

 D^n -finite

Simple functions

Conclusions





DD-finite functions

a computable extension for holonomic functions

Antonio Jiménez-Pastor



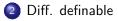


MAX Seminar (Sep. 2021) Joint work with: P. Nuspl and V. Pillwein

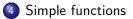


D-finite	Diff. definable	D ⁿ -finite	Simple functions	$\operatorname{Conclusions}_{\circ\circ}$
Outline				





 \bigcirc D^{*n*}-finite







 D^n -finite

Simple functions

 $\underset{\circ\circ}{\text{Conclusions}}$

D-finite functions: the holonomic world





Throughout this talk we consider:

- \mathbb{K} : a **computable** field contained in \mathbb{C} .
- $\mathbb{K}[[x]]$: ring of formal power series over \mathbb{K} .
- ' is the standard derivation w.r.t. x:

$$\left(\sum_{n\geq 0}c_nx^n\right)'=\sum_{n\geq 0}(c_nx^n)'=\sum_{n\geq 0}(n+1)c_{n+1}x^n.$$

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Links to package

Package dd_functions

All results presented in this talk are included in the SageMath package dd_functions.

• Repository:

https://github.com/Antonio-JP/dd_functions

Documentation:

https://antonio-jp.github.io/dd_functions/

• Demo:

https://mybinder.org/v2/gh/Antonio-JP/dd_functions. git/master?filepath=dd_functions_demo.ipynb



D-finite ∞∞•∞∞	Diff. definable	D^n -finite	Simple functions	$\operatorname{Conclusions}_{\circ\circ}$
D-finite fu	nctions			

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that f(x) is D-finite if there exists $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{K}[x]$ (not all zero) such that:

 $p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$

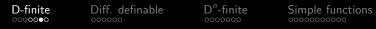


Many special functions are D-finite:

- Exponential functions: e^{x} .
- Trigonometric functions: sin(x), cos(x).
- Logarithm function: log(x + 1).
- Bessel functions: $J_n(x)$.

• Hypergeometric functions:
$${}_{p}F_{q}\left(\begin{array}{c}a_{1},\ldots,a_{p}\\b_{1},\ldots,b_{q}\end{array};x\right)$$
.

- Airy functions: Ai(x), Bi(x).
- Combinatorial generating functions: $F(x), C(x), \ldots$



 $\underset{\circ\circ}{\mathsf{Conclusions}}$

Closure properties

f(x), g(x) D-finite of order d_1, d_2 . a(x) algebraic over $\mathbb{K}(x)$ of degree p.

Property	Function	Order bound
Addition	f(x) + g(x)	$d_1 + d_2$
Product	f(x)g(x)	d_1d_2
Differentiation	f'(x)	d_1
Integration	$\int f(x)$	$d_1 + 1$
Be Algebraic	a(x)	p



Dⁿ-finite

Simple functions

 $\operatorname{Conclusions}_{\circ\circ}$

Working with D-finite functions

There are several implementations of D-finite functions:

- mgfun: Maple package, by F. Chyzak and B. Salvy
- HolonomicFunctions: Mathematica package, by C. Koutschan
- ore_algebra: Sage package, by M. Kauers et al.





Dⁿ-finite

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 $\underset{\circ\circ}{\text{Conclusions}}$

Differentially definable functions





There are power series that **are not** D-finite:

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $tan(x) = \frac{sin(x)}{cos(x)}$.
- \wp -Weierstrass function.
- Gamma function: $f(x) = \Gamma(x+1)$.
- Partition Generating Function: $f(x) = \sum_{n \ge 0} p(n)x^n$.

DD-finite functions

Definition

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$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$

D-finite: NO

• Double exponential: $f(x) = e^{e^x}$.

• Tangent:
$$tan(x) = \frac{sin(x)}{cos(x)}$$
.

