Box representations of embedded graphs

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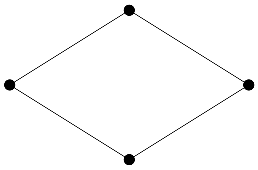

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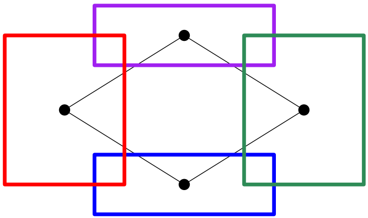

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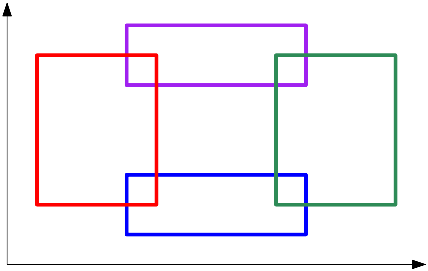

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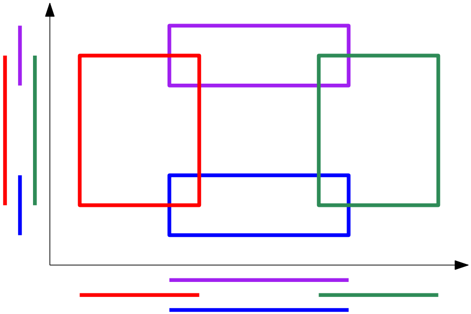

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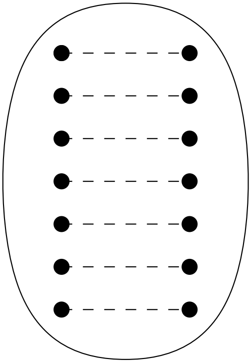

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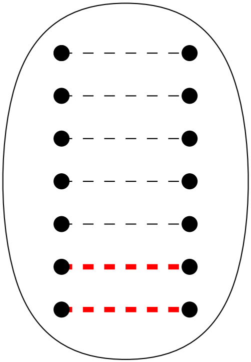

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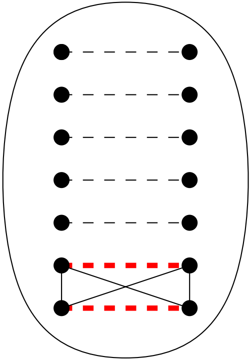

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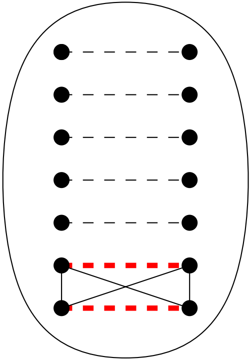

K_n minus a perfect matching

boxicity $n/2$

Boxicity and poset dimension

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If $\mathcal{P}$ is a poset of height 2 and G is its comparability graph, then $\text{box}(G) \leq \dim(\mathcal{P}) \leq 2\,\text{box}(G)$.

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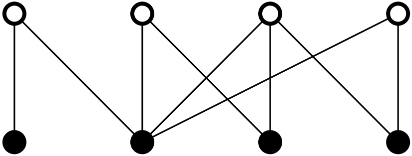
In particular if G is bipartite, it can be viewed as a poset $\mathcal{P}_G$ and we have $\text{box}(G) \leq \dim(\mathcal{P}_G) \leq 2\,\text{box}(G)$:



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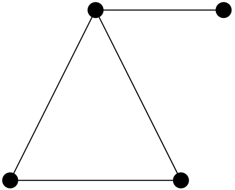

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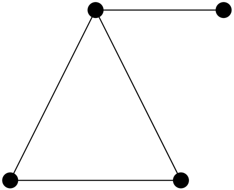

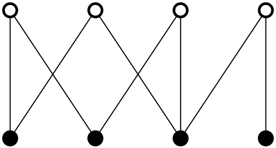

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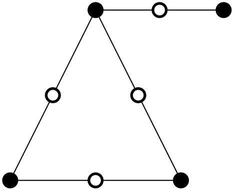

