Edge–colored graphs as higher-dimensional maps

Valentin Bonzom
with Luca Lionni

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GT Combi LIX

Discretization of manifolds

- 2D discrete surfaces: triangulations, p–angulations and combinatorial maps
- 3D triangulations: gluings of tetrahedra

Discretization of manifolds

- 2D discrete surfaces: triangulations, p–angulations and combinatorial maps
- 3D triangulations: gluings of tetrahedra
- How to represent them in a suitable fashion for combinatorics?
- Equivalent of p–angulations?
- Enumeration?

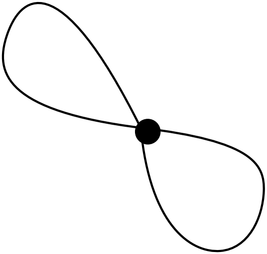
Colored triangulations and colored graphs

Colored triangulations and colored graphs

Bijection with (stuffed colored Walsh) maps

Combinatorial maps

Graph with cyclic ordering of edges incident to each vertex



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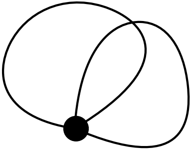

Combinatorial maps

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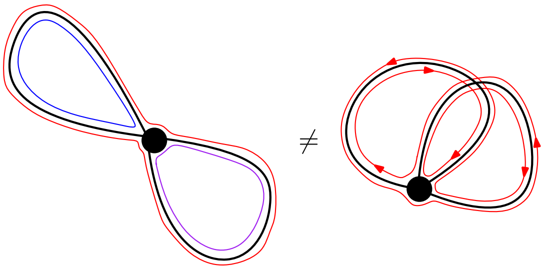
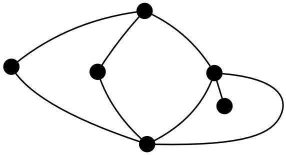


Cyclic ordering defines faces: follow the corners

$2p$–angulation

- Faces of degree $2p$
- Duality: vertices of degree $2p$

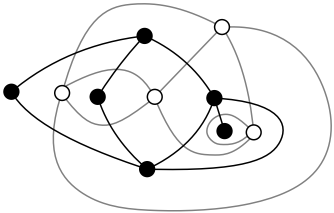


- Euler's relation with $E(M) = pV(M)$

$$F(M) - E(M) + V(M) = F(M) - (p-1)V(M) = 2 - 2g(M)$$

- $g(M) \geq 0 \quad \Rightarrow \quad$ bound on $F(M)$ linear in $V(M)$
- Maximizing $F(M)$ at fixed $V(M)$ equivalent to $g(M) = 0$

$2p$–angulation



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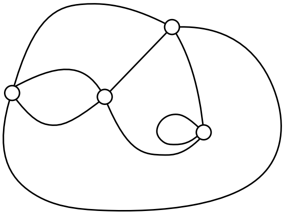

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What do we know?

Maps: from Tutte to today

- Enumeration [Tutte's equations, matrix models]
- Bijections [Cori-Vauquelin-Schaeffer, Bouttier-Di Francesco-Guitter]
- Topological recursion [Eynard]
- Continuum limit [Brownian sphere]
- More being developed nowadays [Hurwitz, integrable hierarchies, etc.]

What do we know?

Simplicial complexes in higher dim

- ??

What do we know?

Simplicial complexes in higher dim

- ??
- (Mostly) numerical works (Ambjorn, Jurkewic, Jonsson, Loll, etc.)
- Why? Combinatorics difficult to control!
- Most recent analytical attempts via

  colored graphs
- Mostly by physicists
  [Gurau, Krajewski, Rivasseau, Tanasa, Vignes-Tourneret and students]
- Try a more systematic combinatorial study
  [Gurau-Schaeffer, Bonzom-Lionni and wip w/ Monteil]

The physics

- Einstein's 2nd revolution

$$\text{Gravitation} = \text{Geometry of space–time}$$

- Quantum physics

$$\text{Quantum} = \text{probabilistic, random}$$

Gravitation and quantum together

Space–time metric is a random variable

$$\text{Quantum gravity} = \text{random geometry}$$

Discrete quantum gravity

Define quantum gravity at the discrete level

Two approaches

Discrete quantum gravity

Define quantum gravity at the discrete level

Two approaches

Regge calculus, LQG, Spin foams	Dynamical triangulations
Fix a discretization	Edges have fixed lengths
Geometry $=$ edge lengths	Geometry $=$ discretization
Quantization $= \int \prod_e d\ell_e$	Quantization
or $\sum_{\text{quantum numbers}}$	$\sum_{\text{Geometries}} = \sum_{\text{Triangulations}}$

In 2nd approach, $\sum_{\text{Triangulations}} \Rightarrow$ generating function!

