Multigraded Castelnuovo-Mumford regularity and Gröbner bases

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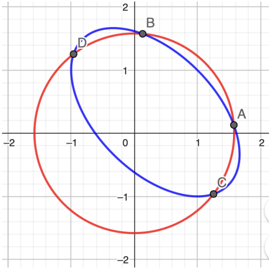
- On-going joint work with
  - Laurent Busé (Inria - Université Côte d'Azur),
  - Carles Checa (ATHENA - NKU Athens), and
  - Elias Tsigardias (Inria, IMJ-PRG).
- Questions:
  - How hard is to compute a Gröbner bases for a multihomogeneous ideal?
  - What determines this hardness?
- Our answer:
  - Multigraded Castelnuovo-Mumford regularity (+ other invariants...)

Computational Algebraic Geometry

Geometry $\xleftrightarrow[\text{regularity}]{\text{Castelnuovo-Mumford}}$ Algebra $\xleftrightarrow[\text{bases}]{\text{Gröbner}}$ Computations

(Projective varieties) (Homogeneous ideals)



$$\begin{cases} 3x_0^2 - 2x_1x_0 - 2x_2x_0 + 2x_1x_2, \\ -5x_0^2 + 2x_1^2 + 2x_2^2, \\ 2x_0^3 - 3x_0^2x_1 - x_0^2x_2 - 2x_0\,x_1^2 + 2x_1^3 \end{cases}$$

$$\begin{cases} 2\,x_1^2 + 2\,x_2^2 - 5\,x_0^2 = 0, \\ x_1^2 + x_1x_2 + x_2^2 - \\ x_1x_0 - x_2x_0 - x_0^2 = 0 \end{cases}$$

Geometry $\leftrightarrow$ Algebra: Empty case

Hilbert's nullstellensatz

Given homogeneous ideal $I \subset S := \mathbb{C}[x_0, \dots, x_n]$,

$$V_{\mathbb{P}^n}(I) = \emptyset \iff \exists \text{ sufficiently big } d_0 \text{ st } (\forall d \geq d_0)\ I_d = S_d$$

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Examples

$$\langle x_0, x_1^2 \rangle_d = C[x_0, x_1]_d \text{ for } d \geq 2 \qquad \langle x_0^3, x_1^2 \rangle_d = C[x_0, x_1]_d \text{ for } d \geq 4$$

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Castelnuovo-Mumford regularity in empty case

Smallest d_0 such that $I_{d_0} = S_{d_0}$

Geometry $\leftrightarrow$ Algebra: Hilbert function = polynomial

Hilbert polynomial

Consider homogeneous $I \subset S := \mathbb{C}[x_0, \ldots, x_n]$.

$$\mathrm{HilbertFunction}_{S/I}(d) := \dim_{\mathbb{C}}\left((S/I)_d\right)$$

There exist a polynomial $\mathrm{HilbertPolynomial}_{S/I}(d) \in \mathbb{Z}[d]$ and a sufficiently big d_0 such that, if $d \geq d_0$,

$$\mathrm{HilbertFunction}_{S/I}(d) = \mathrm{HilbertPolynomial}_{S/I}(d).$$

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If $V_{\mathbb{P}^n}(I)$ is a finite set of δ points (counted multiplicities), then

$$\mathrm{HilbertPolynomial}_{S/I}(d) = \delta.$$

$$\left.\begin{array}{l} I = \langle x_0^2\,(x_0^2 - x_1^2), x_1^2\,(x_0^2 - x_1^2)\rangle \\ V_{\mathbb{P}^n}(I) = \{(1:1),(1:-1)\} \end{array}\right\} \quad \dim_{\mathbb{C}}\left((\mathbb{C}[x_0, x_1]/I)_d\right) = 2, \text{ for } d \geq 5$$

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Castelnuovo-Mumford regularity $\geq$ smallest d_0.

Geometry $\leftrightarrow$ Algebra: Saturation

Equality of projective schemes

Let $\mathfrak{m}_{\boldsymbol{x}} := \langle x_0, \ldots, x_n \rangle$ be irrelevant ideal of $S := \mathbb{C}[x_0, \ldots, x_n]$.
Given homogeneous $I, J \subset S$,

$$\mathrm{Proj}(I) = \mathrm{Proj}(J) \iff \exists \text{ sufficiently big } d_0 \text{ st } (\forall d \geq d_0)\, I_d = J_d$$

In particular, there is big enough d_0 st $(\forall d \geq d_0)$ the ideal I is saturated at degree d, that is,

$$I_d = (I : \mathfrak{m}_{\boldsymbol{x}}^{\infty})_d$$

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Castelnuovo-Mumford regularity $\geq$ smallest d_0.

