An introduction to computer-assisted proofs via a posteriori validation

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- Possible motivation: prove theorems that cannot be proven by "classical" pen-and-paper methods.
- Alternate viewpoint: these computer-assisted techniques can be seen as a way to guarantee/certify the output of some numerical simulations.

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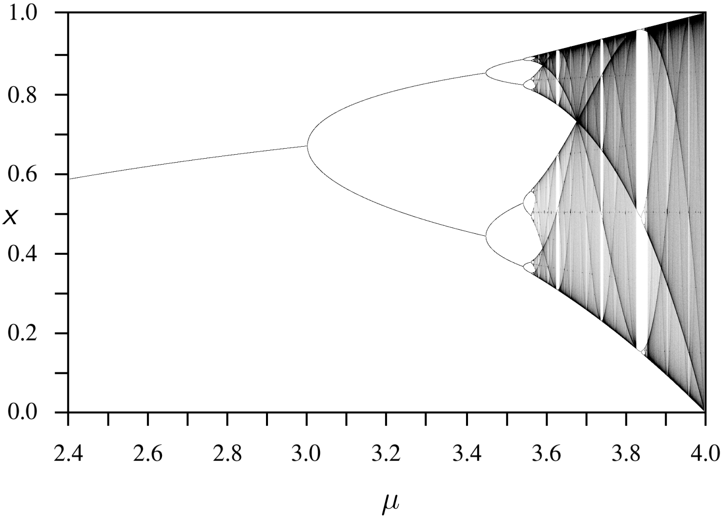

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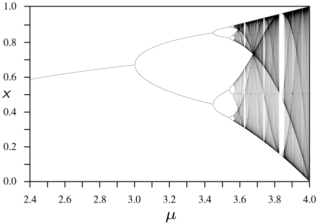

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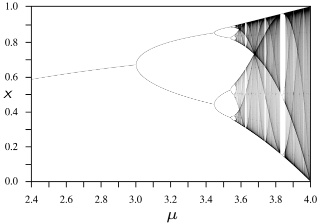

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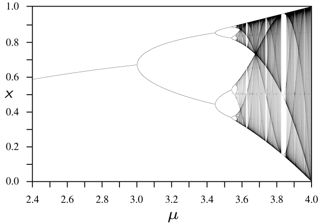

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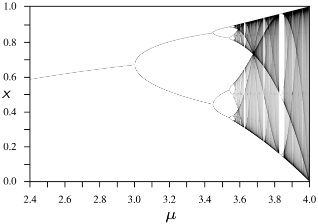

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Theorem [Sharkovsky '64, Li York '75]

"The existence of a period 3 orbit implies chaos"

- For a given value of μ, how can we prove the existence of a period 3 orbit, in order to apply the above theorem?

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- To do so, we can consider the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(x_0, x_1, x_2) = \begin{pmatrix} \mu x_0(1 - x_0) - x_1 \\ \mu x_1(1 - x_1) - x_2 \\ \mu x_2(1 - x_2) - x_0 \end{pmatrix}.$$

  If we manage to find a zero of F (such that $x_0 \neq x_1 \neq x_2$), we then have a period 3 orbit.

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- How to rigorously prove the existence of this zero of F?

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Theorem *(à la Newton-Kantorovich)*

Let $\varepsilon, K, L > 0$ such that

$$\|F(\bar{X})\| \leq \varepsilon$$
$$\|DF(\bar{X})^{-1}\| \leq \kappa$$
$$\|DF(X) - DF(\bar{X})\| \leq L\|X - \bar{X}\| \qquad \forall\, X \in \mathbb{R}^3.$$

If

$$\varepsilon < \frac{1}{2\kappa^2 L},$$

then F has a unique zero X^* satisfying $\|X^* - \bar{X}\| \leq r$, $r = \frac{1-\sqrt{1-2\kappa^2 L\varepsilon}}{\kappa L}$.

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Proof : $T : X \mapsto X - DF(\bar{X})^{-1}F(X)$ is a contraction on the closed ball of center $\bar{X}$ and radius r.

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- In our "proof" of existence of a period 3 orbit, how can we be certain that the quantity ε that we numerically evaluated really bounds $\|F(\bar{X})\|$, or that $\varepsilon < \frac{1}{2\kappa^2 L}$?

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- Example: consider $x = 0.1$. In base 2, x writes
$$x = (1.10011001100110011001100...)_2 \times 2^{-4}.$$
With 8 bits of precision (for the mantissa), we have
$$\triangledown(x) = (1.1001100)_2 \times 2^{-4} \quad \text{and} \quad \triangle(x) = (1.1001101)_2 \times 2^{-4}.$$

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- On can then extend the elementary operations $(+, -, \times, \div)$ to intervals, in such a way that the result always contain the true value:

$$x + y \quad \to \quad [x]\ [+]\ [y],$$

where $[+]$ is defined as follows

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- We then have $\boldsymbol{x} + \boldsymbol{y} \in [\boldsymbol{x}]\ [+]\ [\boldsymbol{y}]$.