How to represent triangulations?

Triangulations

- Gluing of simplices (tetrahedra, pentachora, etc.)
- Defined by attaching maps
- Ensemble of triangulations defined by constraints on attaching maps

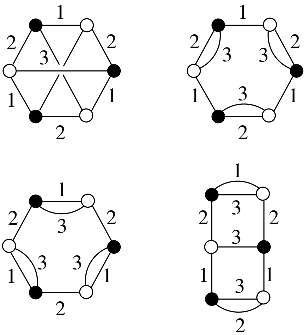
Various ensembles

- Various ensembles in topology (simplicial, CW, Δ–complexes, etc.)
- Not suitable for combinatorics (too wild)
- Digging through old work, found colored triangulations
  [Italian school: crystallization, graph–encoded manifold]
- Represented by edge–colored graphs

Colored graphs

$(d+1)$-colored graphs

- Bipartite graphs
  black and white vertices
- Edges colored with $d+1$ possible colors
- Vertices of degree $d+1$
- All colors incident exactly once at each vertex



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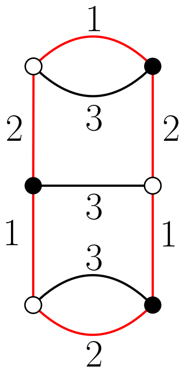

Faces are closed cycles with only two colors.

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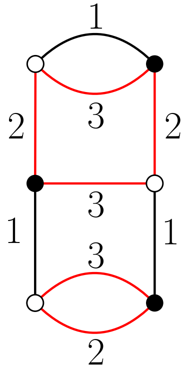

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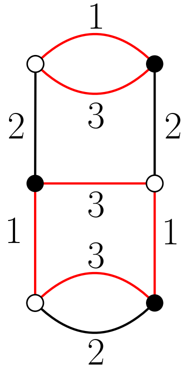
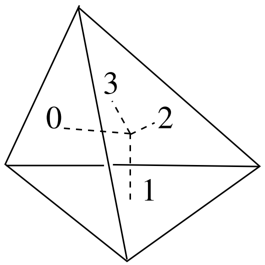


Faces are closed cycles with only two colors.

Triangulations from colored graphs

$$\text{duality} \quad \begin{cases} \text{vertex} & \rightarrow & d\text{–simplex} \\ \text{edge} & \rightarrow & (d-1)\text{–simplex} \\ \text{face} & \rightarrow & (d-2)\text{–simplex} \\ k\text{-bubble} & \rightarrow & (d-k)\text{–simplex} \end{cases}$$

- Boundary triangles labeled by a color $c = 0, \ldots, d$



Colors identify all sub-simplices

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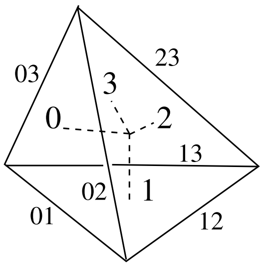
- Boundary triangles labeled by a color $c = 0, \ldots, d$
- Induced colorings
- Edges labeled by pair of colors



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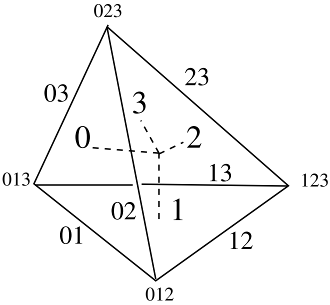
$$

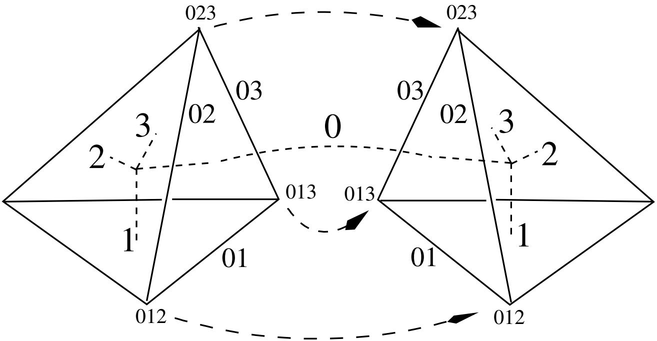
- Boundary triangles labeled by a color $c = 0, \dots, d$
- Induced colorings
- Edges labeled by pair of colors
- Nodes labeled by three colors



