Faster algorithms for symmetric polynomials

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based on joint works with
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MAX team seminar
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Outline

Part I : Computing critical points for invariant algebraic systems

Part II : Deciding the emptiness of invariant algebraic sets over real fields

Polynomial system solving

Let $\mathbb{K}$ be a field and $f_1, \ldots, f_s$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$

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Algebraic sets : the solution set of the ideal $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$

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Folklore procedure :

- compute a Gröbner basis of I
- deduce the Hilbert series $\frac{N(t)}{(1-t)^d}$ of I

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Normally, we have $(v, v_1, \ldots, v_n)$, then exploint information for W.

Critical points

Minimize : $\phi(x_1, x_2, x_3) = x_1 x_2 x_3 - 3(x_1 + x_2 + x_3)$ subject to

$$g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 6 = 0.$$

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The minima satisfy

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$$\operatorname{rank} \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ x_2 x_3 - 3 & x_1 x_3 - 3 & x_1 x_2 - 3 \end{bmatrix} < 2.$$

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- **optimization**
- **real algebraic geometry (decide the emptiness over the reals)**
  (will see in the 2nd half of the talk)
- ...

Critical points and Our goal

Let ϕ and $\mathsf{f} = (f_1, \ldots, f_s)$ be polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ with $s \leqslant n$ s.t.

Assumption (A) **:** the Jacobian matrix of f has full rank at any solution of f

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Then, $V(\mathbf{f})$ is smooth and $(n-s)$-equidimensional and the set of critical points of ϕ restricted to $V(\mathbf{f})$:

$$W(\phi, \mathbf{f}) := \left\{ \boldsymbol{x} \in \overline{\mathbb{K}}^n \; : \; \mathbf{f}(\boldsymbol{x}) = 0 \quad \text{and} \quad \operatorname{rank}(\operatorname{jac}(\mathbf{f}, \phi)(\boldsymbol{x})) < s+1 \right\}$$

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Input : symmetric polynomials ϕ and $(f_1, \ldots, f_s)$ in $\mathbb{K}[x_1, \ldots, x_n]$

Condition : $\mathbf{f} = (f_1, \ldots, f_s)$ satisfies (A) and $W(\phi, \mathbf{f})$ is of zero-dimensional

Output : a representation for $W(\phi, \mathbf{f})$

Main result

Suppose ϕ and $\mathtt{f} = (f_1, \ldots, f_s)$ are symmetric polynomials in $\mathbb{K}[x_1, \ldots, x_n]$

- the Jacobian matrix of $\mathtt{f}$ has full rank at any solution of $\mathtt{f}$
- the degrees of $\mathtt{f}$ and ϕ are at most d
- the set $W(\phi, \mathtt{f}) \subset \overline{\mathbb{K}}^n$ is finite

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Theorem [Labahn-Safey El Din-Schost-**Vu**, 2023]

- There is a randomized algorithm that takes as input $\mathbf{f}$ and ϕ and outputs a representation for $W(\phi, \mathbf{f})$ with the runtime is $\left(d^s \binom{n+d}{n} \binom{n}{s+1}\right)^{O(1)}$.
- The size of the output of our algorithm is at most $d^s \binom{n+d-1}{n}$.

Previous work

[Labahn-Hubert]

- scaling invariants and symmetry reduction of dynamical systems

[Busé-Karasoulou]

- resultant of an equivariant polynomial system

[Riener]

- deciding the emptiness symmetric semi-algebraic sets, fixed degree

[Riener-Safey El Din]

- real root finding for equivariant semi-algebraic systems

[Faugère-Rahmany]

- use SAGBI-Gröbner bases to solve symmetric systems

[Faugère-Svartz]

- globally invariant systems

Determinantal systems

Given $\mathbf{f} = (f_1, \ldots, f_s) \subset \mathbb{K}[x_1, \ldots, x_n]$ and $\mathbf{G} \in \mathbb{K}[x_1, \ldots, x_n]^{p \times q}$

- $\mathrm{wdeg}(x_i) = w_i \geqslant 1$ for $i = 1, \ldots, n$
- $\mathrm{wcdeg}(\mathbf{G}, j) := \max_{1 \leqslant i \leqslant p}(\mathrm{wdeg}(g_{i,j})) = \delta_j$

Compute $V_p(\mathbf{f}, \mathbf{G}) := \{\mathbf{x} \in \overline{\mathbb{K}}^n \,:\, \mathbf{f}(\mathbf{x}) = 0 \text{ and } \mathrm{rank}(\mathbf{G}(\mathbf{x})) < p\}$

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Theorem [Hauenstein-Labahn-Safey El Din-Schost-**Vu**, 2021]

Assume that $n = q - p + s + 1$ and $E_k(\cdot)$ the k-th elementary symmetric function. Then there are at most

$$c = \mathrm{wdeg}(f_1) \cdots \mathrm{wdeg}(f_s) \cdot E_{n-s}(\delta_1, \ldots, \delta_q) / \Delta \text{ with } \Delta = w_1 \cdots w_n$$

isolated points, counted with multiplicities, in $V_p(\mathbf{f}, \mathbf{G})$, which can be computed by a randomized algorithm Homotopy_weighted with runtime being polynomial in c.