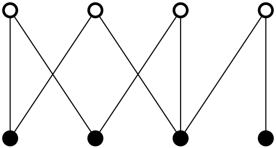

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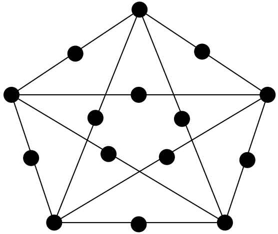

Subdivided K_n

boxicity $\Theta(\log \log n)$

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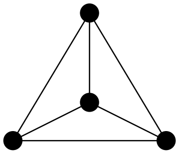

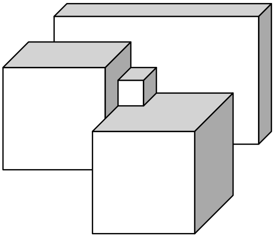

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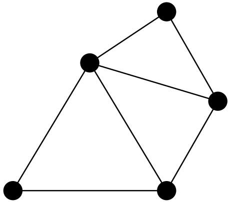

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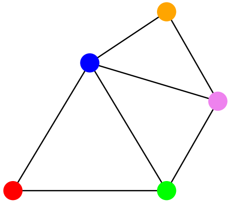

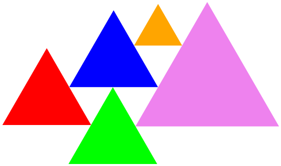

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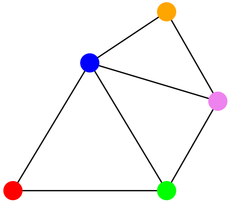

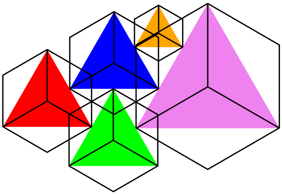

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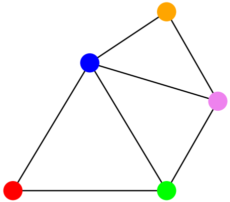

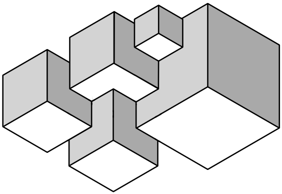

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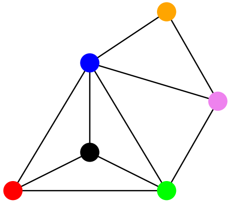

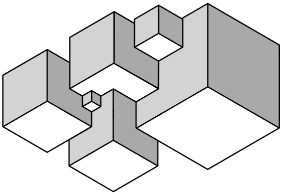

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Theorem (E. 2015)

Graphs with Euler genus g without non-contractible cycles of length at most $40 \cdot 2^g$ have boxicity at most 5.

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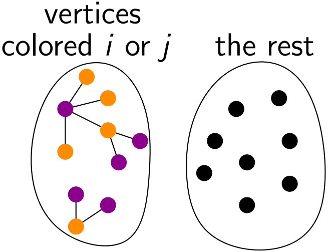

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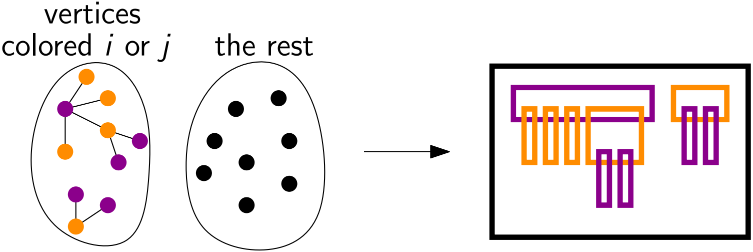

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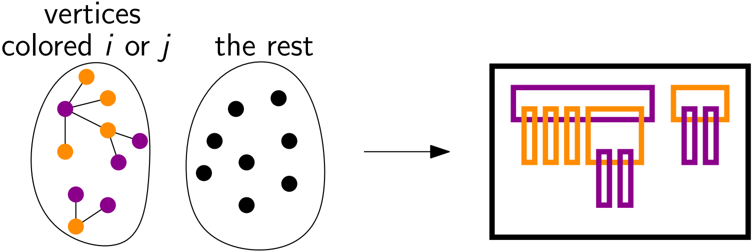