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4. We use these estimates to prove that

$$\boldsymbol{T} : \boldsymbol{X} \mapsto \boldsymbol{X} - \boldsymbol{DF}(\bar{\boldsymbol{X}})^{-1}\boldsymbol{F}(\boldsymbol{X})$$

is a contraction on a small neighborhood of $\bar{X}$.

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   - **The main difficulty lies in controlling $\|DF(\bar{X})^{-1}\|$.**

A new validation criteria

Theorem *à la Newton-Kantorovich* bis

Let $\varepsilon, \kappa, L, \delta > 0$ such that

$$\|F(\bar{X})\| \leq \varepsilon, \quad \|A\| \leq \kappa, \quad \|DF(X) - DF(\bar{X})\| \leq L\|X - \bar{X}\|,$$

$$\|I - ADF(\bar{X})\| \leq \delta < 1.$$

If

$$\varepsilon < \frac{(1-\delta)^2}{2\kappa^2 L},$$

then F has a unique zero X^* satisfying $\|X^* - \bar{X}\| \leq r$, $r = \frac{1-\delta-\sqrt{(1-\delta)^2-2\kappa^2 L}}{\kappa L}$.

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- An equivalent way to interpret this strategy is to say that we replace the former fixed-point operator $T : x \mapsto x - DF(\bar{x})^{-1}F(x)$ by

$$\tilde{T} : x \mapsto x - AF(x).$$

Setting for validated integration of ODEs

$$\begin{cases} u'(t) = f(u(t)) & t \in [0, 2\tau] \\ u(0) = u^{in} \end{cases}$$

with $f : \mathbb{R}^d \to \mathbb{R}^d$ smooth and $\tau > 0$ fixed.

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- Key observation:

$$DF(\bar{u})(h)(t) = h(t) - \int_0^t Df(\bar{u}(s))h(s) \mathrm{d}s,$$

i.e., $DF(\bar{u})$ is a compact perturbation of the identity.

Chebyshev series

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- By plugging the Chebyshev series ansatz into

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- The approximate solution $\bar{\boldsymbol{u}}$ is taken as a truncated Chebyshev series.
- We look for the exact solution in the space $\ell^1_\nu := \{\boldsymbol{u}, \left\|\boldsymbol{u}\right\|_\nu < \infty\}$,

$$\left\|\boldsymbol{u}\right\|_\nu := |u_0| + 2\sum_{n=1}^{\infty} |u_n|\, \nu^n, \qquad \nu \geq 1.$$

Why Chebyshev series?

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  - Simplifies the estimation of $\left\| D^2 F(\boldsymbol{u}) \right\|_\nu$ for $\boldsymbol{u}$ in a neighborhood of $\bar{\boldsymbol{u}}$.

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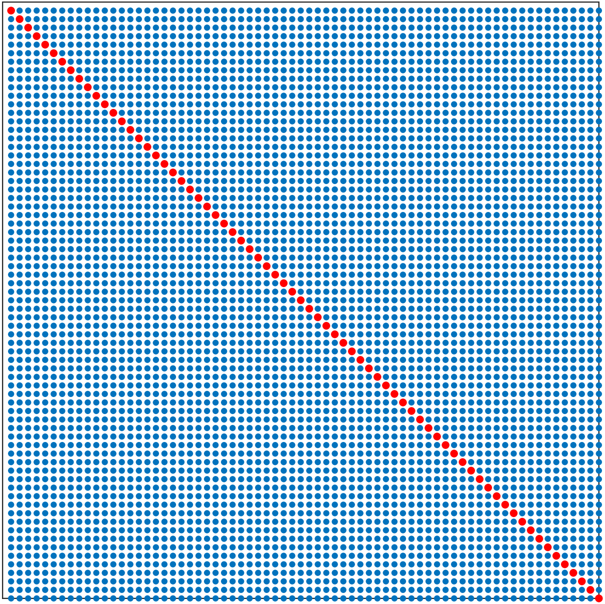