In classical domains, i.e., $\mathrm{wdeg}(x_i) = 1$ for all i.

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Example : $\mathbb{K}[x_1, x_2, x_3]$ with $\mathrm{wdeg}(x_k) = k$

Consider $f_1 = x_1^2 - 3x_1x_2 + 3x_3 - 8$ and $\mathrm{wdeg}(\mathbf{G}) = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$, then

$$\mathrm{wdeg}(f_1) = 3, \;\; \mathrm{wcdeg}(\mathbf{G}) = (3, 2, 1)$$

and $c = 3 \cdot E_2(3, 2, 1)/(1 \cdot 2 \cdot 3) = 3 \cdot (3 \cdot 2 + 3 \cdot 1 + 2 \cdot 1)/(1 \cdot 2 \cdot 3) = 30/6 = 5.$

Determinantal systems and the critical points problem

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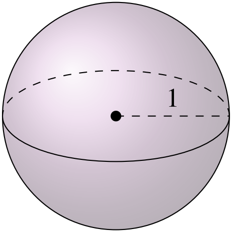

Property : if $\mathbf{f} = (f_1, \ldots, f_s)$ and ϕ are symmetric, then $W(\phi, \mathbf{f})$ is S_n-invariant

Data-structure for invariant sets

A list of positive integers $\lambda = \big(\underbrace{n_1, \ldots, n_1}_{\ell_1}, \ldots, \underbrace{n_r, \ldots, n_r}_{\ell_r}\big)$ is a partition of n if

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Example : $\lambda = (1, 2)$ of $n = 3$, then $n_1 = 1, n_2 = 2, \ell_1 = 1, \ell_2 = 1$, and $\ell = 2$

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Denote $C_\lambda \subset \overline{\mathbb{K}}^n$ contains

$$\mathbf{a} = (\underbrace{a_{1,1}, \dots, a_{1,1}}_{n_1}, \dots, \underbrace{a_{1,\ell_1}, \dots, a_{1,\ell_1}}_{n_1}, \dots, \underbrace{a_{r,1}, \dots, a_{r,1}}_{n_r}, \dots, \underbrace{a_{r,\ell_r}, \dots, a_{r,\ell_r}}_{n_r}) \in \overline{\mathbb{K}}^n \ : \ a_{i,j} \text{ are distinct}$$

Data-structure for invariant sets

A list of positive integers $\lambda = \big(\underbrace{n_1, \ldots, n_1}_{\ell_1}, \ldots, \underbrace{n_r, \ldots, n_r}_{\ell_r}\big)$ is a partition of n if $n_1\ell_1 + n_2\ell_2 + \cdots + n_r\ell_r = n$ with $\ell := \ell_1 + \cdots + \ell_r$ is the length of λ.

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Example : Suppose $n = 3$ and $W = \{(5, 5, 5), (3, 4, 4), (3, 4, 4), (4, 4, 3)\}$. Then

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Data-structure for invariant sets (cont.)

For a partition $\lambda = (n_1^{\ell_1} \ \ldots \ n_r^{\ell_r})$ and $\mathbf{a} = (a_{i,j})_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant \ell_i}$, the compression mapping :

$$E_\lambda(\mathbf{a}) = \big(E_{i,1}(a_{i,1}, \ldots, a_{i,\ell_i}), \ldots, E_{i,\ell_i}(a_{i,1}, \ldots, a_{i,\ell_i})\big)_{1 \leqslant i \leqslant r} \in \overline{\mathbb{K}}^\ell,$$

where $E_{i,j}$'s the j-th elementary symmetric function of$a_{i,1}, \ldots, a_{i,\ell_i}$.

Data-structure for invariant sets (cont.)

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$$E_{(1,2,2)}(3, 4, 4, 5, 5) = (E_{1,1}(3), E_{2,1}(4, 5), E_{2,2}(4, 5)) = (3, 9, 20)$$

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and $W'_{(1,2)} = \{(3, 4)\}$ and $W_{(3)} = \{(5)\}$

Data-structure for invariant sets (cont.)

