Discrete planes: an arithmetic and dynamical approach

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Séminaire de géométrie algorithmique et combinatoire
From discrete geometry to word combinatorics...

...via tilings and quasicrystals
Discrete geometry Digital geometry

Analysis of geometric problems on objects defined on regular lattices

It requires
- a choice of a grid/lattice
- a topology
- basic primitives (lines, circles etc.)
- a dedicated algorithmics
Discrete planes

How to discretize a line in the space?

- There are the usual difficulties related to discrete geometry
- There are further difficulties due to the codimension > 1 for discrete lines

[D. Coeurjoly, Digital geometry in a Nutshell
http://liris.cnrs.fr/david.coeurjolly/doku/doku.php]
Euclid first axiom

Given two points $A$ and $B$, there exists a unique line that contains them.

This is no more true in the discrete case.

[D. Coeurjoly, Digital geometry in a Nutshell]
Intersections

[D. Coeurjoly, Digital geometry in a Nutshell]
[I. Sivignon, D. Coeurjoly, Introduction à la géométrie discrète]
Let $\vec{v} \in \mathbb{R}^d$, $\mu, \omega \in \mathbb{R}$.

The arithmetic discrete plane $\mathcal{P}(\vec{v}, \mu, \omega)$ is defined as

$$\mathcal{P}(\vec{v}, \mu, \omega) = \{ \vec{x} \in \mathbb{Z}^d | 0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega \}.$$

- $\mu$ is the translation parameter.
- $\omega$ is the width.
- If $\omega = \max_i \{ |v_i| \} = \| \vec{v} \|_\infty$, then $\mathcal{P}(\vec{v}, \mu, \omega)$ is said naive.
- If $\omega = \sum_i |v_i| = \| \vec{v} \|_1$, then $\mathcal{P}(\vec{v}, \mu, \omega)$ is said standard.
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- If $\omega = \sum_i |v_i| = \|\vec{v}\|_1$, then $\mathcal{P}(\vec{v}, \mu, \omega)$ is said standard.
Discrete lines and Sturmian words
Discrete lines

A discrete segment

and pixels that do not belong to a same discrete segment

[D. Coeurjoly, Digital geometry in a Nutshell]
[I. Sivignon, D. Coeurjoly, Introduction à la géométrie discrète]
One can code a discrete line (Freeman code) over the two-letter alphabet \( \{0, 1\} \). One gets a Sturmian word \((u_n)_{n \in \mathbb{N}} \in \{0, 1\}^\mathbb{N}\)

\[
0100101001001010010100100101
\]
Discrete lines and Sturmian words

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0100101001001010010100100101

[Lothaire, Algebraic combinatorics on words, N. Pytheas Fogg, Substitutions in dynamics, arithmetics and combinatorics
CANT Combinatorics, Automata and Number theory]
Discrete lines and Sturmian words

Let \( R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} , \ x \mapsto x + \alpha \mod 1 \).

Sturmian words [Morse-Hedlund]

Let \((u_n)_{n \in \mathbb{N}} \in \{0, 1\}^\mathbb{N}\) be a Sturmian word. There exist \( \alpha \in (0, 1) \), \( \alpha \notin \mathbb{Q} \), \( x \in \mathbb{R} \) such that

\[
\forall n \in \mathbb{N}, \ u_n = i \iff R^*_\alpha(x) = n\alpha + x \in I_i \mod 1,
\]

with

\[
l_0 = [0,1-\alpha[, \ l_1 = [1-\alpha,1[\]

or

\[
l_0 = ]0,1-\alpha] , \ l_1 = ]1-\alpha,1].\]
Factors

Theorem
The words 00 et 11 cannot be factors simultaneously of a Sturmian word
Factors

Theorem
The words 00 et 11 cannot be factors simultaneously of a Sturmian word

Preuve : One has

\[ \forall i \in \mathbb{N}, \; u_n = i \iff n\alpha + x \in l_i \pmod{1} \]

Hence

\[ u_n u_{n+1} = 00 \]

iff

\[ \begin{cases} \; n\alpha + x \in [0, 1 - \alpha] \\ (n + 1)\alpha + x \in [0, 1 - \alpha] \end{cases} \]

which requires \( \alpha < \frac{1}{2} \). One thus gets

\[ u_n u_{n+1} = 00 \iff n\alpha + x \in [0, 1 - 2\alpha] \]
From factors to intervals

\[ R_\alpha : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}, \ x \mapsto x + \alpha \mod 1 \]
From factors to intervals

Property  A Sturmian word has 3 factors of length 2
Let $l_0 = [0, 1 - \alpha]$, $l_1 = [1 - \alpha, 1]$. Let $R_\alpha : x \mapsto x + \alpha \mod 1$.

