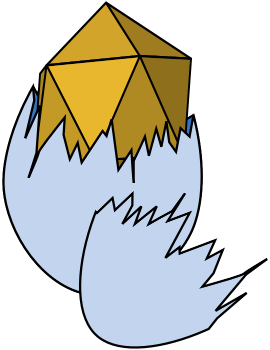


Realizations of neighborly, inscribable and egg-scribable polytopes

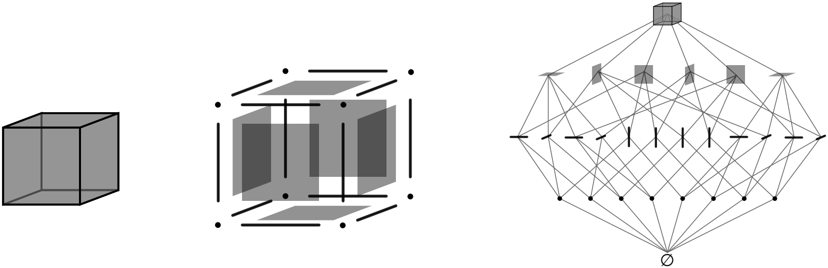
Arnau Padrol
(+ **Karim Adiprasito**, **Bernd Gonska** & **Louis Theran**)
Freie Universität Berlin

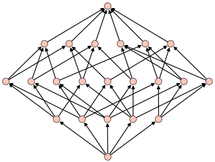
18/12/2014
Séminaire Géométrie Algorithmique et Combinatoire

Getting started

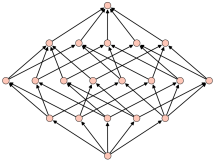
What is a polytope?

Polytope	Convex hull of a finite set of points in $\mathbb{R}^d$
Face	Intersection with a supporting hyperplane
Vertices, Edges, Facets	Faces of dimension 0, 1, $d-1$.
Face lattice	Set of all faces, partially ordered by inclusion.
$P \simeq Q$	P combinatorially equivalent to Q $\Leftrightarrow$ Isomorphic (labeled) face lattices.





Many inscribable neighborly polytopes



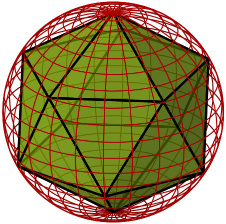
Universality theorems for inscribed polytopes

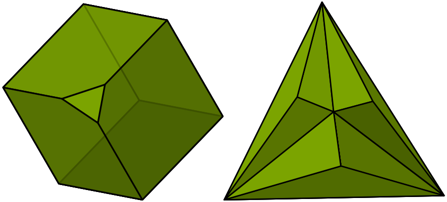
Many inscribable neighborly polytopes

Inscribed polytopes

Definition

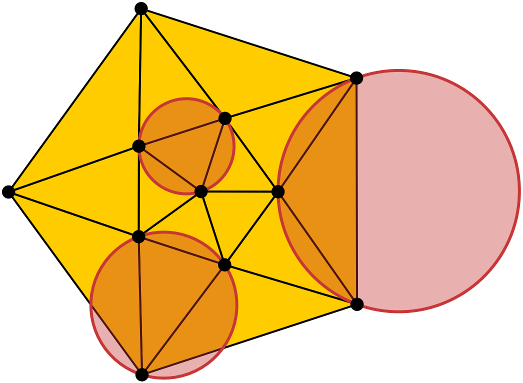
A polytope is *inscribed* if all its vertices lie on a sphere.