DD-finite functions

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that f(x) is DD-finite if there exists $d \in \mathbb{N}$ and D-finite functions $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

DD-finite: **YES**

- Double exponential: $f(x) = e^{e^x} \rightarrow f'(x) e^x f(x) = 0$
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)} \rightarrow \cos^2(x) \tan''(x) 2\tan(x) = 0.$

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Differentially definable functions

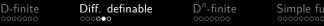
Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that f(x) is differentially definable over R if there exists $d \in \mathbb{N}$ and elements in R $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

We denote the set of all diff. definable functions over R by D(R).

- D-finite functions: $D(\mathbb{K}[x])$.
- DD-finite functions: $D(D(\mathbb{K}[x])) = D^2(\mathbb{K}[x])$.



Simple functions

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Characterization via Linear Algebra

Theorem

The following are equivalent:

f(x) is differentially definable over R ($f(x) \in D(R)$)

€

The **F**-vector space $\langle f(x), f'(x), f''(x), \ldots \rangle$ has finite dimension.

- $R \subset K[[x]]$ is a differential subring
- F is its field of fractions.



 $\underset{\circ\circ}{\text{Conclusions}}$

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D^{*n*}-finite

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 $\operatorname{Conclusions}_{\circ\circ}$

Closure properties

f(x), g(x) in D(R) of order d_1, d_2 . a(x) algebraic over F of degree p.

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Proof for addition:

$$\begin{array}{rcl} \langle (f+g)^{(n)} & : & n \in \mathbb{N} \rangle_F = \langle f^{(n)} + g^{(n)} & : & n \in \mathbb{N} \rangle_F \\ & & \subset \langle f^{(n)} & : & n \in \mathbb{N} \rangle_F + \langle g^{(n)} & : & n \in \mathbb{N} \rangle_F \end{array}$$

 D^n -finite

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 D^n -finite

Simple functions

 $\underset{\circ\circ}{\text{Conclusions}}$

D^n -finite functions: iterating the process





Remark

 $R \subset \mathbb{K}[[x]]$ diff. ring $\Rightarrow \mathsf{D}(R) \subset \mathbb{K}[[x]]$ diff. ring \downarrow Iterate the process



$$\begin{array}{ccc} D\text{-finite} & \text{Diff. definable} & D^{n}\text{-finite} & \text{Simple functions} & \text{Conclusions} \\ \hline D^{n}\text{-finite functions} & \end{array}$$

Remark

 $R \subset \mathbb{K}[[x]]$ diff. ring $\Rightarrow \mathsf{D}(R) \subset \mathbb{K}[[x]]$ diff. ring \downarrow

Iterate the process

D^{*n*}-finite functions

D^{*n*}-finite functions are the *n*th iteration over $\mathbb{K}[x]$, i.e., $\mathsf{D}^n(\mathbb{K}[x])$.

$$\mathbb{K}[x] \subset \mathsf{D}(\mathbb{K}[x]) \subset \mathsf{D}^2(\mathbb{K}[x]) \subset \ldots \subset \mathsf{D}^n(\mathbb{K}[x]) \subset \ldots$$





 $f(x) \in D^{n}(\mathbb{K}[x])$ of order d_{1} . $g(x) \in D^{m}(\mathbb{K}[x])$ of order d_{2} . a(x) algebraic over $D^{m}(\mathbb{K}[x])$ of degree p.

Property	Function	ls in	Order bound
Composition	$f \circ g$	$D^{n+m}(\mathbb{K}[x])$	d_1
Alg. subs.	f ∘ a	$D^{n+m}(\mathbb{K}[x])$	pd_1



 $f(x) \in D^{n}(\mathbb{K}[x])$ of order d_{1} . $g(x) \in D^{m}(\mathbb{K}[x])$ of order d_{2} . a(x) algebraic over $D^{m}(\mathbb{K}[x])$ of degree p.

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Composition	$f \circ g$	$D^{n+m}(\mathbb{K}[x])$	d_1
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• a(x) algebraic over $D^m(K[x])$ implies $a(x) \in D^{m+1}(\mathbb{K}[x])$.



