Longest Common Patterns in Permutations

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Longest Common Subsequences (of Words)

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w_2: CGTTATTCGA

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- The sequence $w =$ ATTCGA is the Longest Common Subsequence (LCS) of w_1 and w_2.
- Can be found in $O(n^2)$ time.
- When words are randomly chosen, length of their LCS is random variable.
- With k letter alphabet, $E(\text{length of LCS}) \to \frac{2n}{\sqrt{k}}$ as $n \to \infty$. [Kiwi, Loebl, Matousek].

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- Their *longest common pattern*, or LCP, is $\sigma = 1\ 3\ 2\ 4\ 5$.
- Finding legnth in general is NP-Hard, but $O(n^8)$ when one permutations is separable [Bouvel, Rossin]

LCP of Random Permutations

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- What is the expected value of L_n?
- How is L_n concentrated around its mean?

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Proposition $E(L_n) < en^{\frac{2}{3}}$.

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- Let S and T be length k subsequences of π_1 and π_2.

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Proposition $E(L_n) < en^{\frac{2}{3}}$.

$$\pi_1: \; 1 \;\; 4 \;\; \color{red}{2} \;\; 5 \;\; \color{red}{6} \;\; 3 \;\; \color{red}{8} \;\; \color{red}{7}$$
$$\pi_2: \; 2 \;\; 7 \;\; \color{blue}{1} \;\; \color{blue}{4} \;\; 3 \;\; \color{blue}{6} \;\; \color{blue}{5} \;\; 8$$

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$$P(L_n \geq k) = P\left(\bigcup X_{S,T}\right) \leq \sum P(X_{S,T}) = \frac{1}{k!}\binom{n}{k}^2$$

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When $k > en^{\frac{2}{3}}$,

$$P(L_n \geq k) < \frac{1}{k^{\frac{3}{2}}}$$

Upper bound on L_n taking high values $\rightarrow$ upper bound on mean.

Lower Bound

Theorem

$$\liminf_{n\to\infty} \frac{E(L_n)}{n^{2/3}} \geq C$$

where $C = 2 - \log(2e - 1) \approx 0.51$.

Proof Outline of Lower Bound

Divide the permutations into blocks of size $\sim \sqrt[3]{n}$.

$$\begin{array}{l|l|l|l|l}
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Divide the numbers $1, 2, \ldots, n$ into groups of size $\sqrt[3]{n}$:

$$1, 2, 3, 4, 5, 6, 7, 8$$

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π_1:	1	4	2	5	6	3	8	7
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- A *match* occurs when entry in π_1 is in same block and group as one in π_2.

Proof Outline of Lower Bound

$$
\begin{array}{l|cc|cc|cc|cc}
\pi_1: & \boxed{1} & 4 & 2 & 5 & 6 & 3 & 8 & 7 \\
\pi_2: & \boxed{2} & 7 & 1 & 4 & 3 & 6 & 5 & 8
\end{array}
$$

- A *match* occurs when entry in π_1 is in same block and group as one in π_2.

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$$\pi_1: \quad \boxed{1}\ \ 4 \ \Big| \ 2 \ \ 5 \ \Big| \ 6 \ \ 3 \ \Big| \ 8 \ \ 7$$

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- Two matches are *compatible* if they are in different blocks and groups.

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π_1:	$\boxed{1}$	4	$\boxed{2}$	5	6	3	8	7
π_2:	$\boxed{2}$	7	$\boxed{1}$	4	3	6	5	8

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- A *match* occurs when entry in π_1 is in same block and group as one in π_2.
- Two matches are *compatible* if they are in different blocks and groups.
- To find a common pattern, search for set of pairwise compatible matches.

Proof Outline for Lower Bound

Step 1: Search for matches in first group (**black**).

π_1:	1 4	2 5	6 3	8 7
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Proof Outline for Lower Bound

Step 2: Search for matches in second group (red).

π_1:	1 4	2 5	6 3	8 7
π_2:	2 7	1 4	3 6	5 8

Proof Outline for Lower Bound

Step 3: Search for matches in third group (green).

π_1:	1 4	2 5	6 3	8 7
	↓		↙	
π_2:	2 7	1 4	3 6	5 8

Proof Outline for Lower Bound

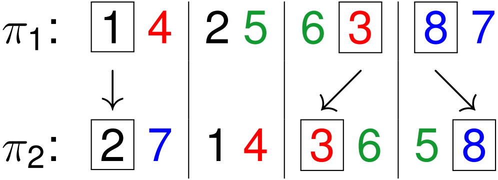
Step 4: Search for matches in fourth group (blue).

π_1: [1] 4 | 2 5 | 6 [3] | [8] 7

π_2: [2] 7 | 1 4 | [3] 6 | 5 [8]

Proof Outline for Lower Bound

Step 4: Search for matches in fourth group (blue).



Proof consists of proving that this algorithm finds, on average, at least $0.51n^{\frac{2}{3}}$ matches.

More Permutations

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Proposition. $E(L_{n,m}) < en^{\frac{m}{2m-1}}$.

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Conjecture. $E(L_{n,m}) \in \Theta\left(n^{\frac{m}{2m-1}}\right)$.

Future Work

- We showed that, asymptotically,

$$0.51 < \frac{E(L_n)}{n^{\frac{2}{3}}} < e$$

- Conjecture: There exists a constant κ such that

$$\lim_{n\to\infty} \frac{E(L_n)}{n^{\frac{2}{3}}} = \kappa$$

- Find κ, improve bounds.
- Variations on problem (consecutive patterns, permutations with specific properties).

Thanks for listening!

Questions?