Confluence in string rewriting systems compatible with a crystal structure

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Abstract

A crystal is a kind of directed labeled graph arising in the field of representation theory. We consider an adapted abstract notion of crystals, called $\mathcal{K}$-graphs. A $\mathcal{K}$-graph structure on a free monoid is a directed colored graph structure satisfying certain conditions for the product of the monoid. A $\mathcal{K}$-congruence on the free monoid is one that identifies isomorphic connected components of the $\mathcal{K}$-graph. In this work we define a notion of $\mathcal{K}$-string rewriting systems which generate such $\mathcal{K}$-congruences. For a class of $\mathcal{K}$-graphs called proper, the interaction of the string rewriting system with the $\mathcal{K}$-graph structure reduces the proofs of the rewriting properties of termination and (local) confluence to a family of reduced words in the free monoid, called words of highest weight. From this we deduce $\mathcal{K}$-versions of Newman's lemma, critical pair lemma, and Squier's coherent completion theorem. Finally we illustrate these constructions with an example for the plactic monoid of type A. The constructions in this work are phrased in terms of $\mathcal{K}$-graphs, though many of their applications lie in the original context of crystals.

1 Introduction

A central approach in the study of confluence in rewriting theory is to reduce the problem to a subset of branchings namely to local-confluence, and to critical branchings. This approach is detailed in two results: Newman's Lemma [8], where the property of confluence of a terminating string rewriting system is equated to the property of local confluence; and the Critical Pair Lemma (CPL) [6], where the local confluence of a string rewriting system is equated to confluence of its critical branchings, which are pairs of overlapping rules on a minimal source.

In an algebraic context, rewriting theory has found applications in the higher dimensional study of objects like monoids, small categories, and algebras over a field. This consists of realizing the object in question by a presentation with generators and oriented relations compatible with the defining axioms of the algebraic object. In this context, a confluence diagram may be regarded as a *relation* between two rewriting sequences, which in turn are relations in the presentation itself, thus one may view the confluence diagrams as *relations between relations*. A question in this direction is how to obtain a set of generating relations between relations from the given presentation, so that any confluence diagram of the rewriting system can be described in terms of this specified set. This question is answered by Squier in [10] in the context of monoids and small categories. Namely, given a presentation of a monoid or a small category by generators and oriented relations, the rewriting system of which is convergent, then the generating relations between relations are the confluence diagrams of critical branchings. The data of generators, generating relations, and generating relations between relations is called a *coherent presentation*. An important aspect of the study of relations between relations in presentations of monoids, is that it provides an algorithmic approach to the study of the monoid's lower-dimensional homology. Note however that a critical branching may admit several confluence diagrams in Squier's construction. To complete this algorithmic point of view, Malbos and Guiraud in [3] employ a notion of a *normalization strategy* which is a deterministic way to choose confluence diagrams for critical branchings. In their work, they take the ideas of Squier even further: they show that for a small category presented by generators and oriented relations, whose string rewriting system is convergent, one can construct a cofibrant replacement for the category. In a grander scheme these constructions facilitate an algorithmic approach to the study of the homology of small categories.

The works of Squier, and Guiraud and Malbos provide a way of determining coherent presentations of small categories and monoids from convergent ones. Finding a convergent presentation in the first place and computing the confluence diagrams of critical branchings remains a difficult task, as in general this problem is heavily dependent on the intrinsic properties of the monoid and the presentation.

In this work, we consider a notion of a string rewriting system adapted to the theory of *crystals*, and give corresponding versions of three classical results in rewriting theory. The notion of crystals was first defined by Kashiwara in [5] in his study of representations of complex semisimple Lie algebras. We phrase our constructions in terms of an abstracted version of crystals called $\mathcal{K}$-*graphs*, which have been adapted from [1]. This work emerged from a study of coherence of the plactic monoid of type C in [7], and forms part of a forthcoming PhD thesis by the author.

In Section 2 we introduce the notion of a $\mathcal{K}$-graph as a directed colored graph satisfying certain conditions. We then consider a $\mathcal{K}$-graph structure on the free monoid generated by the vertices of a $\mathcal{K}$-graph. A $\mathcal{K}$-*congruence* is one that identifies isomorphic connected components of the $\mathcal{K}$-graph on the free monoid. We then introduce a notion of a $\mathcal{K}$-string rewriting system. This is a string rewriting system that is compatible with the $\mathcal{K}$-graph structure, and such that the congruence generated by it is a $\mathcal{K}$-congruence. In Section 3 we show that if $\mathcal{K}$ is *proper*, then the study of rewriting properties of termination, and local confluence is reduced to a subfamily of words called *words of highest weight*. In particular we obtain $\mathcal{K}$-versions of Newman's lemma, the critical pair lemma, and of Squier's coherent completion theorem. In Appendix A we illustrate these constructions and results with an example of the *plactic monoid of type* A. Finally in Section 4 we briefly discuss how this approach could be extended to higher dimensions in accordance with the work of Guiraud and Malbos [3].