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Alg. subs.	<i>f</i>	$D^{n+m}(\mathbb{K}[x])$	pd_1

- a(x) algebraic over $D^m(K[x])$ implies $a(x) \in D^{m+1}(\mathbb{K}[x])$.
- Then f(a(x)) is in $D^{n+m+1}(\mathbb{K}[x])$.



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- a(x) algebraic over $D^m(K[x])$ implies $a(x) \in D^{m+1}(\mathbb{K}[x])$.
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$\mathcal{K}[x] \subsetneq \mathcal{D}(\mathcal{K}[x]) \subset \mathcal{D}^2(\mathcal{K}[x]) \subset \ldots \subset \mathcal{D}^n(\mathcal{K}[x]) \subset \ldots$

 $e^x \notin K[x]$



$$\begin{array}{ccc} D\text{-finite} & \text{Diff. definable} & D^n\text{-finite} & \text{Simple functions} & \text{Conclusions} \\ D^n \subsetneq D^{n+1} \text{: Iterated exponentials} \end{array}$$

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Iterated Exponentials

- $e_0(x) = 1$,
- $\hat{e}_n(x) = \int_0^x e_n(t) dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x)).$

D-finite Diff. definable
$$D^n$$
-finite Simple functions Conclusions

$D^n \subsetneq D^{n+1}$: Iterated exponentials

$$\mathcal{K}[x] \subsetneq \mathsf{D}(\mathcal{K}[x]) \subsetneq \mathsf{D}^2(\mathcal{K}[x]) \subsetneq \ldots \subsetneq \mathsf{D}^n(\mathcal{K}[x]) \subsetneq \ldots$$

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Iterated Exponentials

• $e_0(x) = 1$, • $\hat{e}_n(x) = \int_0^x e_n(t) dt$,

•
$$e_{n+1}(x) = \exp(\hat{e}_n(x)).$$

$$\left(\begin{array}{c} e_{n+1}(x)\in\mathsf{D}^{n+1}(K[x])\\ e_{n+1}(x)\notin\mathsf{D}^n(K[x])\end{array}\right)$$



Diff. Algebraic functions

Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that f(x) differentially algebraic over R if there is $n \in \mathbb{N}$ and $P(y_0, \ldots, y_n) \in R[y_0, \ldots, y_n]$ such that

$$P(f(x), f'(x), \ldots, f^{(n)}(x)) = 0.$$

We denote by DA(R) the set of all differentially algebraic functions over R.



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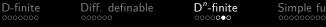
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```
Diff. definable D(R) Diff. algebraic DA(R)

