# Shortest Paths in Intersection Graphs of Unit Disks

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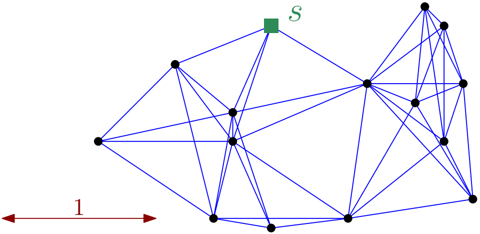
Material based on joint work with
Miha Jejčič and Panos Giannopoulos

# Setting

$P$: $n$ points in the plane
$G(P)$: connect two points when distance $\leq 1$
intersection graph congruent disks



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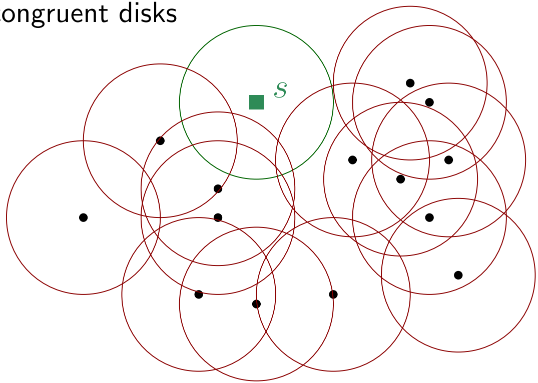


Objective: fast computation of sssp in $G(P)$

# Motivation

Bounded communication range:

- minimize hops/links $\rightarrow$ unweighted $G(P)$
- minimize energy $\rightarrow$ weighted $G(P)$

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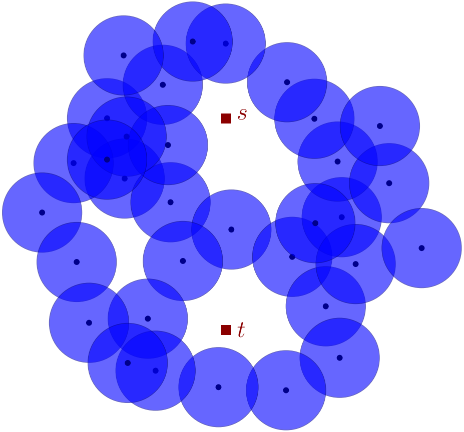
Separation in the plane:

- set $D$ of unit disks
- $s$ and $t$ in $\mathbb{R}^2 \setminus \bigcup D$
- $\min |D'|$ s.t. $D' \subseteq D$, $D'$ separates $s$ and $t$



# Overview

- Setting/Motivation
- Related work for sssp
- Unweighted
  - $O(n \log n)$ time
  - implementable: Delaunay, Voronoi, point location
- Weighted:
  - $O(n^{1+\varepsilon})$ time
  - unimplementable: dynamic bichromatic closest pair, shallow cuttings
- Separation with unit disks:
  - $O(n^2 \operatorname{polylog} n)$ time
  - Implementable, but many ingredients

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# Related work

exact SSSP

- Roditty and Segal, 2011
  - unweighted: $O(n \log^6 n)$ expected time via Chan's dynamic NN DS
  - weighted: $O(n^{4/3+\varepsilon})$ time
- C. and Jejčič, 2014
  - unweighted: $O(n \log n)$ time; implementable
  - weighted: $O(n^{1+\varepsilon})$ time

# More related work

- Roditty and Segal, 2011
  - $(1+\varepsilon)$-approximate distance oracles, improving Bose, Maheshwari, Narasimhan, Smid, and Zeh, 2004.
- Gao and Zhang, 2005
  - WSPD of size $O(n \log n)$ for unit-disk metric
  - $(1+\varepsilon)$-approximate sssp distance in $O(n \log n)$ time
- Chan and Efrat, 2001 (Fuel consumption)
  - distances $\ell : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{>0}$
  - $O(n \log n)$ time when $\ell(p,q) = f(|pq|) \cdot |pq|^2$, $f$ increasing.
  - $O(n^{4/3+\varepsilon})$ time when $\ell$ has constant size description.

