ZIGZAGS IN TRIANGULATIONS AND SIMPLICIAL COMPLEXES

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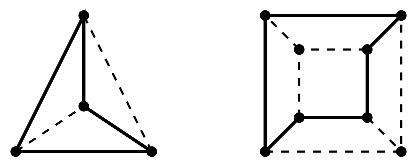
1 Zigzags in polyhedra.
2 Z-knotted triangulations.
3 Zigzags and distances between facets in simplicial complexes.

1. Zigzags in polyhedra

1.1. **Definition.** A *zigzag* in a polyhedron is a sequence of vertices $\{x_i\}_{i\in\mathbb{N}}$ satisfying the following conditions:

(1) x_i, x_{i+1} are distinct vertices on a certain edge,
(2) x_i, x_{i+1}, x_{i+2} are mutually distinct and there is the unique face containing them,
(3) the faces containing x_i, x_{i+1}, x_{i+2} and $x_{i+1}, x_{i+2}, x_{i+3}$ are distinct.

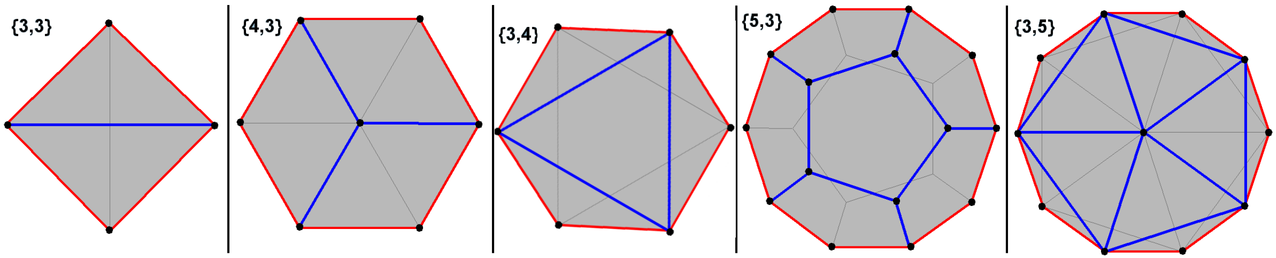
Examples:



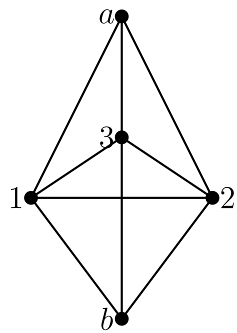
Remark. Zigzags are cyclic sequences of vertices: if $\{x_i\}_{i\in\mathbb{N}}$ is a zigzag, then there is n such that $x_{n+i} = x_i$ for every i. The smallest $n > 0$ satisfying this condition is called the *length* of the zigzag.

1.2. **Historical remark.** Zigzags in regular polytopes are called *Petrie polygons*. See Coxeter's book *Regular polytopes*.

The Petrie polygons in Platonic solids (red):



1.3. **Example: a 3-gonal bipyramid.**



There is the unique zigzag:

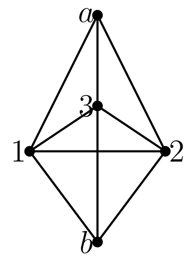
$$a12b31a23$$
$$b12a31b23$$

1.4. A polyhedron is called *z-knotted* if it contains the unique zigzag.

Example: Every n-gonal bipyramid is z-knotted if n is odd.
Let $1, \ldots, n$ be the vertices of the n-gon. If $n = 3$, then the unique zigzag is

$$a12b31a23$$

$$b12a31b23.$$

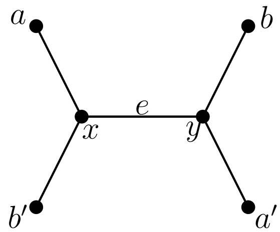


If $n = 5$, then the unique zigzag is

$$a12b34a51b23a45$$

$$b12a34b51a23b45.$$

1.5. Two types of edges in z-knotted polyhedra.



Suppose that the unique zigzag goes from x to y, i.e. it is a sequence of type $a, x, y, a', \ldots$.

