Enumeration schemes to count according to permutation statistics

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Overall Goal

Example

Let $\mathrm{des}(\pi)$ be the number of descents in π, and consider $\mathfrak{S}_3(1\text{-}2\text{-}3) = \{132, 213, 231, 312, 321\}$. The distribution is:

$$\sum_{\pi \in \mathfrak{S}_3(1\text{-}2\text{-}3)} q^{\mathrm{des}(\pi)} = 4\,q^1 + 1\,q^2$$

I want to repurpose enumeration schemes to answer:
Given a set of pattern-avoiding permtuations $\mathfrak{S}_n(B)$ and a permutation statistic $f : \mathfrak{S}_n \to \mathbb{Z}$, find the distribution of f over $\mathfrak{S}_n(B)$:

$$F(\mathfrak{S}_n(B), f, q) := \sum_{\pi \in \mathfrak{S}_n(B)} q^{f(\pi)}$$

Vincular patterns

(Also called "generalized patterns" or "dashed patterns".)
Definition by examples: Permutation $\pi \in \mathfrak{S}_n$ contains a copy of 23-1 if there are indices $1 \leq i < i+1 < j \leq n$ such that $\pi_i\pi_{i+1}\pi_j \approx 231$.
Example: 24315 contains a copy of 23-1
Example: 31524 avoids 23-1.
Example: 31524 contains a copy of 2-31 (and 2-3-1).

Absence of a dash indicates adjacency required.
Presence of a dash indicates space is allowed.

Notation

For set of patterns B, let $\mathfrak{S}_n(B)$ be the set of permutations of length n which avoid every pattern in B.

Overview of Enumeration Schemes

An enumeration scheme encodes a recurrence to compute $\left|\mathfrak{S}_n(B)\right|$ for a given set of patterns B.

Enumeration schemes may be constructed algorithmically:

Input A set of patterns B, two search parameters.

Output An enumeration scheme to compute $\left|\mathfrak{S}_n(B)\right|$, *or* a proof that one does not exist within the search parameters.

Theorem (B.-Pudwell, 2012)

Enumeration schemes can be constructed algorithmically for sets B which contain only vincular patterns.

Previous Work

Theorem (Zeilberger 1998, Vatter 2008, Zeilberger 2007)

If pattern-set B admits a finite enumeration scheme, then $\left|\mathfrak{S}_n(B)\right|$ can be computed in polynomial time.

Theorem (B. 2011)

If pattern-set B admits a finite enumeration scheme, then $\sum_{\pi \in \mathfrak{S}_n(B)} q^{\mathrm{inv}(\pi)}$ can be computed in polynomial time, where $\mathrm{inv}(\pi)$ is the number of inversions.

Today: Finite enumeration schemes compute $\sum_{\pi \in \mathfrak{S}_n(B)} q^{f(\pi)}$ for other permutation statistics f.

Outline of Talk

- ✓ Introduction and Statement of Goal
- How schemes work
- ES-compatibility
- Application

Schemes Overview

Schemes follow a “divide and conquer” approach to build a recurrence.

1. Divide (partition) $\mathfrak{S}_n(B)$ according to prefix-pattern.
2. Conquer (enumerate) using gap vectors and reversibly-deletable letters.

Dividing

Notation

For permutation $p \in \mathfrak{S}_k$*, let* $\mathfrak{S}_n(B)[p]$ *be the set of all* $\pi \in \mathfrak{S}_n(B)$ *so that* $\pi_1 \cdots \pi_k \approx p$*.*

Example: $\mathfrak{S}_4(1\text{-}3\text{-}2)[12] = \{1234, 2314, 2341, 3412, 3421\}$.

Notation

For permutation $p \in \mathfrak{S}_k$ *and word* $w \in [n]^k$ *so that* $w \approx p$*, let* $\mathfrak{S}_n(B)[p; w]$ *be the set of all* $\pi \in \mathfrak{S}_n(B)[p]$ *so that* $\pi_1 \cdots \pi_k = w$*.*

Example: $\mathfrak{S}_4(1\text{-}3\text{-}2)[12; 34] = \{3412, 3421\}$.

