On the Relationship Between Differential Algebra and Tropical Differential Geometry (joint work with S. Falkensteiner and M. P. Noordman)

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DA vs TDAG

- Differential Algebra (DA) founded by J. F. Ritt (1932, 1950). Developed by E. R. Kolchin (1973). An algebraic theory for polynomial differential equations.
- Tropical Differential Algebraic Geometry (TDAG) founded by D. Grigoriev (2015). A differential analogue of tropical algebra (aka min-plus algebra) for the study of formal power series (FPS) solutions of polynomial differential equations.

Common topic : FPS solutions of ODE systems

Fundamental Theorem of TDAG (2016) Aroca, Garay, Toghani

Autonomous Equations

$$\dot{y}^2 + 8\,y^3 - 1 \quad = \quad 0\,.$$

Differentiate
$$\begin{aligned} &\dot{y}^2 + 8\,y^3 - 1\,, \\ &2\,\dot{y}\,\ddot{y} + 24\,y^2\,\dot{y}\,, \\ &\quad\vdots \end{aligned}$$

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Rename $y^{(k)}$ as a_k.

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$\vdots$

Rename $y^{(k)}$ as a_k. Solve and get some (truncated) arc $\underline{a}$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots) \quad = \quad (0, 1, 0, 0, -24, 0, 0, 2880, \ldots)\,.$$

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Plug in $\Psi(\underline{a}) \quad = \quad \sum \frac{a_i}{i\,!}\,x^i\,.$

A FPS solution (centered at the origin) is obtained

$$\bar{y}(x) \quad = \quad x - x^4 + 4/7\,x^7 + \cdots$$

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$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots) \quad = \quad (0, 1, 0, 0, -24, 0, 0, 2880, \ldots)\,.$$

Plug in $\Psi_\alpha(\underline{a}) \quad = \quad \sum \frac{a_i}{i\,!}\,(x-\alpha)^i\,.$

A FPS solution (centered at $x = \alpha$) is obtained

$$\bar{y}(x) \quad = \quad (x-\alpha) - (x-\alpha)^4 + 4/7\,(x-\alpha)^7 + \cdots$$

Non Autonomous Equation

$$x\,\dot{y}^2 + 8\,x\,y^3 - 1 \;=\; 0\,.$$

Differentiate $x\,\dot{y}^2 + 8\,x\,y^3 - 1\,,$

$$2\,x\,\dot{y}\,\ddot{y} + \dot{y}^2 + 24\,x\,y^2\,\dot{y} + 8\,y^3\,,$$

$$\vdots$$

Rename $y^{(k)}$ as a_k. Replace x by the expansion point (say) $\alpha = 1$. Solve and get some (truncated) arc $\underline{a}$

$$(a_0, a_1, a_2, a_3, a_4, a_5, \ldots) \;=\; (0, 1, -\tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{207}{8}, \tfrac{489}{16}, \ldots)\,.$$

Plug in $\Psi_{\alpha=1}(\underline{a}) \;=\; \sum \frac{a_i}{i\,!}\,(x-1)^i\,.$

A FPS solution (centered at $x = \alpha = 1$) is obtained

$$\bar{y}(x) \;=\; (x-1) - \frac{1}{4}\,(x-1)^2 + \frac{1}{8}\,(x-1)^3 - \frac{69}{64}\,(x-1)^4 + \cdots$$

Summary — The Issue

1. An issue on the expansion point ($x = \alpha$) or on the initial values ($y(\alpha) = a_0, \dot{y}(\alpha) = a_1, \ldots$) arises if the leading coefficients of the differentiated system (the initial and separant of the differential polynomial) vanish at these values
2. Such a cancellation may prevent FPS solutions to exist or to be unique at $x = \alpha$
3. Reduction to the autonomous case is always possible but transforms issues on the expansion point into issues on initial values

Developed on the 2 next slides

Existence Problem of Solutions

In the PDE case, Hilbert's 10th Problem

Start with any polynomial $f \in \mathbb{Q}[z]$

$$f(z) \;=\; z^2 - 2\,.$$

Obtain $p \in \mathbb{Q}\{y\}$ by
$$\begin{aligned} p &= f\left(x\,\frac{\mathrm{d}}{\mathrm{d}x}\right) y\,. \\ &= x^2\,\ddot{y} + x\,\dot{y} - 2\,y\,. \end{aligned}$$

Fact 1. If $\underline{a} = (a_0, a_1, \ldots)$ is any arc $\quad p(\Psi(\underline{a})) = \sum_{i \geq 0} a_i\, f(i)\, x^i\,.$

Fact 2. The following identity holds $\quad \frac{1}{1-x} = \sum_{i \geq 0} x^i\,.$

at the origin!

Thus $(1-x)\,p - 1 = 0$ has a FPS solution iff $a_i = 1/f(i)$.