2 $\mathcal{K}$-string rewriting systems

A $\mathcal{K}$-*graph* is a directed colored graph Γ with vertex set $V(\Gamma)$, and with edges colored from a set I, satisfying the following conditions

(P1) for any $x \in V(\Gamma)$ and $i \in I$, there exists at most one edge e with *source (target)* x and color i,

(P2) for any $i \in I$, there exists no infinite directed path in Γ with edges colored by i.

It is practical to realize the $\mathcal{K}$-graph structure via the *Kashiwara operators*, which are partial maps e_i and f_i on $V(\Gamma)$ defined by setting

$$x \xrightarrow{i} y \quad \text{if and only if} \quad y = f_i.x, \text{ and } x = e_i.y.$$

Remark 2.1. *The notion of crystals was introduced by Kashiwara in [5] in his study of the representation theory of quantum groups. In this work the constructions are phrased in terms of* $\mathcal{K}$*-graphs, which are an abstract graph-theoretic adaptation of crystals as introduced in [1]. We remark that a large class of crystals satisfies* **(P1), (P2)**.

Given a $\mathcal{K}$-graph Γ, the graph structure extends to the free monoid on the vertices $V(\Gamma)$ of Γ, denoted Γ^*. The Kashiwara operators e_i and f_i extend to Γ^* inductively on the lengths of words $w = uv \in \Gamma^*$ as follows

$$e_i.(uv) = \begin{cases} (e_i.u)v & \text{if } \varphi_i(u) \geq \varepsilon_i(v), \\ u(e_i.v) & \text{if } \varphi_i(u) < \varepsilon_i(v), \end{cases} \tag{1}$$

and

$$f_i.(uv) = \begin{cases} (f_i.u)v & \text{if } \varphi_i(u) > \varepsilon_i(v), \\ u(f_i.v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v), \end{cases} \tag{2}$$

where $\varepsilon_i, \varphi_i : \Gamma^* \longrightarrow \mathbb{N}$ are also defined inductively via

$$\varepsilon_i(w) = \#\{e_i.w, e_i^2.w, \dots\}, \quad \varphi_i(w) = \#\{f_i.w, f_i^2.w, \dots\},$$

which are finite quantities by an iteration of **(P2)**. We remark here a few consequences of these definitions:

i) e_i and f_i are partial operators on Γ^*: e.g. if $\varphi_i(u) \geq \varepsilon_i(v)$ and $e_i.u$ is undefined, then $e_i.(uv)$ is also undefined;

ii) the definition of e_i and f_i on a word w is independent of the factorization $w = uv$;

iii) e_i and f_i are inverse operators: $f_i.(e_i.w) = w$ and $e_i.(f_i.w) = w$.

Thus the free monoid Γ^* carries a directed colored graph structure, and we call it the *free $\mathcal{K}$-monoid generated by* Γ. As Γ^* is a graph, we have a notion of *connected components* in Γ^*. The connected component of $w \in \Gamma^*$ is denoted by $\mathbf{B}(w)$. Using this notion, we specify a type of congruence on the free $\mathcal{K}$-monoid.

Definition 2.2. *Let Γ be a $\mathcal{K}$-graph, and Γ^* the corresponding free $\mathcal{K}$-monoid. A $\mathcal{K}$-congruence on Γ^* is a congruence $\sim$ such that if $w \sim w'$, then*

i) *there exists a directed colored graph isomorphism $p : \mathbf{B}(w) \longrightarrow \mathbf{B}(w')$ such that $p(w) = w'$,*

ii) *if $e_i.w$ (resp. $f_i.w$) is defined, then so is $e_i.w'$ (resp. $f_i.w'$) and we have*

$$e_i.w \sim e_i.w' \qquad (\textit{resp. } f_i.w \sim f_i.w').$$

The largest such congruence, denoted $\sim_\Gamma$ is defined by setting $w \sim_\Gamma w'$ if and only if there exists an isomorphism p as in Definition 2.2 **i)**.