**Lemma** The word $w = w_1 \cdots w_n$ over the alphabet $\{0, 1\}$ is a factor the Sturmian word $u$ iff

$$l_{w_1} \cap R^{-1}_\alpha l_{w_2} \cap \cdots R^{-n+1}_\alpha l_{w_n} \neq \emptyset.$$
A key lemma

Let \( l_0 = [0, 1 - \alpha] \), \( l_1 = [1 - \alpha, 1] \). Let \( R_\alpha : x \mapsto x + \alpha \mod 1 \).

Lemma The word \( w = w_1 \cdots w_n \) over the alphabet \( \{0, 1\} \) is a factor the Sturmian word \( u \) iff

\[
\begin{align*}
I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n} &\neq \emptyset.
\end{align*}
\]

Proof

\( \forall i \in \mathbb{N}, \ u_n = i \iff n\alpha + x \in I_i \ (\mod 1) \).

- One first notes that \( u_k u_{k+1} \cdots u_{n+k-1} = w_1 \cdots w_n \) iff

\[
\begin{cases}
  k\alpha + x \in I_{w_1} \\
  (k + 1)\alpha + x \in I_{w_2} \\
  \cdots \\
  (k + n - 1)\alpha + x \in I_{w_n}
\end{cases}
\]

- One then applies the density of \( (k\alpha)_{\in \mathbb{N}} \) in \( \mathbb{R}/\mathbb{Z} \).
A key lemma

Let $I_0 = [0, 1 - \alpha[$, $I_1 = [1 - \alpha, 1[$. Let $R_\alpha : x \mapsto x + \alpha \mod 1$.

Lemma The word $w = w_1 \cdots w_n$ over the alphabet $\{0, 1\}$ is a factor the Sturmian word $u$ iff

$$I_{w_1} \cap R_{\alpha}^{-1} I_{w_2} \cap \cdots R_{\alpha}^{-n+1} I_{w_n} \neq \emptyset.$$

Application One deduces combinatorial properties on the
- number of factors of given length/enumeration of local configurations
- densities of factors/statistical properties of local configurations
- powers of factors, repetitions, palindromes/symmetries
A key lemma

Let \( I_0 = [0, 1 - \alpha], \ I_1 = [1 - \alpha, 1]. \) Let \( R_\alpha : x \mapsto x + \alpha \mod 1. \)

**Lemma** The word \( w = w_1 \cdots w_n \) over the alphabet \( \{0, 1\} \) is a factor the Sturmian word \( u \) iff

\[
I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n} \neq \emptyset.
\]

**Fact** The sets \( I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n} \) are intervals of \( \mathbb{R}/\mathbb{Z}. \)

The factors of \( u \) are in one-to-one correspondence with the \( n + 1 \) intervals of \( \mathbb{T} \) whose end-points are given by

\[
-k\alpha \mod 1, \text{ for } 0 \leq k \leq n
\]

**Theorem** [Coven-Hedlund]

A word \( u \in \{0, 1\}^\mathbb{N} \) is Sturmian iff it admits exactly \( n + 1 \) factors of length \( n. \)
A key lemma

Let $l_0 = [0, 1 - \alpha]$, $l_1 = [1 - \alpha, 1]$. Let $R_\alpha : x \mapsto x + \alpha \mod 1$.

**Lemma** The word $w = w_1 \cdots w_n$ over the alphabet $\{0, 1\}$ is a factor the Sturmian word $u$ iff

$$l_{w_1} \cap R_\alpha^{-1} l_{w_2} \cap \cdots R_\alpha^{-n+1} l_{w_n} \neq \emptyset.$$
To summarize...

We have used

- A **coding** as an infinite binary word
- A **dynamical system**: the rotation of $\mathbb{R}/\mathbb{Z}$, $R_\alpha : x \mapsto x + \alpha$
- The **key lemma**: bijection between intervals and factors
To summarize...

We have used

- A **coding** as an infinite binary word
- A **dynamical system**: the rotation of $\mathbb{R}/\mathbb{Z}$, $R_\alpha : x \mapsto x + \alpha$
- The **key lemma**: bijection between intervals and factors

**Discrete dynamical system**

A **dynamical system** $(X, T)$ is defined as the action of a continuous and onto map $T$ on a compact space $X$. 
From a discrete plane to a tiling by projection....

