Quantum invariants of knots and 3-manifolds

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June 2015
I. Topology of knots and manifolds
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Knots, links and ribbons

- **Knot**: embedding of $S^1 \rightarrow \mathbb{R}^3$.
- **Link**: embedding of $S^1 \times \ldots \times S^1 \rightarrow \mathbb{R}^3$.
- **Ribbon**: knot/link with orientation and framing.
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- Link: embedding of $S^1 \times \ldots \times S^1 \to \mathbb{R}^3$.
- Ribbon: knot/link with orientation and framing.
- *d-manifold*: every point is locally homeomorphic to $\mathbb{B}^d$.
- Generalized 3-triangulation: set of tetrahedra with triangle gluings.
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Quantum invariants of knots
Construction of the invariant
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\[ f: V_1 \otimes \ldots \otimes V_n \rightarrow W_1 \otimes \ldots \otimes W_m \]
Construction of the invariant

\[ f \otimes g : V_1 \otimes \ldots \otimes V_p \to W_1 \otimes \ldots W_q \]
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\[ V \downarrow \triangleq \text{id}_V : V \to V \quad V \uparrow \triangleq \text{id}_{V^*} \]

\[ V \xrightarrow{\theta_V} : V \to V \]

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Construction of the invariant

\[ V \downarrow \equiv \text{id}_V : V \to V \quad V \uparrow \equiv \text{id}_{V^*} \]

\[ V \quad \theta_V : V \to V \]

\[ c_{V,W} : V \otimes W \to W \otimes V \]

\[ f \otimes g : V_1 \otimes \ldots \otimes V_p \to W_1 \otimes \ldots \otimes W_q \]

\[ g \circ f : U_1 \otimes \ldots \otimes U_\ell \]

\[ f : V_1 \otimes \ldots \otimes V_n \to W_1 \otimes \ldots \otimes W_{m+1} \]

\[ g : V_{n+1} \otimes \ldots \otimes V_p \to W_{m+1} \otimes \ldots \otimes W_q \]
Construction of the invariant

\[ f \otimes g : V_1 \otimes \ldots \otimes V_p \rightarrow W_1 \otimes \ldots \otimes W_q \]

\[ (f \otimes g)(v_1, \ldots, v_p) = (f(v_1, \ldots, v_p), g(v_1, \ldots, v_p)) \]

\[ g \circ f : U_1 \otimes \ldots \otimes U_\ell \rightarrow W_1 \otimes \ldots \otimes W_m \]

\[ (g \circ f)(u_1, \ldots, u_\ell) = g(f(u_1, \ldots, u_p), u_{p+1}, \ldots, u_\ell) \]
Construction of the invariant

\[ \begin{align*}
V & \xrightarrow{\text{id}_V} V \\
V^* & \xrightarrow{\text{id}_{V^*}} V \\
c_{V,W} & : V \otimes W \rightarrow W \otimes V \\
\theta_V & : V \rightarrow V \\
d_v & : V^* \otimes V \rightarrow \mathbb{1} \\
b_v & : \mathbb{1} \rightarrow V \otimes V^* \\
c_{V,W} & : V \otimes W \rightarrow W \otimes V
\end{align*} \]
Construction of the invariant

\[ V \xmapsto{\id_V} V \xmapsto{\id_{V^*}} V \]

\[ V \xmapsto{\theta_V} V \xmapsto{d_v} V^* \otimes V \rightarrow 1 \]

\[ V \xmapsto{b_v} 1 \rightarrow V \otimes V^* \]

\[ c_{V,W} : V \otimes W \rightarrow W \otimes V \]

\[ c_{W,V} \]

\[ f \]

\[ \id_{V^*} \otimes W \]
Construction of the invariant

\[ V \xrightarrow{id} V \to V \quad V \xrightarrow{id} V^* \]

\[ V \xrightarrow{\theta} V \to V \quad V \xleftarrow{d} V^* \otimes V \to 1 \]

\[ V \xleftarrow{b} 1 \to V \otimes V^* \]

\[ c_{V,W} : V \otimes W \to W \otimes V \]

\[ g \xrightarrow{\cdot} V \]

\[ g \xrightarrow{\cdot} V \]

\[ g \xrightarrow{\cdot} V \]
Construction of the invariant

\[ V \xRightarrow{\theta_V} V \]
\[ V \xRightarrow{g} V \]
\[ V \xRightarrow{id_V} V \]

\[ V \xRightarrow{c_{V,W}} V \otimes W \to W \otimes V \]

\[ V \xRightarrow{d_v} V^* \otimes V \to 1 \]
\[ V \xRightarrow{b_v} 1 \to V \otimes V^* \]
Ribbon category and ribbon diagrams

A ribbon category $\mathcal{V}$ is a category with:

- tensor product $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$,
- braiding $\{c_{V,W} : V \otimes W \to W \otimes V\}$,
- twist $\{\theta_V : V \to V\}$,
- duality $\{V^*, b_V : \mathbb{1} \to V \otimes V^*, d_V : V^* \otimes V \to \mathbb{1}\}$,

satisfying a set of natural axioms.

