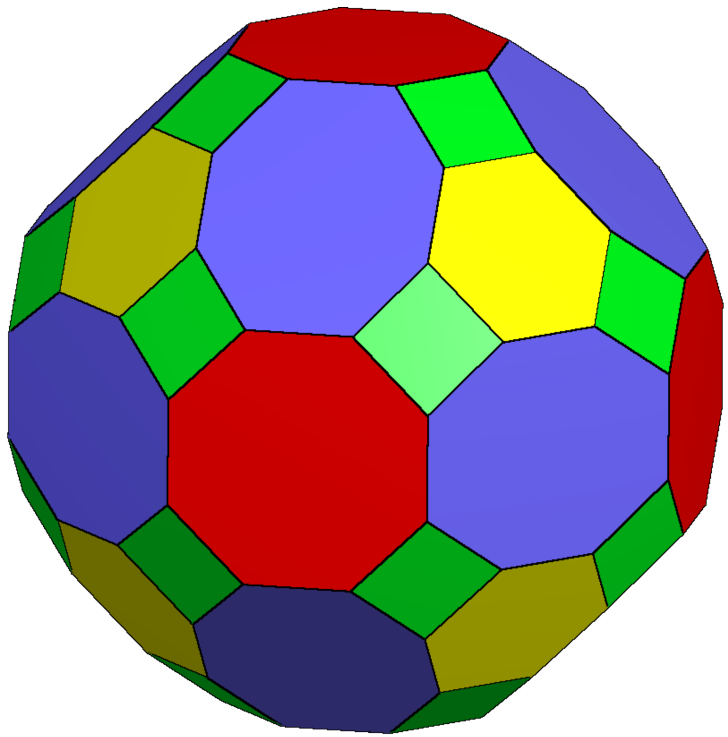
Euler polytopes and convex matroid optimization



Antoine Deza, McMaster

based on joint works with

George Manoussakis, University Paris Sud

Shmuel Onn, Technion

McMaster University

Advanced Optimization Laboratory

Séminaire Modèles Combinatoires, LIX, March 2016

Linear Optimization?

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** matrix ***A***
find, in any, a ***d***-dimensional vector ***x*** such that :

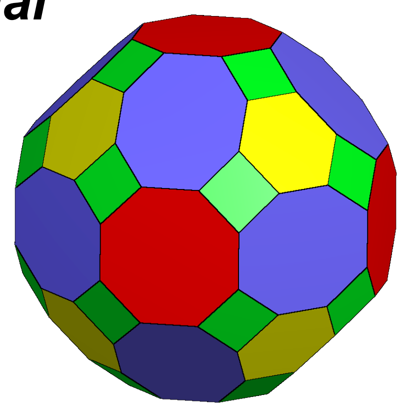
Ax* = *b	***Ax* = *b*** ***x*** ≥ 0
linear algebra	**linear optimization**

Linear Optimization?

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** matrix ***A*** find, in any, a ***d***-dimensional vector ***x*** such that :

$\boldsymbol{Ax} = \boldsymbol{b}$	$\boldsymbol{Ax} = \boldsymbol{b}$ $\boldsymbol{x} \geq 0$
linear algebra	**linear optimization**

Can linear optimization be solved in ***strongly polynomial*** time? is listed by Smale (Fields Medal 1966) as one of the top mathematical problems for the XXI century



Strongly polynomial *:* algorithm ***independent*** from the ***input data length*** and polynomial in ***n*** and ***d****.*

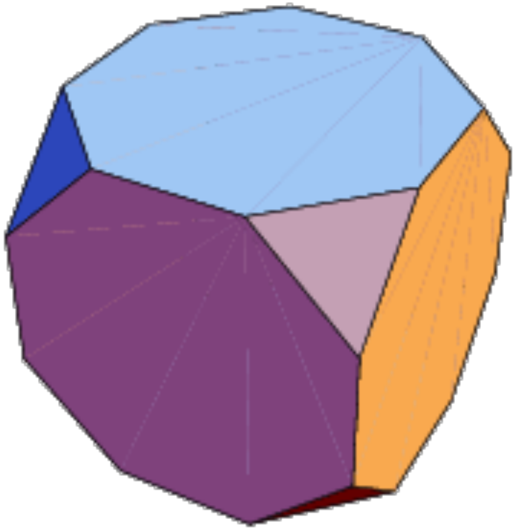
Lattice polytopes with large diameter

lattice (***d***,***k***)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope ***P*** : smallest number such that **any two vertices** of ***P*** can be connected by a **path with at most** $\delta(P)$ **edges**