\downarrow \downarrow \downarrow

Linear diff. equations Polynomial diff. equations
```



Simple functions

Conclusions

Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow \mathsf{DA}(R) \subset \mathsf{DA}(S)$.
- $\mathsf{DA}(\mathbb{K}[x]) = \mathsf{DA}(\mathbb{K}).$





 $\underset{\circ\circ}{\text{Conclusions}}$

Inclusion into Diff. Algebraic

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- $\mathsf{DA}(\mathbb{K}[x]) = \mathsf{DA}(\mathbb{K}).$

Proposition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring. Then DA(D(R)) = DA(R).

The proof is constructive.





 D^n -finite

Simple functions

 $\operatorname{Conclusions}_{\circ\circ}$

Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow \mathsf{DA}(R) \subset \mathsf{DA}(S)$.
- $\mathsf{DA}(\mathbb{K}[x]) = \mathsf{DA}(\mathbb{K}).$
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.

Theorem

For all $n \in \mathbb{N}$, if $f(x) \in \mathsf{D}^n(\mathbb{K}[x])$, then $f(x) \in \mathsf{DA}(\mathbb{K})$.





 D^n -finite

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Conclusions

Inclusion into Diff. Algebraic

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- $\mathsf{DA}(\mathbb{K}[x]) = \mathsf{DA}(\mathbb{K}).$
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.
- Theorem: $D^n(\mathbb{K}[x]) \subset DA(\mathbb{K})$.

Example: double exponential

$$\exp(\exp(x) - 1) \longrightarrow f'(x) - \exp(x)f(x) = 0$$

$$\downarrow$$

$$f''(x)f(x) - f'(x)^2 - f'(x)f(x) = 0$$



Simple functions

Conclusions

Inclusion into Diff. Algebraic

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- $DA(\mathbb{K}[x]) = DA(\mathbb{K}).$
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R)).$
- Theorem: $D^n(\mathbb{K}[x]) \subset DA(\mathbb{K})$.

Example: tangent

$$\tan(x) \longrightarrow \cos(x)^{2} f''(x) - 2f(x) = 0$$

$$\downarrow$$

$$-2f^{(5)}(x)f''(x)^{2}f(x) + 12f^{(4)}(x)f'''(x)f''(x)f(x) - 6f^{(4)}(x)f''(x)^{2}f'(x) - 12f'''(x)^{3}f(x) + 12f'''(x)^{2}f''(x)f'(x) - 4f'''(x)f''(x)^{3} - 8f'''(x)f''(x)^{2}f(x) + 8f''(x)^{3}f'(x) = 0$$

D-finite	Diff. definable	D^n -finite	Simple functions	$\operatorname{Conclusions}_{\circ\circ}$

Reverse inclusion

Is the other inclusion true? Can we have $DA(\mathbb{K}[x]) = D^{\infty}(\mathbb{K}[x])$?



 D^n -finite

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Conclusions

Reverse inclusion

Is the other inclusion true? Can we have $DA(\mathbb{K}[x]) = D^{\infty}(\mathbb{K}[x])$?

For some diff. algebraic functions, we can find an $n \in \mathbb{N}$:

Riccati differential equation

Let y(x) be a solution to the Riccati differential equation

$$y'(x) = c(x)y(x)^2 + b(x)y(x) + a(x),$$

where $a(x), b(x) \in D^{n}(\mathbb{K}[x])$ and $c(x) \in D^{n-1}(\mathbb{K}[x])$. Then $y(x) \in D^{n+2}(\mathbb{K}[x])$. Dⁿ-finite Simple functions Conclusions

Reverse inclusion

D-finite

Is the other inclusion true? Can we have $DA(\mathbb{K}[x]) = D^{\infty}(\mathbb{K}[x])$?

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But that is not always the case

Theorem (Noordman, Top, van der Put)

Let y(x) be a non-constant solution to the differential equation

$$y'(x) = y(x)^3 - y(x)^2.$$

Then, there is no $n \in \mathbb{N}$ with $y(x) \in D^n(\mathbb{K}[x])$.



 $\underset{\circ\circ}{\text{Conclusions}}$

Simple functions: handling singularities



Simple functions

 $\operatorname{Conclusions}_{\circ\circ}$

Singularities on differential equations

Zero and Singular set