Geometry $\leftrightarrow$ Algebra: Castelnuovo-Mumford (CM) regularity

CM regularity in terms of the Betti numbers [Eisenbud-Goto '84]

Let $S := \mathbb{C}[x_0, \ldots, x_n]$ and $\{\beta_{i,j}\}_{i,j}$ be the graded Betti numbers of I (shifts in minimal free resolution).

$$0 \to \bigoplus_j S(-j)^{\beta_{r,j}} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}} \to \bigoplus_j S(-j)^{\beta_{0,j}} \to I \to 0$$

CM regularity $\sim$ maximal shift in minimal free resolution

$$\mathrm{reg}(I) = \max_{i,j}(j - i : \beta_{i,j} \neq 0)$$

$I = \langle x_0^4 - x_0^2 x_1^2, x_0^2 x_1^2 - x_1^4 \rangle$

$$0 \to S(-6) \xrightarrow{\begin{pmatrix} -x_1^2 & x_0^2 \end{pmatrix}} S(-4)^2 \xrightarrow{\begin{pmatrix} x_0^4 - x_0^2 x_1^2 \\ x_0^2 x_1^2 - x_1^4 \end{pmatrix}} I \to 0$$

$$\mathrm{reg}(I) = \max(\mathbf{6} - 1, \mathbf{4} - 0) = 5$$

- Betti numbers: $\mathrm{reg}(I) = \max_{i,j}(j - i : \beta_{i,j}(I) \neq 0)$

CM regularity in terms of linear resolution [Eisenbud-Goto '84]

Given a degree$d$ and an ideal homogeneous $I \subset S$, its d-truncated ideal is

$$I_{\geq d} := \bigoplus_{i \geq d} I_d.$$

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CM regularity = minimal d st d-truncation has linear resolution.

$$\mathrm{reg}(I) = \min(d : I_{\geq d} \text{ has a linear resolution})$$

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$$0 \to S(-\mathbf{5}-1)^3 \xrightarrow{\begin{pmatrix} -x_1 & x_0 & 0 & 0 \\ 0 & -x_1 & x_0 & 0 \\ 0 & 0 & -x_1 & x_0 \end{pmatrix}} S(-\mathbf{5})^4 \xrightarrow{\begin{pmatrix} x_0^5 - x_0^3 x_1^2 \\ x_0^4 x_1 - x_0^2 x_1^3 \\ x_0^3 x_1^2 - x_0 x_1^4 \\ x_0^2 x_1^3 - x_1^5 \end{pmatrix}} I_{\geq \mathbf{5}} \to 0$$

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CM reg via local cohomology [Castelnuovo'1896] [Mumford'66]

CM regularity $\sim$ minimal shift st local cohomology wrt $\mathfrak{m}_x$ vanishes.

$$\mathrm{reg}(I) = \min\left(d : (\forall i)\left(H^i_{\mathfrak{m}_x}(I)\right)_{d+i-1} = 0\right)$$

Vanishing of first local cohomology module = Ideal is saturated wrt $\mathfrak{m}_x$

$$H^1_{\mathfrak{m}_x}(I) = (I : \mathfrak{m}_x^\infty)/I$$

$$(\langle x_0^2 - x_1^2\rangle / \langle x_0^4 - x_0^2\, x_1^2, x_0^2\, x_1^2 - x_1^4\rangle)_5 = 0$$

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CM regularity in terms of colon ideals [Bayer-Stillman '87]

CM regularity of an ideal I generated in degree k, is the minimal degree $d \geq k$ st there are (generic) linear forms $\ell_0, \ldots, \ell_n \in S$ satisfying

$$\begin{cases} (\forall i \geq 0)\ (\langle I, \ell_0, \ldots, \ell_{i-1} \rangle : \ell_i)_d = \langle I, \ell_0, \ldots, \ell_{i-1} \rangle_d, \\ \langle I, \ell_0, \ldots, \ell_n \rangle_d = S_d \end{cases}$$

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Consider $I = \langle x_0^4 - x_0^2\, x_1^2, x_0^2\, x_1^2 - x_1^4\rangle$ and let $\ell_0 = x_0, \ell_1 = x_1$.