Colors identify all sub-simplices

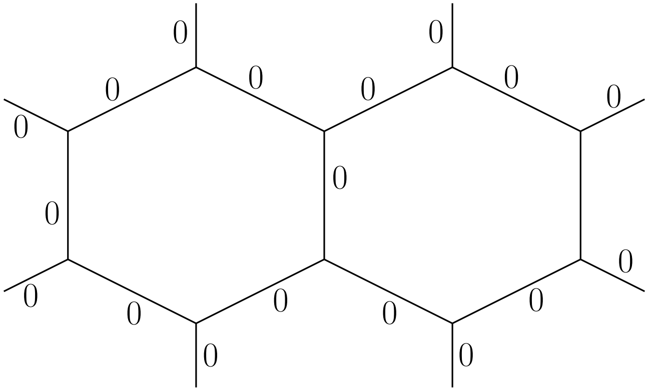
Colored attaching maps

Gluing respecting all induced colorings



Theory of crystallization and GEMs (graph–encoded manifolds): $(d+1)$-colored graphs are dual to triangulations of pseudo-manifolds of dimension d [Pezzana, Ferri, Cagliardi, Lins].

The 2D case



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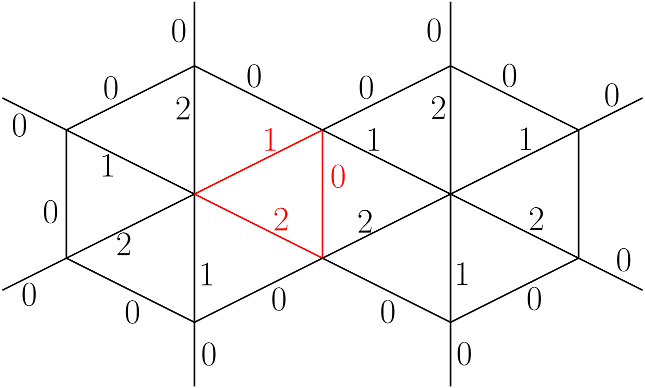

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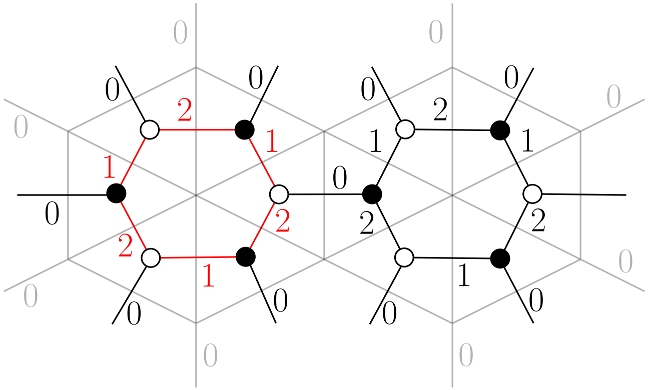

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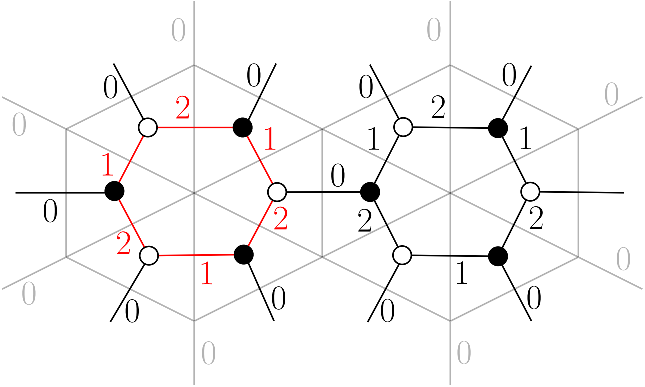
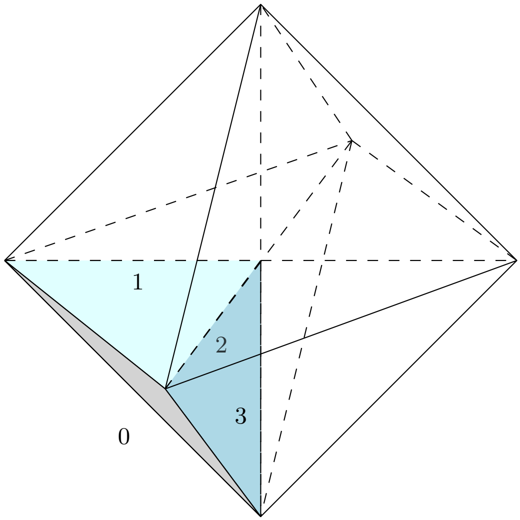