↪ iff f has no positive integer root

Reduction to Autonomous Equations

$$p(x, y, \dot{y}, \ddot{y}, \ldots) \quad = \quad 0\,,$$

View x and y as two unknown functions $x(\xi)$ and $y(\xi)$ and add

$$\dot{x} \quad = \quad 1\,.$$

Compute a FPS solution

$$\begin{aligned} y(\xi) \quad &= \quad a_0 + a_1\,\xi + (a_2/2)\,\xi^2 + \cdots \\ x(\xi) \quad &= \quad b_0 + \xi\,. \end{aligned}$$

$\xi = (x - b_0)$

If $x = \alpha$ was a problematic expansion point before reduction then $b_0 = \alpha$ is a problematic initial value after reduction.

Summary — Ritt DA Approach

- Ritt only considers autonomous systems $\Sigma \subset \mathscr{F}\{y_1, \ldots, y_n\}$ (say $\mathscr{F} = \mathbb{Q}$)
- The existence problem of FPS solutions at any α and unspecified initial values is equivalent to the decision problem $1 \in [\Sigma]$ (the differential ideal generated by Σ). It is algorithmic.

For non autonomous systems

- Thanks to the reduction process, the existence problem of FPS solutions at unspecified α and unspecified initial values is equivalent to the decision problem $1 \in [\Sigma]$

We feel free to change the value if an issue arises

Summary — The TDAG Approach

TDAG considers general systems $\Sigma \subset \mathscr{F}[[x]]\{y_1, \ldots, y_n\}$

$\mathscr{F}$ field of constants, characteristic zero

Reduction to the autonomous case is impossible

FPS solutions are sought at the origin

- This problem (over $\mathbb{Q}[x]$) is decidable (I have not understood the proof).
- The existence problem of nonzero solutions is undecidable.

The fundamental theorem of TDAG only states an equivalence.

- Provided that Σ is a differential ideal and the base field is an algebraically closed, uncountable, field of characteristic zero
- The tropicalization of the FPS solutions of Σ exactly is the solution set of the tropicalization of Σ:

$$\operatorname{trop}(\operatorname{sol}(\Sigma)) = \operatorname{sol}(\operatorname{trop}(\Sigma))$$

new result on last slide

Tropicalization of a FPS

$$\varphi = \sum_{i\in\mathbb{N}} a_i\, x^i\,.$$

- The support of φ is the set $\{i \in \mathbb{N} \mid a_i \neq 0\}$.
- The valuation of φ is ∞ if $\varphi = 0$ else it is the minimal element of its support.
- Def The tropicalization of φ is its support.

Thus $\mathrm{trop}(\mathrm{sol}(\Sigma))$ is the set of supports of all FPS solutions of Σ.

Tropicalization of a differential monomial

Let

- $y_1, \ldots, y_n$ be n differential indeterminates
- $S_1, \ldots, S_n$ be n supports
- $m = c\, v_1^{d_1} \cdots v_r^{d_r}$ be a monomial ($c \in \mathscr{F}[[x]]$ and each v_i a derivative $(y^{(j)})^k$)

Then $\mathrm{trop}(m)$ [at $S_1, \ldots, S_n$] is the valuation of the FPS obtained by evaluating m at any tuple of n FPS with supports $S_1, \ldots, S_n$.

Examples

$y(x) = a_0 + a_1 x + a_2 x^2 \quad (a_i \neq 0)$

```
> TropicalizePolynomial (x^2*y, {y = {0,1,2}}, R);
```

↳ valuation $(x^2 y) = ?$

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$y(x) = a_0 + a_3 x^3 \quad (a_0 \neq 0)$

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> TropicalizePolynomial (y[x]^3, {y = {0,3}}, R);
```

valuation of $y^3 = ?$

Examples

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$\hookrightarrow$ valuation of $\ddot{y}^3 = ?$

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> TropicalizePolynomial (y[x,x]^3, {y = {0,1}}, R);
                                [infinity]
```

Tropicalization of a polynomial. Support of solutions

The differential polynomial $\dot{y}^2 - 4\,y$ admits as FPS solutions

- $y(x) = 0$ (support $\varnothing$) and
- $y(x) = (x + c)^2$ (supports $\{0, 1, 2\}$ and $\{2\}$)

all supports are supports of solutions !

