Trapezoidal Diagrams, Upward Triangulations, and Prime Catalan Numbers

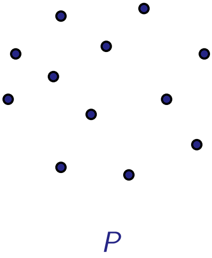
Manuel Wettstein

ETH Zürich

December 6, 2016

The Geometric Problem

Let P be a set of n points in the plane.



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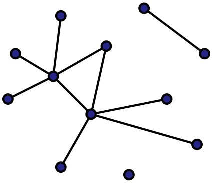

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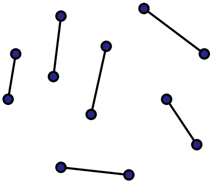

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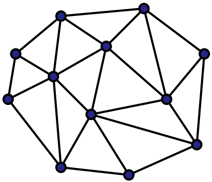

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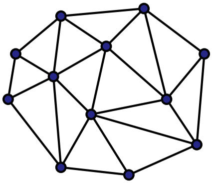

Question: How many are there?

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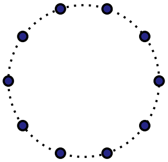
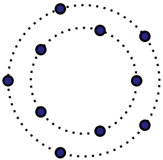
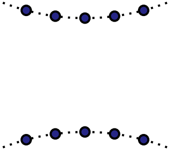


convex position



double circle



double chain

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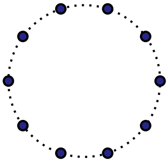

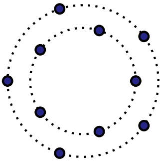

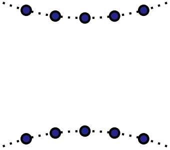

pm(P)	$\approx 2^n$	$\approx 2^n$	$\approx 3^n$

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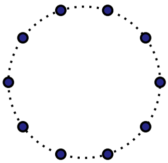

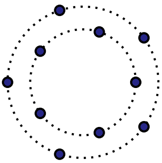

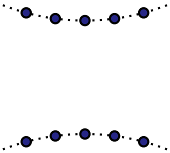

tr(P)	$\approx 4^n$	$\approx 3.464^n$	$\approx 8^n$

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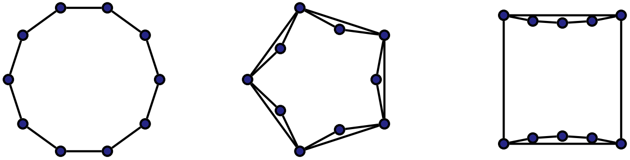

Easy to analyze because of *unavoidable* edges

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Theorem (Sharir, Sheffer 11)

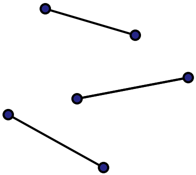
The number of triangulations on any P is at most 30^n.

Theorem (Dumitrescu, Schulz, Sheffer, Tóth 11)

There are sets of n points with at least 8.65^n triangulations.

Trapezoidal Diagrams

Trapezoidal Diagram:



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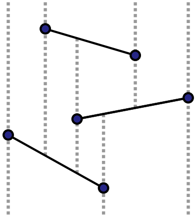
- Augment plane graph with its trapezoidal decomposition.



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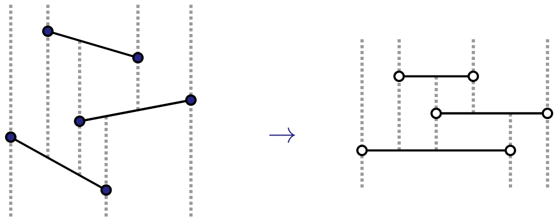

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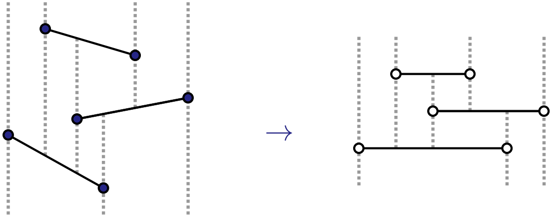

Intuition: Every edge knows the order of vertices that it sees above, and the order of vertices that it sees below, but not more.