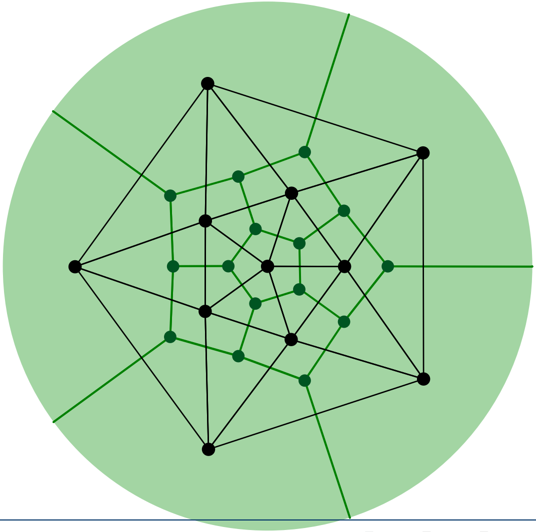


Not all polytopes are *inscribable*

Delaunay triangulation: empty circumsphere condition

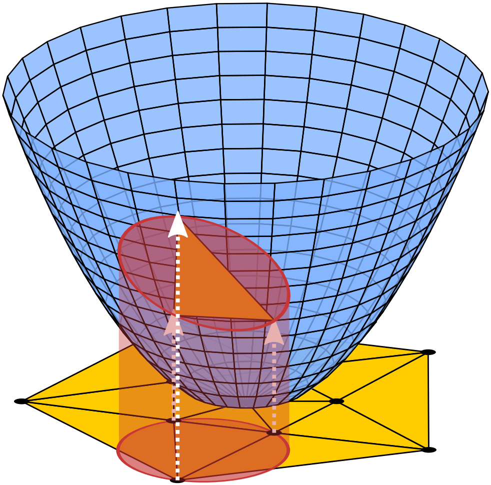


Voronoi diagram

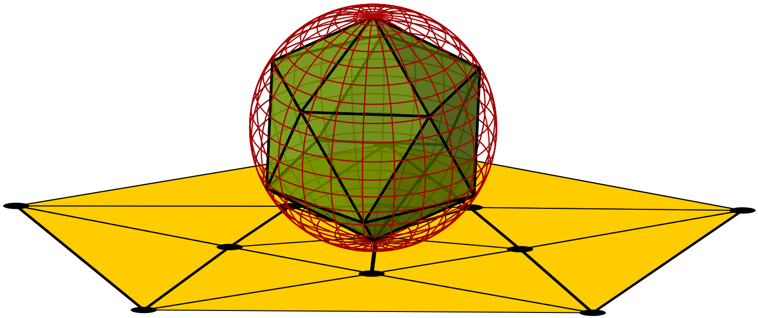


Delaunay triangulations are regular

$$(x_1, x_2, \ldots, x_d) \mapsto (x_1, x_2, \ldots, x_d, x_1^2 + x_2^2 + \cdots + x_d^2)$$



Delaunay ⟷ Inscribed



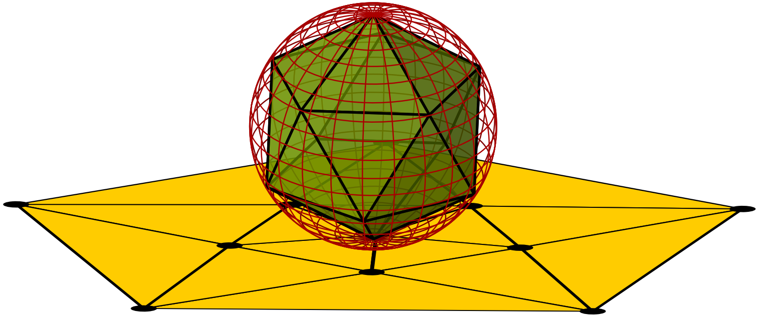
Delaunay ⟷ Inscribed

Inscribed → Delaunay:

Stereographic projection sp_v from a vertex v

Delaunay → Inscribed:

$$\text{Faces of } \mathrm{sp}_v^{-1}(A) = \begin{cases} S & \text{for } S \in \mathcal{T} \\ v \cup S & \text{for } S \in \partial\mathcal{T} \end{cases}$$



Some questions about inscribable polytopes

- Which polytopes are inscribable?

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  - ????

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  - Some → Many

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  - Some → Many
- How many faces can an inscribed polytope have?
  - This we know!

Neighborly polytopes

How many faces can a polytope have?