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$$\frac{\operatorname{card}(W)}{\operatorname{card}(W_\lambda)} = \binom{n}{n_1, \ldots, n_1, \ldots, n_r, \ldots, n_r} = \frac{n!}{n_1!^{\ell_1} \cdots n_r!^{\ell_2}} \text{ and } \frac{\operatorname{card}(W_\lambda)}{\operatorname{card}(W'_\lambda)} = \ell_1! \cdots \ell_r!$$

Back to polynomial systems

Let $\lambda = (n_1^{\ell_1} \ \ldots \ n_r^{\ell_r})$ a partition of n and $\mathbf{z}_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of ℓ_i variables

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Define the $\mathbb{K}$-algebra homomorphism $\mathbb{T}_\lambda : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[\mathbf{z}_1, \ldots, \mathbf{z}_r]$

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$$\mathbb{T}_{(1,2,2)}(x_1, x_2, x_3, x_4, x_5) = (z_{1,1}, z_{2,1}, z_{2,1}, z_{2,2}, z_{2,2})$$

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$$\mathbb{T}_{(1,2,2)}(\mathbf{G}) = (\mathbb{T}_\lambda(g_{i,j}))_{i,j} \text{ for } \mathbf{G} = (g_{i,j}) \in \mathbb{K}[x_1, \ldots, x_n]^{p \times q}$$

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Example : $\lambda = (1, 2, 2)$ of $n = 5$, then

$$\mathbb{T}_{(1,2,2)}(x_1, x_2, x_3, x_4, x_5) = (z_{1,1}, z_{2,1}, z_{2,1}, z_{2,2}, z_{2,2})$$

Properties : Denote $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and let f be a S_n-invariant. Then

- $\mathbb{T}_\lambda(f)$ is S_λ-invariant if f is S_n-invariant

Back to polynomial systems

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Properties : Denote $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$ and let f be a S_n-invariant. Then

- $\mathbb{T}_\lambda(f)$ is S_λ-invariant if f is S_n-invariant
- discarding some **duplicated** columns from $\mathbb{T}_\lambda(\nabla f)$ gives a S_λ-**equivariant** system,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Back to polynomial systems

Let $\lambda = (n_1^{\ell_1} \ \ldots \ n_r^{\ell_r})$ a partition of n and $\mathbf{z}_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ sequence of ℓ_i variables

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- discarding some **duplicated** columns from $\mathbb{T}_\lambda(\nabla f)$ gives a S_λ-**equivariant** system,

  $$\mathbb{T}_{(1,2,2)}(\nabla x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3) = 3\mathbb{T}_{(1,2,2)}(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2) = 3(z_{1,1}^2, z_{2,1}^2, z_{2,1}^2, z_{2,2}^2, z_{2,2}^2)$$

  the sequence $(z_{1,1}^2, z_{2,1}^2, z_{2,2}^2)$ is $S_1 \times S_2$-equivariant but **NOT** $S_1 \times S_2$-invariant

From S_λ-equivariant to S_λ-invariant

Recall

- $\lambda = (n_1^{\ell_1} \;\ldots,\; n_r^{\ell_r})$ a partition of n of length ℓ and $S_\lambda := S_{\ell_1} \times \cdots \times S_{\ell_r}$
- $\mathbf{z}_i = (z_{i,1}, \ldots, z_{i,\ell_i})$ for $i = 1, \ldots, r$

We index $(\mathbf{z}_1, \ldots, \mathbf{z}_r) = (z_1, \ldots, z_\ell)$.

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A sequence of polynomials $\mathbf{q} = (q_1, \ldots, q_\ell)$ in $\mathbb{K}[\mathbf{z}_1, \ldots, \mathbf{z}_r]$ is S_λ-equivariant if

$$q_i(z_{\sigma(1)}, \ldots, z_{\sigma(\ell)}) = q_{\sigma(i)}(z_1, \ldots, z_\ell) \text{ for all } i = 1, \ldots, \ell \text{ and } \sigma \in S_\lambda$$

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Proposition : Suppose $\mathbf{q}$ is S_λ-equivariant and $z_i - z_j$ divides $q_i - q_j$. Then there exists an algorithm Symmetrize$(\lambda, \mathbf{q})$ which returns $\mathbf{p} = (p_1, \ldots, p_\ell)$ s.t.