2p–angle

- Gluing of 2p triangles with boundary of color 0
- Dually: Components with all colors but 0

The 3D case



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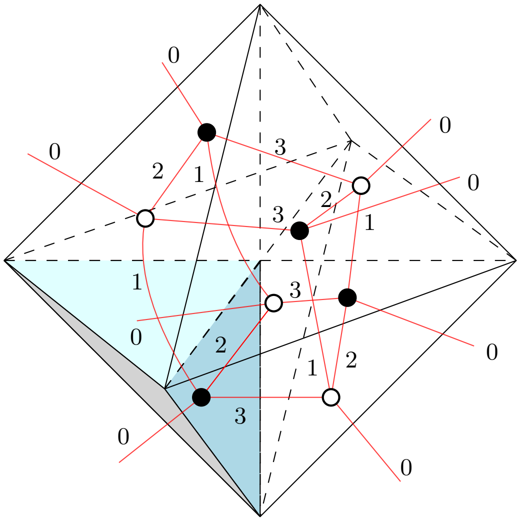

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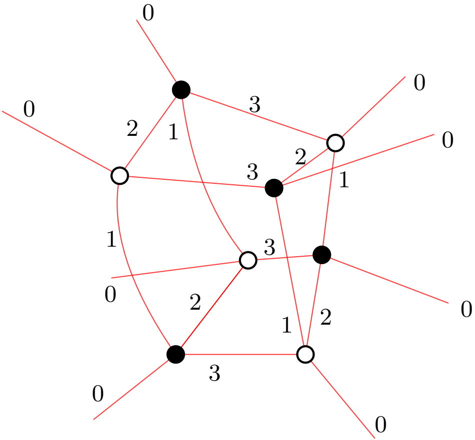

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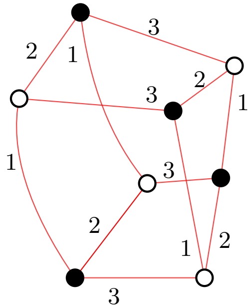

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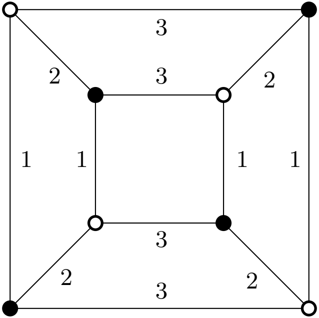
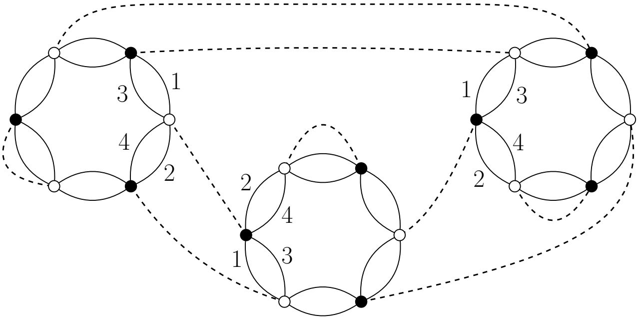


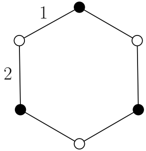
Bubbles

- Colored graph with colors $0, 1, \ldots, d$
  (triangulation in dim d)
- Bubble: connected piece with colors $1, \ldots, d$
  Obtained by removing the color 0
- All graphs obtained by gluing bubbles along edges of color 0
- $\mathcal{G}(B)$ set of $(d+1)$-colored graphs where all bubbles are B