A d-polytope P is

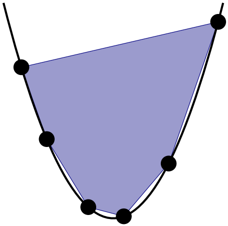
- *k-neighborly* if every $\leq k$ vertices form a face,
- *neighborly* if it is $\lfloor \frac{d}{2} \rfloor$-neighborly.

Moment curve

$$\gamma : t \mapsto (t, t^2, \dots, t^d)$$

Cyclic polytope

$$\mathcal{C}_d(n) = \operatorname{conv}\{\gamma(t_1), \dots, \gamma(t_n)\}$$
$$t_1 < t_2 < \cdots < t_n$$



The Upper Bound Theorem

Theorem (Upper bound theorem [McMullen 1970])

If P is a d-polytope with n vertices

$$f_i(P) \leq f_i(\mathcal{C}_d(n))$$

with equality if and only if P is simplicial and neighborly.

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[Carathéodory 1911]

Trigonometric moment curve

$$\tau : t \mapsto (\sin(t), \cos(t), \sin(2t), \cos(2t) \dots)$$

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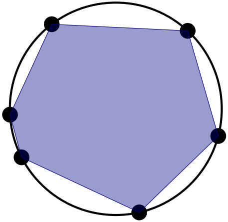

The number of polytopes, neighborly polytopes and inscribable polytopes

$$1 \leq 2 \leq 4^n \leq n^{\frac{1}{2}n} \leq \text{nei}(n,d)$$

$$\wedge$$

$$n^{\frac{1}{4}dn} \leq \text{pol}(n,d) \leq n^{d^2 n}$$

[Grünbaum 1967]
[Motzkin 1957]

[Shemer 1982]
[Barnette 1981]

[Alon 1986]
[Goodman & Pollack 1986]

(approximate behaviors when $n \gg d$)

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(approximate behaviors when $n \gg d$)

Lexicographic liftings

Lexicographic lifting

$$A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$$

$$\downarrow$$

$$\widehat{A} = \{\widehat{a}_1, \dots, \widehat{a}_n\} \subset \mathbb{R}^{d+1}$$

where $\widehat{a}_i = \binom{a_i}{h_i}$ and $\exists\, K$ such that:

- if $i < K$ $h_i = 0$
- if $i \geq K$, then $|h_i|$ is *large enough*: $\widehat{a}_i$ *above*/*below* every hyperplane spanned by $\{\widehat{a}_1, \dots, \widehat{a}_{i-1}\}$

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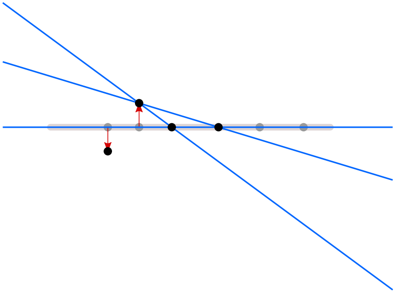

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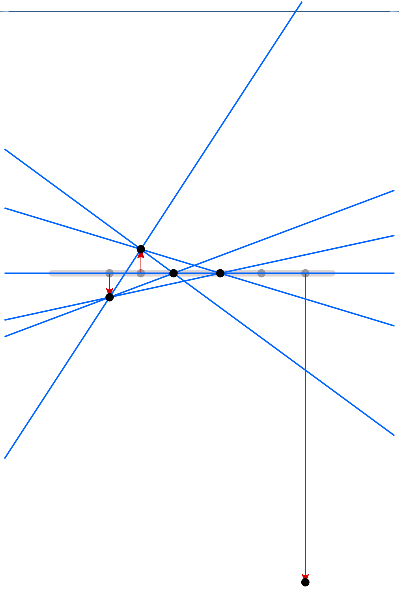