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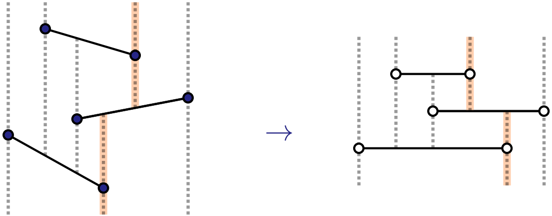

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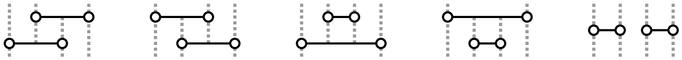

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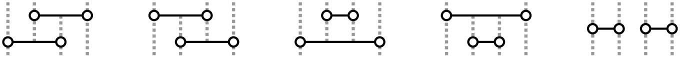

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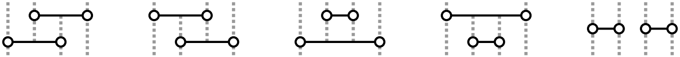

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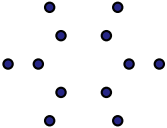

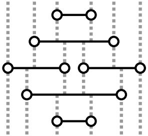

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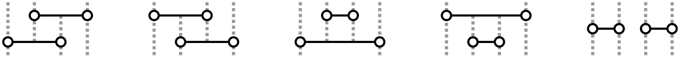

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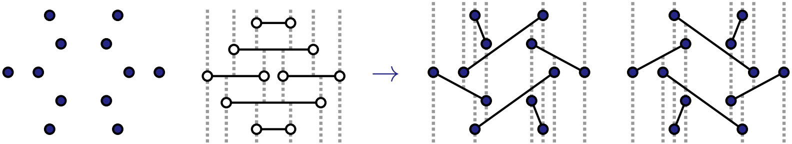

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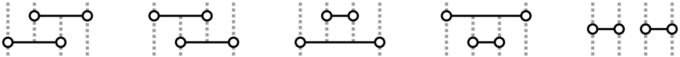
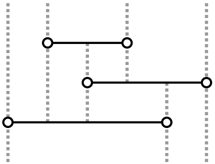
- **Interpretation 3:** Trapezoidal diagrams (of perfect matchings) with $n = 2k$ points.



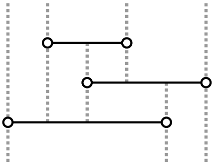
Bijection from Diagrams to Bracket Expressions



$\mapsto$?

Bijection from Diagrams to Bracket Expressions

- Enumerate trapezoids "bottom-up" and "left-to-right".



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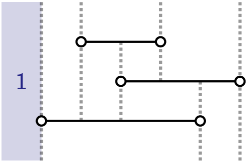

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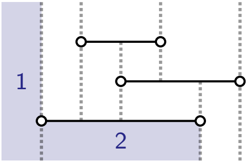

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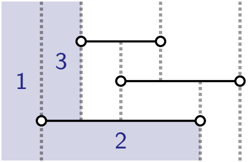

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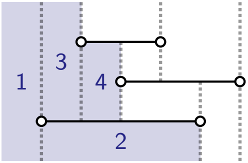

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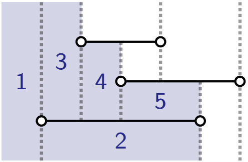

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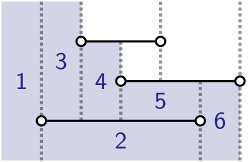

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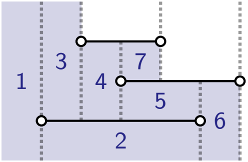

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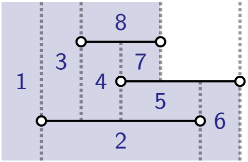

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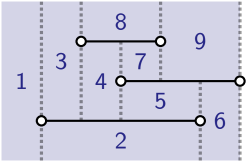

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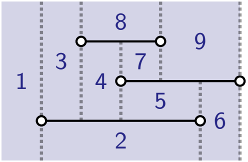
Bijection from Diagrams to Bracket Expressions

- Enumerate trapezoids "bottom-up" and "left-to-right".
- Associate each trapezoid with edge that defines right boundary.



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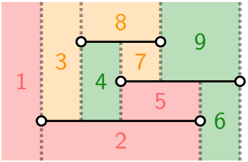

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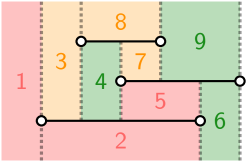

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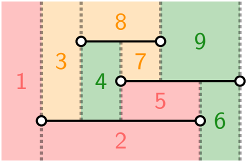

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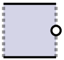
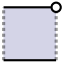
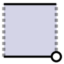
Simple Rule:

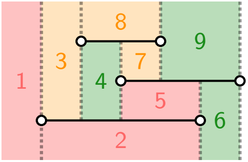
$\langle$ = 

$|$ = 

$\rangle$ = 

Bijection from Diagrams to Bracket Expressions

- Enumerate trapezoids "bottom-up" and "left-to-right".
- Associate each trapezoid with edge that defines right boundary.
- Apply simple substitution rule.



