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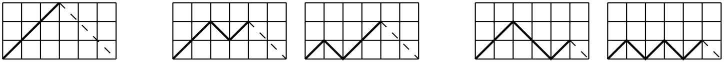

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The proof follows from standard bijections from Dyck paths to $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$.

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Lemma (Explicit formula for $P_n(j,k)$)

For every $1 \le j,k \le n$,

$$P_n(j,k) = \begin{cases} b(n-j+1,k-1)\,b(n-k+1,j-1) & j+k \le n+1, \\ b(j,n-k)\,b(k,n-j) & j+k > n+1. \end{cases}$$

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- Any σ counted by $P_n(j,k)$ is determined by $(red(\tau), red(\rho))$, where $\tau = (\sigma(j), \sigma(j+1), \ldots, \sigma(n))$ and $\rho = (\sigma^{-1}(k), \sigma^{-1}(k+1), \ldots, \sigma^{-1}(n))^{-1}$ (the elements of σ larger than k in their same relative order)

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- $P_n(j,k) = P_{n-j+1}(1,k) \times P_{n-k+1}(j,1)$

Sketch of proofs for $P_n(j,k)$ and $Q_n(j,k)$

Example: $P_7(2,3) = P_6(1,3)P_5(2,1), \sigma = (6,3,7,2,1,5,4)$

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

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$$red(\rho) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Sketch of proofs for $P_n(j,k)$ and $Q_n(j,k)$

Lemma (Explicit formula for $P_n(j,k)$)

For every $1 \le j,k \le n$,

$$P_n(j,k) = \begin{cases} b(n-j+1,k-1)\, b(n-k+1,j-1) & j+k \le n+1, \\ b(j,n-k)\, b(k,n-j) & j+k > n+1. \end{cases}$$

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For every $1 \le j,k \le n$,

$$Q_n(j,k) = \sum_r b(n-j+1,k-r-1)\, b(n-k+1,j-r-1)\, C_r,$$

where r *satisfies*

$$\max\{0, j+k-(n+1)\} \le r \le \min\{j,k\} - 1.$$

Sketch of proofs for $P_n(j,k)$ and $Q_n(j,k)$

Stirling's formula:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as} \quad n \to \infty.$$

Stirling's formula implies

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Lemmas + Stirling's formula → proof of theorems on asymptotic behavior of $P_n(j,k)$ and $Q_n(j,k)$.

Fixed points

Definition:

Let $\sigma \in S_n$. Denote $\mathrm{fp}_n(\sigma)$, the number of fixed points in σ, to be $\#\{i \text{ s.t. } \sigma(i) = i, 1 \leq i \leq n\}$.

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Example: For $\sigma \in S_n$ chosen uniformly,

$$\mathbb{E}[\mathrm{fp}_n(\sigma)] = \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = 1.$$

Fixed points - expectation

Let E_τ be $\mathbb{E}[\mathrm{fp}_n(\sigma)]$, for $\sigma \in \mathcal{S}_n(\tau)$ chosen uniformly.

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Theorem (M-Pak '13+)

$$E_{231} \to \frac{2\,\Gamma\left(\frac{1}{4}\right)}{\sqrt{\pi}} n^{\frac{1}{4}}, \quad \textit{as} \quad n \to \infty.$$

Fixed points - expected location

Theorem (M-Pak '13+)

Let $\varepsilon > 0$. Let $\sigma \in \mathcal{S}_n(123), \pi \in \mathcal{S}_n(132), \rho \in \mathcal{S}_n(321)$ be chosen uniformly at random. If $\sigma(i) = i$, then w.h.p.

$$i \in \left[\left(\frac{1}{2} - \varepsilon\right) n, \left(\frac{1}{2} + \varepsilon\right) n\right].$$

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If $\rho(i) = i$, w.h.p.

$$i \in [1, \varepsilon n] \cup [(1-\varepsilon)n, n].$$

Fixed points of **123**-avoiding permutations

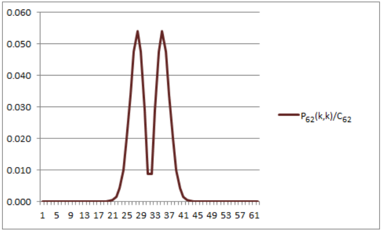

Figure: Values of $P_{62}(k,k)/C_{62}$.