$y(x) = 0$

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> TropicalizePolynomial (y[x]^2 - 4*y, {y = {}}, R);
```

what is the list of the trop (m_i) ?

↳ should be a min !

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Both monomials vanish

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Both monomials may cancel each other

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                       [infinity, infinity]
> TropicalizePolynomial (y[x]^2 - 4*y, {y = {2}}, R);
                       [2, 2]
> dp := Differentiate (y[x]^2 - 4*y, x, x, R);
                                                2
         dp := 2 y[x] y[x, x, x] + 2 y[x, x]  - 4 y[x, x]
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$y(x) = a_2 x^2\ (a_2 \neq 0)$

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```

One monomial vanishes

```
                    [infinity, 0, 0]
```

Two monomials may cancel each other

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You don't see it on the ODE

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                                        [0, 0]
> dp := Differentiate (y[x]^2 - 4*y, x, R);
                    dp := 2 y[x] y[x, x] - 4 y[x]
> TropicalizePolynomial (dp, {y = {0,1}}, R);
                                        [infinity, 0]
```

You see it on the first derivative of the ODE

Solution of a Tropical Expression

$$f = m_1 + m_2 + \cdots + m_r$$

be a differential polynomial in expanded form. View

$$\mathrm{trop}(f)$$

as a function of n unknown supports.

Then $S = (S_1, \ldots, S_n)$ is said to be a *solution* of $\mathrm{trop}(f)$ if either

1. each $\mathrm{trop}(m_i) = \infty$ or
2. there exists m_i, m_j $(i \neq j)$ such that

$$\mathrm{trop}(m_i) = \mathrm{trop}(m_j) = \min_{k=1}^{r}(\mathrm{trop}(m_k)).$$

the observed phenomenon on supports of solutions!

The Fundamental Theorem of TDAG

If $\Sigma \subset \mathscr{F}[[x]]\{y_1, \dots, y_n\}$ then

$$\mathrm{trop}(\mathrm{sol}(\Sigma)) \quad \subset \quad \mathrm{sol}(\mathrm{trop}(\Sigma))\,.$$

The converse inclusion is difficult. It needs

- Σ to be a differential ideal (i.e. $\dot{p} \in \Sigma$ whenever $p \in \Sigma$)
- $\mathscr{F}$ to be algebraically closed and uncountable (if we look for FPS with coefficients in $\mathscr{F}$)

New result on the last slide

The Key Proposition

- Present $\Sigma \subset \mathscr{F}[[x]]\{y_1, \ldots, y_n\}$ by finitely many differential polynomials g_i.
- Enumerate the derivatives of the y_i as $v_0, v_1, v_2, \ldots$
- Enumerate the derivatives of the g_i, evaluated at $x = 0$, as $f_0, f_1, f_2, \ldots$
- For any k let κ and Σ_k be such that

$$\Sigma_k \underset{\text{def}}{=} \{f_i \mid 0 \leq i \leq k\} \subset \mathscr{F}[v_0, \ldots, v_\kappa].$$

- Define (S being any support)

alg. var. of Σ_k

$$A_k = \{a \in \mathscr{F}^{\kappa+1} \mid f_0(a) = \cdots = f_k(a) = 0\},$$
$$A_{k,S} = \{a \in A_k \mid a_i \neq 0 \text{ if and only if } i \in S \cap [0, \kappa]\}.$$

$\equiv$ compatible with S

Prop

Then if $A_{k,S} \neq \varnothing$ for each k then $A_{\infty,S} \neq \varnothing$.

Sketch of Proof

Assume $A_{k,S} \neq \varnothing$ for each k

Key prop proved in [AGT16]
This proof is due to M. P. Noordman

$$A_{k,S} \quad = \quad \{a \in A_k \mid a_i \neq 0 \text{ if and only if } i \in S \cap [0,\kappa]\}\,.$$

DL84

- Pick one solution in each $A_{k,S}$ and deduce a solution of $\Sigma_\infty = \{f_0, f_1, f_2, \ldots\}$ in an ultrapower of $\mathscr{F}$.
- Thus the following ring is not the null ring

quotient by ideal + localization

$$\mathscr{R} \quad = \quad \mathscr{F}[v_i, v_j^{-1} \mid i \in \mathbb{N}, j \in S]/(f_i,\ v_j \mid i \in \mathbb{N}, j \notin S)\,.$$

- Thus $\mathscr{R}$ contains a maximal ideal $\mathfrak{m}$.
- Since $\mathscr{F}$ is uncountable and $\mathscr{R}$ has countable dimension as a vector space over $\mathscr{F}$, the field $\mathscr{R}/\mathfrak{m}$ is algebraic over $\mathscr{F}$.
- Since $\mathscr{F}$ is algebraically closed $\mathscr{R}/\mathfrak{m} \simeq \mathscr{F}$.

Sktech of Proof (new, countable version)

Assume $A_{k,S} \neq \varnothing$ for each k

Key prop proved in [AGT16]
This proof is new

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- Thus $\mathscr{R}$ contains a maximal ideal $\mathfrak{m}$.
- The field $\mathscr{R}/\mathfrak{m}$ contains a solution and is a field extension of $\mathscr{F}$ with at most countable transcendence degree over $\mathscr{F}$.

new argument

Bibliography

DA

DL84

Tropical

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