- $\mathbf{p}$ is S_λ-invariant
- $\mathbf{p}$ and $\mathbf{q}$ generate the same ideal in a suitable localization of $\mathbb{K}[\mathbf{z}_1, \ldots, \mathbf{z}_r]$, that is, $\mathbf{p}\mathbf{U} = \mathbf{q}$, where $\mathbf{U}$ has a determinant unit in $\mathbb{K}[\mathbf{z}_1, \ldots, \mathbf{z}_r, 1/\Delta]$ with $\Delta = \prod_{1 \leqslant i < j \leqslant \ell} (z_i - z_j)$
- $\deg(p_i) \leqslant \delta - \ell + i$ with $\delta = \deg(\mathbf{q})$ $\quad p_i = 0$ if $\ell \geqslant \delta + i$
- the runtime is $O\tilde{}(\ell^3 \binom{\ell+\delta}{\delta})$ operations in $\mathbb{K}$

From S_λ-equivariant to S_λ-invariant

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- $\mathbf{p}$ is S_λ-invariant
- $\mathbf{p}$ and $\mathbf{q}$ generate the same ideal in a suitable localization of $\mathbb{K}[\mathbf{z}_1, \ldots, \mathbf{z}_r]$, that is, $\mathbf{p}\mathbf{U} = \mathbf{q}$, where $\mathbf{U}$ has a determinant unit in $\mathbb{K}[\mathbf{z}_1, \ldots, \mathbf{z}_r, 1/\Delta]$ with $\Delta = \prod_{1 \leqslant i < j \leqslant \ell}(z_i - z_j)$
- $\deg(p_i) \leqslant \delta - \ell + i$ with $\delta = \deg(\mathbf{q})$ $\quad p_i = 0$ if $\ell \geqslant \delta + i$
- the runtime is $O\tilde{}(\ell^3 \binom{\ell+\delta}{\delta})$ operations in $\mathbb{K}$

Note [Hubert, 2009] has an algorithm which symmetrizes polynomials constructed via a generating set of **rational** invariants; **but we wish to avoid rational functions**

Sketch of the main algorithm

Input : symmetric polynomials ϕ and $(f_1, \ldots, f_s)$ in $\mathbb{K}[x_1, \ldots, x_n]$

Condition : $\mathsf{f} = (f_1, \ldots, f_s)$ satisfies (A) and $W(\phi, \mathsf{f})$ is finite

Output : a representation for $W(\phi, \mathsf{f})$

Assumption (A) **:** the Jacobian matrix of f has full rank at any solution of f

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with $\deg(E_{i,k}) = k$; so $\mathrm{wdeg}(e_{i,k}) = k$

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5. find $\mathscr{R}_\lambda = \text{Homotopy\_weighted}(\zeta_{\mathbf{g}}, \zeta_{\mathbf{H}})$

Part II :

Emptiness decision and Computing sample points

Let $\mathbb{Q}$ be a field and $f_1, \ldots, f_s$ be polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$

Input: $f_1 = \cdots = f_s = 0$ that defines $S \subset \mathbb{R}^n$

Output: true iff $S \neq \emptyset$ else false

This is a decision problem.

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Output: Some points in S whenever they exist

- how to encode them? What to do if $|S| = \infty$?
- representative points in all the connected components of S
- quantitative results on the number of connected components of S?

Exact/Symbolic computation.

State-of-the-art

Collins' Cylindrical Algebraic Decomposition algorithm

- complexity doubly exponential in n
- implementations are limited to small n

[Hong, McCallum, Arnon, Brown, Strzebonski, Anai, Sturm, Weispfenning]

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$\rightsquigarrow$ Quest for algorithms singly exponential in n

The critical point method

[Grigoriev-Vorobjov], [Canny] [Renegar], [Heintz-Roy-Solerno], [Basu-Pollack-Roy], [Bank-Giusti-Heintz-Mbakop], [Aubry-Rouillier-Safey El Din], [Rouillier-Roy-Safey El Din] [Safey El Din-Schost]

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Reduction of the dimension through Global Optimization

Main idea : studying a map that

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Existence : from n-variate to univariate problems

Our goal with symmetry

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Theorem [Labahn-Riener-Safey El Din-Schost-**Vu**, preprint 2023]

There exists a randomized algorithm that takes $\mathbf{f}$ as input and decides the existence of real points in $V(\mathbf{f})$. The runtime is polynomial in d^s, $\binom{n+d}{d}$, and $\binom{n}{s+1}$.