$\delta(d,k)$: largest diameter over all **lattice** (***d***,***k***)-polytopes

ex. $\delta(3,3) = 6$ and is achieved by the ***truncated cube***



Lattice polytopes with large diameter

lattice (***d***,***k***)-polytope : convex hull of points drawn from $\{0,1,\dots,k\}^d$

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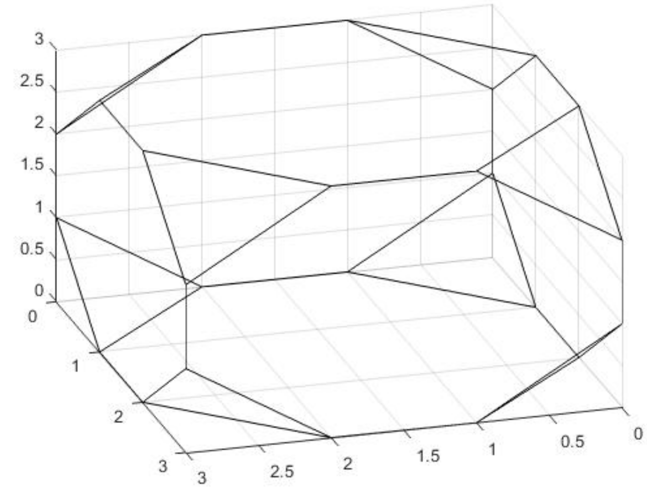


Lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

$\delta(d,1) = d$	[Naddef 1989]
$\delta(2,k) = O(k^{2/3})$	[Balog, Bárány 1991]
$\Rightarrow \delta(d,k) = \Omega(k^{2/3} d)$	[Del Pia, Michini 2013]
$\delta(d,k) \leq kd$	[Kleinschmid, Onn 1992]
$\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k)$	[Thiele 1991] [Acketa, Žunić 1995]
$\delta(d,2) = \lfloor 3d/2 \rfloor$	[Del Pia, Michini 2015]
$\delta(d,k) \leq kd - \lceil d/2 \rceil$ for $k \geq 2$	[Del Pia, Michini 2015]

Lattice polytopes with large diameter

$\delta(d,k)$		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	?	?	?	?	?	?
	4	4	6	?	?	?	?	?	?	?
	5	5	7	?	?	?	?	?	?	?

$\delta(d,k) = \Omega(k^{2/3}\, d)$ [Del Pia, Michini 2015]

Lattice polytopes with large diameter

$\delta(d,k)$		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2	3	**4**	4	5	6	6	7	8
	3	3	4	6	?	**9+**	?	?	?	?
	4	4	6	?	?	?	?	**16+**	?	?
	5	5	7	?	?	?	?	?	?	**25+**

$\delta(d,k) = \Omega(k^{2/3} d)$ [Del Pia, Michini 2015]

$\delta(d,k) \geq (k+1)d/2$ for infinitely many d for each odd k [Deza, Manoussakis, Onn 2015]

Conjecture: $\delta(d,k) \leq (k+1)d/2$

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	**4**	4	5	6	6	7	8
	3	3	4	6	?	**9+**	?	?	?	?
	4	4	6	?	?	?	?	**16+**	?	?
	5	5	7	?	?	?	?	?	?	**25+**

$\delta(d,k) = \Omega(k^{2/3} d)$ [Del Pia, Michini 2015]

$\delta(d,k) \geq k(d-k)/2$ [Deza, Manoussakis, Onn 2015]

Conjecture: $\delta(d,k) \leq (k+1)d/2$

Lattice polytopes with many vertices

Motivation : convex matroid optimization [Melamed, Onn 2014]

The optimal solution of max { **f**(**W***x*) : $x \in$ **S**} is attained at a vertex of the projection integer polytope in $\mathbb{R}^d$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^n$ (in the talk **S** $\in \{0,1\}^n$)
W : integer ***d*** x ***n*** matrix (in the talk **W** is {0,1}-valued)
f : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Q. What is the maximum number **v**(***d***,***n***) of vertices of conv(**WS**) when **S** $\in \{0,1\}^n$ and **W** is a {0,1}-valued ***d*** x ***n*** matrix ?