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- Faster algorihtms for geometric intersection graphs

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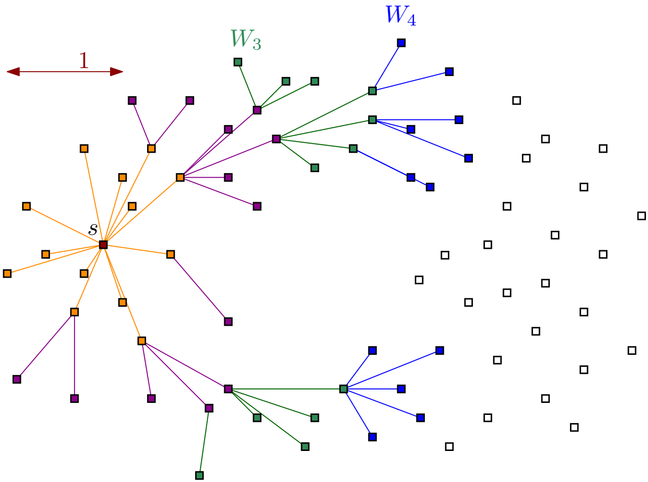
# Unweighted

- BFS in $G(P)$ without building $G(P)$
- $W_i = \{p \in P \mid d_{G(P)}(s,p) = i\}$
- Build $W_0 = \{s\}$
- Iteratively build $W_i$ from $W_{i-1}$
- Edge connecting $p$ to $NN(p, W_{i-1})$ for all $p \in W_i$
- Until $W_i$ empty

# Unweighted - Growing $W_i$



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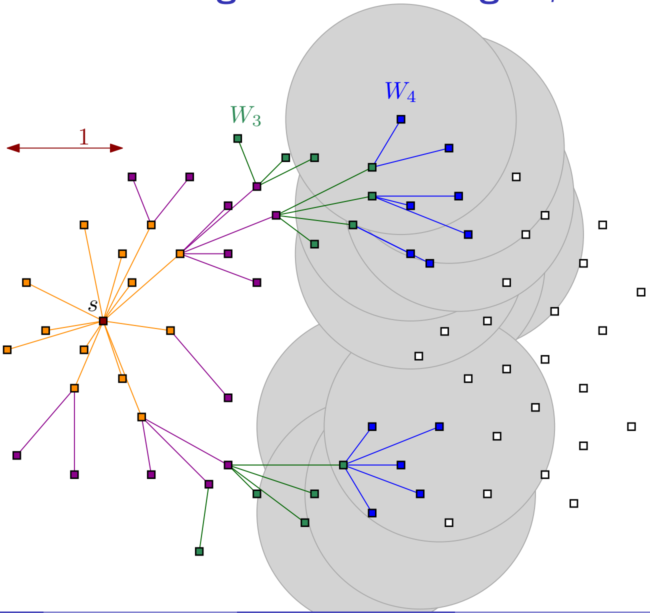

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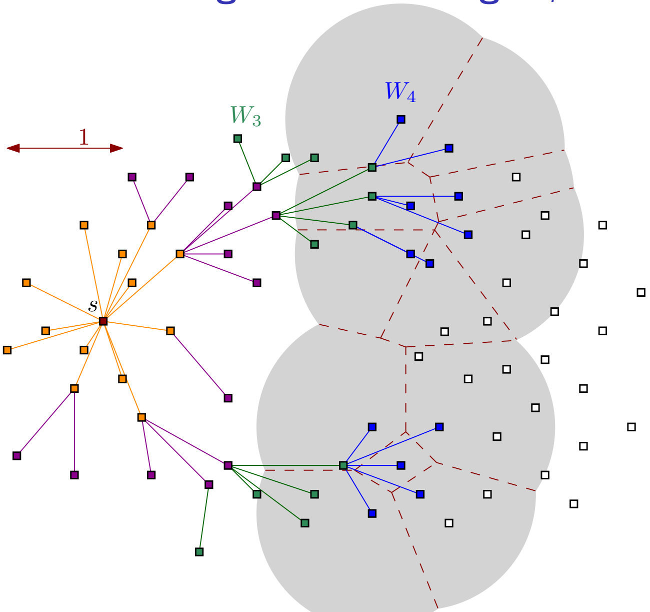