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Partition $\mathfrak{S}_n(B)$ into these $\mathfrak{S}_n(B)[p]$.

$$\mathfrak{S}_n(B) = \mathfrak{S}_n(B)[1] = \mathfrak{S}_n(B)[12] \cup \mathfrak{S}_n(B)[21]$$

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$$\mathfrak{S}_n(B)[12] = \bigcup_{1 \le a < b \le n} \mathfrak{S}_n(B)[12; ab]$$

Conquering

How do we compute $\left|\mathfrak{S}_n(B)[p]\right|$?

For prefix $p \in \mathfrak{S}_k$, one of the following events might be true:

(a) $\mathfrak{S}_n(B)[p] = \{p\}$ or $\emptyset$ (only when $n \leq k$).

(b) For each $w \in [n]^k$ such that $w \approx p$, one of the following happens:
   - $\mathfrak{S}_n(B)[p; w]$ is empty
   - $\mathfrak{S}_n(B)[p; w]$ is in bijection with some $\mathfrak{S}_{n'}(B)[p'; w']$ for $n' < n$.

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- $\mathfrak{S}_n(B)[p; w]$ is empty **(Gap vector criteria)**
- $\mathfrak{S}_n(B)[p; w]$ is in bijection with some $\mathfrak{S}_{n'}(B)[p'; w']$ for $n' < n$. **(Reversible deletions)**

Event (b) is detected through **gap vector criteria** and **reversible deletions**.

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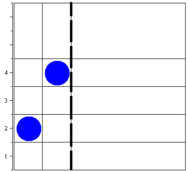
If neither (a) or (b) occurs, then $\mathfrak{S}_n(B)[p]$ must be partitioned further.

Gap Vectors

For what w does $\mathfrak{S}_n(B)[p; w] = \emptyset$?

Example: No permutation avoids 1-3-2 and has the first two letters $\pi_1\pi_2 = 24$. *The vertical "gap" between these first two letters is too large.*



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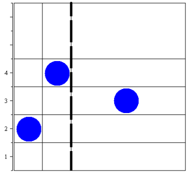

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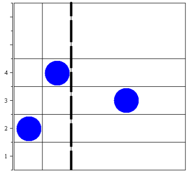

For prefix p, a *gap vector* $\vec{g}$ encodes a vertical space condition on w for when $\mathfrak{S}_n(B)[p; w] = \emptyset$.

Example: $\mathfrak{S}_n(1\text{-}3\text{-}2)[12; w_1 w_2] = \emptyset$ if $w_2 - w_1 > 1$. This is encoded as $\langle 0, 1, 0 \rangle$.

Reversible Deletions

Let $d_R : \mathfrak{S}_n \to \mathfrak{S}_{n-|R|}$ be the map deleting π_r for each $r \in R$ and "reducing" the resulting word.

Example: $d_{\{2,4\}} : 2\underline{5}7\underline{3}641 \mapsto 2\ 7\ 641$

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For domain $\mathfrak{S}_n[p; w]$ and $R \subseteq \{1, \ldots, |p|\}$, d_R is a bijection:

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R is *reversibly deletable for prefix* p if d_R restricts to a bijection:

$$d_R : \mathfrak{S}_n(B)[p; w] \to \mathfrak{S}_{n-|R|}(B)[p'; w']$$

whenever $\mathfrak{S}_n(B)[p; w] \neq \emptyset$.

Example: $d_{\{1\}} : \mathfrak{S}_n(1\text{-}3\text{-}2)[21; ab] \to \mathfrak{S}_{n-1}(1\text{-}3\text{-}2)[1; b]$ is bijective when $a > b$.