We specify here a class of $\mathcal{K}$-graphs that occurs often and has practical combinatorial advantages. A word $w \in \Gamma^*$ is called *a word of highest weight* if $e_i.w$ is undefined for all $i \in I$. If Γ is such that every connected component $\mathbf{B}(w) \subset \Gamma^*$ contains a unique word of highest weight, the $\mathcal{K}$-graph Γ is called *proper*.

Next we introduce a notion of string rewriting which is compatible with a $\mathcal{K}$-graph structure. One may view the next definition simply as an oriented generating data for a $\mathcal{K}$-congruence, hence the similarity with Definition 2.2.

Definition 2.3. *A $\mathcal{K}$-string rewriting system is a string rewriting system (Γ^*, R) where Γ is a $\mathcal{K}$-graph, and such that if $w \Longrightarrow w'$ is a rewriting rule in R, then*

i) *there exists a directed colored graph isomorphism $p : \mathbf{B}(w) \longrightarrow \mathbf{B}(w')$ such that $p(w) = w'$,*

ii) *if $e_i.w$ (resp. $f_i.w$) is defined, then so is $e_i.w'$ (resp. $f_i.w'$) and we have*

$$(e_i.w \Longrightarrow e_i.w') \in R \qquad (\textit{resp. } (f_i.w \Longrightarrow f_i.w') \in R).$$

For a $\mathcal{K}$-string rewriting system (Γ^*, R), the congruence in Γ^* generated by R is a $\mathcal{K}$-congruence. Thus $\mathcal{K}$-string rewriting systems are well-adapted at studying such congruences. We call a $\mathcal{K}$-string rewriting system *proper* if Γ^* is proper.

3 Confluence for $\mathcal{K}$-string rewriting systems

Here we interpret Newman's lemma and the critical pair lemma in the context of $\mathcal{K}$-string rewriting systems.

For a $\mathcal{K}$-string rewriting system (Γ^*, R), denote by $\mathrm{Seq}(\Gamma^*, R)$ the set of rewriting sequences of (Γ^*, R); by $\mathrm{Br}(\Gamma^*, R)$ the set of branchings of (Γ^*, R); and by $\mathrm{Crit}(\Gamma^*, R)$ the set of critical pairs of (Γ^*, R). We denote the length function on $\mathrm{Seq}(\Gamma^*, R)$ by $|\cdot|$. We then have the following result.

Theorem 3.1. *Let (Γ^*, R) be a $\mathcal{K}$-string rewriting system. Then the Kashiwara operators e_i and f_i extend to Seq(Γ^*, R), Br(Γ^*, R), and Crit(Γ^*, R) and commute with the source and target maps of the $\mathcal{K}$-rewriting system. In particular*

i) *for $\mathfrak{s} \in$ Seq(Γ^*, R) and $i \in I$ such that $e_i.\mathfrak{s}$ (resp. $f_i.\mathfrak{s}$) is defined, we have a commutative square*

$$\begin{array}{ccc} w_1 & \overset{\mathfrak{s}}{\Longrightarrow} & w_2 \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ e_i.w_1 & \underset{e_i.\mathfrak{s}}{\Longrightarrow} & e_i.w_2 \end{array} \qquad \textit{resp.} \qquad \begin{array}{ccc} f_i.w_1 & \overset{f_i.\mathfrak{s}}{\Longrightarrow} & f_i.w_2 \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ w_1 & \underset{\mathfrak{s}}{\Longrightarrow} & w_2 \end{array}$$

and $|e_i.\mathfrak{s}| = |\mathfrak{s}|$ (resp.$|f_i.\mathfrak{s}| = |\mathfrak{s}|$),

ii) *for a branching $(\alpha, \beta) \in$ Br(Γ^*, R) and $i \in I$ such that $e_i.(\alpha, \beta)$ (resp. $f_i.(\alpha, \beta)$) is defined, we have*

$$(\alpha, \beta) \textit{ is confluent if and only if } e_i.(\alpha, \beta) \textit{ is confluent}$$
$$\big(\textit{resp.} \quad (\alpha, \beta) \textit{ is confluent if and only if } f_i.(\alpha, \beta) \textit{ is confluent}\big).$$

This result shows that the property of termination and of confluence of a $\mathcal{K}$-string rewriting system is independent of the action of the Kashiwara operators. If the $\mathcal{K}$-graph Γ is proper, we can use this result to obtain reduced versions of classical rewriting results in our context as follows. Let (Γ^*, R) be a proper $\mathcal{K}$-string rewriting system and consider an abstract rewriting system $((\Gamma^*)^0, R^0)$ where

$$(\Gamma^*)^0 := \{w \in \Gamma \mid w \text{ a word of highest weight in } \Gamma^*\},$$

and

$$R^0 := \{tuv \overset{t\alpha v}{\Longrightarrow} tu'v \mid tuv \in (\Gamma^*)^0,\ u \overset{\alpha}{\Longrightarrow} u' \in R\}.$$

We have the following consequence of Theorem 3.1.