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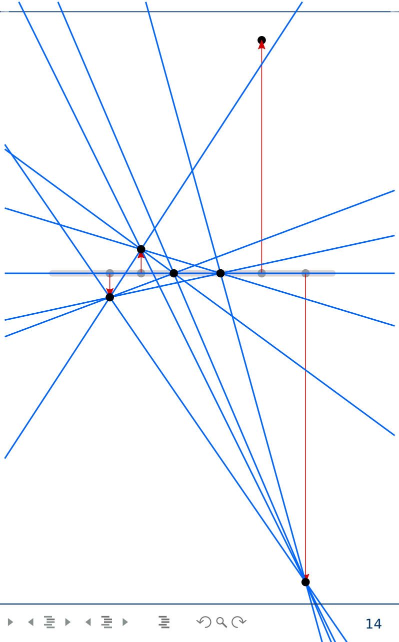

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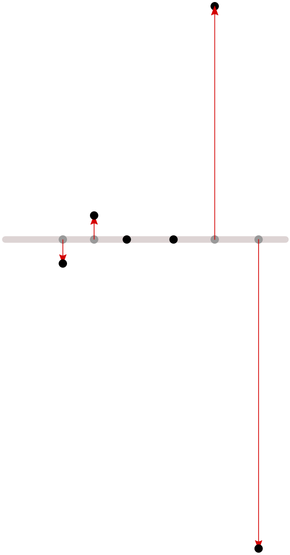

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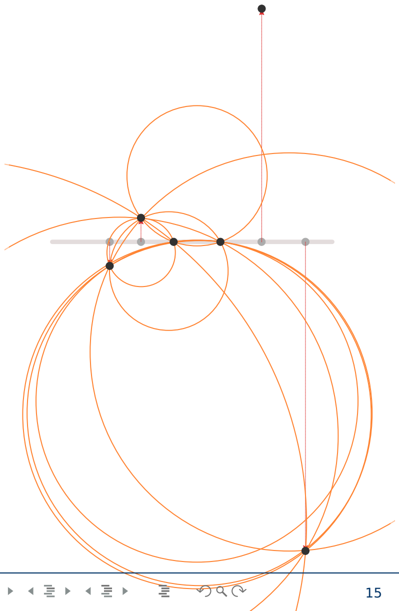
Delaunay Lexicographic lifting

A lexicographic lifting $\widehat{A}$

with $|h_i|$ *large enough*:

- $\widehat{a}_i$ *outside* each *circumsphere* spanned by $\{\widehat{a}_1, \ldots, \widehat{a}_{i-1}\}$.



Many Neighborly Delaunay Triangulations

Theorem

If

- $A = \text{vert}(P) \subset \mathbb{R}^d$
- *P is k-neighborly*
- *$\mathcal{T}$ Delaunay triangulation of $\widehat{A} \subset \mathbb{R}^{d+1}$ (Del. lex. lift. of A)*

Then

- *$\partial\mathcal{T}$ is k-neighborly*
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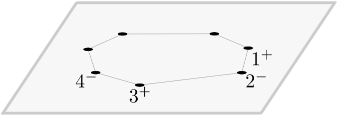

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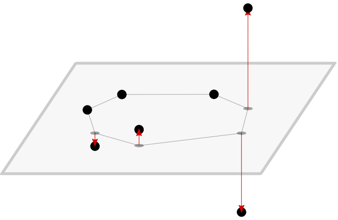

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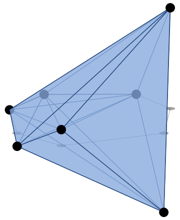

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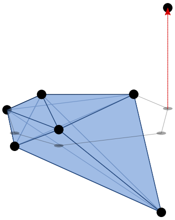

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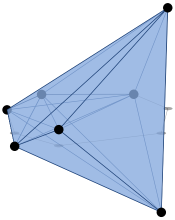

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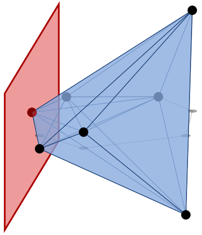