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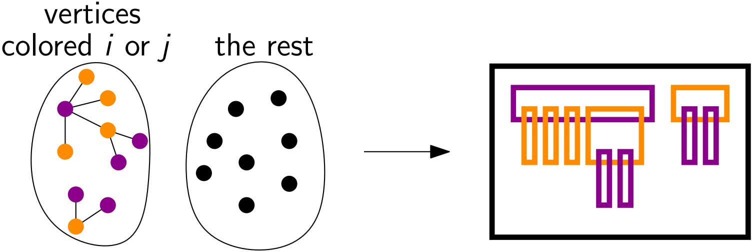

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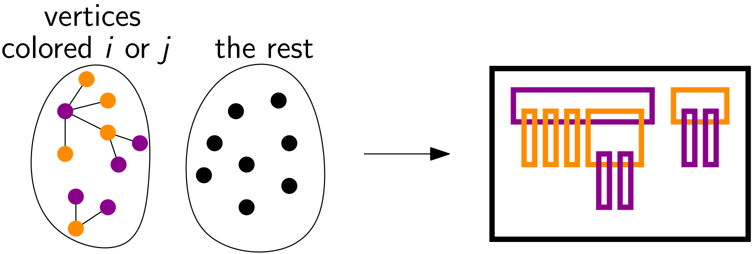

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$\Rightarrow \text{box}(G) \leq k(k-1)$

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Theorem (Kawarabayashi, Thomassen 2012)

If a graph G has Euler genus g, then there is a set A of $O(g)$ vertices such that $G - A$ has an acyclic coloring with 7 colors.

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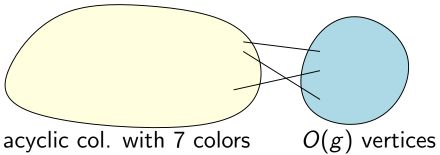

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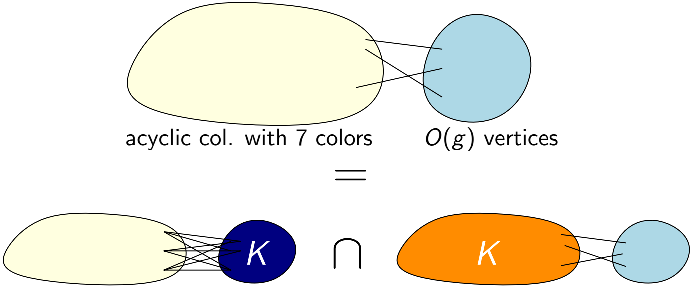

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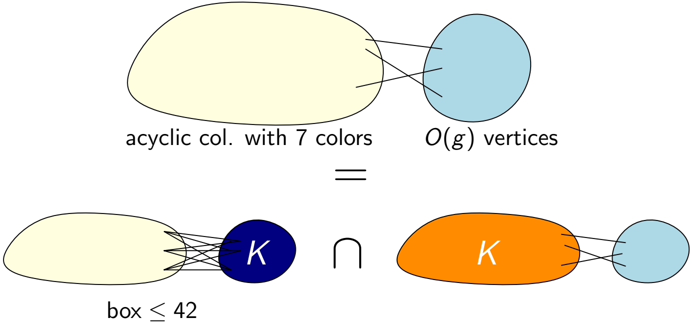

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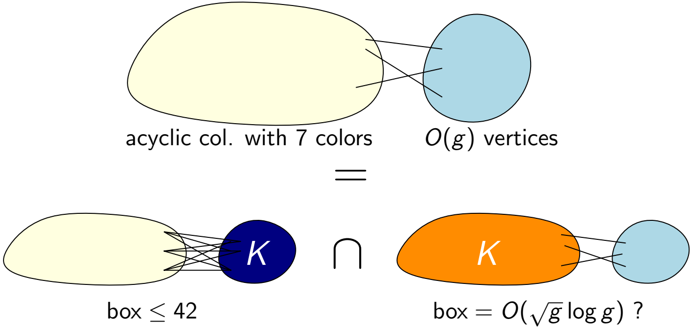