$(I : x_0) = \langle x_0^3 - x_0\, x_1^2, x_0^2\, x_1^2 - x_1^4\rangle, \quad \langle I, x_0\rangle = \langle x_0, x_1^4\rangle, \quad (\langle I, x_0\rangle : x_1) = \langle x_0, x_1^3\rangle$

CM Regularity is 5 as $\begin{cases} x_1\,(x_0^3 - x_0\, x_1^2) \in (I : x_0)_4 \not\subseteq I_4 \\ (I : x_0)_5 = I_5 \\ (\langle I, x_0\rangle : x_1)_5 = \langle I, x_0\rangle_5 \end{cases}$

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CM regularity is independent of the coordinates $x_0, \ldots, x_n$

Geometry $\leftrightarrow$ Algebra: Bounds on CM regularity

The Castelnuovo-Mumford regularity can be big...

...very big [Galligo '79] [Giusti '84]

Consider homogeneous $I \subset \mathbb{C}[x_0, \ldots, x_n]$ generated in degree $\leq d$. Then,

$$\mathrm{reg}(I) \leq (2\,d)^{2^{n-1}}.$$

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...and it can not be avoided [Mayr-Meyer '82]

There is an ideal generated in degree 4 st its regularity $\geq 2^{2^{n/10}} + 1$.

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...but generically is small

If $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$ is a regular sequence of degs $\leq d$, then

$$\mathrm{reg}(\langle f_1, \ldots, f_r \rangle) \leq \sum_i \mathrm{degree}(f_i) - r + 1 \leq d\,(n+1)$$

Algebra $\leftrightarrow$ Computations: Gröbner bases

We fix degree reverse lexicographical monomial order $>$ ($\textsc{GRevLex}$) st

$$x_0 < \cdots < x_n$$

Initial ideal of I wrt $\textsc{GRevLex}$

$$\mathrm{in}_>(I) := \langle \mathrm{LeadingMonomial}_>(f) : f \in I \rangle$$

A set of generators $\{f_1, \ldots, f_r\}$ of an ideal I is a Gröbner basis (GB) if

$$\mathrm{in}_>(I) = \langle \mathrm{LeadingMonomial}_>(f_i) : 1 \leq i \leq r \rangle$$

Algebra $\leftrightarrow$ Computations: Maximal degrees in GB

$\mathrm{reg}_0(J) :=$ max degree in a min generating set of homogeneous ideal J.

How hard is to compute a GB for I

$$\mathrm{reg}_0(\mathrm{in}_>(I)) = \text{ maximal degree of an element in a GB}$$

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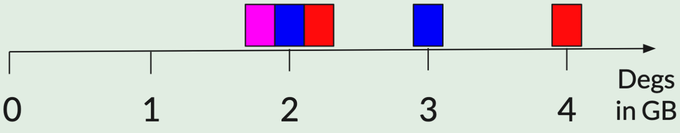
Maximal deg in GB vs Castelnuovo-Mumford regularity

- $I := \langle x_2^2 + x_0^2, x_2 x_1 + x_0^2 \rangle$
- $V * I$, where V be change of coord:$\{x_0 = x_1', x_1 = x_0', x_2 = x_2'\}$
- $U * I$, U change of coord:$\{x_0 = x_0' + x_1' + x_2', x_1 = x_1' + x_2', x_2 = x_2'\}$

We have that, $\mathrm{reg}(I) = \mathrm{reg}(V * I) = \mathrm{reg}(U * I) = 3$; but



Geometry $\leftrightarrow$ Computations: Generic initial ideal

The maximal degree $\mathrm{reg}_0(\mathrm{in}_>(I))$ depends on the coordinates $x_0, \ldots, x_n$.

Generic initial ideal [Galligo '74]

For each homogeneous ideal I, there exist a monomial ideal $\mathrm{gin}_>(I)$ st, for every generic change of coordinates $U \in \mathrm{GL}_{n+1}$, we have that

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Regularity and maximal degree of GB [Bayer-Stillman '87]

Consider homogeneous $I \subset \mathbb{C}[x_0, \ldots, x_n]$ and monomial order $\textsc{GRevLex}$.

$$\mathrm{reg}(I) = \mathrm{reg}(\mathrm{gin}_>(I)) = \mathrm{reg}_0(\mathrm{gin}_>(I))$$

In particular, if I is in generic coordinates, $\mathrm{in}_>(I) = \mathrm{gin}_>(I)$ and $reg(I)$ is the maximal degree of a polynomial in a minimal GB of I.