Bubbles II

- 2D: only bubbles with $2p$ vertices
  Cycles of colors $(1,2)$



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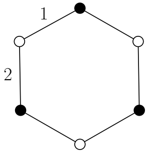
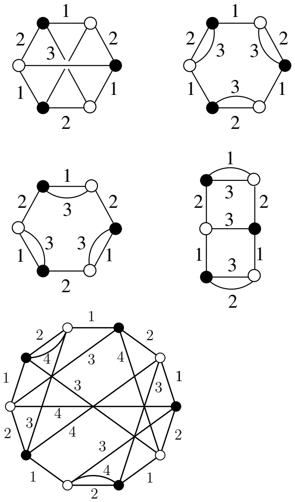
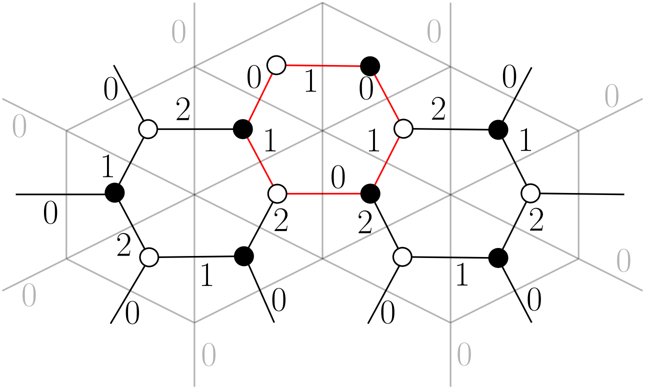


- Many more in higher dimensions
- Vast world to explore



Faces



Vertices: two types

- cycle with colors $(0, 1)$
- cycle with colors $(0, 2)$

The problem

- Set B a bubble, $G \in \mathcal{G}(B)$
- Enumerate w.r.t.
  - # bubbles $b(G)$
  - # subsimplices of codimension 2 which belong to bubble boundary
- Face of colors $(0, c)$: cycle with colors $(0, c)$

$$\text{Number of faces} \qquad F(G) = \sum_{c=1}^{d} F_{0c}(G)$$

- Classify graphs according to $F(G)$ at fixed $b(G)$

$$\mathcal{G}_b(B) = \bigcup_F \mathcal{G}_b^{(F)}(B)$$

- Focus on $\mathcal{G}_b^{(F)}(B)$
  How to maximize $F(G)$ at fixed number of bubbles $b(G)$?

Gurau's degree theorem

Bound on $F(G)$

There exists $\omega(G) \geq 0$

$$F(G) - (d-1)(p(B)-1)b(G) = d - \omega(G) \leq d$$

- $d = 2$
  $F(G) - (p(B)-1)b(G) = 2 - \omega(G) \quad \Rightarrow \quad \omega(G) = 2g(G)$
- For $d \geq 3$, bound can be saturated only for certain type of bubbles
- Maximizing graphs (melonic) are series–parallel
  - Bijection with trees
  - Expected from numerics
- Gurau–Schaeffer classification according to the degree
- Need to investigate more generic bubbles

Colored triangulations and colored graphs

Bijection with (stuffed colored Walsh) maps

Mechanism

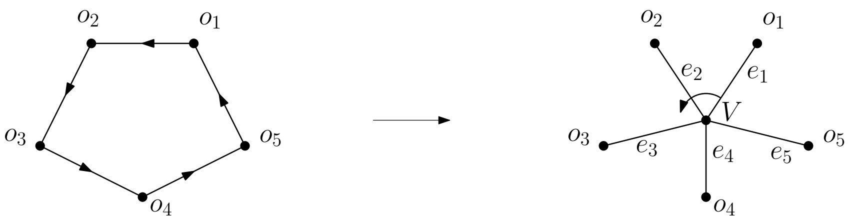
- How to control faces?

Mechanism

- How to control faces? Maps!
- Same mechanism as Tutte's bijection between bipartite quadrangulations and generic maps, in the dual picture

Cycle of graph to star–maps

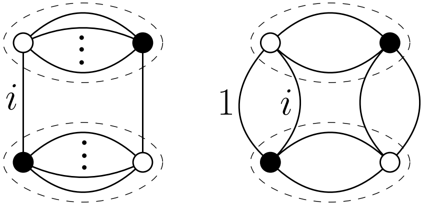
Cyclically ordered list of objects $(o_1, \ldots, o_n)$