The scheme for $\left|\mathfrak{S}_n(1\text{-}3\text{-}2)\right|$

Let $B = \{1\text{-}3\text{-}2\}$. The scheme for $\left|\mathfrak{S}_n(B)\right|$ looks like:

$$E = \left\{(1, \emptyset, \emptyset), (12, \{\langle 0, 1, 0\rangle\}, \{1\}), (21, \emptyset, \{1\})\right\}$$

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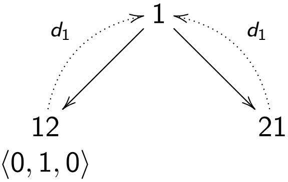

The recurrence for $\left|\mathfrak{S}_n(1\text{-}3\text{-}2)\right|$ from the scheme

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$$\begin{aligned}\left|\mathfrak{S}_n(B)\right| &= \left|\mathfrak{S}_n(B)[1]\right| \\ &= \sum_{a=1}^{n} \left|\mathfrak{S}_n(B)[1;a]\right|\end{aligned}$$

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&= \sum_{b=1}^{a-1} \left|\mathfrak{S}_{n-1}(B)[1;b]\right| + \left|\mathfrak{S}_{n-1}(B)[1;a]\right| + 0
\end{aligned}$$

The recurrence for $|\mathfrak{S}_n(1\text{-}3\text{-}2)|$ from the scheme

Let $B = \{1\text{-}3\text{-}2\}$. The scheme for $|\mathfrak{S}_n(B)|$ encodes the recurrence:

$$\begin{aligned}
|\mathfrak{S}_n(B)| &= |\mathfrak{S}_n(B)[1]| \\
&= \sum_{a=1}^{n} |\mathfrak{S}_n(B)[1;a]| \\
|\mathfrak{S}_n(B)[1;a]| &= \sum_{b=1}^{a-1} |\mathfrak{S}_n(B)[21;ab]| + \sum_{b=a+1}^{n} |\mathfrak{S}_n(B)[12;ab]| \\
&= \sum_{b=1}^{a-1} |\mathfrak{S}_{n-1}(B)[1;b]| + |\mathfrak{S}_{n-1}(B)[1;a]| + 0 \\
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\end{aligned}$$

But we want to compute more than just the cardinality $\left|\mathfrak{S}_n(B)\right|$.

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- ✓ Introduction and Statement of Goal
- ✓ How schemes work
- ES-compatibility
- Application

R-deletion difference

Definition

Let $f : \bigcup_{n\geq 0} \mathfrak{S}_n \to \mathbb{Z}$ be a permutation statistic. For a permutation π, define the *R-deletion difference* to be

$$(\Delta_R f)(\pi) := f(\pi) - f(d_R(\pi)).$$

i.e., $(\Delta_R f)(\pi)$ measures how f changes when applying d_R.

Example:
$(\Delta_{\{1\}}\mathrm{des})(621534) = \mathrm{des}(621534) - \mathrm{des}(21534) = 1.$
i.e., 1 descent is lost.

ES-compatibility

Definition

A permutation statistic f is *ES-compatible with margin m* if for any prefix pattern $p \in \mathfrak{S}_k$ for $k \geq m$ and any $R \subseteq \{1, \ldots, k-m\}$ the difference $\Delta_R f$ is constant over any nonempty set $\mathfrak{S}_n[p; w]$.

i.e., the deletion d_R changes the value of f by the same amount for all permutations in $\mathfrak{S}_n[p; w]$ as long as R does not cut too close to the right edge of p.

Trivial Example: The length of a permutation $\ell(\pi_1 \cdots \pi_n) = n$ is ES-compatible with margin 0, since $\Delta_R \ell(\pi) = |R|$ for any $\pi \in \mathfrak{S}_n$ with $n \geq |R|$.

Non-example: The "final letter" $f(\pi_1 \cdots \pi_n) = \pi_n$ is **not** ES-compatible for any margin. 625431 and 621345 both lie in $\mathfrak{S}_n[21; 62]$, but $\Delta_{\{2\}} f(625431) = 0$ and $\Delta_{\{2\}} f(621345) = 1$.