Multihomogeneous systems

Generalized Eigenvalue Problem

$$\left(x_0 \cdot \begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{20} \end{bmatrix} + x_1 \cdot \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{0} & \mathbf{20} \end{bmatrix}\right) \cdot \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = 0$$

$$\left\{ \begin{array}{l} \mathbf{2}\, x_0\, y_0 + \mathbf{6}\, x_0\, y_1 \mathbf{-2}\, x_1\, y_0 + \mathbf{4}\, x_1\, y_1 = 0 \\ \mathbf{-1}\, x_0\, y_0 + \mathbf{20}\, x_0\, y_1 + \mathbf{0}\, x_1\, y_0 + \mathbf{20}\, x_1\, y_1 = 0 \end{array} \right. \quad \in \mathbb{C}[x_0, x_1]_1 \otimes \mathbb{C}[y_0, y_1]_1$$

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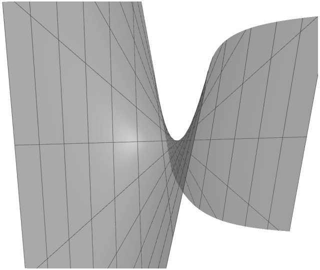
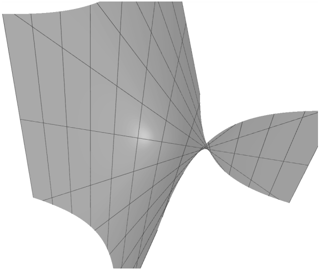
We look for solutions in $\mathbb{P}^1 \times \mathbb{P}^1$





Multihomogeneous systems

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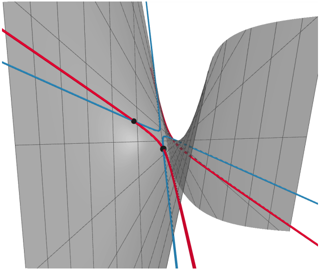

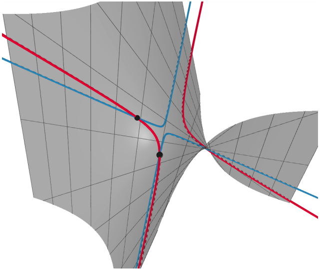

The bihomogeneous coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$ is

$$R := \bigoplus_{(d,e)\in\mathbb{Z}^2} \mathbb{C}[x_0, \ldots, x_n]_d \otimes \mathbb{C}[y_0, \ldots, y_m]_e$$

Irrelevant ideal of R is $\mathfrak{b} = \mathfrak{m}_{\boldsymbol{x}} \cap \mathfrak{m}_{\boldsymbol{y}}$, where

$$\mathfrak{m}_{\boldsymbol{x}} = \langle x_0, \ldots, x_n \rangle,\ \mathfrak{m}_{\boldsymbol{y}} = \langle y_0, \ldots, y_m \rangle.$$

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Geometry $\leftrightarrow$ Algebra

$$I = \langle \boldsymbol{x_0 - x_1, x_1\, y_0, x_1\, y_1, x_0\, y_1, x_0^2\, y_0, x_0\, y_0^2} \rangle \qquad V_{\mathbb{P}^n \times \mathbb{P}^m}(I) = \emptyset$$

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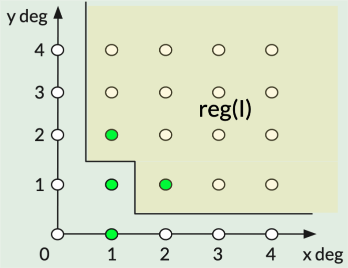

$$\begin{cases} I_{0,e} = 0 & \text{for } e \geq 0 \\ I_{d,0} = \langle x_0 - x_1\rangle_{d,0} & \text{for } d \geq 1 \\ I_{1,1} = \mathrm{Span}_{\mathbb{C}}(\{x_0\, y_1, x_1\, y_0, x_1\, y_1\}) & \\ \qquad \not\ni x_0\, y_1 & \\ I_{2,1} = R_{2,1}, I_{1,2} = R_{1,2}, & \\ I_{e,d} = R_{e,d}, \text{ for } d \geq 2 \text{ and } e \geq 2. & \end{cases}$$