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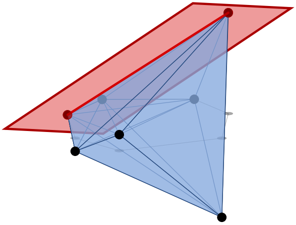

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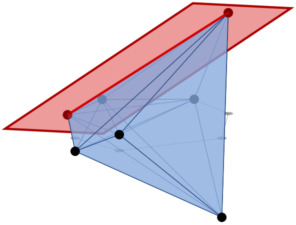


- P neighborly n vertices dim d
- $\sigma \in \mathcal{S}_n$
- $\varepsilon \in \{+,-\}^{n-d-1}$

$\Rightarrow$ $\text{sp}^{-1}(\widehat{P})$ inscribed neighborly $n+1$ vertices dim $d+2$

Many insribable neighborly polytopes

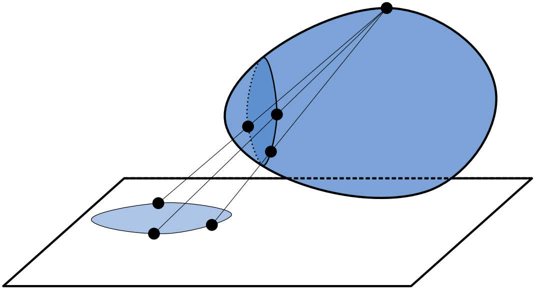
Theorem ([P][GP])

The number of labeled inscribable neighborly d-polytopes with n vertices is at least:

$$n^{\lfloor \frac{d}{2} \rfloor n(1-o(1))}$$

[P] A. Padrol, *Many neighborly polytopes and oriented matroids*, Disc. & Comp. Geom., 2013
[GP] B. Gonska & A. Padrol, *Neighborly inscribed polytopes and Delaunay triangulations*, arXiv:1308.5798

Many egg-scribable neighborly polytopes



The same method works on any smooth strictly convex body (*egg*):

Theorem ([GP])

At least

$$n^{\lfloor \frac{d}{2} \rfloor n(1+o(1))}$$

d-polytopes with n vertices are inscribable in any egg!

Universality theorems for inscribed polytopes

Realization spaces

Definition

The *realization space* of a d-polytope P with n vertices is

$$\mathcal{R}_{\mathrm{pol}}(P) := \{Q \subseteq \mathbb{R}^d \ : \ Q \text{ combinatorially equivalent to } P\} \,/\, \text{Aff. transf.}$$

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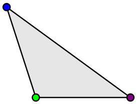
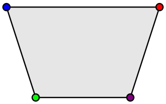
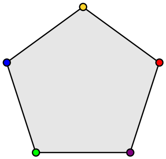
Examples:

$\mathcal{R}_{\mathrm{pol}}(\triangle) \cong \bullet$ $\mathcal{R}_{\mathrm{pol}}(\square) \cong \mathbb{R}^2$ $\mathcal{R}_{\mathrm{pol}}(\pentagon) \cong \mathbb{R}^4$







Realization spaces

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The *realization space* of a d-polytope P with n vertices is

$$\mathcal{R}_{\text{pol}}(P) := \{Q \subseteq \mathbb{R}^d \;:\; Q \text{ combinatorially equivalent to } P\} \,/\, \text{Aff. transf.}$$

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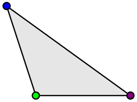

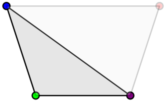

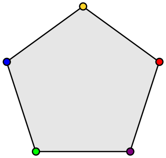

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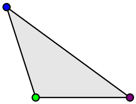

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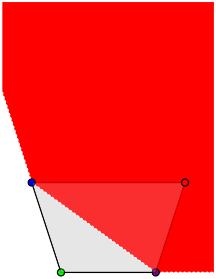

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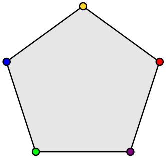