Obviously v(*d*,*n*) ≤ |**WS**| = $O(n^d)$
In particular v(2,*n*) = $O(n^2)$, and v(2,*n*) = $\Omega(n^{0.5})$

Lattice polytopes with many vertices

Motivation : convex matroid optimization [Melamed, Onn 2014]

S : set of feasible point $\in \mathbb{Z}^n$ (in the talk **S** $\in \{0,1\}^n$)
W : integer d x n matrix (in the talk **W** is {0,1}-valued)
f : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Assume **S** in $\{0,1\}^n$ is a ***matroid*** of order n; that is, the set of indicating vectors of bases of a matroid with ground set $\{1,\ldots,n\}$

Given a matroid **S** of order n, {0,1}-valued d x n matrix **W**, the maximum number **m**(d) of vertices of conv(**WS**) is independent of n and **S**

Given a matroid **S** of order n, $\{0, \pm1,\ldots,\pm p\}$-valued d x n matrix **W**, the maximum number **m**(d,p) of vertices of conv(**WS**) is independent of n and **S**

Lattice polytopes with many vertices

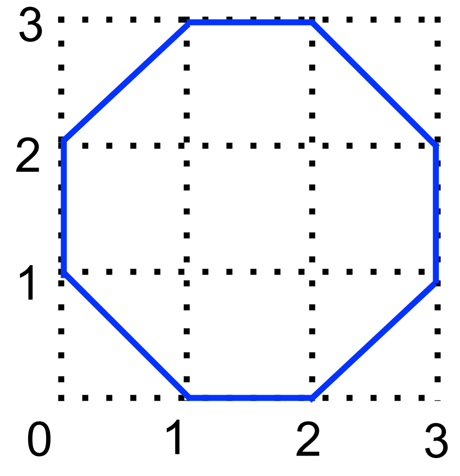
Motivation : convex matroid optimization [Melamed, Onn 2014]

Given a matroid **S** of order ***n***, {0,1}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***) of vertices of conv(**WS**) is independent of ***n*** and **S**

Example : the maximum number **m**(2) of vertices of a planar projection conv(**WS**) of matroid **S** by a binary matrix **W** is attained by the following matrix and uniform matroid of rank 3 and order 8:

W = $\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$

S = U(3,8) = $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$



conv(**WS**)

Lattice polytopes with many vertices

Motivation : matroid optimization

m(d): maximum number of vertices of conv(**WS**) over matroid **S** of order n, and {0,1}-valued d x n matrix **W**

$$d\,2^d \leq \mathbf{m}(d) \leq 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$$

24 ≤ **m**(3) ≤ 158
64 ≤ **m**(4) ≤ 19840

[Melamed, Onn 2014]

Lattice polytopes with many vertices

Motivation : matroid optimization

m(d): maximum number of vertices of conv(**WS**) over matroid **S** of order n, and {0,1}-valued d x n matrix **W**

$$d\,2^d \le \mathbf{m}(d) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$$

$$2d! \le \mathbf{m}(d) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i} - 2\binom{(3^{d-1}-3)/2}{d-1}$$

24 ≤ **m**(3) ≤ 158
64 ≤ **m**(4) ≤ 19840

48 ≤ **m**(3) ≤ **96**
672 ≤ **m**(4) ≤ **5376**

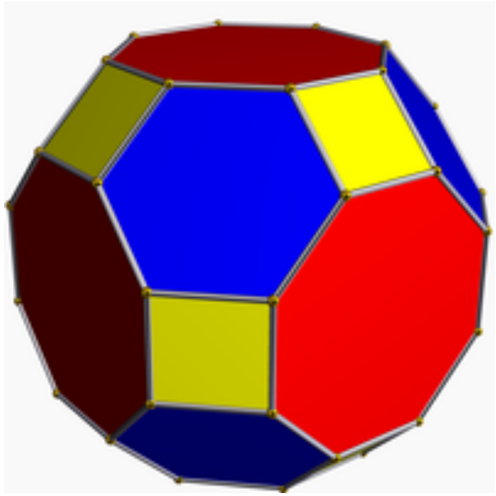
[Melamed, Onn 2014]

[Deza, Manoussakis, Onn 2015]

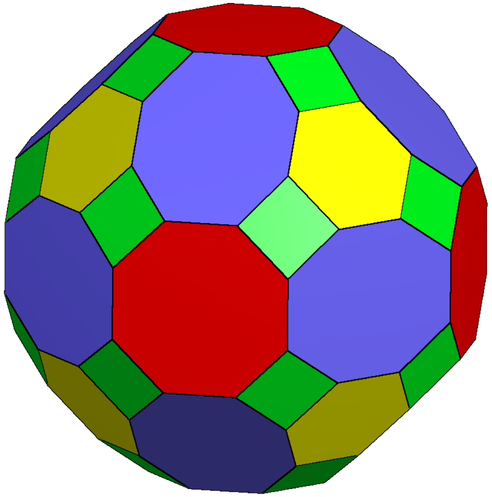
Lattice polytopes with many vertices

m(*d*): maximum number of vertices of conv(**WS**) over matroid **S** of order *n*, and {0,1}-valued *d* x *n* matrix **W**