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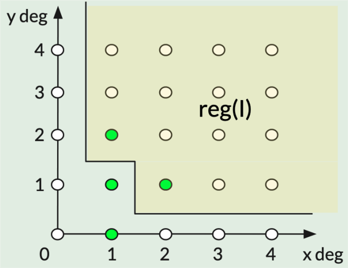
$$I = \langle x_0 - x_1, x_1 y_0, x_1 y_1, x_0 y_1, x_0^2 y_0, x_0 y_0^2\rangle \qquad V_{\mathbb{P}^n\times\mathbb{P}^m}(I) = \emptyset$$



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Regularity is a region (unbounded complement)

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Geometry $\leftrightarrow$ Algebra

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Algebra $\leftrightarrow$ Computations

Change of coordinates $U \in GL_{n+m+2}$ destroy multigraded structure.

$$U = \{x_0 = x_0' + x_1' + y_0' + y_1', \ldots\}$$

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Change of coordinates $U \in GL_{n+m+2}$ destroy multigraded structure.

$$U = \{x_0 = x'_0 + x'_1 + y'_0 + y'_1, \dots\}$$

We need to restrict to $(U,V) \in \mathrm{GL}_{n+1} \times \mathrm{GL}_{m+1}$

$$\begin{cases} U = \{x_0 = x'_0 + x'_1, x'_0 - x'_1\} \\ V = \{y_0 = 3\,y'_0 - y'_1, 2\,y'_0 + y'_1\} \end{cases}$$

Geometry ↔ Algebra: Multigraded CM regularity

- Defined in terms of vanishing of local cohomology wrt $\mathfrak{b}$

Bigraded Castelnuovo-Mumford regularity [Maclagan-Smith '04]

We say that $(a, b) \in \mathrm{reg}(I) \subset \mathbb{Z}^2$ iff, for every $i \geq i$ and every shift $\lambda_x, \lambda_y \in \mathbb{Z}_{\geq 0}$ st $\lambda_x + \lambda_y = i - 1$

$$(\forall (a', b') \geq (a - \lambda_x, b - \lambda_y)) \quad H^i_{\mathfrak{b}}(I)_{(a',b')} = 0$$

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(there are other candidate definitions...)

Algebra $\leftrightarrow$ Computations: Bigeneric initial ideals

Bigeneric initial ideal [Aramovaa-Crona-De Negri '00]

For each bihomogeneous ideal I, $\exists$ monomial ideal $\mathrm{bigin}(I)$ st, for every bigeneric change of coordinates $U \in \mathrm{GL}_{n+1} \times \mathrm{GL}_{m+1}$, we have that

$$\mathrm{bigin}_{>}(I) = \mathrm{in}(U * I)$$

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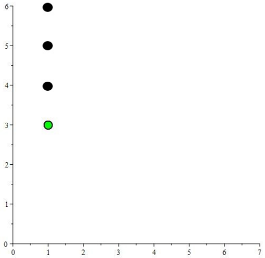
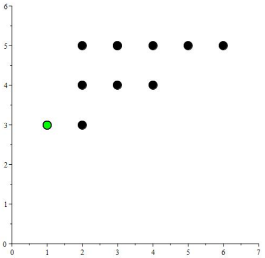
Careful, not unique bigin even for GREVLEX.



GREVLEX st $x_i < y_j$



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Betti numbers of bigin(I) and GB [Aramovaa-Crona-De Negri '00]

$$\mathfrak{R}_x(J) = \max(a \in \mathbb{Z} : \beta_{i,(a+i,b)}(J) \text{ for some } i, b \in \mathbb{Z})$$
$$\mathfrak{R}_y(J) = \max(b \in \mathbb{Z} : \beta_{i,(a,b+i)}(J) \text{ for some } i, a \in \mathbb{Z})$$

For GRevLex order, $\mathfrak{R}_x(\mathrm{bigin}(I))$ and $\mathfrak{R}_y(\mathrm{bigin}(I))$ bound GB of I.

