Additional Structure on Baxter Permutations

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Outline

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What is a Baxter Permutation?

Definition

A *Baxter permutation* is a permutation that avoids the vincular patterns 3-14-2 and 2-41-3. This is to say that there are no instances of the patterns 3142 or 2413 where the letters representing 1 and 4 are adjacent in the original word.

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(I'm not married to the name sub-Baxter, open to suggestions)

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- Taking the inverse of a permutation / flipping the permutation matrix across its diagonal.
- Combining one of the first two with the third leads to a quarter-turn of the permutation matrix.

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- Easier to look at Aztec Diamond/floorplan tilings.

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Goal

Look at what happens when you consider how many permutations/Baxter permutations/sub-Baxter permutations are fixed under the actions $w \mapsto w$ (enumeration), $w \mapsto w^{-1}$, $w \mapsto w_0 w w_0$, $w \mapsto w_0 w^{-1}$.

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$n!$

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One can analyze how the permutations are distributed with respect to a statistic, for example, descents.

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Conjecture

$A_n(x, y) = \sum \gamma_{i,j} (xy)^i (x + y)^j (1 + xy)^{n-1-2i-j}$
with $\gamma_{i,j} \geq 0$

Number of Baxter Permutations

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Theorem (Chung, Graham, Hoggatt, Kleiman)

The number of Baxter permutations of length n is

$$B(n) := \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}}$$

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For $n = 1, 2, 3 \ldots$, $B(n) = 1, 2, 6, 22, 92, 422, 2074, 10754 \ldots$.

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Theorem (Mallows)

The number of Baxter permutations with k ascents is given by the k^{th} summand.

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Theorem

$$B(n,x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n+3)_i(1-n)_{2i}}{(1)_i(2)_i(3)_i} x^i(1+x)^{n-1-2i}$$

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Combinatorial intepretation for γ_i?

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Corollary

$$B(n,x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n+3)_i (1-n)_{2i}}{(1)_i (2)_i (3)_i} (xy)^i (1+xy)^{n-1-2i}$$

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- Twisted Baxter permutations (Av(2-41-3,3-41-2))

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- Twisted Baxter permutations (Av(2-41-3,3-41-2))
- Standard $3 \times n$ Young tableaux with no consecutive entries in any row

Twin Trees

Definition

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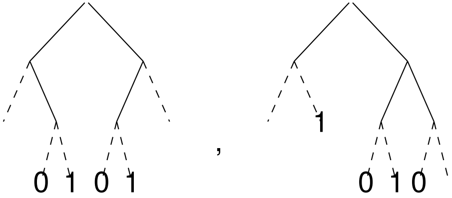

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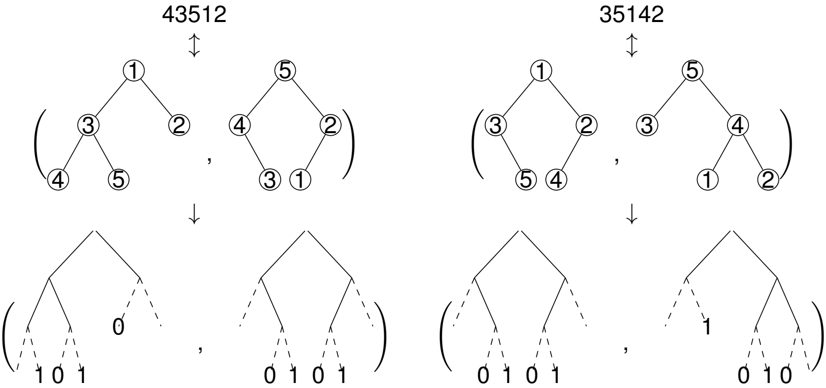
Theorem (Dulucq and Guibert)

There is a bijection between Baxter permutations and pairs of truncated twin trees