ES-compatibility and schemes

Notation: For a set of permutations S and permutation statistic f, let $F(S, f, q) = \sum_{\pi \in S} q^{f(\pi)}$.

Theorem (B., 2013)

Let f be an ES-compatible permutation statistic with margin m. If R is reversibly-deletable for p and $\max R + m \leq |p|$, then there is an integer $\delta(f, n, R, w)$ such that

$$F(\mathfrak{S}_n(B)[p; w], f, q) = q^{\delta(f,n,R,w)} F(\mathfrak{S}_{n-|R|}(B)[p'; w'], f, q),$$

Remark: The analogous result holds when f is replaced by a multistatistic $\mathbf{f} = \langle f_1, \ldots, f_s \rangle$ and the weights are given by $\mathbf{q}^{\mathbf{f}} = {q_1}^{f_1(\pi)} \ldots {q_s}^{f_s(\pi)}$

Example: Descents over $\mathfrak{S}_n(1\text{-}3\text{-}2)$

Example: des is an ES-compatible statistic with margin 1. Let $B = \{1\text{-}3\text{-}2\}$. The scheme from earlier implies:

$$F(\mathfrak{S}_n(B)[12; a(a+1)], \mathrm{des}, q) = q^0\, F(\mathfrak{S}_{n-1}(B)[1; a], \mathrm{des}, q)$$
$$F(\mathfrak{S}_n(B)[21; ab], \mathrm{des}, q) = q^1\, F(\mathfrak{S}_{n-1}(B)[1; b], \mathrm{des}, q)$$

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Let $G(n, a) := F(\mathfrak{S}_n(B)[1; a], \operatorname{des}, q)$. Then the above implies:

$$F(\mathfrak{S}_n(B), \operatorname{des}, q) = \sum_{a=1}^{n} G(n, a)$$

$$G(n, a) = G(n-1, a) + \sum_{b=1}^{a-1} q\, G(n-1, b)$$

Non-trivial examples of ES-compatible statistics

Theorem (B., 2013)

Let $\sigma \in \mathfrak{S}_t$ and $f(\pi)$ be the number of copies of the (consecutive) pattern $\sigma_1 \cdots \sigma_t$ in π. Then f is an ES-compatible statistic with margin $t-1$.

Remark: $\mathrm{des}(\pi) = f(\pi)$ for $\sigma = 21$.

Theorem (B., 2013)

Let $\sigma \in \mathfrak{S}_t$ and let $g(\pi)$ be the number of copies of the pattern $\sigma_1 \cdots \sigma_{t-1}\text{-}\sigma_t$ in π. Then g is an ES-compatible statistic with margin $t-2$.

Remark: $\mathrm{inv}(\pi) = g(\pi)$ for $\sigma = 2\text{-}1$.

Non-trivial examples of ES-compatible statistics

Theorem (B., 2013)

The following statistics are ES-compatible with margin 0*:*

- *The number of right-to-left maxima*
- *The number of right-to-left minima*

Non-trivial examples of ES-compatible statistics

We may use the multistatistic versions to prove:

Corollary (B., 2013)

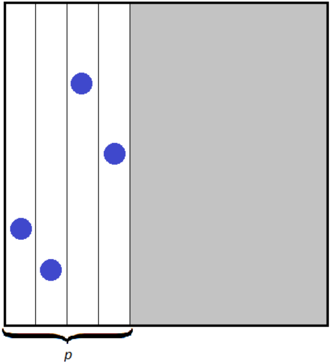
Any linear combination of ES-compatible statistics is also ES-compatible.