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Observe : The runtime is

- polynomial in n when n and d are fixed
- equal to $n^{O(1)} 2^n$ when $d = n$
- subexponential in n when $d \simeq n^\alpha$ with $\alpha < 1$

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Then, $\mathsf{g} := \mathbb{T}_\lambda(\mathsf{f})$ **is S_λ-invariant and also satisfies (A)**

$$\mathbb{T}_\lambda(\mathrm{jac}(\mathsf{f})) = \mathrm{jac}(\mathsf{g}) \cdot \mathbf{M}, \text{ where } \mathbf{M} = \mathrm{diag}(\mathbf{M}_1, \dots, \mathbf{M}_r) \in \mathbb{K}^{\ell \times n}$$

$$\mathbf{M}_i = \begin{pmatrix} \frac{1}{n_i} & \cdots & \frac{1}{n_i} & \cdots & \mathbf{0} \\ & \vdots & & \ddots & \vdots \\ & \mathbf{0} & & \cdots & \frac{1}{n_i} & \cdots & \frac{1}{n_i} \end{pmatrix} \in \mathbb{K}^{\ell_i \times n_i \ell_i} \text{ of rank } \ell_i; \text{ so } \mathrm{rank}(\mathbf{M}) = \ell$$

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Example : $n = 7$ and $\lambda = (2, 2, 3)$. Then $\boldsymbol{M} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & \\ & & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ of rank $2 + 1$

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and for $\mathbf{c} \in V(\mathbb{T}_\lambda(\mathbf{f})) \cap \mathbb{C}^\ell$, there exists $\mathbf{u} \in V(\mathbf{f}) \cap \mathbb{C}^n$ s.t. $\mathbb{T}_\lambda(\mathrm{jac}(\mathbf{f}))(\mathbf{c}) = \mathrm{jac}(\mathbf{f})(\mathbf{u})$. Thus

$$\mathrm{jac}(\mathbf{f})(\mathbf{u}) = \mathrm{jac}(\mathbf{g})(\mathbf{c}) \cdot \boldsymbol{M}$$

The left kernel of $\mathrm{jac}(\mathbf{f})(\mathbf{u})$ is trivial by (A), so is $\mathrm{jac}(\mathbf{g})(\mathbf{c})$.

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Then, **ζ_{g} in $\mathbb{K}[\mathbf{e}_1, \dots, \mathbf{e}_r]$ is also satisfies (A), where $\zeta_{\mathsf{g}}(E_{i,j}) = \mathsf{g}$**

$$\mathrm{jac}(\mathsf{g}) = \mathrm{jac}(\zeta_{\mathsf{g}})(E_{i,j}) \cdot \mathbf{V}, \text{ where } \mathbf{V} = \mathrm{diag}(\mathbf{V}_1, \dots, \mathbf{V}_r)$$

with $\mathbf{V}_i$ the Vandermonde matrix of $(E_{i,1}, \dots, E_{i,\ell_i})$

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Sketch of the main algorithm

Input : $f_1 = \cdots = f_s = 0$ that defines $S \subset \mathbb{R}^n$; all are symmetric

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for a partition λ of n of length at least s

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   - $W =$ Homotopy_weighted$\left(\zeta_\mathbf{g}, \mathrm{jac}(\zeta_\mathbf{g}, \zeta_\phi)\right)$
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5. existence of real roots of bi-variate polynomial systems $(v, v_{i,1}, \ldots, v_{i,\ell_i})$
   - from $e_{i,j}$ coordinates back to $(\mathbf{z}_1, \ldots, \mathbf{z}_r)$ then to $(x_1, \ldots, x_n)$
   - use Vieta polynomials $\rho_i := u^{\ell_i} - v_{i,1}(t)u^{\ell_i - 1} + \cdots + (-1)^{\ell_i} e_{i,\ell_i}(t) \in \mathbb{C}[t][u]$

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$$\phi := \sum_{i=1}^{r} c_i P_{i,\ell_i+1} + \sum_{i=1}^{r} \sum_{k=0}^{\ell_i} c_{i,j} P_{i,k} \quad \text{where} \quad P_{i,k} := z_{i,1}^k + \cdots + z_{i,\ell_i}^k$$

with $c_{i,j}$ are random numbers in $\mathbb{Q}$ and $c_i = 1$ if ℓ_i is odd and $c_i = 0$ if ℓ_i is even

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  $$v = 200t^4 - 360t^3 + 62t^2 + 60t - 27,\ v_{1,1} = t,\ \text{and } v_{1,2} = -1/6t^3 + 9/20t^2 - 31/600t - 1/20$$

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- check the system

  $$\rho_1 = v = 0, \quad \text{with} \quad \rho_1 = v'u^2 - v_{1,1}u + v_{2,1} \in \mathbb{Q}[t,u],$$

  has real solutions

See you in Tromsø this summer !