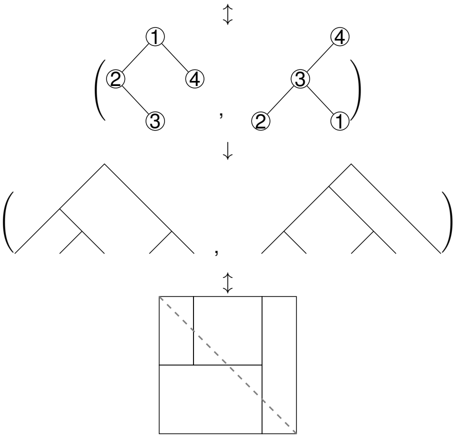
The map is given by taking the increasing and decreasing trees of the permutation, and then forgetting their labels.

Example



Map From Baxter to Diagonal Rectangulations

$w = 2314$



sub-Baxter Permutations

Theorem (Asinowski, Barequet, Bousquet-Melou, Mansour, Pinter)

The number of sub-Baxter permutations is

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- Is descent the right statistic?

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53827164

Naturally in bijection with signed permutations.

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5**3**8271**6**4

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Map+Spin=Spin+Map

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Upshot

Only need to find one Baxter object where it's easy to count the rotationally invariant ones.

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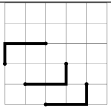
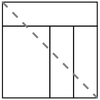
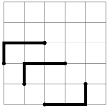
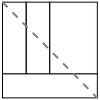
- Complementation of plane partition in the box.

Chart of Examples

Baxter Perms	Baxter Paths	Baxter Tableaux	Diagonal Rects	Baxter Plane Partitions
2341		1 4 6 9 2 5 8 11 3 7 10 12		2 2
↕	↕	↕	↕	↕
4123		1 3 6 10 2 5 8 11 4 7 9 12		1 1

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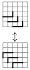
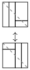
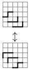
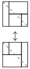
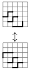
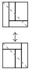
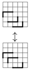
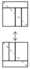
Baxter Permutations	Twisted Baxter Permutations	Baxter Paths	Baxter Tableaux	Diagonal Rectangulations	Baxter Plane Partitions
1243 ↕ 2134	1243 ↕ 2134		1 3 6 9 / 2 5 7 11 / 4 8 10 12 ↕ 1 3 5 9 / 2 6 8 11 / 4 7 10 12		3 3 ↕ 0 0
1342 ↕ 3124	1342 ↕ 3124		1 3 6 9 / 2 4 8 11 / 5 7 10 12 ↕ 1 3 6 8 / 2 5 9 11 / 4 7 10 12		3 2 ↕ 1 0
↕ 1423 ↕ 2314	↕ 1423 ↕ 2314	↕ 	↕ 1 3 5 7 / 2 6 9 11 / 4 8 10 12 ↕ 1 3 5 9 / 2 4 7 11 / 6 8 10 12	↕ 	↕ 3 1 ↕ 2 0
2341 ↕ 4123	2341 ↕ 4123		1 4 6 9 / 2 5 8 11 / 3 7 10 12 ↕ 1 3 6 10 / 2 5 8 11 / 4 7 9 12		2 2 ↕ 1 1
1324 ↺	1324 ↺	 ↺	1 3 5 7 / 2 4 9 11 / 6 8 10 12 ↺	 ↺	3 0 ↺
3412 ↺	3142 ↺	 ↺	1 3 7 9 / 2 5 8 11 / 4 6 10 12 ↺	 ↺	2 1 ↺

q-analog

For $\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}$, $[m]!_q = [m]_q[m-1]_q \dots [1]_q$, and $[j]_q = 1 + q + \dots + q^{j-1}$, we have

$$\sum_{\pi} q^{|\pi|} = \frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q}{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q} := \Theta_{k,n-k-1}(q)$$

where π runs over all plane partitions in the $k \times (n-k-1) \times 3$ box.

q=-1 Phenomenon

Theorem (Stembridge)

The number of self complementary plane partitions in an $a \times b \times c$ box is equal to $\sum_\pi q^{|\pi|}|_{[q=-1]}$, where π runs over all plane partitions in an $a \times b \times c$ box.