Examples:

- $\mathrm{maj}^r(\pi) := \mathrm{maj}(\pi^r) =$ $(23\text{-}1)(\pi) + (13\text{-}2)(\pi) + (12\text{-}3)(\pi) + (12)(\pi)$ is ES-compatible with margin 1.
- The peak-number $\mathrm{peak}(\pi) = (231)(\pi) + (132)(\pi)$ is ES-compatible with margin 2.

Concept of proof for consecutive pattern function

Let $f(\pi)$ be the number of copies of 123 in π. Why is f ES-compatible with margin 2?



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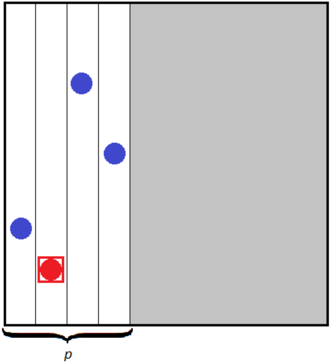
Suppose $R = \{2\}$ is reversibly-deletable for a prefix p of length at least 4.



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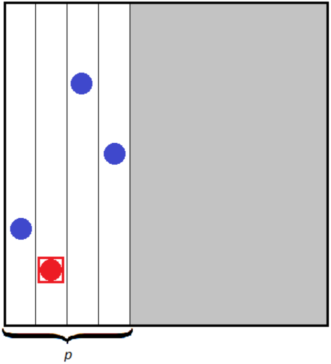

Concept of proof for vincular pattern function

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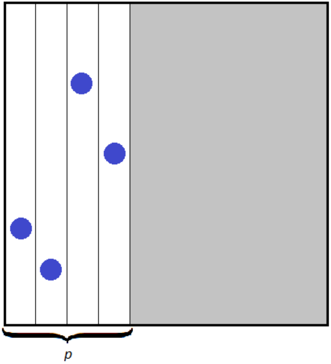

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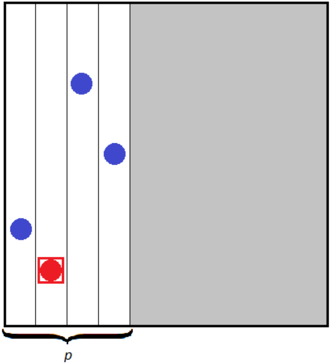

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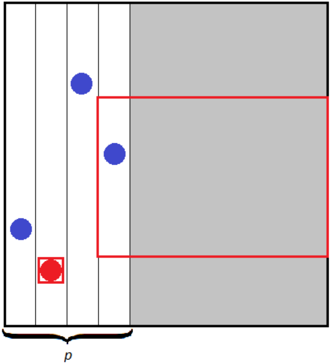

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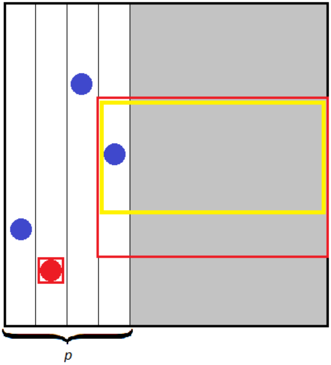
Let $g(\pi)$ be the number of copies of 13-2 in π. Why is g ES-compatible with margin 1?
Suppose $R = \{2\}$ is reversibly-deletable for a prefix p of length at least 4. Then we know how many copies of 13-2 π_2 is part of, and how many will be present after applying $d_{\{2\}}$.



Accommodating margins

Potential Problem: Not all enumeration schemes accommodate statistics with "large" margins.
Example: The earlier scheme for 1-3-2 can only accommodate statistics with margin at most 1.

Accommodating margins

Potential Problem: Not all enumeration schemes accommodate statistics with “large” margins.
Example: The earlier scheme for 1-3-2 can only accommodate statistics with margin at most 1.

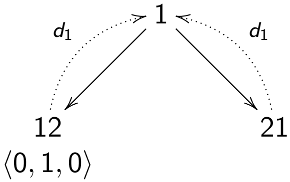
Theorem (B., 2013)

Let $m \geq 0$. If B admits a finite enumeration scheme E, then B admits a finite enumeration scheme E' which can accommodate a ES-compatible statistic of margin m.