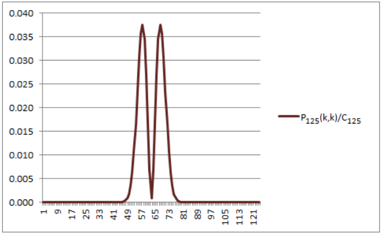

Figure: Values of $P_{125}(k,k)/C_{125}$.

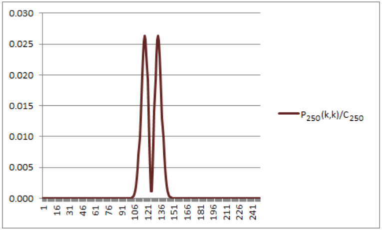

Figure: Values of $P_{250}(k,k)/C_{250}$.

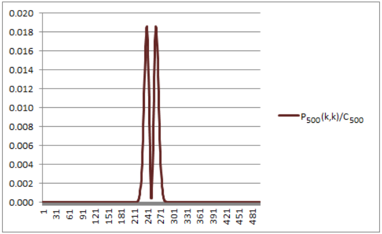

Figure: Values of $P_{500}(k,k)/C_{500}$.

Rank

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Example: For $\sigma = (3, 5, 1, 2, 4)$, $rank(\sigma) = 2$.

$$\begin{pmatrix} \color{red}{0} & \color{red}{0} & 1 & 0 & 0 \\ \color{red}{0} & \color{red}{0} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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Theorem (Knuth-Richards '88, Claesson-Kitaev '09)

For $\sigma \in \mathcal{S}_n(123)$ and $\pi \in \mathcal{S}_n(132)$ chosen uniformly at random, $rank(\sigma)$ is equidistributed with $rank(\pi)$.

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Let $\lambda > 0$. We define *rank*$_\lambda$ of a permutation $\sigma \in S_n$ as the largest integer r s.t. $\sigma(i) > \lambda r$ for all $1 \leq i \leq r$.

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Theorem (M-Pak, '13+)

Let $\lambda > 0$. For $\sigma \in \mathcal{S}_n(123)$ and $\pi \in \mathcal{S}_n(132)$ chosen uniformly at random, we have

$$\mathbb{E}[rank_\lambda(\sigma)] \sim \frac{n}{\lambda+1} - \Theta(\sqrt{n}), \quad \textit{and}$$
$$\mathbb{E}[rank_\lambda(\pi)] \sim \frac{n}{\lambda+1} - \Theta(\sqrt{n}),$$

as $n \longrightarrow \infty$.

Further extensions

Remark:

For $\tau \in S_4$,

$$\begin{aligned} \mathcal{S}_n(1234) &\approx 9^n \text{ (Gessel '90)}, \\ \mathcal{S}_n(1342) &\approx 8^n \text{ (Bóna '97)}, \\ \mathcal{S}_n(1324) &\approx \alpha^n \text{ for some } \ 9.42 < \alpha < 7 + 4\sqrt{3} \\ & \qquad\qquad \text{(AERWZ '06, Bóna '12)}, \end{aligned}$$

where $f_n \approx g_n$ if $\log_n f_n \to \log_n g_n$ as $n \to \infty$.

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(1) Open Problem:

What do the limiting shapes look like for $\mathcal{S}_n(\tau)$ for $\tau \in S_m$, $m > 3$?

Further extensions

Remark:

Enumeration of $\{0,1\}$-pattern-avoiding matrices in $M_n(\{0,1\})$:

Kitaev '04	3-tile L-shaped patterns,
Kitaev-Mansour-Vella '05	2×2 patterns,
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(2) Open Problem:

Can we generalize our results to $\sigma \in M_n(\{0,1\})$, and $\tau \in M_r(\{0,1\})$?