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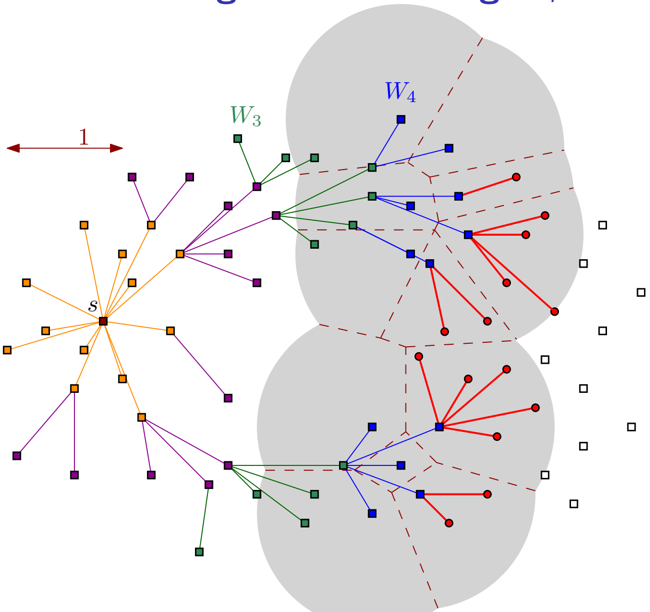
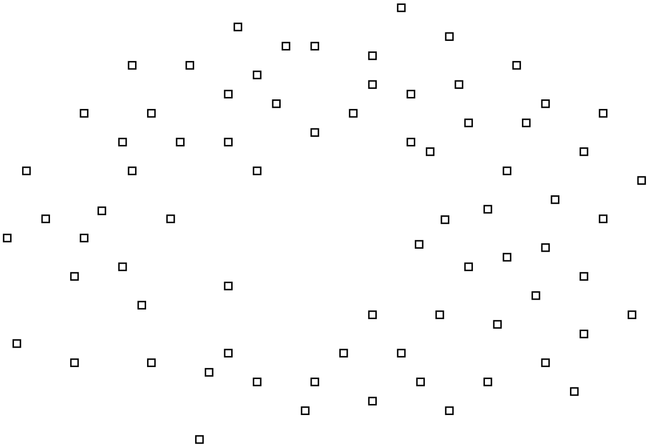


# Unweighted

- BFS in $G(P)$ without building $G(P)$
- $W_i = \{p \in P \mid d_{G(P)}(s,p) = i\}$
- Build $W_0 = \{s\}$
- Iteratively build $W_i$ from $W_{i-1}$
- edge connecting $p$ to $NN(p, W_{i-1})$ for all $p \in W_i$
- Until $W_i$ empty
- Use $DT(P)$ to guide the search of candidate points for $W_i$
- Candidate points for $W_i$:
  - points adjacent to $W_{i-1}$ in $DT(P)$
  - points adjacent to $W_i$ in $DT(P)$
- Is this good enough?

# Unweighted - Growing $W_i$



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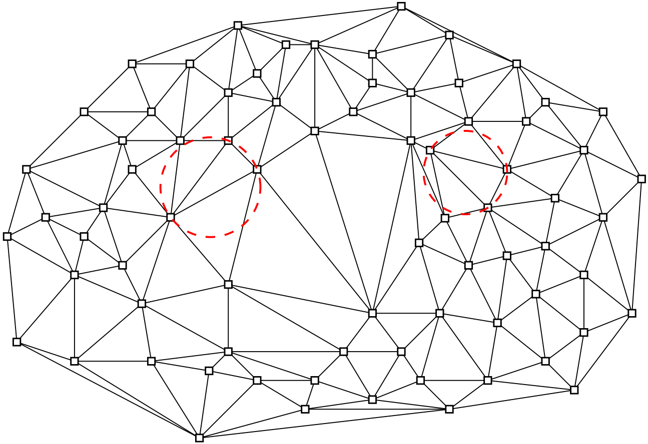

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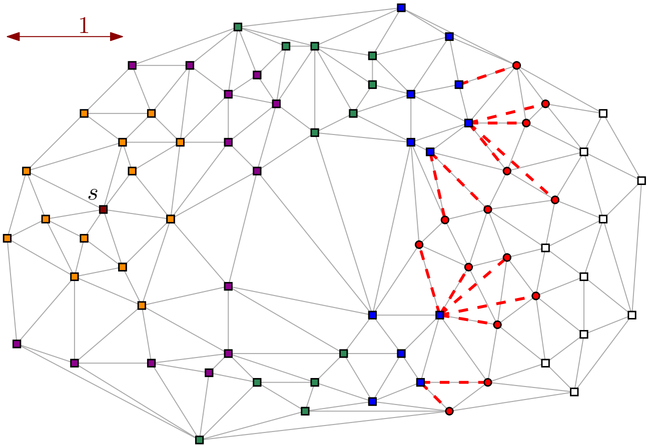