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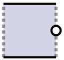
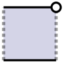
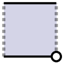
$\langle =$  $| =$  $\rangle =$ 

Mapping from Bracket Expression to Trap. Diagram

$$\langle | \langle \langle \rangle | | \rangle \rangle \qquad \mapsto \qquad ?$$

Mapping from Bracket Expression to Trap. Diagram

- Go over symbols and reconstruct diagram using the simple rule.

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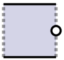
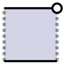
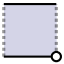
$\langle =$  $| =$  $\rangle =$ 

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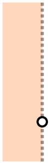
$\langle | \langle \langle \rangle | | \rangle \rangle \quad \mapsto$ 

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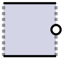
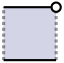
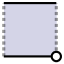
$\langle =$  $\quad | =$  $\quad \rangle =$ 

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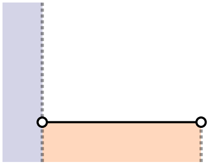

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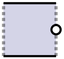
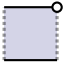
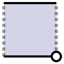
$\langle =$ 

$| =$ 

$\rangle =$ 

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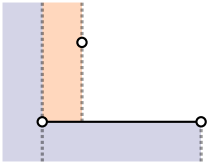
$\langle | \langle \langle \rangle | | \rangle \rangle$ $\mapsto$ 

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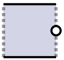
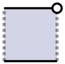
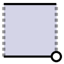
$\langle =$ 

$| =$ 

$\rangle =$ 

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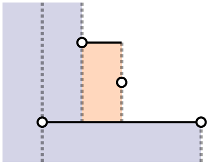

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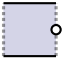
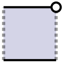
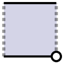
$\langle =$ 

$| =$ 

$\rangle =$ 

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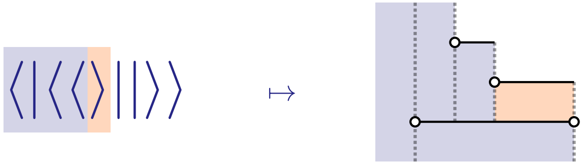

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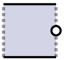
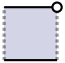
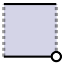
$\langle =$ 

$| =$ 

$\rangle =$ 

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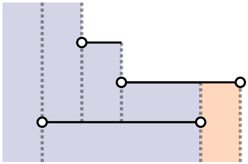

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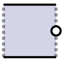
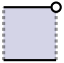
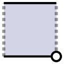
$\langle =$ 

$| =$ 

$\rangle =$ 

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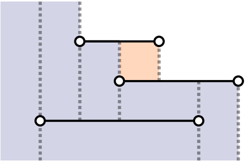

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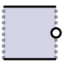
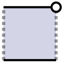
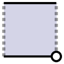
$\langle =$ 

$| =$ 

$\rangle =$ 

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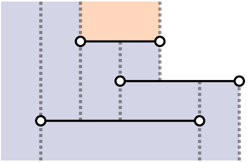

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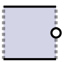
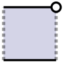
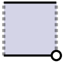
$\langle =$ 

$\mid =$ 

$\rangle =$ 

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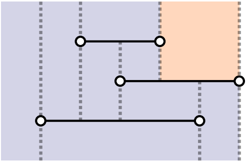

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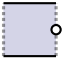
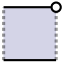
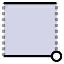
$\langle =$ 

$| =$ 

$\rangle =$ 

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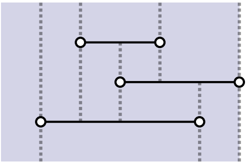

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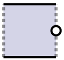
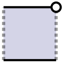
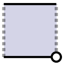
$\langle =$ 

$| =$ 

$\rangle =$ 

Number of Diagrams of Perfect Matchings

Theorem

The number of trapezoidal diagrams of perfect matchings over $n = 2k$ points is

$$C_k^{(3)} \approx 5.196^n.$$

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There is a set P of n points and a trapezoidal diagram with approximately 1.071^n embeddings on P.