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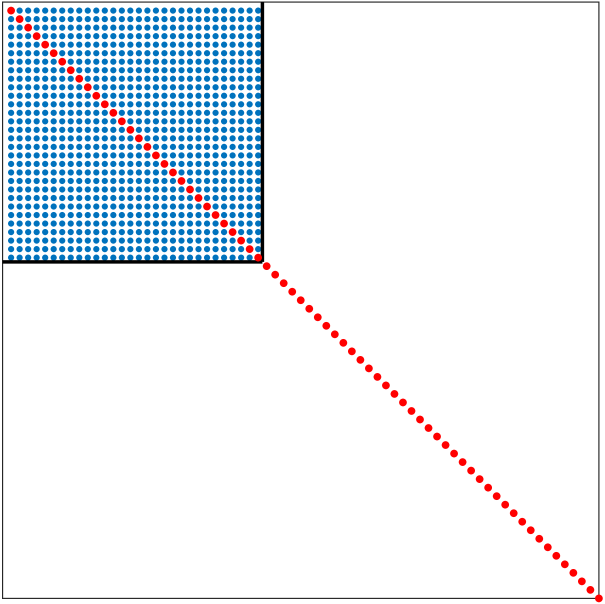
$DF(\bar{\boldsymbol{u}}) \approx$ 

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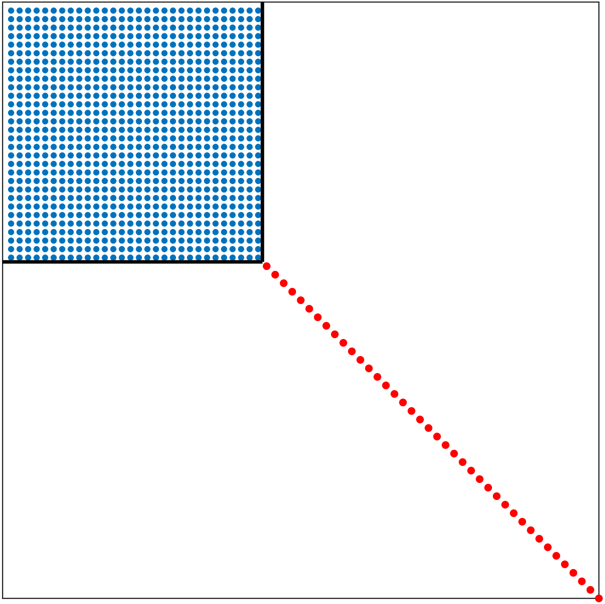
$A :=$ 

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- [Lessard Reinhardt '14]

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Some examples from [van den Berg Sheombarsing '21]

$$x' = 10(x - y)$$
$$y' = 28x - y - xz$$
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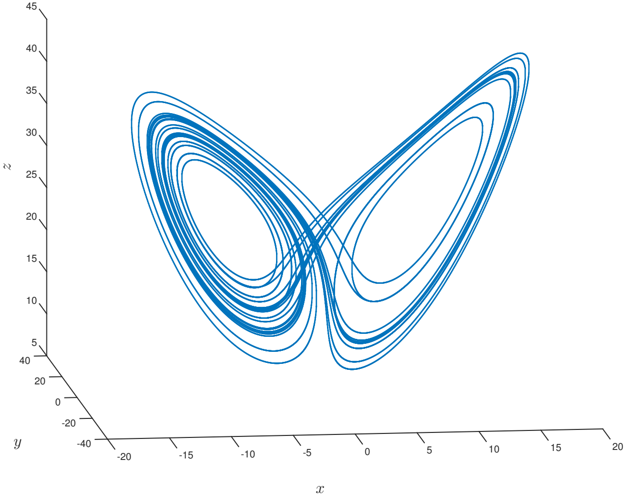

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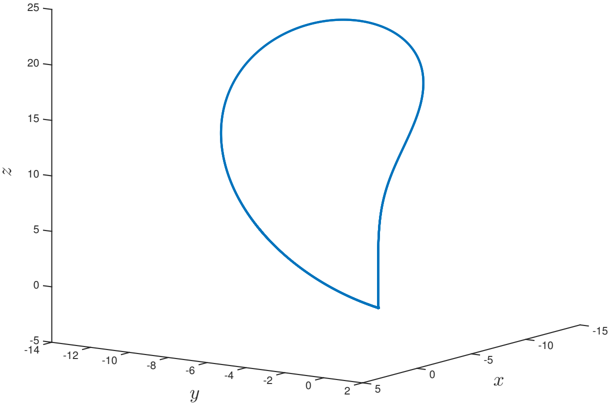