....and from a tiling by lozenges to a ternary coding
Two-dimensional word combinatorics

An arithmetic discrete plane can be coded as

\[
\begin{array}{cccccccccccc}
1 & 2 & 1 & 2 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 3 & 1 & 3 \\
3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 1 & 2 \\
2 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 & 1 & 2 \\
3 & 1 & 2 & 1 & 2 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 3 & 1 \\
2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 3 & 1 & 2 & 1 & 2 & 3 & 1 & 3 & 1 & 2 \\
3 & 1 & 2 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 & 1 \\
\end{array}
\]
Discrete planes and two-dimensional Sturmian words

Theorem [B.-Vuillon]

Let \((U_{m,n})_{(m,n)\in \mathbb{Z}^2} \in \{1, 2, 3\}^{\mathbb{Z}^2}\) be a 2d Sturmian word, that is, a coding of an arithmetic discrete plane. Then there exist \(x \in \mathbb{R}\), and \(\alpha, \beta \in \mathbb{R}\) such that \(1, \alpha, \beta\) are \(\mathbb{Q}\)-linearly independent and \(\alpha + \beta < 1\) such that

\[\forall (m, n) \in \mathbb{Z}^2, \ U_{m, n} = i \iff R^m_{\alpha} R^n_{\beta}(x) = x + n\alpha + m\beta \in I_i \pmod{1},\]

with

\[I_1 = [0, \alpha], \ I_2 = [\alpha, \alpha + \beta[, \ I_3 = [\alpha + \beta, 1[\]

or

\[I_1 = ]0, \alpha], \ I_2 = ]\alpha, \alpha + \beta], \ I_3 = ]\alpha + \beta, 1].\]
Combinatorial properties of 2d Sturmian words

- They key lemma still holds: rectangular factors are in one-to-one correspondence with intervals of $\mathbb{R}/\mathbb{Z}$.

Theorem [B.-Vuillon]

There exist exactly $mn + m + n$ rectangular factors of size $m \times n$ in a 2d Sturmian word.

- Two discrete planes with the same normal vector have the same configurations.
- We also deduce information on the frequencies of configurations

[B.-Vuillon, Daurat-Tajine-Zouaoui]
Tilings of the line

- By projecting the vertices of the discrete line, one gets a tiling of the line.
- This corresponds to a cut-and-project scheme in quasicrystallography.
Quasiperiodicity and quasicrystals

Quasicrystals are solids discovered in 84 with an atomic structure that is both ordered and aperiodic [Shechtman-Blech-Gratias-Cahn]

An aperiodic system may have long-range order
(cf. Aperiodic tilings [Wang'61, Berger'66, Robinson'71,...])
Quasiperiodicity and quasicrystals

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An aperiodic system may have long-range order
(cf. Aperiodic tilings [Wang’61, Berger’66, Robinson’71,...)

- Quasicrystals produce a discrete diffraction diagram (=order)
- Diffraction comes from regular spacing and local interactions of the point set \( \Lambda \) (consider the relative positions \( \Lambda - \Lambda \))
Quasiperiodicity and quasicrystals

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- Quasicrystals produce a discrete diffraction diagram (=order)
- Diffraction comes from regular spacing and local interactions of the point set $\Lambda$ (consider the relative positions $\Lambda - \Lambda$)

There are mainly two methods for producing quasicrystals
- Substitutions
- Cut and project schemes

[What is... a Quasicrystal? M. Senechal]
Cut and project schemes

Projection of a “plane” slicing through a higher dimensional lattice

- The order comes from the lattice structure
- The nonperiodicity comes from the irrationality of the normal vector of the “plane”
Recurrence and frequencies
Frequencies

The frequency \( f_w \) of a word \( w \) in \( u = (u_n)_{n \in \mathbb{N}} \) is defined as the following limit, if it exists

\[
f_w = \lim_{n \to \infty} \frac{|u_0 \cdots u_{N-1}|_w}{N}
\]

where \( |x|_j \) stands for the number of occurrences of \( w \) in \( x \)

By uniform distribution of \( (k\alpha)_k \) modulo 1, the frequency of a factor \( w \) of a Sturmian word/discrete line is equal to the length of \( I_w \)

\[
I_w = I_{w_1} \cap R_{\alpha}^{-1} I_{w_2} \cap \cdots R_{\alpha}^{-n+1} I_{w_n}
\]
Three-length theorem

Let $0 < \alpha < 1$ be an irrational number.