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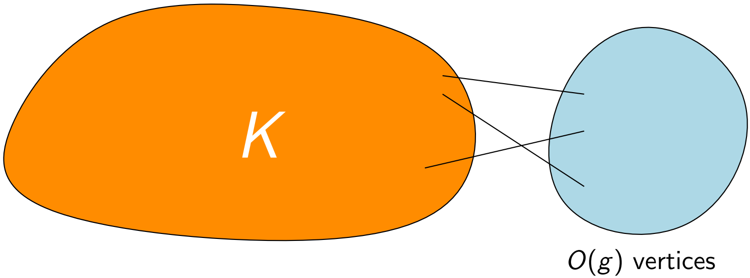

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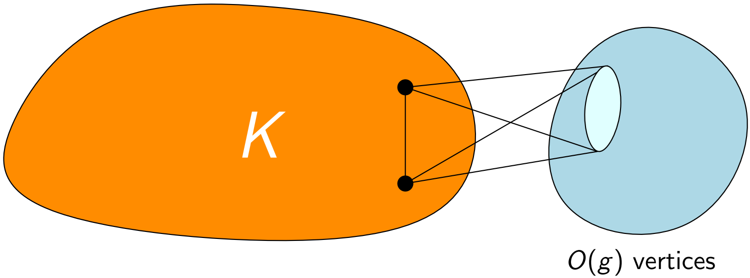

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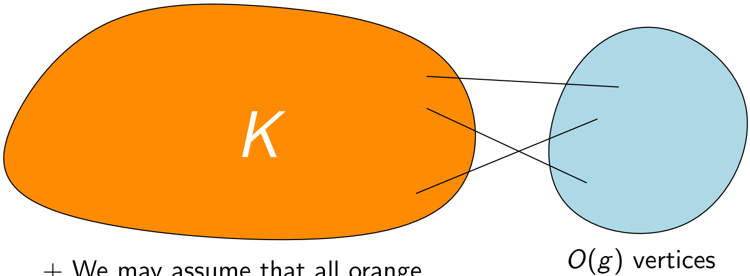

+ We may assume that all orange vertices have distinct blue neighborhoods

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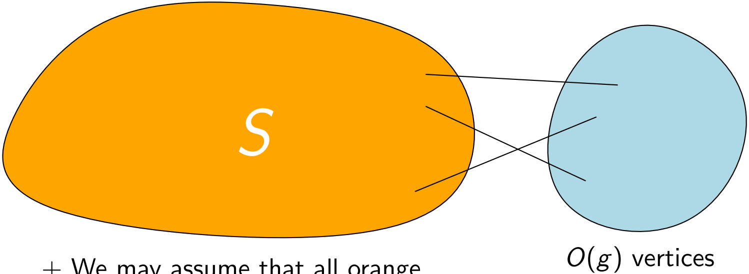

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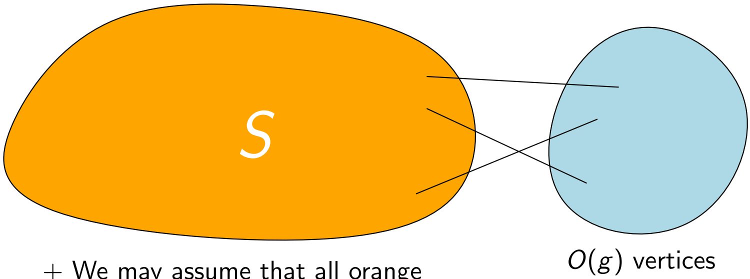

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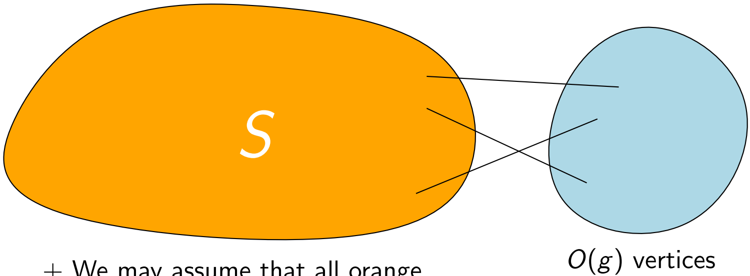