Algebra ↔ Computations: Bigeneric initial ideals

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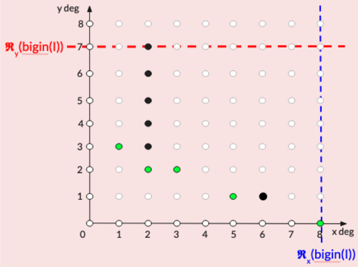

- minimal generator of I
- minimal generator of $\mathrm{bigin}(I)$

Algebra ↔ Computations: Bigeneric initial ideals

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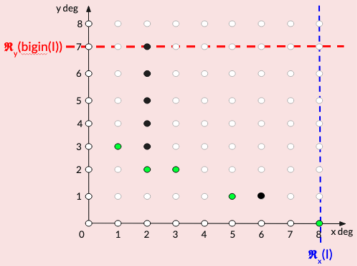

Betti of I and GB [Römmer '01]

If GRevLex st $x_0 < \cdots < x_n < y_0 < \cdots < y_m$,

$$\mathfrak{R}_x(I) = \mathfrak{R}_x(\mathrm{bigin}(I)),$$
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Algebra ↔ Computations: Bigeneric initial ideals

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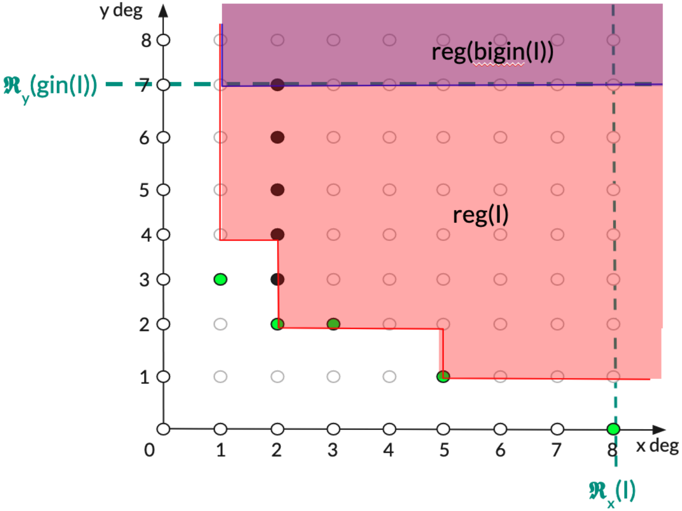

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Geometry $\leftrightarrow$ Computations: x-regularity and GB

Definition of x-regularity

Consider bihomogeneous I. The $\text{x-reg}(I)$ is the region of bi-degrees $(a, b) \in \mathbb{Z}^2$ st for every $i \geq 1$ and $(a', b') \geq (a - i + 1, b)$, $H^i_{\mathfrak{m}_x}(I)_{(a',b')} = 0$.

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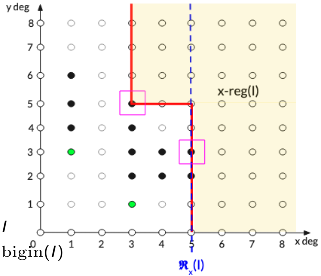
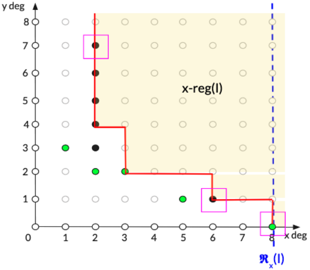
Relation between GB and x-reg [B.-Busé-Checa-Tsigaridas '24+]

Consider bihomogeneous I and $\mathrm{GRevLex}$ st $x_0 < \cdots < x_n < y_0 < \cdots < y_m$.
If $(a,b) \in \mathrm{x\text{-}reg}(I)$ and $a \geq 0$, then





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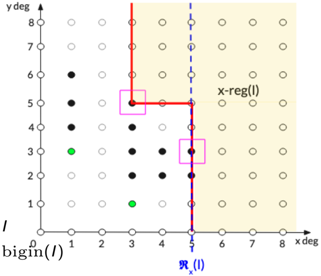
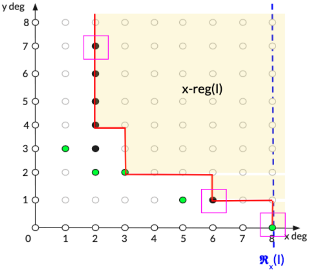
Consider bihomogeneous I and GRevLex st $x_0 < \cdots < x_n < y_0 < \cdots < y_m$.
If $(a, b) \in \text{x-reg}(I)$ and $a \geq 0$, then

- For every $(a', b') \geq (a + 1, b)$, there is no generator of $\text{bigin}(I)$ of degree (a', b') .