Edge (o_k, o_{k+1}) maps to corner between e_k and e_{k+1}

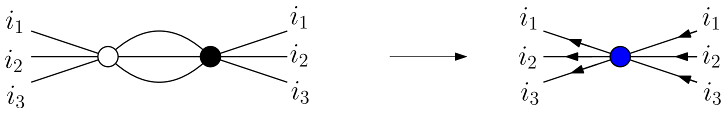
From bubble to map

- Choose a pairing of B



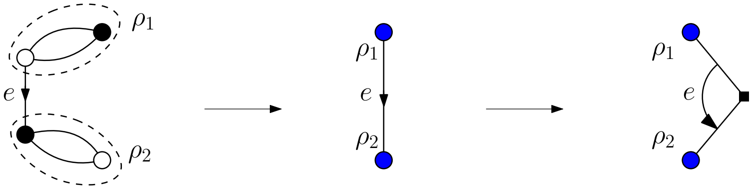
From bubble to map

- Choose a pairing of B
- Orient edges from white to black; merge pairs to blue vertices
- Oriented cycle of color $j \to$ counter–clockwise star–map

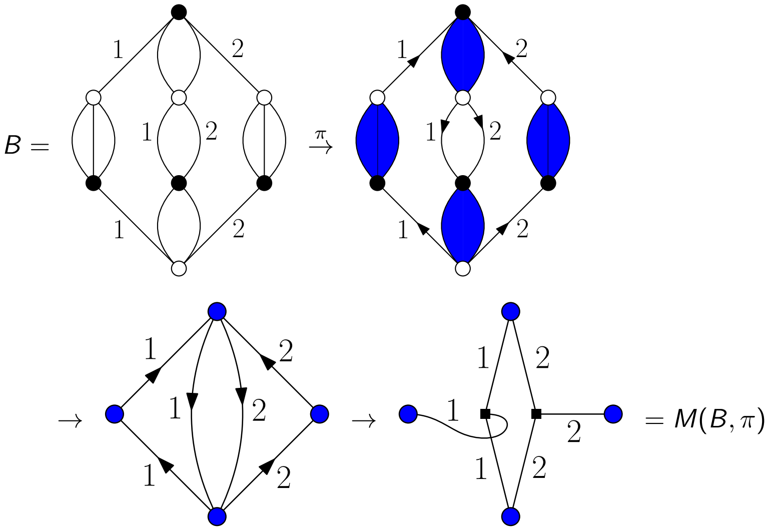


From bubble to map

- Choose a pairing of B
- Orient edges from white to black; merge pairs to blue vertices
- Oriented cycle of color $j \to$ counter–clockwise star–map
- Edge of color $j \to$ counter–clockwise corner of color j

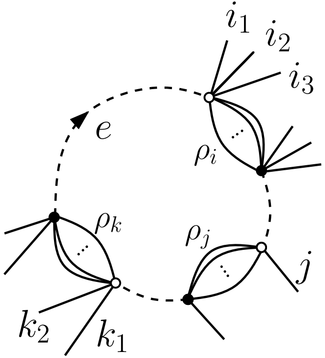


Example

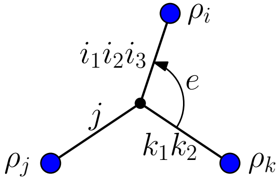


Universal part

Cycles of color 0 and pairs of vertices $\rightarrow$ counter–clockwise star–map



$\rightarrow$

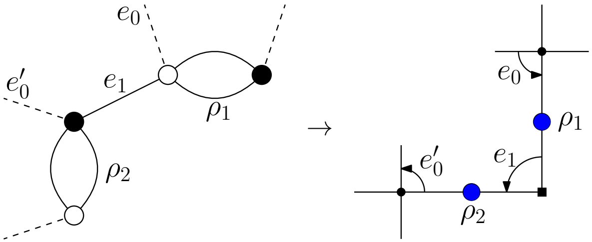


Use $M(B, \pi)$ as "*colored hyper–edge*"

Faces

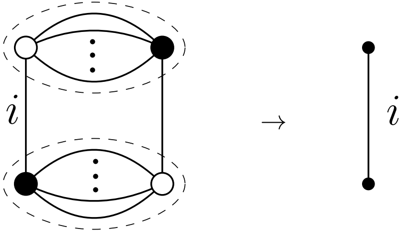
- Black vertices of arbitrary degree
- Blue vertices and box–vertices form

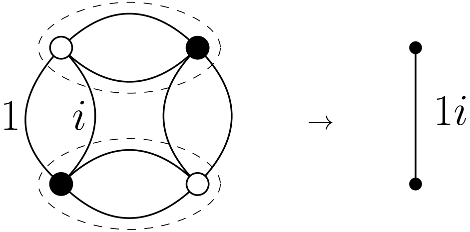
Cycle of colors $(0c)$ of graph $\to$ face of color c in map



Quartic case, $d = 4$

Simplification!