Theorem (Reshetikhin, Turaev)

A ribbon category associates to every $\mathcal{V}$-coloured ribbon diagram a morphism $\mathbb{1} \to \mathbb{1}$. It is an isotopy invariant.
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satisfying a set of natural axioms.

Theorem (Reshetikhin, Turaev)

A ribbon category associates to every $\mathcal{V}$-coloured ribbon diagram a morphism $\mathbb{1} \to \mathbb{1}$. It is an isotopy invariant.

Proof: any isotopy of ribbon diagrams may be described by a sequence of Reidemeister moves.
Quantum invariants of 3-manifolds
Surgery presentation

Let $k \subseteq S^3$. A surgery on the 3-sphere along $k$ consists in "drilling" $k$ out of $S^3$ and glue back a solid torus along the toric boundary.

Theorem (Lickorish-Wallace)

Every 3-manifold may be obtained by surgery on $S^3$ along a link.
Invariant of 3-manifold

Let $M$ be a 3-manifold, obtained by surgery on $S^3$ along a link $k$ with $m$ components $\{L_1, \ldots, L_m\}$.

Let $\mathcal{V}$ be a ribbon category \(^1\). For a colouring $\lambda: \{L_1, \ldots, L_m\} \rightarrow \mathcal{V}$, denote by $F(k, \lambda)$ the associated ribbon invariant. Finally, sum over all colourings:

$$\tau(M, \mathcal{V}) = A_\mathcal{V} \sum_{\lambda: \{L_1, \ldots, L_m\} \rightarrow \mathcal{V}} D_\lambda \times F(k, \lambda)$$

\(^1\)with an extra notion of "decomposability" of objects.
Invariant of 3-manifold

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### Invariant of 3-manifold

**Theorem (Reshetikhin, Turaev)**

*For a manifold $M$ obtained by surgery on $S^3$ along $k$, and a ribbon category $\mathcal{V}$,*

$$\tau(M, \mathcal{V}) = A_\mathcal{V} \sum_{\lambda: \{L_1, \ldots, L_m\} \to \mathcal{V}} D_\lambda \times F(k, \lambda)$$

*is a 3-manifold invariant.*
Invariant of 3-manifold

Theorem (Reshetikhin, Turaev)

For a manifold \( M \) obtained by surgery on \( S^3 \) along \( k \), and a ribbon category \( \mathcal{V} \),

\[
\tau(M, \mathcal{V}) = A_{\mathcal{V}} \sum_{\lambda: \{l_1, \ldots, l_m\} \rightarrow \mathcal{V}} D_\lambda \times F(k, \lambda)
\]

is a 3-manifold invariant.

Proof: Two ribbons leading to the same manifold via surgery on \( S^3 \) are related by a sequence of Reidemeister moves and Kirby moves.

\[
\begin{align*}
\begin{array}{c}
\vdash \\
\end{array} & \begin{array}{c}
\end{array} \\
\end{align*}
\]
Why "quantum"?

It is easy to find algebraic objects (vector spaces, modules) with the structure of a ribbon category (usual tensor product, duality).

These simple examples however lead to trivial knots invariants.

Ex: vector spaces $c_{V,W}(v \otimes w) = w \otimes v$

Quantum groups (in the representation theory of Lie algebras) lead to non-trivial ribbon categories. And powerful invariants in $\mathbb{C}$.
Algorithmic aspects of quantum invariants
Computation of the invariants

Pushing a bit more the construction, we get the \textit{Turaev-Viro invariant} \((\equiv |\tau|^2)\) defined directly on the triangulation:

Quantum groups lead to invariants parameterised by an integer \(r \geq 3\).

- \(r = 3\), polynomial time algorithm (reduced to homology),
- \(r = 4\), \# P hard,
- fully parameterised algorithm in treewidth: \(O((r + 1)^{6k} \times \text{poly}(n))\)

[Burton, M., Spreer '15]
Conclusion
Take away

Turn a qualitative theory into a quantitative computation via Reidemeister moves, surgery, Kirby moves, Pachner moves, etc.

Interesting complexity theory for the computation of quantum invariants.
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Interesting complexity theory for the computation of quantum invariants.

Thank you!