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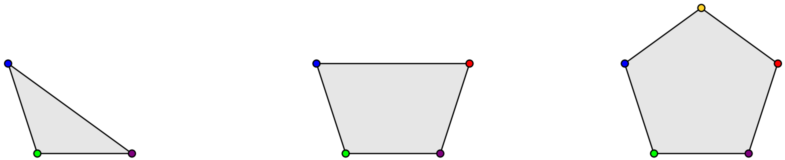

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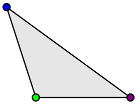

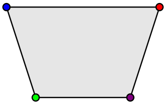

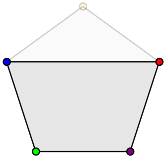

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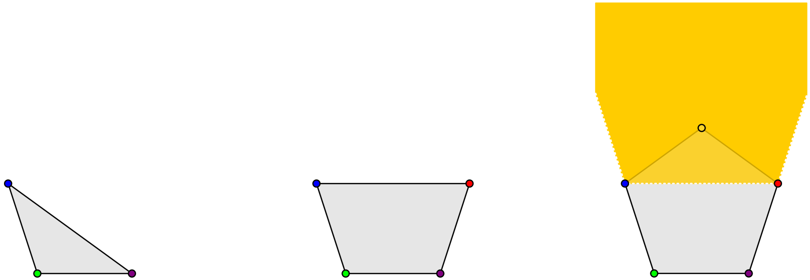

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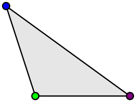
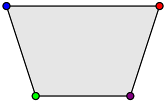
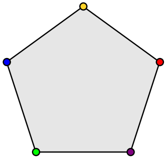
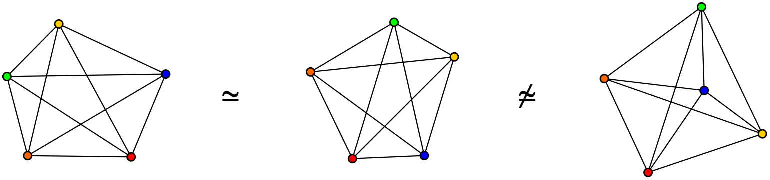






Oriented matroids



Definition (Realization space of an oriented matroid)

$$\mathcal{R}_{\mathrm{om}}(M) := \{A \text{ set of } n \text{ points in } \mathbb{R}^d \;:\; A \text{ realizes } M\} \,/\, \text{Aff. transf.}$$

Universality Theorem ([Mnëv 1988])

$\forall S$ primary basic *semi-algebraic set* over $\mathbb{Z}$

$\exists\, A \subset \mathbb{R}^2$ with $\mathcal{R}_{\mathrm{om}}(A)$ *stably equivalent* to S.

If S is *open*, then $\exists\, A$ in *general position*.

N. Mnëv

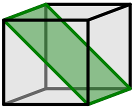
"Unless there is some a priori reason otherwise, the deformation space may be as bad as possible."

Ravi Vakil

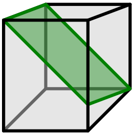
Some consequences of Universality

- The *Realizability problem* for oriented matroids (deciding if a realization exists) is polynomially equivalent to *Existential Theory of the Reals* (ETR). In particular, *NP*-hard.
- All algebraic numbers are needed to *coordinatize* oriented matroids.
- There are oriented matroids whose realization space has an *exponential* number of *connected components* (on the number of points).

Polytopes vs Oriented Matroids: Rigidity



$\not\simeq$



In general,

$$\mathcal{R}_{\mathrm{om}}(A) \subseteq \mathcal{R}_{\mathrm{pol}}(A)$$

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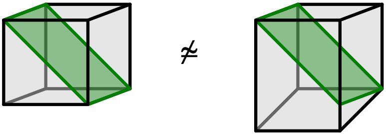

In general,

$$\mathcal{R}_{\mathrm{om}}(A) \subseteq \mathcal{R}_{\mathrm{pol}}(A)$$

Definition

P is *rigid* if $\mathcal{R}_{om}(A) = \mathcal{R}_{pol}(P)$, where $P = \mathrm{conv}(A)$.