There are the following two possibilities:

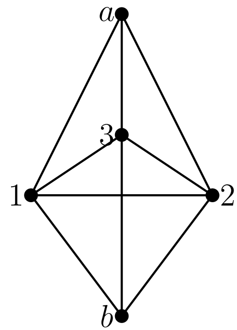
(1) the zigzag is $a, \underbrace{x, y}_{\rightarrow}, a', \ldots, b, \underbrace{y, x}_{\leftarrow}, b', \ldots$ and passes through the edge e twice in different directions.

(2) the zigzag is $a, \underbrace{x, y}_{\rightarrow}, a', \ldots, b', \underbrace{x, y}_{\rightarrow}, b, \ldots$ and passes through the edge e twice in the same direction.

We say that the edge e is of *first* or *second type*, respectively.

Example: an n-gonal bypiramid, n is odd.
If $n = 3$, then the unique zigzag is

$$a12b31a23b12a31b23$$



The edges ai and bi, $i \in \{1, 2, 3\}$ are of first type.
The edges 12, 23, 13 are of second type.
The same holds for every odd n.

2. z-KNOTTED TRIANGULATIONS

We consider a triangulation of a closed surface (not necessarily orientable) and suppose that this triangulation is z-knotted.

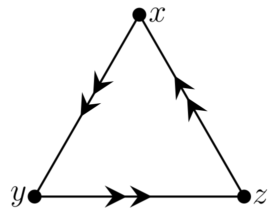
2.1. Two types of faces in z-knotted triangulations. Let Γ be a z-knotted triangulation.

Lemma 1. *For every face of* Γ *one of the following possibilities is realized:*

- *it is a* $(1,1,2)$*-face, i.e. there are two edges of first type and one edge of second type;*
- *it is a* $(2,2,2)$*-face, i.e. all edges are of second type.*

Example: all faces in the n-gonal bypiramid (n is odd) are $(1,1,2)$-faces.

2.2. $(2,2,2)$**-faces.** Let F be a $(2,2,2)$-face in Γ whose vertices are x, y, z.



The zigzag contains the sequences x, y, z and y, z, x and z, x, y, and there are only the following two possibilities for the zigzag:

(1) $x, y, z, \dots, z, x, y, \dots, y, z, x, \dots,$
(2) $x, y, z, \dots, y, z, x, \dots, z, x, y, \dots.$

There are z-knotted triangulations containing (2,2,2)-faces of first type.

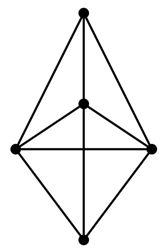
Problem. *Are there* $(2,2,2)$*-faces of second type?*

2.3. The connected sum of triangulations. Let Γ and Γ' be triangulations.
Let F and F' be faces in Γ and Γ', respectively.
Let also g be a bijection between vertices of F and F'.

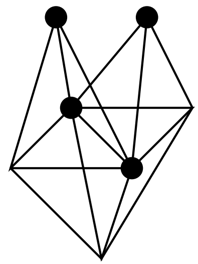
The *connected sum* $\Gamma \#_g \Gamma'$ is the triangulation is obtained as follows:

- we remove the interiors of F and F' from Γ and Γ', respectively,
- every vertex $x \in F$ is identified with the vertex $g(x) \in F'$.

Example: the connected sum of two tetrahedrons is a 3-gonal bipyramid.



Example: one of the possible connected sums of two 3-gonal bypiramids



It is z-knotted and contains two $(2,2,2)$-faces of first type (formed the vertices with big dots).

There are connected sums of two 3-gonal bypiramids which are not z-knotted.

2.4. Connected sums of z-knotted triangulations. Suppose that Γ and Γ' are z-knotted triangulations.
As above, F and F' are faces in Γ and Γ', respectively.

Theorem 1 (M.P., Adam Tyc)**.** *The following assertions are fulfilled:*

(1) *If both F and F' are not $(2,2,2)$-faces of second type, then there is a bijection g between vertices of F and F' such that the connected sum $\Gamma\#_g\Gamma'$ is z-knotted.*
(2) *If F is a $(2,2,2)$-face of first type, then for every bijection g between vertices of F and F' the connected sum $\Gamma\#_g\Gamma'$ is z-knotted.*
(3) *Suppose that F is a $(2,2,2)$-face of second type and the connected sum $\Gamma\#_g\Gamma'$ is z-knotted. Then F' is a $(2,2,2)$-face of first type or a $(1,1,2)$-face of special type.*

2.5. Deza conjecture.

Conjecture (M. Deza)**.** *In any z-knotted triangulation the number of edges of second type is even.*

This holds trivially for an n-gonal bipyramid (n is odd).