**Theorem** The points $\{i\alpha\}$, for $0 \leq i \leq n$, partition the unit circle into $n + 1$ intervals, the lengths of which take at most three values, one being the sum of the other two.

[Steinhaus, Sós, Świerczkowski, Surányi]
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Theorem The frequencies of factors of length $n$ of a Sturmian word take at most 3 values.
Three-gap theorem [Slater]

Let $\alpha$ be an irrational number in $]0,1[$ and let $I$ be an interval of $\mathbb{R}/\mathbb{Z}$.

The sequence $(n\alpha)_{n \in \mathbb{N}} \mod 1$ enters the interval $I$ with bounded gaps, that is, there exists $N \in \mathbb{N}$ such that any sequence of $N$ successive values of the sequence contains a value in $I$. 
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**Theorem** The gaps between the successive integers $j$ such that $\{\alpha j\} \in I$ take at most three values, one being the sum of the other two.
Recurrence function

The recurrence function $R$ of a uniformly recurrent word $u$ is defined by

$$R_u(n) = \min\{m \in \mathbb{N} \text{ such that } \forall B \in L_m, \forall A \in L_n, A \text{ is a factor of } B\}$$

where $L_n$ denotes the set of factors of $u$ of length $n$.

$R(n)$ is the size of the smallest window that contains all factors of length $n$ whatever its position on the word.
Recurrence function

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$R(n)$ is the size of the smallest window that contains all factors of length $n$ whatever its position on the word.

**Theorem** Let $u$ be a Sturmian word with angle $\alpha$. Let $(q_k)_{k \in \mathbb{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of $\alpha$.

$$R(n) = n - 1 + q_k + q_{k-1}, \text{ where } q_{k-1} \leq n < q_k.$$

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$
Recurrence function

- Let $\delta_n$ be the **smallest length** of the nonempty intervals $I_{w_1 \ldots w_n}$.
- Let $\ell_n$ be the **greatest gap** between the successive integers $k$ such that $\{k\alpha\} \in [0, \delta_n[$.
- We have
  
  $$R(n) = n - 1 + \ell_n.$$
Recurrence function

- Let $\delta_n$ be the smallest length of the nonempty intervals $I_{w_1 \ldots w_n}$.
- Let $\ell_n$ be the greatest gap between the successive integers $k$ such that $\{k\alpha\} \in [0, \delta_n[$.
- We have

$$R(n) = n - 1 + \ell_n.$$  

- Assume $q_{k-1} \leq n < q_k$. Then we have

$$\delta_n = \eta_{k-1} \text{ and } \ell_n = q_k + q_{k-1}.$$  

- A Sturmian word is linearly recurrent iff its slope has bounded partial quotients

$$R(n) \leq Cn \quad \text{for all } n.$$
For any real $\alpha$ one has
\[
\liminf_{n \to \infty} \frac{R_\alpha(n)}{n} \leq 3
\]

[**Morse-Hedlund**] For almost any real $\alpha$, one has
\[
\limsup \frac{R_\alpha(n)}{n \log n} = +\infty, \quad \text{and} \quad \limsup \frac{R_\alpha(n)}{n(\log n)^{1+\varepsilon}} = 0 \text{ for } \varepsilon > 0.
\]
Recurrence and frequencies

Consider an infinite word such that all its factors admit frequencies.

- Let $e_n$ be the smallest frequency of factors of length $n$.
- **Theorem** Linear recurrence is equivalent to
  \[ \exists C > 0, \; ne_n > C \text{ for all } n. \]
- One has for all $n$
  \[ e_n \geq \frac{1}{R(n)} \]
Back to discrete planes
Back to 2d Sturmian words

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Factors

• The block $\mathcal{W} = [w_{i,j}]$, defined on $\{1, 2, 3\}$ and of size $(m, n)$, is a factor of $U$ if and only if

$$l_{\mathcal{W}} := \bigcap_{1 \leq i \leq m, 1 \leq j \leq n} R_{\alpha}^{-i+1} R_{\beta}^{-j+1} I_{w_{i,j}} \neq \emptyset.$$  

• The sets $l_{\mathcal{W}}$ are connected.