Examples:

- *Lawrence polytopes*
- *neighborly polytopes* in even dimension **[Shemer 1982]**
- ...

Theorem (Universality Theorem [Mnëv 1988])

$\forall S$ *primary basic semi-algebraic set over* $\mathbb{Z}$

$\exists$ P *d-polytope with* $\mathcal{R}_{pol}(P)$ *stably equivalent to* S.

If S *is open, then* P *is simplicial.*

N. Mnëv

Mnëv's Universality Theorem

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- Lawrence polytopes **[Mnëv 1988]**
- 4-dimensional **[Richter-Gebert 1996]**

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- Neighborly polytopes **[AP 2014]**

N. Mnëv

Freie Universität Berlin

Lemma

$\widehat{A}$ *lexicographic lifting of* A

$$\Downarrow$$

$\mathcal{R}_{om}(\widehat{A})$ *stably equivalent* *to* $\mathcal{R}_{om}(A)$.

Realization spaces and lexicographic lifting

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Theorem ([Kortenkamp 1997])

$d+4$ *points in* $\mathbb{R}^d$

$\downarrow$ *lexicographic liftings*

neighborly $(2d+4)$*-polytope with* $2d+8$ *vertices.*

Universality for neighborly polytopes

Corollary ([AP'])

Realization spaces of neighborly polytopes present universality.

Proof.

open primary basic semi-algebraic set	S
$\downarrow$ **[Mnëv]**	$\wr$
planar point configuration	$\mathcal{R}\mathrm{om}$
$\downarrow$ Gale dual	$\wr$
$d+4$ points in $\mathbb{R}^d$	$\mathcal{R}\mathrm{om}$
$\downarrow$ **[Kortenkamp]**	$\wr$
neighborly point configuration	$\mathcal{R}\mathrm{om}$
$\downarrow$ rigidity	$\parallel$
neighborly polytope	$\mathcal{R}\mathrm{pol}$

□

[AP'] K. A. Adiprasito & A. Padrol, *The universality theorem for neighborly polytopes*, arXiv:1402.7207, to appear in Combinatorica

Definition

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$$\mathcal{R}_{\text{ins}}(P) := \{A \text{ set of } n \text{ points in } \mathbb{S}^{d-1} \ : \ P \simeq \operatorname{conv}(A)\} \,/\, \text{Möbius transf.}$$

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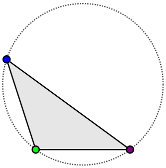

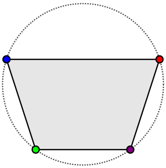

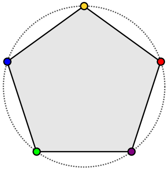

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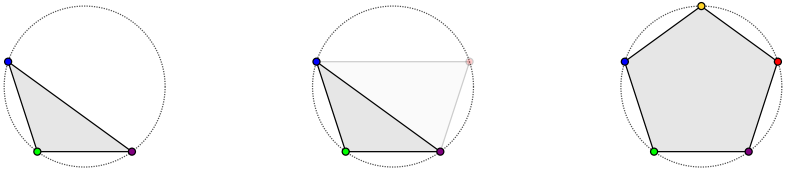

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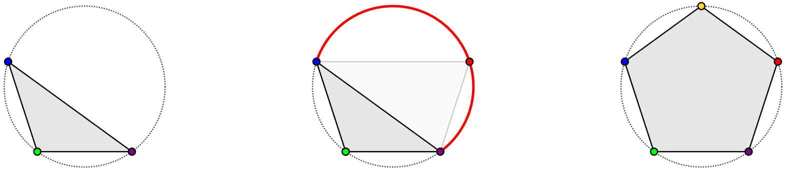

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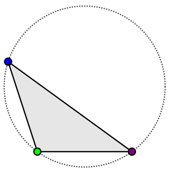

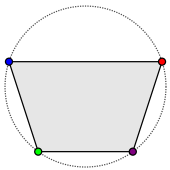

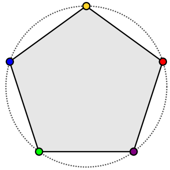

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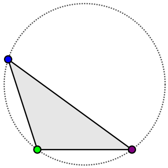