48 ≤ **m**(3) ≤ **96** [Deza, Manoussakis, Onn 2015]



truncated cuboctahedron
(great rhombicuboctahedron)



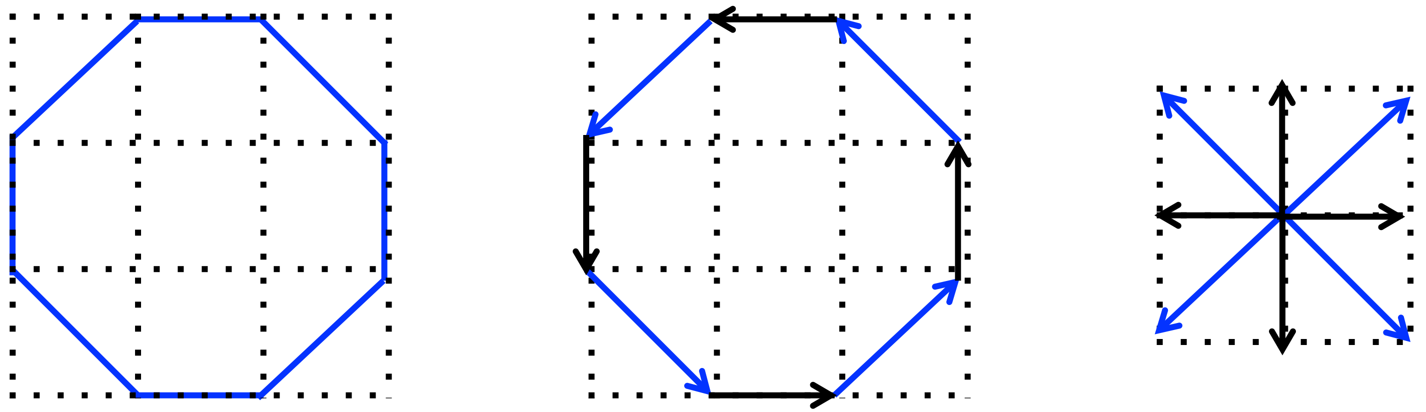
truncated small rhombicuboctahedron

- both are ***zonotopes***

Lattice polygons with large diameter (i.e. many vertices)

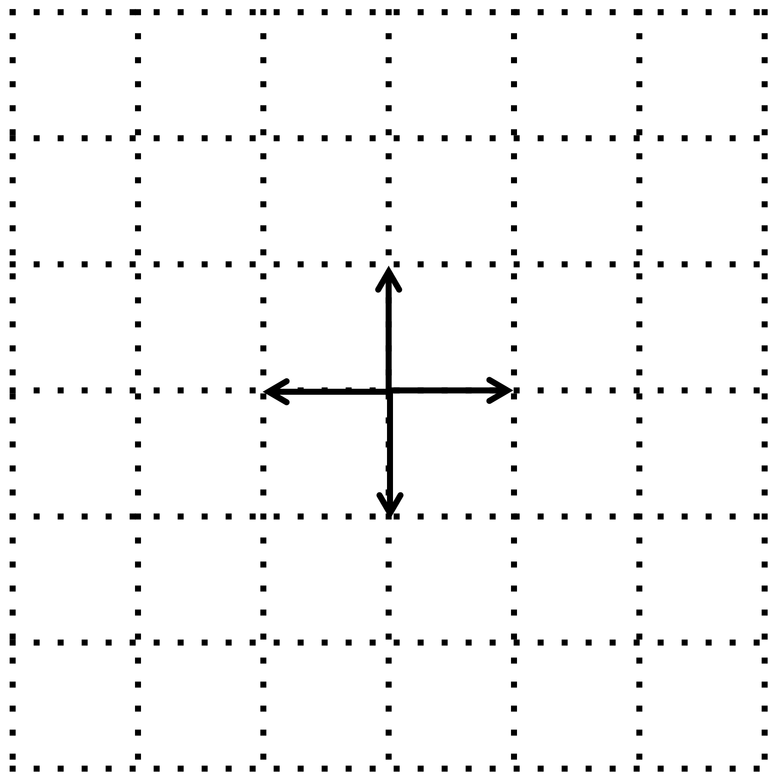
Q. What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