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Corollary

The number of self complementary Baxter plane partitions in a $k \times n-k-1 \times 3$ box is equal to $\Theta_{k,n-k-1}(-1)$.

q=-1 Phenomenon

Theorem (D.)

The number of Baxter objects with parameter k fixed under their natural involution is given by
$$\frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q}{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q}\Bigg|_{[q=-1]}$$

n=4, k=1

$$\Theta_{k,n-k-1}(q) = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$

$$\lim_{q \to 1} \Theta_{k,n-k-1}(q) = 10$$

$$\lim_{q \to -1} \Theta_{k,n-k-1}(q) = 2$$

Half Rotation on sub-Baxter Permutations

- Nothing obvious.

Half Rotation on sub-Baxter Permutations

- Nothing obvious.
- Possibly $q = -1$ phenomenon, but what is q?

Outline

Self Involutive Permutations

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Theorem (Robinson, Schensted)

There is a bijection between permutations of length n and pairs of standard Young tableaux of the same shape (and size n).

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There's a geometric version of RS due to Viennot that makes this fact obvious.

Self Involutive Permutations

Theorem

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$$\sum_{\lambda \vdash n} f^{\lambda},$$

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Other Interesting Fact

Number of fixed points of w is number of odd columns in $sh(P)$.

Self Involutive Baxter permutations

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- Unlike conjugation by w_0, inverse action not carried to all Baxter objects.

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- Only have univariate formulas for fixed-point free self-involutive, and multivariate formulas.

Self Involutive Baxter permutations

Theorem (Bousquet-Melou)

The number of involutive fixed-point free Baxter permutations of length $2n$ is

$$\frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}$$

Self Involutive Baxter permutations

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- Combinatorial proof of this fact, and multivariate formula given by Fusy by looking at plane bipolar orientations.

Theorem (Fusy)

The number $a_{n,k,p,r}$ of self-involutive Baxter permutations with $2n$ non-fixed points, $2k$ descents not crossing the diagonal, p fixed points, and r descents crossing the diagonal is

$$\frac{\binom{p+r}{r}\binom{n+p-1}{k}^2\binom{n}{t}}{nq^2(q+1)(k+1)(t+1)}\begin{vmatrix} q(q+1) & q(q-1) & s(s-1) \\ k(q+1) & (k+1)q & s(t+1) \\ k(k-1) & k(k+1) & t(t+1) \end{vmatrix}$$

for $q := n+p-k$, $s := n-k-r$, $t := k+r$.

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- Grand Dyck paths of length $2n$ are words with exactly n 0's and n 1's.
- Self involutive sub-Baxter permutations appear to be in bijection with Grand Dyck paths that avoid the consecutive sequences 101 and 010 (zig-zag avoiding).
- Fixed point free self-involutive sub-Baxter permutations of length $2n$ appear to be equinumerous with sub-Baxter permutations of length $n - 1$.

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Quarter Rotation on Permutations

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- Decomposes into 4-cycles of the form (i j n+1-i n+1-j), possibly with central fixed point.

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- Counted by $\frac{(2k)!}{k!}$,quadruple factorial numbers.
- Other objects counted by this, unsure if there is a bijection.

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Proof.

Generating trees □

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- Consider all trees with positive integer weights on the nodes so that the weight of a parent is the sum of the weight of their children.

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Quarter Rotation on sub-Baxter Permutations

- Consider all trees with positive integer weights on the nodes so that the weight of a parent is the sum of the weight of their children.
- The weight of a tree is the weight of its root.
- It appears that sub-Baxter permutations of length $n = 4k$ or $4k + 1$ fixed under order 4 rotation are equinumerous with trees of weight k.

Thanks for sticking around!