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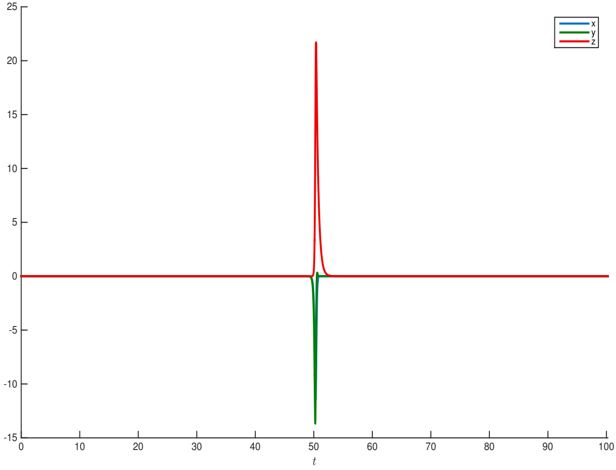

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- Many other methods, some of which are more in the spirit of *traditional* numerical methods for ODEs. A particularly successful one is the CAPD::DynSys library [Kapela Mrozek Wilczak Zgliczynski '21].

Outline

A different fixed point reformulation

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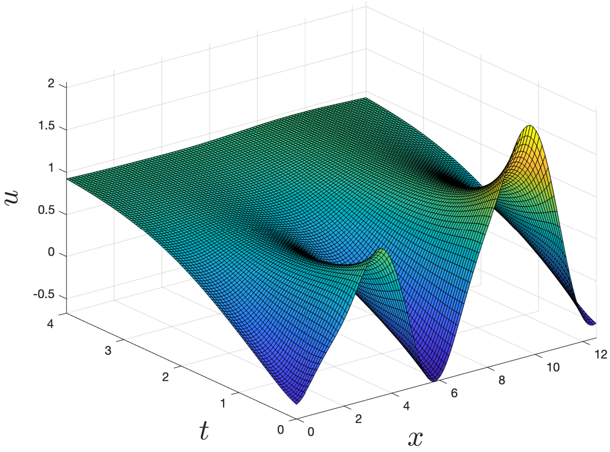
- We again split the time interval $0 = \tau_0 < \tau_1 < \ldots < \tau_M = \tau$, and take a different approximation on each smaller subinterval:

$$L^{(m)} \approx Df(\bar{u}^{(m)})(s), \qquad s \in [\tau_m, \tau_{m+1}].$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u(1-u) & (t,x) \in (0,4] \times \mathbb{T}_{4\pi}, \\ u(0,\cdot) = u^{in}. \end{cases}$$



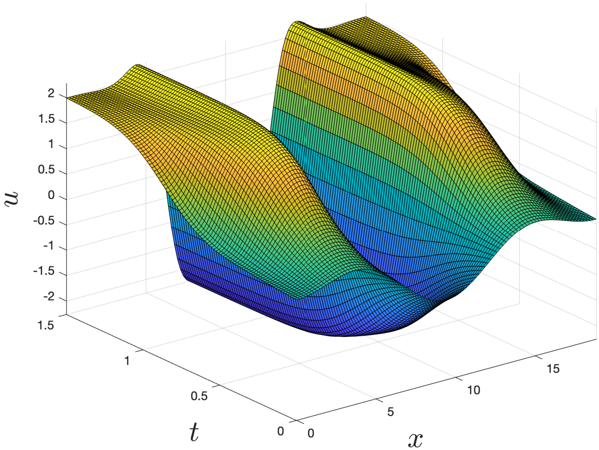
Theorem

$$\|\bar{u} - u\| \leq 5e^{-2}$$

$N = 14$
$K = 2$
$M = 25$

Application to parabolic PDEs 2: Swift-Hohenberg

$$\begin{cases} \frac{\partial u}{\partial t} = -\left(\frac{\partial^2}{\partial x^2} + 1\right)^2 u + 5u - u^3 & (t,x) \in (0, 1.5] \times \mathbb{T}_{6\pi}, \\ u(0,\cdot) = u^{in}. \end{cases}$$



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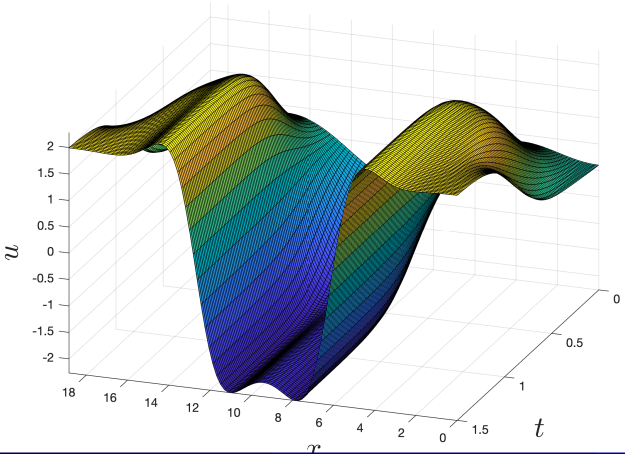

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THANK YOU FOR YOUR ATTENTION!