Permutation Statistics and Multiple Pattern Avoidance

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Permutation Patterns '13

Pattern Avoidance

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Notations:

$\mathfrak{S}_n(\pi) := \{\sigma \in \mathfrak{S}_n : \sigma \text{ avoids } \pi\}$, $\quad \mathfrak{S}(\pi) := \bigcup_{n \geq 0} \mathfrak{S}_n(\pi)$,

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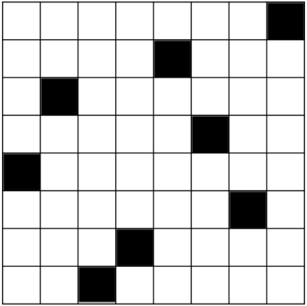
Two sets of patterns Π and Π' are *Wilf equivalent*, written $\Pi \equiv \Pi'$, if $|\mathfrak{S}_n(\Pi)| = |\mathfrak{S}_n(\Pi')|$ for all n.

Permutation Matrices

Example: The permutation 46127538.



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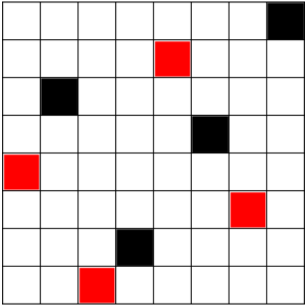
Example: The permutation 46127538 contains 3142.



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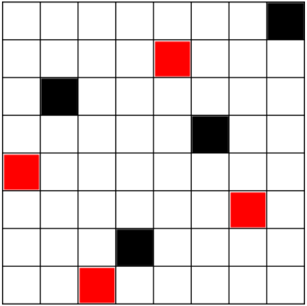
Example: The permutation 46127538 contains 3142 but avoids 4321.



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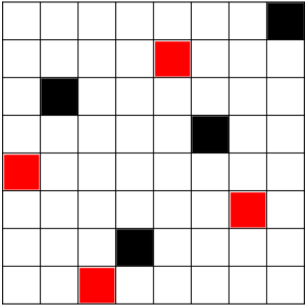

Remark: We have a D_4-action on $\mathfrak{S}_n$.

st-polynomial

For a permutation statistic $\mathrm{st} : \mathfrak{S} \to \mathbb{N}$, we define the *st-polynomial* with respect to Π as

$$F_n^{\mathrm{st}}(\Pi; q) := \sum_{\sigma \in \mathfrak{S}_n(\Pi)} q^{\mathrm{st}(\sigma)}.$$

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Theorem (DDJSS '12)

The inv-Wilf equivalent classes in $\mathfrak{S}_3$ are

$$\{123\}, \{321\}, \{132, 213\}, \{231, 312\}.$$

Conjecture (DDJSS '12)

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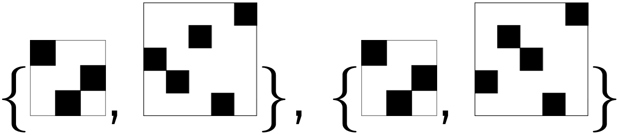

Inflation

Definition

Given permutations $\pi \in \mathfrak{S}_r$ and $\sigma_1, \ldots, \sigma_r \in \mathfrak{S}$, the *inflation* $\pi[\sigma_1, \ldots, \sigma_r]$ of π by the σ_i is the permutation whose permutation matrix is obtained by putting the permutation matrices of σ_i in the relative order of π.