# Unweighted - Growing $W_i$

Lemma

*Let* $p \in W_i$*.*
*There exists a path* $q_1, \dots, q_k = p$ *in* $G(P) \cap DT(P)$ *with* $q_1 \in W_{i-1}$ *and* $q_2, \dots, q_k \in W_i$*.*

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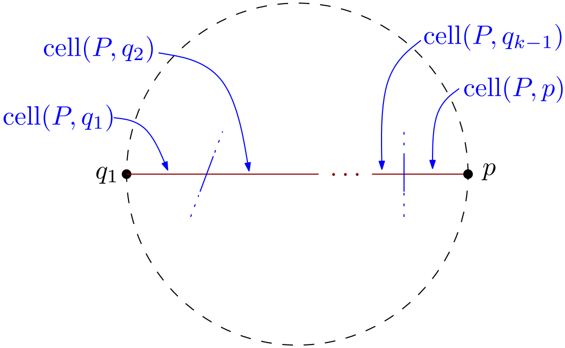

# Unweighted - Growing $W_i$

**Lemma**

*Let $p \in W_i$.*
*There exists a path $q_1, \ldots, q_k = p$ in $G(P) \cap DT(P)$ with $q_1 \in W_{i-1}$ and $q_2, \ldots, q_k \in W_i$.*

- Data structure to decide whether candidate $q$ is $\in W_i$
  - DS for $NN(q, W_{i-1})$
  - check if distance $\leq 1$
- each edge of $DT(P)$ explored twice
- building $W_i$ takes time

$$O\Big(\big(|W_{i-1}| + |W_i| + \sum_{p \in W_{i-1} \cup W_i} \deg_{DT(P)}(p)\big) \log n\Big)$$

```
for p ∈ P do
    dist[p] ← ∞;
dist[s] ← 0
build the Delaunay triangulation DT(P)
W_0 ← {s}
i ← 1
while W_{i−1} ≠ ∅ do
    build data structure for nearest neighbour queries in W_{i−1}
    Q ← W_{i−1}     (* generator of candidate points *)
    W_i ← ∅
    while Q ≠ ∅ do
        q an arbitrary point of Q
        remove q from Q
        for qp edge in DT(P) do
            w ← NN(W_{i−1}, q)
            if dist[p] = ∞ and |pw| ≤ 1 then
                dist[p] ← i
                add p to Q and to W_i
    i ← i + 1
return dist[·]
```

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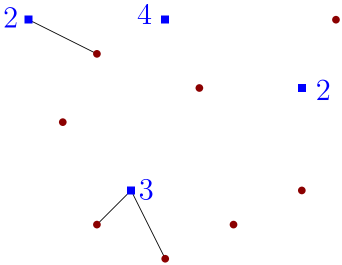
# Weighted - Ingredient - BCP

Bichromatic closest pair (BCP)

- weighted Euclidean
- red points $R$
- blue points $B$
- weights $w_b$ for each $b \in B$
- $\delta\colon B \times R \to \mathbb{R} \qquad \delta(b, r) = w_b + |br|$



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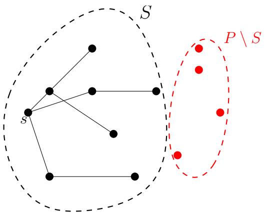
- Eppstein 1995 + Agarwal, Efrat, Sharir 1999: dynamic BCP in $O(n^\varepsilon)$ amortized per operation
  - insertion/deletion
  - query for minima $\min_{r,b} \delta(r, b)$

# Weighted - Idea

- Modification of Dijkstra's algorithm
- Standard Dijsktra's algorithm
  - keep an array dist[·]
  - dist[$v$] is an (over)estimate of $d_{G(P)}(s, v)$
  - keep partition $P$ into $S$ and $P \setminus S$
  - $S$ contains vertices with dist[$s$] $= d_{G(P)}(s, v)$



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  - $S$ contains vertices with dist[$s$] $= d_{G(P)}(s, v)$
  - an iteration: find a vertex

$$q^* \in \arg \min_{q \in P \setminus S} \min_{p \in S,, |pq| \leq 1} \text{dist}[p] + |pq|$$