Let $f(x) \in \mathbb{K}[[x]]$:

- Zero set: $Z(f) = \{ \alpha \in \mathbb{C} : f(\alpha) = 0 \}.$
- Singular set: $S(f) = \{ \alpha \in \mathbb{C} : \alpha \text{ singularity of } f \}.$

Theorem

Let $f(x) \in \mathbb{K}[[x]]$ that satisfy the linear diff. equation

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0,$$

for some $r_0(x), \ldots, r_d(x) \in \mathbb{K}[[x]]$. Then:

$$S(f) \subseteq Z(r_d) \bigcup_{i=0}^d S(r_i).$$

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Singularities on differential equations

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 $\underset{\circ\circ}{\overset{\mathsf{Conclusions}}{\overset{}}}$

Creating new singularities

Closure properties computations may create **new** singularities





Simple functions

 $\operatorname{Conclusions}_{\circ\circ}$

Creating new singularities

Closure properties computations may create **new** singularities

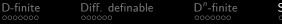
Adding two D-finite functions I

•
$$e^x \longrightarrow \partial - 1$$
.

•
$$Ai(x) \longrightarrow \partial^2 - x$$
.

•
$$e^x + Ai(x)$$
:

$$(x-1)\partial^3 - x\partial^2 - (x^2 - x)\partial + (x^2 - x + 1)$$



Simple functions

 $\underset{\circ\circ}{\mathsf{Conclusions}}$

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D-finite

Diff. definable

 D^n -finite

Conclusions

Creating new singularities

Closure properties computations may create **new** singularities

Adding two D-finite functions II

- $\log(x+1) \longrightarrow (x+1)\partial^2 + \partial$.
- $\sin(x) \longrightarrow \partial^2 + 1$.
- $\log(x+1) + \sin(x)$:

$$(x+1)(x^2+2x+3)\partial^4 + (x^2+2x+7)\partial^3 + (x+1)(x^2+2x+3)\partial^2 + (x^2+2x+7)\partial$$



 D^n -finite

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Adding two D-finite functions II

- $\log(x+1) \longrightarrow (x+1)\partial^2 + \partial$.
- $\sin(x) \longrightarrow \partial^2 + 1$.
- $\log(x+1) + \sin(x)$:

$$(x+1)(x^2+2x+3)\partial^4 + (x^2+2x+7)\partial^3 + (x+1)(x^2+2x+3)\partial^2 + (x^2+2x+7)\partial$$

D-finite

Diff. definable

 D^n -finite

Simple functions

 $\operatorname{Conclusions}_{\circ\circ}$

Creating new singularities

Closure properties computations may create **new** singularities

Multiplying two D-finite functions

•
$$Ai(x+1) \longrightarrow \partial^2 - x$$
.

•
$$J_1(x) \longrightarrow x^2 \partial^2 + x \partial + (x-1)^2$$
.

•
$$Ai(x)J_1(x)$$
:

$$\begin{aligned} x^{2}(4x^{3}+4x^{2}-3)\partial^{4}+4x(x-1)(x^{2}+3x+3)\partial^{3}-\\ &2x^{2}(4x^{4}-4x^{2}+6x+9)\partial^{2}-\\ &2x^{2}(6x^{3}+10x^{2}-4x-27)\partial-\\ &(4x^{7}+12x^{6}+12x^{5}-x^{4}-28x^{3}-11x^{2}+6x-9) \end{aligned}$$

 $\underset{\circ\circ\circ\bullet}{\text{Simple functions}}$

Conclusions

Desingularization

Desingularization

- **INPUT:** a differential equation $\mathcal{L} \cdot f(x) = 0$.
- **OUTPUT:** a differential equation $\tilde{\mathcal{L}} \cdot f(x) = 0$ such that:
 - For all g(x) with $\mathcal{L} \cdot g(x) = 0$, $\tilde{\mathcal{L}} \cdot g(x) = 0$.
 - 2 $\tilde{\mathcal{L}}$ has no apparent singularities.

Previous work on desingularization

- For differential systems over $\mathbb{Q}(x)$: M. Barkatou et al.
- For Ore Operators: S. Chen, M. Jaroschek, M. Kauers, M. F. Singer

Input \rightarrow Closure property \rightarrow Desing. \rightarrow Output w/o sing.

 $\underset{\circ\circ\circ\bullet}{\text{Simple functions}}$

Conclusions

Desingularization

Desingularization

- **INPUT:** a differential equation $\mathcal{L} \cdot f(x) = 0$.