• The frequency $f(\mathcal{W})$ of a factor $\mathcal{W}$ of the word $U$ is defined as the limit, if it exists, of the number of occurrences of this block in the “central” square factor

$$U_{-n,n} \ldots U_{n,n}$$

$$\vdots \quad \vdots$$

$$U_{-n,-n} \ldots U_{n,-n},$$

of the word divided by $(2n + 1)^2$. 
Factors

- The block $W = [w_{i,j}]$, defined on $\{1, 2, 3\}$ and of size $(m, n)$, is a factor of $U$ if and only if

$$I_W := \bigcap_{1 \leq i \leq m, 1 \leq j \leq n} R_\alpha^{-i+1} R_\beta^{-j+1} I_{w_{i,j}} \neq \emptyset.$$ 

- The sets $I_W$ are connected.

- The frequency $f(W)$ of a factor $W$ of the word $U$ is defined as the limit, if it exists, of the number of occurrences of this block in the “central” square factor

$$U_{-n,n} \ldots U_{n,n}$$
$$\vdots \quad \vdots$$
$$U_{-n,-n} \ldots U_{n,-n},$$

of the word divided by $(2n + 1)^2$.

- The frequency of every factor $W$ of $U$ exists and is equal to the length of $I(W)$. 


Number of frequencies

Theorem [Geelen and Simpson] The set of points
\[ \{i\alpha + j\beta + \rho, \ 0 \leq i \leq m - 1, \ 0 \leq j \leq n - 1\} \]
partitions the unit circle into intervals having at most \( \min\{m, n\} + 3 \) lengths.

Frequencies The frequencies of rectangular factors of size \((m, n)\) of a \(2d\) Sturmian word take at most \( \min\{m, n\} + 5 \) values.
Number of frequencies

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Frequencies The frequencies of rectangular factors of size \((m, n)\) of a \(2d\) Sturmian word take at most \( \min\{m, n\} + 5 \) values.

Theorem [Chevallier] Let \( d \geq 3 \). Let \( \alpha_1, \ldots, \alpha_d \in \mathbb{T}^1 \) and \( 2 \leq n_1 \leq \ldots \leq n_d \) integers. The set
\[ \left\{ \sum_{i=1}^{d} k_i \alpha_i, \ 0 \leq k_i < n_i, \ i = 1, \ldots, d \right\} \]
divides \( \mathbb{T}^1 \) into intervals whose lengths take at most
\[ \prod_{i=1}^{d-1} n_i + 3 \prod_{i=1}^{d-2} n_i + 1 \] values.
Are there finitely many frequencies?

Are there finitely many lengths for the intervals obtained by taking points on $\mathbb{R}/\mathbb{Z}$

$$i\alpha + j\beta \quad 0 \leq i \leq m, \ 0 \leq j \leq n$$

- Is there a finite uniform upper bound on the number of distinct frequencies for rectangular factors of size $(m, n)$ for some parameters $(\alpha, \beta)$?
- Can one characterize those parameters?
- There are finitely many lengths for badly approximable numbers [Boshernitzan]

$$|(r, s)|^2||r\alpha + s\beta|| \geq C \text{ for all } (r, s) \neq 0$$

- What is the generic behavior?
- Same questions for squares?
Repetitivity

Fact Arithmetic discrete planes are repetitive (factors occur with bounded gaps)

Recurrence function Let $N$ be the smallest integer $N$ such that every square factor of radius $N$ contains all square factors of size $n$. We set $R(n) := N$.

Linear recurrence There exists $C$ such that $R(n) \leq Cn$ for all $n$.

Discrete planes [A. Haynes, H. Koivusalo, J. Walton] Linearly recurrent discrete planes are the planes that have a badly approximable normal vector

$$|(r, s)|^2 ||r\alpha + s\beta|| \geq C \text{ for all } (r, s) \neq 0$$

Discrete lines One has linear recurrence iff and the slope of the line has bounded partial quotients in its continued fraction expansion.
Substitutions

- Substitutions on words and symbolic dynamical systems
- Substitutions on tiles: inflation/subdivision rules, tilings and point sets

Tilings Encyclopedia [E. Harriss, D. Frettlöh]

http://tilings.math.uni-bielefeld.de/
Back to tilings and long-range aperiodic order

Discrete planes with irrational normal vector are
- repetitive (uniform recurrence)
- aperiodic

The corresponding tilings are obtained by a cut and project scheme and yield quasicrystals
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Assume we have a “substitutive” arithmetic discrete plane

Multidimensional substitutive tilings $\rightsquigarrow$ Local/matching rules
[S. Mozes, C. Goodman-Strauss]

One can recognize a given “substitutive” arithmetic discrete plane by local inspection [N. Bedaride-Th. Fernique]