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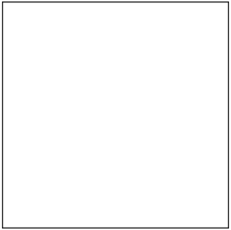

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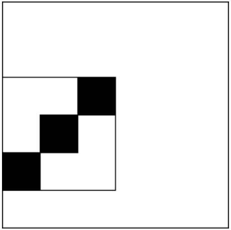

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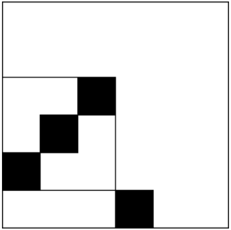

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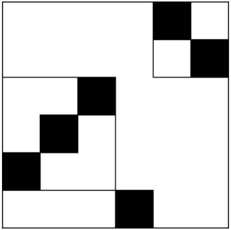

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Block decomposition

Let $\pi_* := 21[\pi, 1]$ and $\iota_r := 12 \cdots r$.
Every $\pi \in \mathfrak{S}(312)$ can be written as $\pi = \iota_r[\pi_{1*}, ..., \pi_{r*}]$ where $\pi_j \in \mathfrak{S}(312)$.

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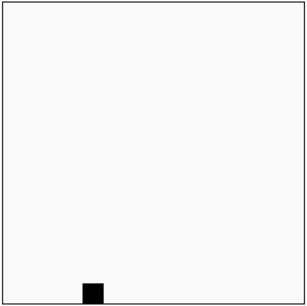

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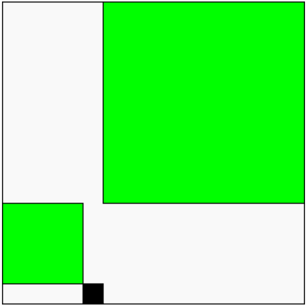

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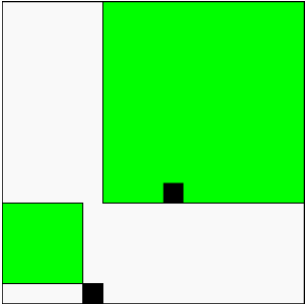

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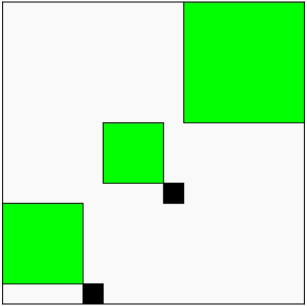

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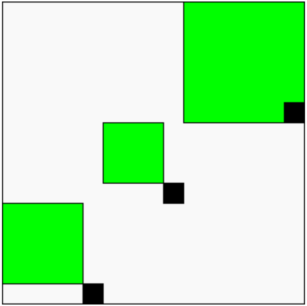

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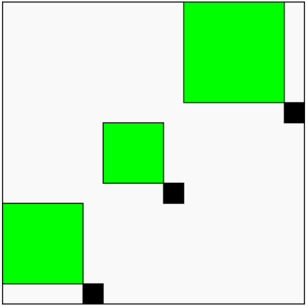

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For each $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}_{n+1}$ with $|\sigma_1| = k$,

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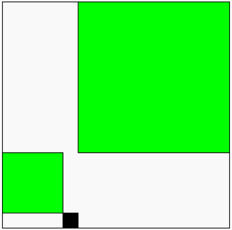

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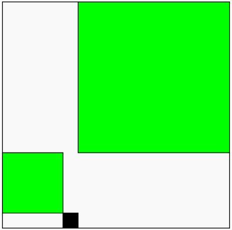

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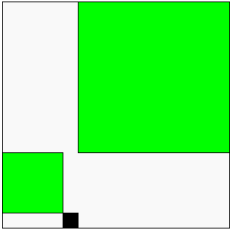

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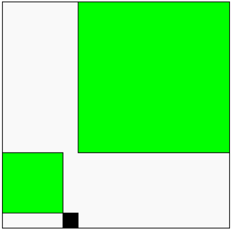

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$$F(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{qx}{1 - \cfrac{q^2 x}{1 - \cfrac{q^3 x}{\cdots}}}}}.$$

We will consider permutation statistics $\mathrm{st} : \mathfrak{S} \to \mathbb{N}$ satisfying

$$\mathrm{st}(\sigma) = f(k, n-k) + \mathrm{st}(\sigma_1) + \mathrm{st}(\sigma_2) \qquad (☺)$$

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Example: inv, des, and $\underline{213}$.