Drawback: The encoded recurrence for E' is more complicated (but still polynomial time).

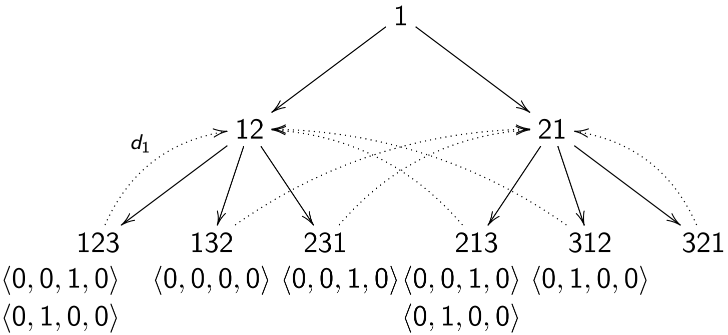
A deepened scheme

Scheme for $\mathfrak{S}_n(1\text{-}3\text{-}2)$ for statistics with margin ≤ 1.



A deepened scheme

Scheme for $\mathfrak{S}_n(1\text{-}3\text{-}2)$ for statistics with margin ≤ 2.



Outline of Talk

- ✓ Introduction and Statement of Goal
- ✓ How schemes work
- ✓ ES-compatibility
- Applications

Implementation

The Maple package `Statter`, available for download, can do the following:

1. **Discover** an enumeration scheme for $\mathfrak{S}_n(B)$ (if one exists within search parameters) which can accomodate given margins for ES-compatible statistics.
2. **Read** enumeration schemes to compute the distribution $\sum_{\pi \in \mathfrak{S}_n(B)} q^{f(\pi)}$ for (multi)statistics f chosen from:
   - Number of copies of a consecutive pattern
   - Number of copies of a vincular pattern of the form $\sigma_1 \cdots \sigma_{k-1}\text{-}\sigma_k$
   - Number of right-to-left maxima or minima.

Applications

Data from `Statter` suggested the following information about the distributions of the number of peaks over classical avoidance classes, which were proven by other methods.

Theorem (B., 2013)

$$\sum_{\pi \in \mathfrak{S}_n(1\text{-}2\text{-}3)} q^{\mathrm{peak}(\pi)} = \sum_{\pi \in \mathfrak{S}_n(3\text{-}2\text{-}1)} q^{\mathrm{peak}(\pi)} = \sum_{\pi \in \mathfrak{S}_n(3\text{-}1\text{-}2)} q^{\mathrm{peak}(\pi)}$$

Theorem (B., 2013)

The distribution of peaks over $\mathfrak{S}_n(1\text{-}3\text{-}2)$ equals the distribution of subfactors DDU over Dyck words of length n.

Other directions

The distributions $F(\mathfrak{S}_n(B), f, q) = \sum_{\pi \in \mathfrak{S}_n(B)} q^{f(\pi)}$ can be mined for many other qualities:

- deg $F(\mathfrak{S}_n(B), f, q)$ corresponds to "pattern-packing" questions.
- $\frac{d}{dq}\left[F(\mathfrak{S}_n(B), f, q)\right]\Big|_{q=1}$ corresponds to "total number of copies" questions.
- For what combinations of B and f are the distributions $F(\mathfrak{S}_n(B), f, q)$ symmetric? unimodal? log-concave? asymptotically normal?
- Do the distributions exhibit cyclic-sieving properties?

(I will take requests for data.)

Conclusion

Enumeration schemes can now compute (in polynomial time) the number of B-avoiding permutations of length n with k copies of a consecutive or "nearly-consecutive" pattern. In some cases, the recurrences themselves lead to proofs of these conjectures.

These procedures are implemented in the Maple package `Statter`, available for download.

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Thank you!