Theorem (Sharir, Welzl 06)

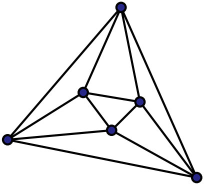
The number of plane perfect matchings on any set of n points is at most 10.05^n.

Theorem (Asinowski, Rote 15)

There are sets of n points with at least 3.093^n plane perfect matchings.

Upward Triangulations

Trapezoidal Diagram (of triangulation):



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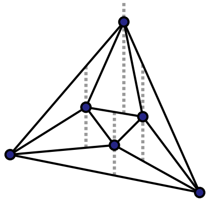

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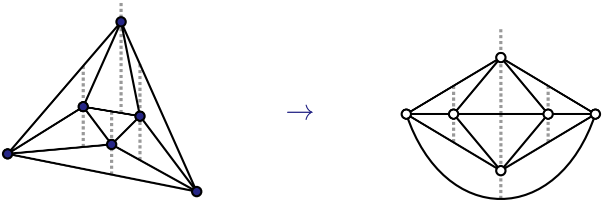

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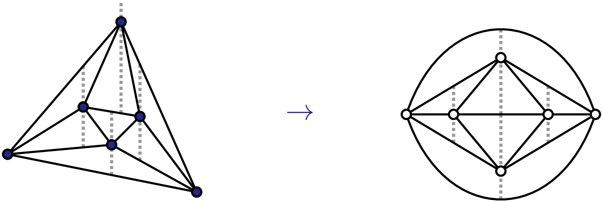

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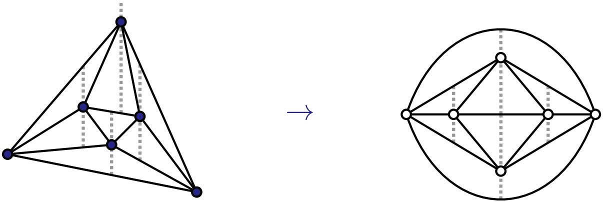
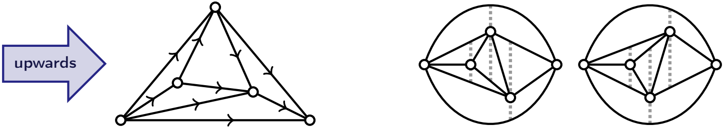


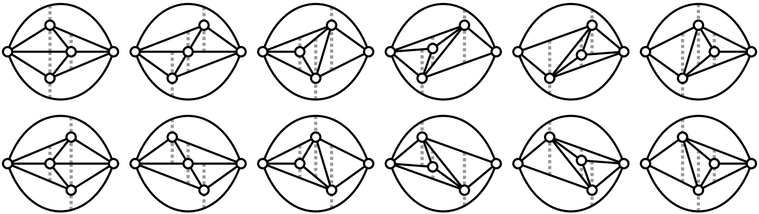
Remark: Abstract upward triangulations correspond to 1 or 2 diagrams.



The Roadmap

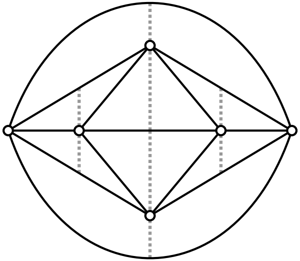
- **Step 1:** Calculate the number of diagrams.

n	2	3	4	5	6	7	8	...
#	1	1	2	12	107	1178	14805	...



- **Step 2:** Bound the number of embeddings.

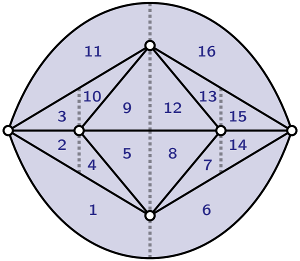
Injection from Diagrams to Bracket Expressions



$\mapsto$?

Injection from Diagrams to Bracket Expressions

- Enumerate trapezoids "bottom-up" and "left-to-right".



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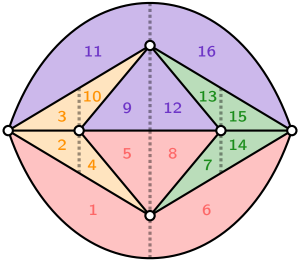
Injection from Diagrams to Bracket Expressions

- Enumerate trapezoids "bottom-up" and "left-to-right".
- Associate trapezoids with vertex that defines vertical boundary.