  - move $q^*$ from $P \setminus S$ to $S$
  - usually we keep dist[$q$] $= \min_{p \in S} \text{dist}[p] + |pq|$

# Weighted - Idea

- Modification of Dijkstra's algorithm
  - array dist[·], dist[$v$] is an (over)estimate of $d_{G(P)}(s, v)$
  - keep partition $P$ into $S$ and $R = P \setminus S$
  - partition $S$ into $D$ and $B$
  - $D$ are "dead" points, irrelevant when min dist[$p$] $+ |pq|$
  - an iteration: find a pair

$$(b^*, r^*) \in \arg \min_{(b,r) \in \in B \times R} \text{dist}[b] + |br|$$

  - if $|b^* r^*| > 1$, move $b^*$ from $B$ to $D$
  - else normal Dijsktra's step

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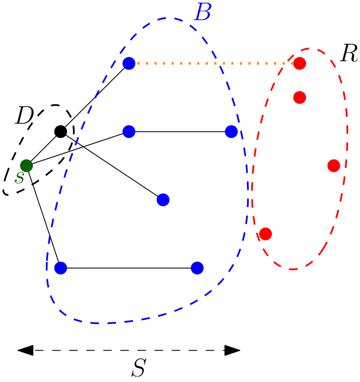

**for** $p \in P$ **do**
$\text{dist}[p] \leftarrow \infty$
$\text{dist}[s] \leftarrow 0$
$B \leftarrow \{s\}$
$D \leftarrow \emptyset$
$R \leftarrow P \setminus \{s\}$
store $R \cup B$ in a BCP dynamic DS wrt $\delta(b, r) = \text{dist}[b] + |br|$
**while** $R \neq \emptyset$ **do**
$(b^*, r^*) \leftarrow \text{BCP}(B, R)$
**if** $|b^* r^*| > 1$ **then**
$\text{delete}(B, b^*)$
$D \leftarrow D \cup \{b^*\}$
**else**
$\text{dist}[r^*] \leftarrow \text{dist}[b^*] + |b^* r^*|$
$\text{delete}(R, r^*)$
$\text{insert}(B, r^*)$
**return** $\text{dist}[\cdot]$

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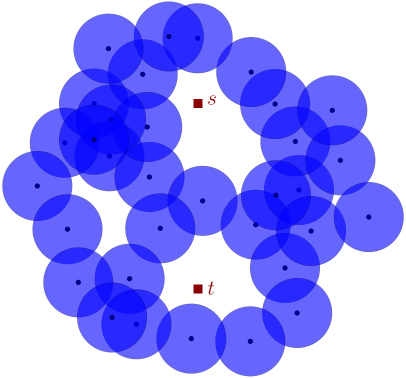
# Setting

- set $D$ of unit disks
- $s, t$ points in $\mathbb{R}^2 \setminus \bigcup D$
- $P$ centers of the disks
- $G(P)$ as before, with distance 2
- C. and Giannopoulos
  - $O(n^2 + n \cdot |E(G(D))|)$
  - general objects



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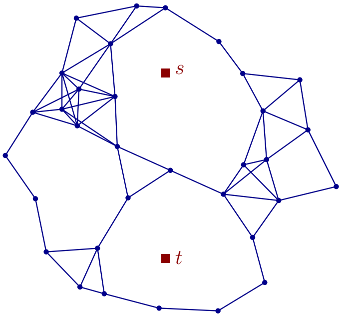
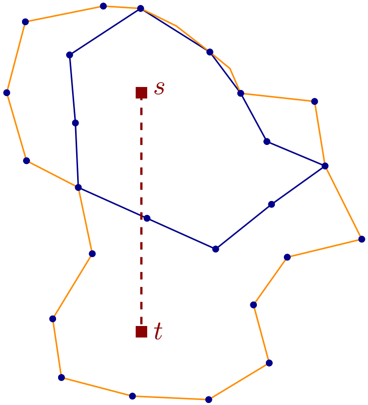
  - general objects
- today: $O(n^2 \log^4 n)$ for unit disks
- also easier to explain & understand