Theorem 2 (M.P., Adam Tyc)**.** *If a z-knotted triangulation is the connected sum of $B_1, \dots, B_t$ where every B_i is a n_i-gonal bipyramid and n_i is odd, then Deza conjecture holds for this triangulation.*

3. Zigzags and distances between facets in simplicial complexes

Let Δ be an n-dimensional polytope whose facets are $(n-1)$-simplices.
Interesting examples:

- Coxeter complexes,
- nested complexes.

A *zigzag* in Δ is a sequence of vertices $\{x_i\}_{i\in\mathbb{N}}$ satisfying the following conditions:

(1) $x_i, x_{i+1}, \ldots, x_{i+n-1}$ form a facet,
(2) the facets containing $x_i, \ldots, x_{i+n-1}$ and $x_{i+1}, \ldots, x_{i+n}$ are distinct.

3.1. Graph of facets. The graph $\Gamma(\Delta)$: the vertices are facets of Δ, two facets X, Y are adjacent vertices if $X \cap Y$ is $(n-2)$-face.

Let X, Y be facets.
The *distance* $d(X, Y)$ between X, Y is the smallest number of edge in a path joining X with Y in $\Gamma(\Delta)$.
A path joining X with Y is *geodesic* if it contains precisely $d(X, Y)$ edges.

Let $Z = \{x_i\}_{i \in \mathbb{N}}$ be a zigzag of length t.
Denote by X_i the facet containing $x_i, \ldots, x_{i+n-1}$.
The sequence $X_1, \ldots, X_t$ is a closed path in $\Gamma(\Delta)$, it is called the *shadow* of Z.

Question. For each pair of facets X, Y there is a geodesic joining X with Y and contained in the shadow of a zigzag?

Example: the *cross-polytope* β_n.

The vertices are

$$1, \dots, n, -1, \dots, -n.$$

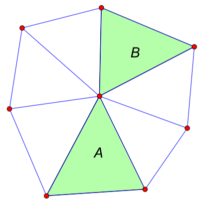
$X \subset \{\pm 1, \dots, \pm n\}$ is a face if for every $i \in X$ we have $-i \in X$.
The graph $\Gamma(\beta_n)$ is the hypercube graph Q_n.
If X, Y are facets, then

$$d(X, Y) = n - \dim(X \cap Y) - 1.$$

Every geodesic is contained in the shadow of a zigzag.

In the general cases, for any two facets X, Y of Δ we have

$$d(X,Y) \geq n - \dim(X \cap Y) - 1.$$



$d(A,B) = 3$ and $\dim(A \cap B) = 0$.

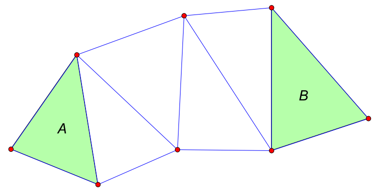
3.2. Distance normal pairs. Let X, Y be facets of Δ.

The case $d(X, Y) \leq n$.
We say that X, Y is a *distance normal pair* if

$$d(X, Y) = n - \dim(X \cap Y) - 1.$$

In this case, every geodesic connecting X, Y is called *distance normal*.

The case $d(X, Y) > n$.



$d(A, B) = 5$ and $A \cap B = \emptyset$.

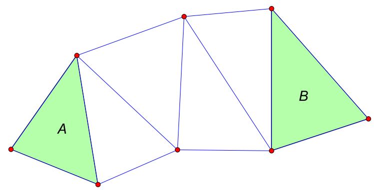
X, Y is a *distance normal pair* if there is a geodesic

$$X = X_0, X_1, \ldots, X_t = Y$$

such that $d(X_i, X_j) \leq n$ implies that X_i, X_j is a distance normal pair.
Every such geodesic is called *distance normal*.

Remark. If $d(X, Y) > n$ and X, Y is a distance normal pair, then we cannot state that every geodesic joining X with Y is distance normal.

Simple fact: If Z is a simple zigzag (i.e. without self-intersections), then every geodesic contained in the shadow of Z is distance normal.



Theorem 3 (M. Deza, M.P.). *Every distance normal geodesic is contained in the shadow of a zigzag.*