Corollary 3.2. *Let (Γ^*, R) be a proper $\mathcal{K}$-string rewriting system. Then (Γ^*, R) is terminating respectively (locally) confluent if and only if $((\Gamma^*)^0, R^0)$ is terminating respectively (locally) confluent.*

Using this result, we then obtain corresponding $\mathcal{K}$-versions of two classical results in rewriting theory.

Theorem 3.3 (Newman's lemma for $\mathcal{K}$-SRS)**.** *Let (Γ^*, R) be a proper $\mathcal{K}$-string rewriting system. Then (Γ^*, R) is confluent if and only if $((\Gamma^*)^0, R^0)$ is terminating and locally confluent.*

To state the Critical Pair Lemma, we remark that the notion of critical pairs descends to the abstract rewriting system $((\Gamma^*)^0, R^0)$. These are the branchings (α, β) with $\alpha, \beta \in R^0$ which are critical in R.

Theorem 3.4 ($\mathcal{K}$-Critical Pair Lemma)**.** *Let (Γ^*, R) be a proper $\mathcal{K}$-string rewriting system. Then (Γ^*, R) is locally confluent if and only if the critical pairs of $((\Gamma^*)^0, R^0)$ are confluent.*

3.5 Squier's coherent extension for $\mathcal{K}$-string rewriting systems

Given a convergent string rewriting system X, Squier's theorem [10] asserts that the confluence diagrams of X can be interpreted in terms of a *homotopy basis*, which is a set Ω consisting of confluence diagrams of critical pairs. Note that one may choose different confluence diagrams for Ω. The work of Guiraud and Malbos in [3] gives a deterministic procedure of constructing these base confluence diagrams via normalization strategies, in the case when X is reduced.

In the context of $\mathcal{K}$-string rewriting systems, we have the following interpretation of Squier's coherent completion theorem.

Theorem 3.6 (Squier's theorem for $\mathcal{K}$-string rewriting systems)**.** *Let (Γ^*, R) be a convergent $\mathcal{K}$-string rewriting system. Then one can choose a coherent completion (Γ^*, R, Ω) such that Ω admits a $\mathcal{K}$-graph structure. Moreover if Γ is a proper $\mathcal{K}$-graph, then this Ω is entirely determined by the confluence diagrams of $((\Gamma^*)^0, R^0)$.*

This result, along with Theorems 3.3 and 3.4 reduce the study of confluence of a proper $\mathcal{K}$-string rewriting system (Γ^*, R) to the study of confluence of the abstract rewriting system $((\Gamma^*)^0, R^0)$. In practice, the combinatorics of $\mathcal{K}$-graphs is simplified at words of highest weight, hence the task of studying confluence is easier for $((\Gamma^*)^0, R^0)$.

4 Conclusions

The study of monoids via rewriting theory hinges on two parameters: The first consists of identifying a well-behaved string rewriting system that presents the given monoid; and the second consists of using the combinatorics of the rewriting system and the corresponding monoid to obtain computational results, as for instance expliciting Squier's coherent completion.

The notion of $\mathcal{K}$-string rewriting systems provides a framework for studying $\mathcal{K}$-congruences via adapted string rewriting systems. Firstly, if the $\mathcal{K}$-graph is proper, we obtain versions of Newman's lemma and critical pair lemma in this context, which reduce the verification of properties of termination and confluence of the given $\mathcal{K}$-SRS. Secondly, given a $\mathcal{K}$-convergent string rewriting system, the expliciting of Squier's coherent extension is reduced to computations with words of highest weight.

In [3], Guiraud and Malbos construct a cofibrant replacement for a monoid presented by a convergent presentation. The fact that a $\mathcal{K}$-graph structure and $\mathcal{K}$-string rewriting systems interact well on free monoids, especially manifested in Squier's coherence theorem, suggests that this behaviour extends to higher dimensions in the context of [3].