Quartic case, $d = 4$

- Maps of arbitrary degree
- Monocolored edges, colors $1, \dots, d$
- Bicolored edges, colors $1c$ for $c = 2, 3, 4$

Maximizing faces $\sum_{c=1}^{d} F_{0c}$

- Monocolored edges are bridges
- Bicolored form planar components
- Bicolored types $1c$ and $1c'$ touch on cut–vertices (similar to $O(n)$ model on planar maps)

The quartic case

- Generating function of (rooted) maps for k types of bicolored edges

$$f_k(t,\lambda) = \sum_M t^{\#\text{edges}} \; \lambda^{\#\text{monocol. edges}}$$

- Algebraicity

$$f_k(t,\lambda) = 1 - k + t\lambda f_k(t,\lambda)^2 + kP(tf_k(t,\lambda)^2)$$

implies

$$\begin{cases} tf^2 = u(1-u)^2 \\ f = k(1-u)(1+3u) - k + 1 + \lambda u(1-u)^2 \end{cases}$$

- Generic planar maps for $\lambda = 0$ and $k = 1$

$$27t^2 A(t)^2 + (1-18t)A(t) + 16t - 1 = 0$$

Explicit singularity analysis for $k = 1$

- Quartic eq on $f(t, \lambda)$
- For $\lambda < 3$, singularity at $t_1(\lambda) = \frac{27}{4(\lambda+9)^2}$

$$f(t,\lambda) = \frac{4}{27}(\lambda+9) + \frac{16(\lambda+3)(\lambda+9)^3}{729(\lambda-3)}(t_1(\lambda)-t) + \frac{64(\lambda+9)^{11/2}}{6561(3-\lambda)^{5/2}}(t_1(\lambda)-t)^{3/2} + o\big((t_1(\lambda)-t)^{3/2}\big)$$

- For $\lambda > 3$, singularity at $t_2(\lambda) = \frac{\lambda}{4(1+\lambda)^2}$

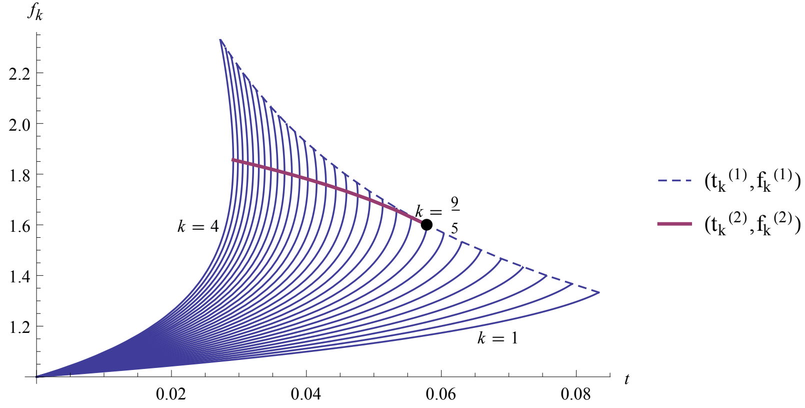
$$f(t,\lambda) = 2\frac{\lambda^2-1}{\lambda^2} - \frac{4(1+\lambda)^2}{\lambda^{5/2}}\sqrt{\lambda^2-2\lambda-3}\,(t_2(\lambda)-t)^{1/2} + o\big((t_2(\lambda)-t)^{1/2}\big)$$

- $\lambda = 3$, proliferation of baby universes

$$f(t,\lambda=3) = \frac{16}{9} - \frac{128}{3^{5/3}}\left(\frac{3}{64}-t\right)^{2/3} + o\left(\left(\frac{3}{64}-t\right)^{2/3}\right)$$

Same results with respect to k

- k small enough: universality class of maps
- k large enough: branching process and square–root singularity
- k critical: singularity exponent $2/3$



Conclusion

- (At least some) Enumeration is feasible in dim $d > 2$!
- To appear with L. Lionni: enumeration of gluings of octahedra which maximize the number of edges
- Beyond maximizing number of faces in quartic case
  $\rightarrow$ Topological recursion! [to appear w/ S. Dartois]
- More to be studied
- Harer–Zagier formula equivalent for unicellular maps?