$$\mathrm{des}(\sigma) = \#\{i \in [n-1] : \sigma(i) > \sigma(i+1)\},$$
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We have

$$\mathrm{inv}(\sigma) = k + \mathrm{inv}(\sigma_1) + \mathrm{inv}(\sigma_2),$$
$$\mathrm{des}(\sigma) = 1 - \delta_{0,k} + \mathrm{des}(\sigma_1) + \mathrm{des}(\sigma_2),$$
$$\underline{213}(\sigma) = (1 - \delta_{0,k})(1 - \delta_{k,n}) + \underline{213}(\sigma_1) + \underline{213}(\sigma_2).$$

Suppose $\pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}]$. We write

$$\underline{\pi}_i := \begin{cases} \pi_1 \ \text{(not } \pi_{1*}\text{)} & \text{if } i = 1, \\ \iota_i[\pi_{1*}, \ldots, \pi_{i*}] & \text{otherwise} \end{cases}$$

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Let $\Pi = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\}$ where $\pi^{(j)} = \iota_{r_j}[(\pi_1^{(j)})_*, \ldots, (\pi_{r_j}^{(j)})_*]$. For $I = (i_1, \ldots, i_m)$, we denote

$$\underline{\Pi}_I = \{312, \underline{\pi^{(1)}}_{i_1}, \ldots, \underline{\pi^{(m)}}_{i_m}\}$$

and

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Main theorem

Theorem (T. '13+)

Let $\Pi = \{312, \pi^{(1)}, \dots, \pi^{(m)}\}$ *where* $\pi^{(j)} = \iota_r[\pi^{(j)}_{1*}, \dots, \pi^{(j)}_{r_j*}]$. *Suppose* $st : \mathfrak{S} \to \mathbb{N}$ *satisfies (☺). Then* $F^{st}_0(\Pi) = 0$ *if some* $\pi^{(j)} = \epsilon$ *and* 1 *otherwise, and for* $n \geq 1$

$$F^{st}_{n+1}(\Pi; q) = \sum_{k=0}^{n} q^{f(k,n-k)} \left[\sum_{S \subseteq [m]} (-1)^{|S|} \sum_{\substack{I=(i_1,\dots,i_m): \\ 1 \leq i_j \leq r_j - \delta_j}} F^{st}_k(\underline{\Pi}_I) \cdot F^{st}_{n-k}(\overline{\Pi}_{I+\delta}) \right],$$

where $\delta = (\delta_1, \dots, \delta_m)$ *with* $\delta_j = 1$ *if* $j \in S$ *and* 0 *if* $j \notin S$.

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Remark: The special case $q = 1$ yields the number $|\mathfrak{S}_n(\Pi)|$, which generalizes a result by Mansour and Vainshtien.

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	(1,2)	{312,12,2143}	{312,1,21}	$\delta_{0,n-k}$
{2}	(1,1)	{312,12,1}	{312,2314,21}	$\delta_{0,k}$
	(2,1)	{312,2314,1}	{312,1,21}	$\delta_{0,k} \cdot \delta_{0,n-k}$

Example

Compute $a_n = F_n^{\text{inv}}(\Pi)$, where $\Pi = \{312, 2314, 2143\}$.

Block decomposition:

$$\pi^{(1)} = 2314 = 12[12_*, \epsilon_*] \text{ and } \pi^{(2)} = 2143 = 12[1_*, 1_*].$$