Further extensions

Remark:

Enumeration of $\sigma \in S_n$ s.t. σ has exactly k τ-patterns:

Noonan-Zeilberger '98	$\tau = \mathbf{123}$, all k,
Bóna '98	$\tau = \mathbf{132}$, $k = 1, 2$,
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(3) Open Problem:

Let $a_\tau(\sigma)$ be the number of occurrences of τ within σ. Define

$$P_n(j,k,q) = \sum_{\sigma \in S_n} q^{a_{123}(\sigma)} M(\sigma)_{j,k}.$$

Observe that $P_n(j,k,0) = P_n(j,k)$, and $P_n(j,k,1) = (n-1)!$ for all $1 \le j,k \le n$. Can we say anything about $P_n(j,k,q)$ in general for $0 < q < 1$?

Thank you!

Shape of **123**-avoiding permutations

Theorem (M.-Pak, 2013+)

Let $(a, b) \in [0, 1]^2, c \in \mathbb{R}$ *and* $\alpha \in [0, 1)$ *be fixed constants. Then there exist* $\delta = \delta(a, b), \varepsilon = \varepsilon(a, c) \in (0, 1)$ *such that* $\frac{P_n(an - cn^\alpha, bn - cn^\alpha)}{C_n}$ *is asymptotically equivalent to*

$$\begin{cases} \delta^n & a + b \neq 1, \\ \varepsilon^{n^{2\alpha-1}} & a + b = 1, \alpha > \frac{1}{2}, \\ \eta(a, c) n^{-\frac{3}{2}+2\alpha} & a + b = 1, 0 < \alpha < \frac{1}{2}, \\ \xi(a, c) n^{-\frac{3}{2}} & a + b = 1, \alpha = 0, \\ \eta(a, c)\kappa(a, c) n^{-\frac{1}{2}} & a + b = 1, \alpha = \frac{1}{2} \end{cases}$$

as $n \to \infty$, *where* $\xi(a, c) = \frac{(2c+1)^2}{4\sqrt{\pi}(a(1-a))^{\frac{3}{2}}}$, $\eta(a, c) = \frac{c^2}{\sqrt{\pi}(a(1-a))^{\frac{3}{2}}}$ *and* $\kappa(a, c) = \exp\left[\frac{-c^2}{a(1-a)}\right]$.

Shape of **132**-avoiding permutations

Theorem (M.-Pak, 2013+)

Let $(a,b) \in [0,1]^2$, $c \in \mathbb{R}$ *and* $\alpha \in [0,1)$ *be fixed constants. Then there exist* $\delta = \delta(a,b), \varepsilon = \varepsilon(a,c) \in (0,1)$ *such that* $\frac{Q_n(an - cn^\alpha, bn - cn^\alpha)}{C_n}$ *is asymptotically equivalent to*

$$\begin{cases} (y(a,c)\,\kappa(a,c))n^{-\frac{1}{2}} & a+b=1, c>0, \alpha=\frac{1}{2}, \\ z(a)n^{-\frac{3}{4}} & a+b=1, 0\le\alpha<\frac{3}{8}, \\ (z(a)+y(a,c))n^{-\frac{3}{4}} & a+b=1, c>0, \alpha=\frac{3}{8} \\ y(a,c)n^{-\frac{3}{2}+2\alpha} & a+b=1, c>0, \frac{3}{8}<\alpha<\frac{1}{2}, \\ w(c)n^{-\frac{3}{2}\alpha} & a+b=1, c<0, \frac{1}{2}<\alpha, \\ v(a,b)n^{-\frac{3}{2}} & 1<a+b<2, \\ w(c)n^{-\frac{3}{2}\alpha} & a+b=2, \end{cases}$$

as $n \to \infty$.

Shape of **132**-avoiding permutations

where

$$z(a) = \frac{\Gamma(\frac{3}{4})}{2^{\frac{3}{2}}\pi a^{\frac{3}{4}}(1-a)^{\frac{3}{4}}}, \quad v(a,b) = \frac{1}{2\sqrt{\pi}(2-a-b)^{\frac{3}{2}}(a+b-1)^{\frac{3}{2}}},$$

$$w(c) = \frac{1}{2^{\frac{5}{2}}\, c^{\frac{3}{2}}\sqrt{\pi}},$$

$$y(a,c) = \frac{2c^2}{\sqrt{\pi}a^{\frac{3}{2}}(1-a)^{\frac{3}{2}}},$$

and $\kappa(a,c) = \exp\left[\frac{-c^2}{a(1-a)}\right]$ as above.