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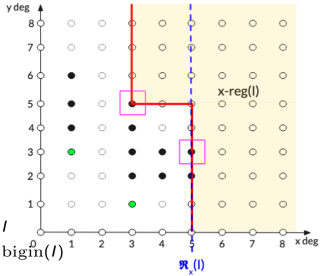
Definition of x-regularity

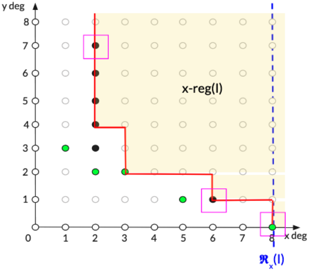
Consider bihomogeneous I. The $\mathrm{x\text{-}reg}(I)$ is the region of bi-degrees $(a, b) \in \mathbb{Z}^2$ st for every $i \geq 1$ and $(a', b') \geq (a - i + 1, b)$, $H^i_{\mathfrak{m}_x}(I)_{(a',b')} = 0$.

Relation between GB and $\mathrm{x\text{-}reg}$ [B.-Busé-Checa-Tsigaridas '24+]

Consider bihomogeneous I and $\mathrm{GRevLex}$ st $x_0 < \cdots < x_n < y_0 < \cdots < y_m$.
If $(a, b) \in \mathrm{x\text{-}reg}(I)$ and $a \geq 0$, then

- For every $(a', b') \geq (a + 1, b)$, there is no generator of $\mathrm{bigin}(I)$ of degree (a', b') .
- If $a \geq 1$ and $(a - 1, b) \notin \mathrm{x\text{-}reg}(I)$, exists $b' \leq b$ and a min gen of $\mathrm{bigin}(I)$ of deg (a, b').





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Geometry $\leftrightarrow$ Computations: GB and CM regularity

Relation x-reg and multigraded CM reg [Chardin-Holanda '22]

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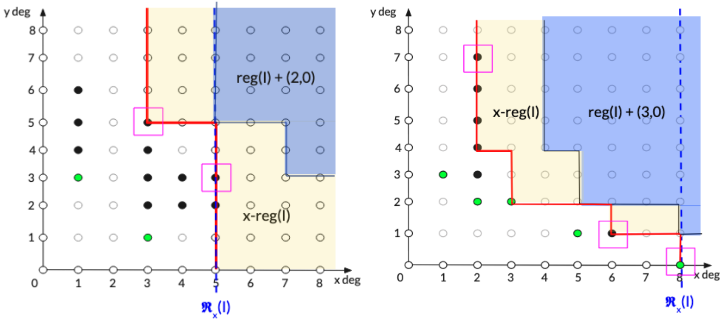
Corollary [B.-Busé-Checa-Tsigaridas '24+]

Fix $\mathrm{GRevLex}$ st $x_0 < \cdots < x_n < y_0 < \cdots < y_m$. There is $1 \leq s < n + 1$ st, if $(a, b) \in \mathrm{reg}(I)$, then there is no generator of $\mathrm{bigin}(I)$ of degree $\geq (a + s, b)$.



● minimal generator of I ● minimal generator of $\mathrm{bigin}(I)$

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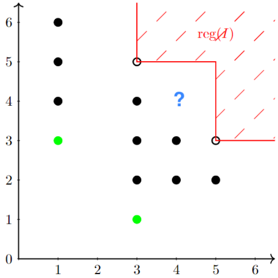
- We do not need generic coordinates wrt every block of variables, only generic coordinates wrt to smallest block (i.e., x_i's).
- It is not clear how to get tight bounds in terms of regularity.



Summing-up

What was known

- In homogeneous setting:
  - Hardness of GB computation = Castelnuovo-Mumford regularity
- In multihomogeneous setting
  - Different notions of Castelnuovo-Mumford regularity
  - No relation with known bounds for degrees in GB

Results

- New region $\mathrm{x\text{-}reg}(I)$ where there are not elements in the GB of I
- Near boundary of $\mathrm{x\text{-}reg}(I)$, there are elements in GB of I
- We relate CM regularity of I with its GB

Questions

- Tighter bound between GB and CM regularity
- Better bound for GB using other invariants of I
- Criterion for multigraded reg. à la Bayer&Stillman

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Thank you!