S	I	$\underline{\Pi}_I$	$\overline{\Pi}_I$	$F_k^{\text{inv}}(\underline{\Pi}_I) \cdot F_{n-k}^{\text{inv}}(\overline{\Pi}_{I+\delta})$
$\emptyset$	(1,1)	{312,12,1}	Π	$\delta_{0,k} \cdot a_{n-k}$
	(1,2)	{312,12,2143}	{312,2314,21}	1
	(2,1)	{312,2314,1}	{312,1,2143}	$\delta_{0,k} \cdot \delta_{0,n-k}$
	(2,2)	Π	{312,1,21}	$\delta_{0,n-k} \cdot a_k$
{1}	(1,1)	{312,12,1}	{312,1,2143}	$\delta_{0,k} \cdot \delta_{0,n-k}$
	(1,2)	{312,12,2143}	{312,1,21}	$\delta_{0,n-k}$
{2}	(1,1)	{312,12,1}	{312,2314,21}	$\delta_{0,k}$
	(2,1)	{312,2314,1}	{312,1,21}	$\delta_{0,k} \cdot \delta_{0,n-k}$
{1, 2}	(1,1)	{312,12,1}	{312,1,21}	$\delta_{0,k} \cdot \delta_{0,n-k}$

Therefore

$$\begin{aligned} a_{n+1} &= \sum_{k=0}^{n} q^k \left[\delta_{0,k} a_{n-k} + \delta_{0,n-k} a_k + 1 - \delta_{0,k} - \delta_{0,n-k}\right] \\ &= (1+q^n)a_n + \frac{1-q^{n+1}}{1-q} - (1+q^n) \\ &= (1+q^n)a_n + q\left(\frac{1-q^{n-1}}{1-q}\right). \end{aligned}$$

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In particular, by setting $q = 1$ we get $a_{n+1} = 2a_n + n - 1$ with $a_0 = a_1 = 1$. Thus

$$|\mathfrak{S}_n(312, 2314, 2143)| = 2^n - n.$$

Proof:

Restrict everything to the set $\mathfrak{S}_{n+1}^{k}(312) = \{\sigma \in \mathfrak{S}_{n+1}(312) : \sigma(k) = 1\}$.

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Lemma

Let $\sigma = 213[\sigma_1, 1, \sigma_2], \pi = \iota_r[\pi_{1*}, ..., \pi_{r*}] \in \mathfrak{S}(312)$. *Then* σ *avoids* π *if and only if the condition*

$$(C_i) : \sigma_1 \textit{ avoids } \underline{\pi}_i \textit{ and } \sigma_2 \textit{ avoids } \overline{\pi}_i$$

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Let A^j_i, where $1 \leq j \leq m$ and $1 \leq i \leq r_j$, be the set of permutations in $\mathfrak{S}^k_{n+1}(312)$ satisfying the condition

$$(C^j_i)\text{: } \sigma_1 \text{ avoids } \underline{\pi^{(j)}}_i \text{ and } \sigma_2 \text{ avoids } \overline{\pi^{(j)}}_i.$$

(So every $\sigma \in A^j_i$ avoids $\pi^{(j)}$.)

Proof (cont'd):

For $I = (i_1, ..., i_m) \in [r_1] \times [r_2] \times \cdots \times [r_m]$, we define

$$A_I = A_{i_1, i_2, ..., i_m} := A^1_{i_1} \cap A^2_{i_2} \cap A^m_{i_m}.$$

Proof (cont'd):

For $I = (i_1, ..., i_m) \in [r_1] \times [r_2] \times \cdots \times [r_m]$, we define

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So $\mathfrak{S}^k_{n+1}(\Pi)$ is the union

$$\mathfrak{S}^k_{n+1}(\Pi) = \bigcup_{i_1,\dots,i_m} A_{i_1,i_2,\dots,i_m},$$

where the union is taken over all m-tuples $I = (i_1, ..., i_m)$ in $[r_1] \times [r_2] \times \cdots \times [r_m]$.

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Then Inclusion-Exclusion (using Möbius inversion formula)!