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Our approach

Being able to obtain **directly** through closure properties an operator that has **no new singularities** by using *only* linear algebra.

Input \rightarrow Closure property \rightarrow Output w/o new sing.



 D^n -finite

 $\underset{\circ\circ\circ\circ\bullet}{\text{Simple functions}}$

Conclusions

Two key concepts

Noetherian modules

M is a Noetherian module if all its submodules are finitely generated.

Localization ring

Given a multiplicatively closed set $S \subset R$, the localized ring of R over S (denoted by R_S) is the minimal extension to R where we can divide by elements in S.



Dⁿ-finite

 $\underset{\circ\circ\circ\circ\circ}{\text{Simple functions}}$

Conclusions

S-simple diff. definable functions

Definition

Let $R \subset \mathbb{K}[[x]]$ a differential subring and $S \subset R$ multiplicatively closed. We say that $f(x) \in \mathbb{K}[[x]]$ is S-simple differentially definable over R if there are $r_0(x), \ldots, r_{d-1}(x) \in R$ and $s(x) \in S$ such that:

$$s(x)f^{(d)}(x) + r_{d-1}(x)f^{(d-1)}(x) + \ldots + r_0(x)f(x) = 0.$$

We denote the set of all these functions by D(R, S).



 $\underset{\circ\circ\circ\circ\circ}{\text{Simple functions}}$

Conclusions

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D-finite case

The set *S* controls the singularities of the functions in $D(\mathbb{K}[x], S)$!!



Simple functions

 $\operatorname{Conclusions}_{\circ\circ}$

S-simple diff. definable functions

D-finite case

The set *S* controls the singularities of the functions in $D(\mathbb{K}[x], S)$!!

Consider $f(x) \in D(\mathbb{K}[x], S)$ for the following sets:

- $S = \mathbb{K}$: f(x) is analytic in \mathbb{C} .
- ② $S = \{p(x)^n : n \in \mathbb{N}\}$: f(x) can only have singularities in the zeros of p(x).
- S = K[x] \ ℘ where ℘ is a prime ideal. Then f(x) does not have singularities in any α ∈ V(℘).



D-finite

Diff. definable

 D^n -finite

 $\underset{\circ\circ\circ\circ\circ\circ}{\text{Simple functions}}$

Conclusions

Adapting the main characterization

Characterization theorem - diff. def. case

- Let $R \subset \mathbb{K}[[x]]$ and $f(x) \in \mathbb{K}[[x]]$. It is equivalent:
 - $f(x) \in D(R)$
 - **2** The following Fr(R)-vector space has finite dimension

 $\langle f, f'(x), f''(x), ... \rangle$



D-finite

Diff. definable

 D^n -finite

 $\underset{\circ\circ\circ\circ\circ\circ}{\text{Simple functions}}$

Conclusions

Adapting the main characterization

Characterization theorem - S-simple case

Let $R \subset \mathbb{K}[[x]]$, $S \subset R$ m. c. and $f(x) \in \mathbb{K}[[x]]$. It is equivalent:

- $f(x) \in \mathsf{D}(R,S)$
- The following R_S-module is finitely generated

 $\langle f, f'(x), f''(x), ... \rangle$



 $\underset{\circ\circ\circ\circ\circ\circ\circ\bullet\circ\circ\circ}{\mathsf{Simple functions}}$

 $\operatorname{Conclusions}_{\circ\circ}$

Addition is closed

Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R, S)$. Then f(x) + g(x) is again in D(R, S)



D-finite	Diff. definable	D ⁿ -finite	Simple functions	Conclusion
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Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R, S)$. Then f(x) + g(x) is again in D(R, S)

Proof: Let $M(f) = \langle f(x), \dots, f^{(n)}(x) \rangle_{R_S}$.



D-finite Diff. definable D ⁿ -finite Simple functions Conclusion	ons