# Algorithm of C. & Giannopoulos

- for a closed walk $\pi = p_1 \ldots p_k p_1$ in $G(P)$

$$N(\pi) = \pi \cap \overline{st} \pmod 2$$



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- for a closed walk $\pi = p_1 \dots p_k p_1$ in $G(P)$
$$N(\pi) = \pi \cap \overline{st} \pmod 2$$
- if $N(\pi) = 1$ then $\bigcup_{p \in V(\pi)} disk(p, 1)$ separates $s$ and $t$
- shortest closed walk $\pi$ with $N(\pi) = 1$ gives an optimal solution
- shortest closed walk $\pi$ with $N(\pi) = 1$ is actually a cycle
- enough to restrict the search to fundamental cycles: defined by a BFS-tree and an additional edge

$$\begin{aligned} \min\ & |V(cycle(T_r, e))| \\ \text{s.t.}\ & r \in P,\ \ T_r \text{ BFS cycle from } r \\ & e \in E(G(P)) \setminus E(T_r) \\ & N(cycle(T_r, e)) = 1 \end{aligned}$$

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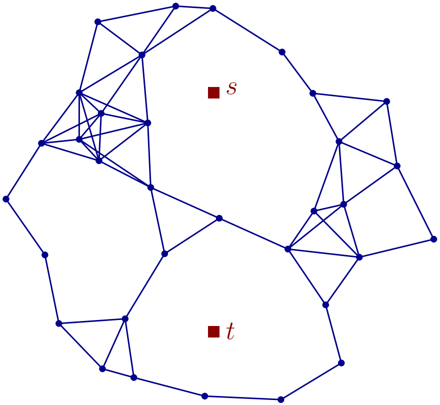

# Adaptation

for each $r$ in $P$

- construct BFS tree $T_r$ from $r$
- attach to each $p \in P$ the label $d[p] = d_{G(P)}(s, p)$
- solve

$$\begin{aligned} \text{min} \ & d[p] + d[q] \\ \text{s.t.} \ & |pq| \leq 1 \\ & N(cycle(T_r, pq)) = 1 \end{aligned}$$

- break $P$ into groups depending on $N(T_r[r, p])$
- use range searching & vertical shooting
  to solve the resulting problems

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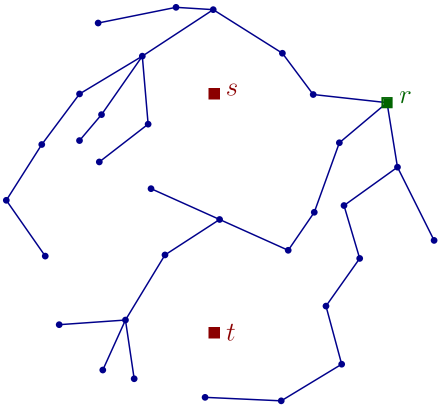

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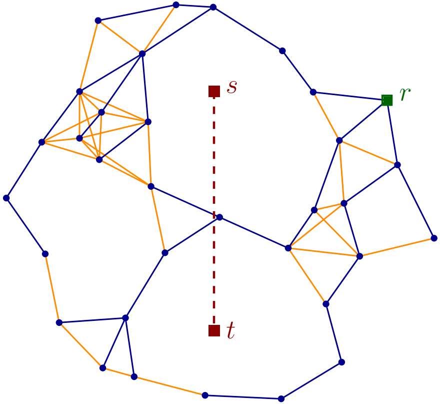
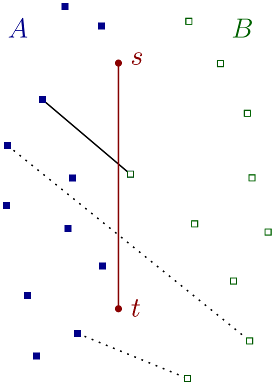


# Resulting problem - Example

- vertical segment $st$
- points $A$ and $B$ with weights $(w_p)_{p \in A \cup B}$

$$\begin{aligned} \min\ & w_a + w_b \\ \text{s.t.}\ & a \in A, b \in B \\ & |ab| \leq 1 \\ & ab \cap st \neq \emptyset \end{aligned}$$

Solvable in $O(n \log^4 n)$



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- shortest paths in unit disk graphs
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- Open problems:
  - Can we compute efficiently a compact representation of the distances in all the graphs $G_{\leq\lambda}(P)$?
  - Given $s, t \in P$ and $k \in \mathbb{N}$,
    find minimum $\lambda$ such that $d_{G_{\leq\lambda}(P)}(s, t) \leq k$.
    Easy in $\tilde{O}(n^{4/3})$.
  - Dual to separation problem – barrier resilience:
    find $(s, t)$-curve that touches as few disks as possible.
    Polynomial? Hard? Both?