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A Example: Plactic monoid of type A

We give here a concrete example of a $\mathcal{K}$-string rewriting system.

Consider the $\mathcal{K}$-graph

$$A_n \ : \ 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n, \tag{3}$$

and set

$$\mathsf{Col}(A_n)_1 := \{w = x_1x_2\cdots x_k \mid x_1 < x_2 < \cdots < x_k,\ x_i \in A_n,\ k \leq n\}.$$

Remark A.1. *The $\mathcal{K}$-graph in (3) is called the crystal base of type A_n.*

Such words whose letters are increasing, are called *column words* in A_n^*. The set $\mathsf{Col}(A_n)_1$ satisfies the conditions **(P1)** and **(P2)** hence is a $\mathcal{K}$-graph itself. Define an order $\preceq$ on $\mathsf{Col}(A_n)_1$ by setting $w \preceq w'$ for two columns $w = x_1 \cdots x_k$ and $w' = y_1 \cdots y_l$ if

i) $k \geq l$,

ii) $x_i \leq y_i$ for $i = 1, 2, \ldots, l$.

Schensted's insertion algorithm, as first introduced in [9], and later adapted to a column approach, see [2], describes a procedure of inserting a letter $x \in A_n$ into a column $c_1 \in \mathsf{Col}(A_n)_1$ as follows.

Schensted's algorithm of inserting a letter into a column (SA):

Input: a column $c = x_1 \cdots x_k$; a letter $x \in A_n$;

if $x > x_k$:

set $c' = x_1 x_2 \cdots x_k x$

return: c'

if $x_l \geq x > x_{l-1}$ for some $l \leq k$:

set $c' = x_1 \cdots x_{l-1} x x_{l+1} \cdots x_k$, and $x' = x_l$

return: $x'c'$

The insertion of a letter x into a column c, denoted $(c \leftarrow x)$, outputs either one column, or two columns, with $c = x'$ being the other column. This notion can be extended to an insertion of a letter into a product of two columns as follows

$$(c_1 c_2 \leftarrow x) = \begin{cases} (c_1(c_2 x)) & \text{if } (c_2 \leftarrow x) \text{ is first case in SA,} \\ (c_1 \leftarrow x_l)c_2' & \text{if } (c_2 \leftarrow x) \text{ is second case in SA.} \end{cases}$$

In [1] Cain, Gray, and Malheiro show that the map $[\ ,\ ] : \mathsf{Col}(A_n)_1^{\times 2} \longrightarrow \mathsf{Col}(A_n)_1^{\times 2}$ defined for $c_1, c_2 \in \mathsf{Col}(A_n)_1$ with $c_2 = x_1 x_2 \cdots x_k$ by setting

$$[c_1, c_2] = (((c_1 \leftarrow x_1) \leftarrow x_2) \leftarrow \cdots) \leftarrow x_k,$$

induces a string rewriting system $\mathsf{Col}(A_n) := (\mathsf{Col}(A_n)_1^*, \mathsf{Col}(A_n)_2)$ where $\mathsf{Col}(A_n)_2$ consists of rewriting rules of the form

$$\mathsf{Col}(A_n)_2 := \{c_1 c_2 \Longrightarrow [c_1, c_2] \mid c_1, c_2 \in \mathsf{Col}(A_n)_1,\ c_1 \not\preceq c_2\}.$$

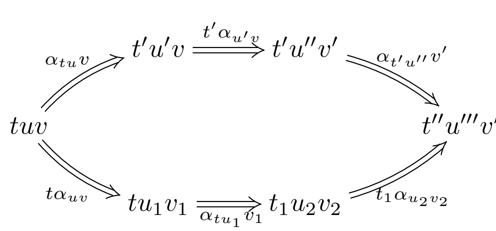
Moreover they prove that this rewriting system is convergent. The monoid presented by $\mathsf{Col}(A_n)$ is called the *plactic monoid of type A*.

We then have the following result.

Theorem A.2 ([1]). *The string rewriting system $\mathsf{Col}(A_n)$ is a finite reduced convergent $\mathcal{K}$-string rewriting system.*

We can then apply Theorem 3.6 to $\mathsf{Col}(A_n)$, and use the combinatorics of $\mathsf{Col}(A_n)$ at highest weight to prove the following.

Theorem A.3. *Squier's homotopy bases for the $\mathcal{K}$-string rewriting system $\mathsf{Col}(A_n)$ consists of confluence diagrams of the form*



We remark that this result has also been proven by Hage and Malbos in [4] using different techniques.