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Proof: Let $M(f) = \langle f(x), \dots, f^{(n)}(x) \rangle_{R_S}$. $M(f+g) \subset M(f) + M(g)$

Theorem (addition)

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Proof: Let $M(f) = \langle f(x), \dots, f^{(n)}(x) \rangle_{R_S}$. $M(f+g) \subset M(f) + M(g)$ R Noetherian $\Rightarrow M(f), M(g)$ Noetherian \Rightarrow M(f) + M(g) Noetherian

Theorem (addition)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R, S)$. Then f(x) + g(x) is again in D(R, S)

Proof: Let $M(f) = \langle f(x), \dots, f^{(n)}(x) \rangle_{R_S}$. $M(f+g) \subset M(f) + M(g)$ R Noetherian $\Rightarrow M(f), M(g)$ Noetherian \Rightarrow M(f) + M(g) Noetherian

Hence, M(f + g) is fin. generated and $f(x) + g(x) \in D(R, S)$.

 D^n -finite

Conclusions

Closure properties

Theorem (product)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x), g(x) \in D(R, S)$. Then f(x)g(x) is again in D(R, S)

Theorem (derivation)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x) \in D(R, S)$. Then f'(x) is again in D(R, S)

Theorem (integration)

Let $R \subset \mathbb{K}[[x]]$ be Noetherian, $S \subset R$ m.c. and $f(x) \in D(R, S)$. Then for any F(x) with F'(x) = f(x), $F(x) \in D(R, S)$.



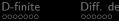
D-finite	Diff. definable	D^n -finite	Simple functions	$\operatorname{Conclusions}_{\circ\circ}$
Closure p	properties			

Theorem

Let $R \subset \mathbb{K}[[x]]$ be Noetherian and $S \subset R$ multiplicatively closed. Then D(R, S) is a differential extension of R closed under integration.

f(x), g(x) S-simple D-finite of order d_1, d_2 .

	in D(<i>R</i> , <i>S</i>)	Order bound
Addition	(f+g)	Unknown
Product	(fg)	Unknown
Differentiation	f'	Unknown
Integration	$\int f$	$d_1 + 1$



 D^n -finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\mathsf{Simple functions}}$

 $\operatorname{Conclusions}_{\circ\circ}$

Closure properties examples

Adding two D-finite functions I

•
$$e^x \longrightarrow \partial - 1.$$

• $Ai(x) \longrightarrow \partial^2 - x.$
• $e^x + Ai(x):$

$$(x-1)\partial^3 - x\partial^2 - (x^2-x)\partial + (x^2-x+1)$$





 $\underset{\circ\circ\circ\circ\circ\circ\circ}{\text{Diff. definable}}$

Dⁿ-finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\mathsf{Simple functions}}$

 $\operatorname{Conclusions}_{\circ\circ}$

Closure properties examples

Adding two D-finite functions I

•
$$e^{x} \longrightarrow \partial - 1.$$

• $Ai(x) \longrightarrow \partial^{2} - x.$
• $e^{x} + Ai(x):$
 $(x-1)\partial^{3} - x\partial^{2} - (x^{2} - x)\partial + (x^{2} - x + 1)$
 \downarrow^{4}
 $\partial^{4} - (x^{2} + 1)\partial^{3} + (x^{2} + 1)\partial^{2} + (x^{3} + x - 2)\partial - (x^{3} - x - 1)$

DD-finite functions



 D^n -finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\text{Simple functions}}$

Conclusions

Closure properties examples

Adding two D-finite functions II

- $\log(x+1) \longrightarrow (x+1)\partial^2 + \partial$.
- $\sin(x) \longrightarrow \partial^2 + 1$.
- $\log(x+1) + \sin(x)$:

$$(x+1)(x^2+2x+3)\partial^4 + (x^2+2x+7)\partial^3 + (x+1)(x^2+2x+3)\partial^2 + (x^2+2x+7)\partial$$

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\text{Simple functions}}$

 $\operatorname{Conclusions}_{\circ\circ}$

Closure properties examples

Adding two D-finite functions II

•
$$\log(x+1) \longrightarrow (x+1)\partial^2 + \partial$$
.

•
$$\sin(x) \longrightarrow \partial^2 + 1$$
.

•
$$\log(x+1) + \sin(x)$$
:

$$(x+1)(x^{2}+2x+3)\partial^{4} + (x^{2}+2x+7)\partial^{3} + (x+1)(x^{2}+2x+3)\partial^{2} + (x^{2}+2x+7)\partial \downarrow \\ \downarrow \\ 4(x+1)^{2}\partial^{5} + p_{4}(x)\partial^{4} + p_{3}(x)\partial^{3} + p_{2}(x)\partial^{2} + p_{1}(x)\partial, \\ \deg(p_{i}(x)) \leq 5$$



Dⁿ-finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\text{Simple functions}}$

 $\underset{\circ\circ}{\text{Conclusions}}$

Closure properties examples

Multiplying two D-finite functions

•
$$Ai(x + 1) \longrightarrow \partial^2 - x.$$

• $J_1(x) \longrightarrow x^2 \partial^2 + x \partial + (x - 1)^2.$
• $Ai(x)J_1(x):$
 $x^2(4x^3 + 4x^2 - 3)\partial^4 + 4x(x - 1)(x^2 + 3x + 3)\partial^3 - 2x^2(4x^4 - 4x^2 + 6x + 9)\partial^2 - 2x^2(6x^3 + 10x^2 - 4x - 27)\partial - (4x^7 + 12x^6 + 12x^5 - x^4 - 28x^3 - 11x^2 + 6x - 9)$





 $\underset{\circ\circ\circ\circ\circ\circ\circ}{\text{Diff. definable}}$

Dⁿ-finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\mathsf{Simple functions}}$

Conclusions

Closure properties examples

Multiplying two D-finite functions

•
$$Ai(x + 1) \longrightarrow \partial^2 - x.$$

• $J_1(x) \longrightarrow x^2 \partial^2 + x \partial + (x - 1)^2.$
• $Ai(x)J_1(x):$
 $195x^3 \partial^5 + p_4(x)\partial^4 + p_3(x)\partial^3 + p_2(x)\partial^2 + p_1(x)\partial deg(p_i(x)) \le 12$





 D^n -finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\mathsf{Simple functions}}$

 $\operatorname{Conclusions}_{\circ\circ}$

Simple DD-finite functions

- All results for S-simple D(R) require R Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?





 D^n -finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\text{Simple functions}}$

Conclusions

Simple DD-finite functions

- All results for S-simple D(R) require R Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?

YES!

Instead of working with the whole $D(\mathbb{K}[x])$, we restrict to a smaller Noetherian subring.



 D^n -finite

 $\underset{\circ\circ\circ\circ\circ\circ\circ\circ\circ}{\text{Simple functions}}$

Conclusions

Simple DD-finite functions

- All results for S-simple D(R) require R Noetherian.
- D-finite functions are not Noetherian.
- Can we extend the result to DD-finite functions?

Definition

Let $S \subset \mathbb{K}[[x]]$ be a multiplicatively closed set. We denote the set of *S*-simple D^n -finite functions with $D^n(\mathbb{K}[x], S)$ and we define it recursively by:

- $D^1(\mathbb{K}[x], S) = D(\mathbb{K}[x], \mathbb{K}[x] \cap S).$
- $\mathsf{D}^n(\mathbb{K}[x], S) = \mathsf{D}(\mathsf{D}^{n-1}(\mathbb{K}[x], S), \mathsf{D}^{n-1}(\mathbb{K}[x], S) \cap S).$

Intuitively, we control the leading coefficient of the equations that define the elements in the whole chain of rings.

 $\underset{\scriptstyle o \circ \circ \circ \circ \circ \circ \circ}{\mathsf{D}^n}\mathsf{-finite}$

Simple functions

Conclusions ••

Conclusions

Achievements

- Extension of the holonomic framework.
- Running implementation of closure properties.
- Relation to differentially algebraic functions.
- Control of singularities throughout closure properties.

Future work

- Fast computation of truncation of Dⁿ-finite functions.
- Development of certified numerical evaluations.
- Combinatorial meaning of the induced sequences.
- Multivariate DD-finite functions.



 D^n -finite

Simple functions

Conclusions

Thank you!

Contact webpage:

- http://www.lix.polytechnique.fr/~jimenezpastor/
- https://www.dk-compmath.jku.at/people/antonio

Code available:

• https://github.com/Antonio-JP/dd_functions