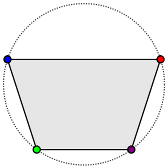

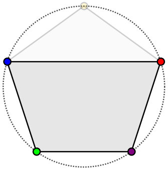

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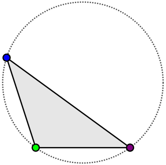

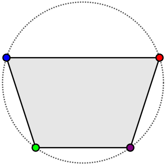

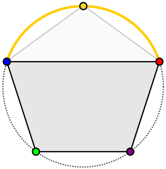

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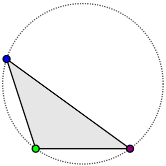
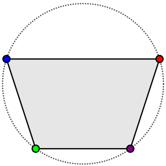
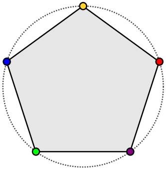

Universality theorem for inscribed polytopes [APT]

$\forall S$ primary basic *semi-algebraic set* over $\mathbb{Z}$

$\exists$ *polytope* P with $\mathcal{R}_{\text{ins}}(P)$ *homotopy equivalent* to S.

Theorem ([AZ])

All algebraic numbers are needed to coordinatize inscribed polytopes.

[APT] K. A. Adiprasito, A. Padrol & L. Theran, *Universality theorems for inscribed polytopes and Delaunay triangulations*, arXiv:1406.7831
[AZ] K. A. Adiprasito and G. M. Ziegler, *Many projectively unique polytopes*, Inventiones Math.

Theorem ([P][GP])

neighborly d-polytope n vertices

$\downarrow$ *lexicographic liftings*

inscribable neighborly $(d+2)$-polytope $n+2$ vertices

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Theorem (Weak universality theorem **[PT,APT]**)

$\forall S$ *open primary basic semi-algebraic set over* $\mathbb{Z}$

$\exists$ *simplicial neighborly polytope* P *such that* S *is a retract of* $\mathcal{R}_{ins}(P)$

Proof.

open primary basic semi-algebraic set	S
$\downarrow$ **[AP]**	$\wr$
neighborly polytope	$\mathcal{R}_{om}$
$\downarrow$ Delaunay lex. lifting	$\twoheaduparrow$
neighborly & inscribable polytope	$\mathcal{R}_{ins}$

□

[PT] A. Padrol & L. Theran, *Delaunay triangulations with disconnected realization spaces*, Proceedings SOCG 2014
[APT] K. A. Adiprasito, A. Padrol & L. Theran, *Universality theorems for inscribed polytopes and Delaunay triangulations*, arXiv:1406.7831

How complicated can $\mathcal{R}_{\text{ins}}(P)$ be?

Theorem ([APT])

In arbitrary dimension (and even for simplicial polytopes):

- *The topology of $\mathcal{R}_{ins}(P)$ can be arbitrarily complicated.*
- *Realizability is as hard as the Existential Theory of the Reals (ETR). (In particular, NP-hard.)*

Theorem ([Rivin 1994])

In $\mathbb{R}^3$:

- *$\mathcal{R}_{ins}(P)$ is either empty or contractible.*
- *Realizability ($\mathcal{R}_{ins}(P) \neq \emptyset$) can be checked in polynomial time.*

Main challenge: fixed dimension

Conjecture

$\exists D (= 4)$ *such that:*

$\forall S$ *primary basic (open) semi-algebraic set over* $\mathbb{Z}$

$\exists$ D*-dimensional (simplicial) polytope* P *with* $\mathcal{R}_{ins}(P)$ *homotopy equivalent to* S.

- $\exists$ inscribable 4-polytope with disconnected realization space.
- Richter-Gebert's Universality for 4-polytopes uses *connected sums* and *non-rigid* polytopes.
- Universality for *simplicial* polytopes in *fixed dimension* is still open.

Merci Beaucoup!