Nontrivial st-Wilf equivalences

Corollary

Let $\pi_i^{(j)}, \pi_i'^{(j)}, 1 \le j \le m, 1 \le i \le r_m$, *be permutations such that*

$$\{312, \pi_{i_1}^{(1)}, \ldots, \pi_{i_m}^{(m)}\} \overset{st}{\equiv} \{312, \pi_{i_1}'^{(1)}, \ldots, \pi_{i_m}'^{(m)}\}$$

for all m-tuples $I = (i_1, \ldots, i_m) \in [r_1] \times \ldots \times [r_m]$.
Setting

$$\pi^{(j)} = \iota_r[\pi_{1*}^{(j)}, \ldots, \pi_{r_j*}^{(j)}] \text{ and } \pi'^{(j)} = \iota_r[\pi_{1*}'^{(j)}, \ldots, \pi_{r_j*}'^{(j)}],$$

we have

$$\{312, \pi^{(1)}, \ldots, \pi^{(m)}\} \overset{st}{\equiv} \{312, \pi'^{(1)}, \ldots, \pi'^{(m)}\}.$$

Corollary (m=1)

Let $\pi_i, \pi'_i, 1 \le i \le r$, *be permutations such that*

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For inversion statistic, we obtain $\{312, \pi_i\} \overset{\text{inv}}{\equiv} \{312, \pi'_i\}$ by simply take each π'_i to be either π_i or π^t_i.

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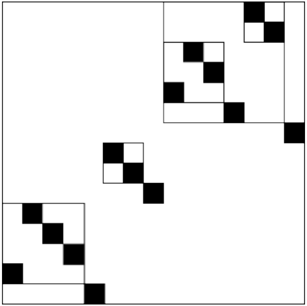
Proposition

For $\sigma \in \mathfrak{S}(312)$*,* $des(\sigma) = des(\sigma^t)$*.*

Example

Suppose $\mathrm{st} : \mathfrak{S} \to \mathbb{N}$ satisfies ☺ and that $\mathrm{st}(\sigma) = \mathrm{st}(\sigma^t)$ for all $\sigma \in \mathfrak{S}(312)$.



{312,25431876CBDAFE9}

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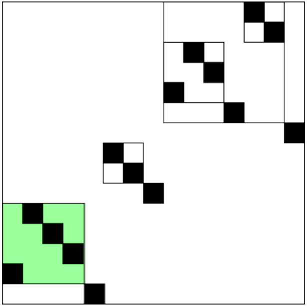

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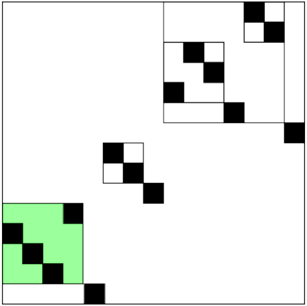

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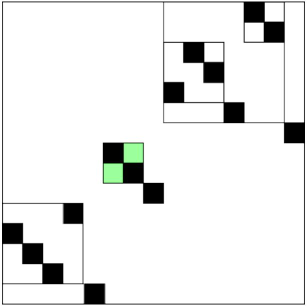

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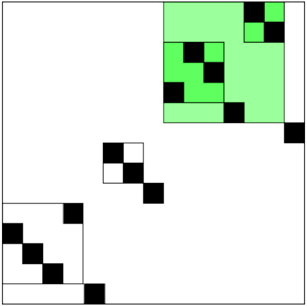

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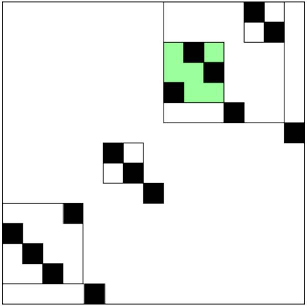

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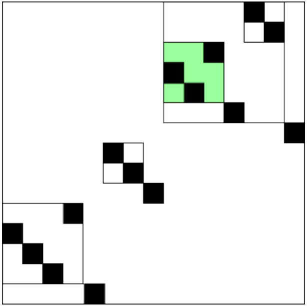

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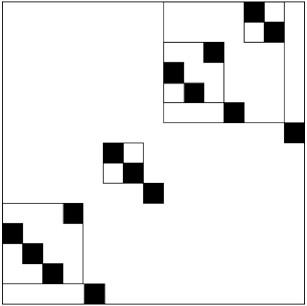

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Thank you.