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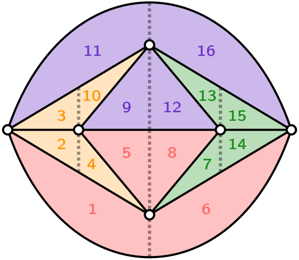

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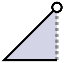
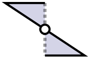
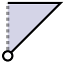
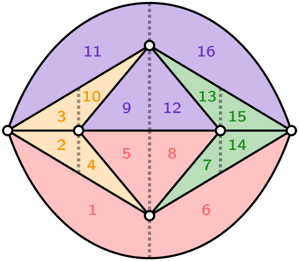
$\langle$ = 

$|$ = 

$\rangle$ = 

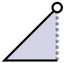
Injection from Diagrams to Bracket Expressions

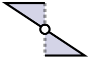
- Enumerate trapezoids “bottom-up” and “left-to-right”.
- Associate trapezoids with vertex that defines vertical boundary.
- Apply simple substitution rule.

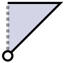


$\mapsto$ $\langle \langle | | \langle \rangle \langle \rangle | | \rangle \rangle$

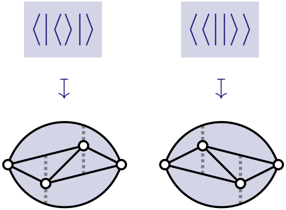
Simple Rule:

$\langle =$ 

$| =$ 

$\rangle =$ 

Prime Catalan Numbers



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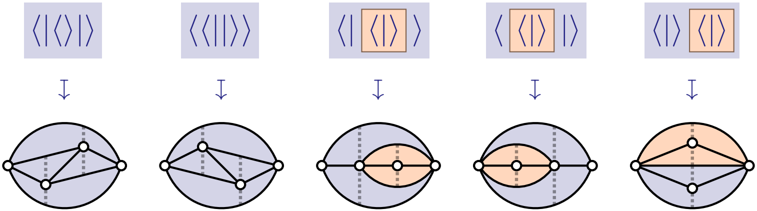

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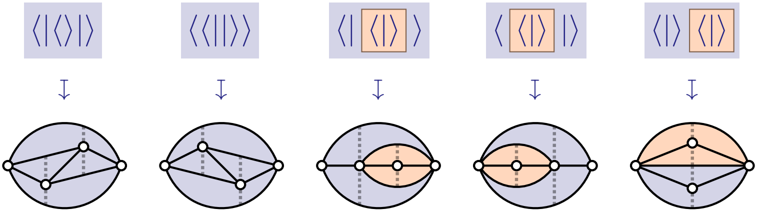

- **Prime Catalan Numbers:** Let $P_k^{(3)}$ be the number of bracket expressions which do not contain any bracket expressions as subsequences.

Number of Diagrams of Triangulations

Theorem

The number of trapezoidal diagrams of triangulations (a.k.a. upward triangulations) over $n = k + 2$ *points is*

$$P_k^{(3)} \approx 23.459^n.$$

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Theorem (Sharir, Sheffer 11)

The number of triangulations on any set of n points is at most 30^n.

Theorem (Dumitrescu, Schulz, Sheffer, Tóth 11)

There are sets of n points with at least 8.65^n triangulations.

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Each balanced bracket expression c *can be written as a unique combination of an expression* p *of size* s *without balanced subexpressions, and an ordered sequence of* $3s$ *expressions* $c_1, \ldots, c_{3s}$.

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$p = \langle\,|\,\langle\,\rangle\,|\,\rangle$ $\quad c_1 = \boxed{\langle\,|\,\rangle}$ $\quad c_2 = \varepsilon$ $\quad c_3 = \boxed{\langle\,|\,\rangle}$

$s = 2$ $\quad c_4 = \varepsilon$ $\quad c_5 = \boxed{\langle\,|\,\langle\,|\,\rangle\,\rangle}$ $\quad c_6 = \boxed{\langle\,|\,\rangle}$

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The power series defined above satisfy

$$C(x) = P(xC(x)^3).$$

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Proof.

$$C(x) = \sum_{c} x^{|c|} = \sum_{p} \sum_{\substack{c_1,\ldots,c_{3s} \\ s=|p|}} x^{|p|+|c_1|+\cdots+|c_{3s}|} = P(xC(x)^3)$$

Asymptotics

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Theorem

The radius of convergence of $P(x)$ is $1/\beta$, and the following limit exists

$$\lim_{k \to \infty} \sqrt[k]{P_k^{(3)}} = \beta.$$

Questions?