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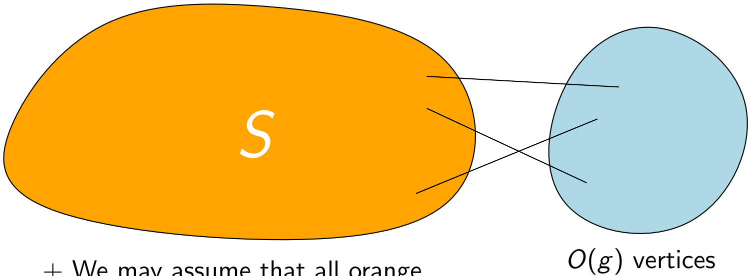

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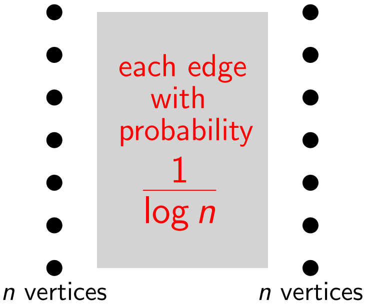
Theorem (Adiga, Chandran, Mathew 2014)

If a graph G with n vertices is k-degenerate, then $\text{box}(G) = O(k \log n)$.

Lower bound

Consider the following random bipartite graph G_n:



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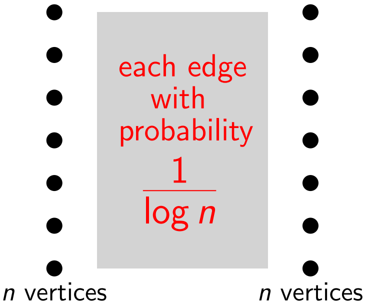
with high probability,

G_n has at most $\frac{2n^2}{\log n}$ edges



Lower bound

Consider the following random bipartite graph G_n:

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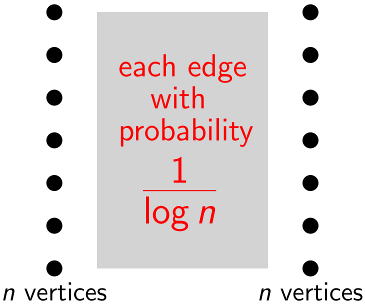

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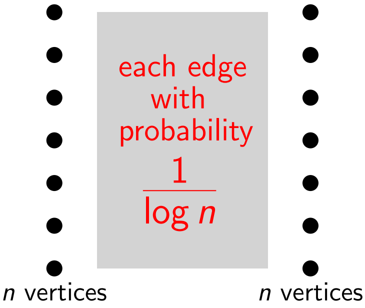

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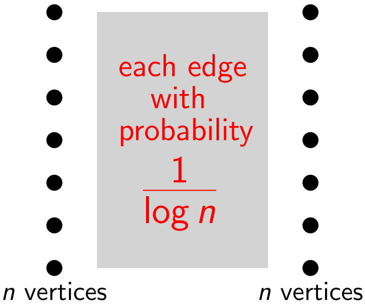

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It follows that $\text{box}(G_n) = \Omega(\sqrt{g \log g})$.

Locally planar graphs

Theorem (E. 2015)

Graphs with genus g, without non-contractible cycles of length at most $40 \cdot 2^g$, have boxicity at most 5.

Locally planar graphs

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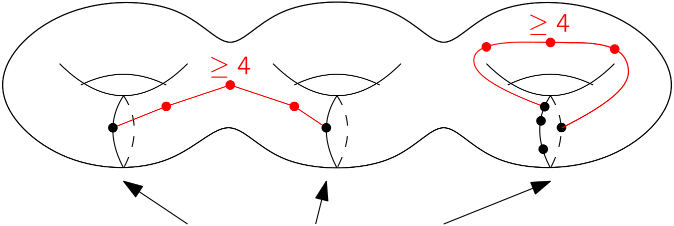
Graphs with genus g, without non-contractible cycles of length at most $40 \cdot 2^g$, have boxicity at most 7.

G triangulation with edge-width at least $40 \cdot 2^g$.



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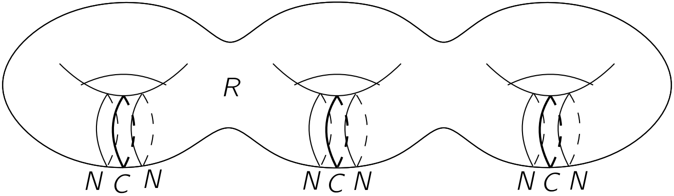

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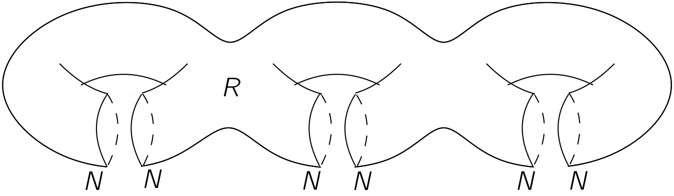

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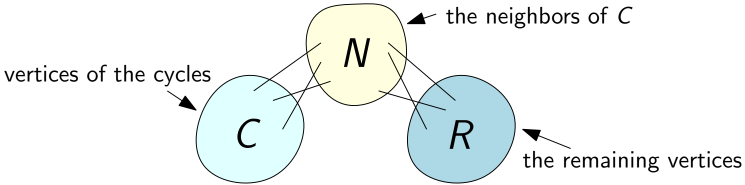

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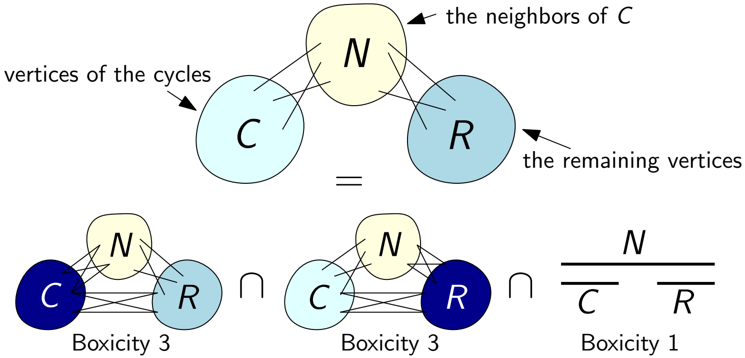

Graphs with large girth

Theorem (E. 2015)

For any proper minor-closed class $\mathcal{F}$, there is an integer $g = g(\mathcal{F})$ such that any graph of $\mathcal{F}$ of girth at least g has boxicity at most 3.

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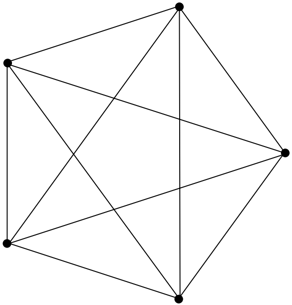

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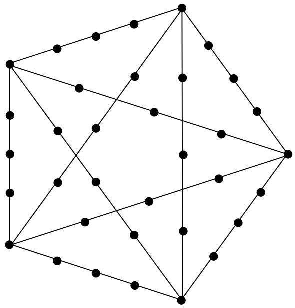

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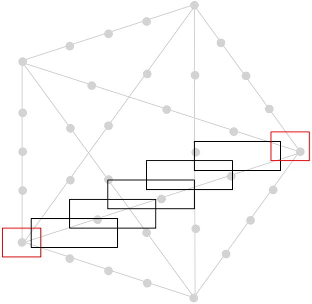

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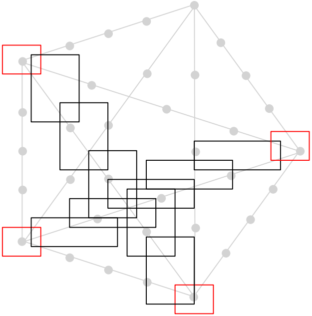

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Theorem (E. 2015)

There is a constant c such that any graph of Euler genus g and girth at least $c \log g$ has boxicity at most 3.

Open problems

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- What is the boxicity of toroidal graphs? (somewhere between 4 and 6)
- Is it true that locally planar graphs have boxicity at most 3?
- Is it true that if G has Euler genus g, then few vertices can be removed from G so that the resulting graph has boxicity at most 3? (it is true with 5 instead of 3)