A polygon can be associated to a set of vectors (*edges*) summing up to zero, and without a pair of positively multiple vectors



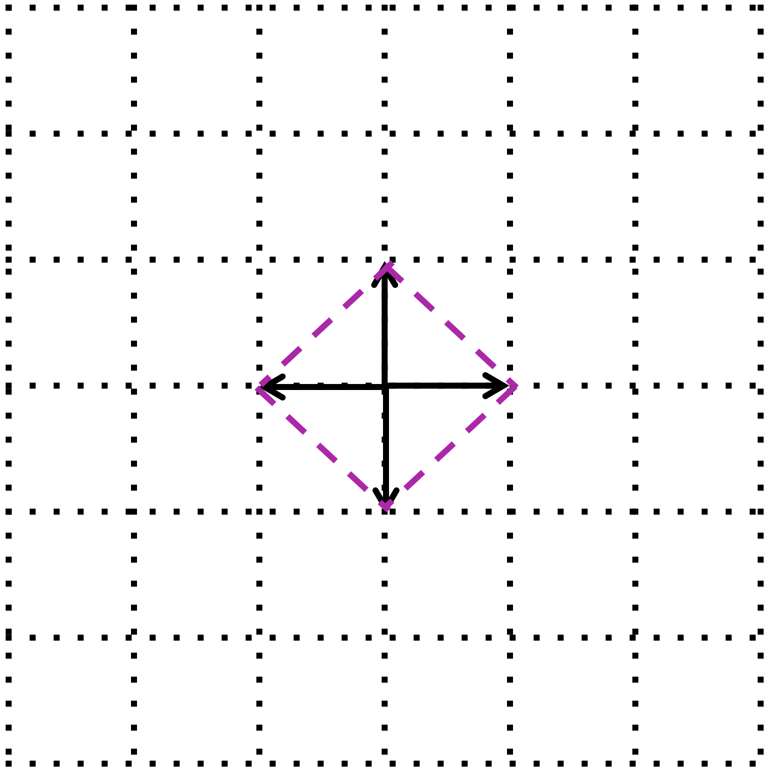
δ(2,3) = 4 is achieved by 8 vectors : (±1,0), (0,±1), (±1,±1)

Lattice polygons with large diameter (i.e. many vertices)



$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

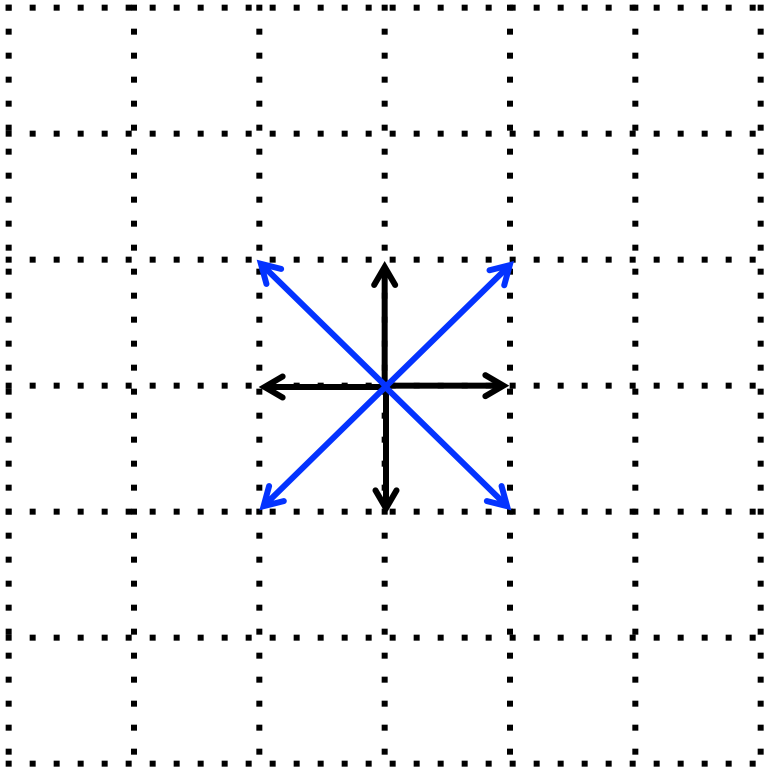
Lattice polygons with large diameter (i.e. many vertices)



$||x||_1 \leq 1$

$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

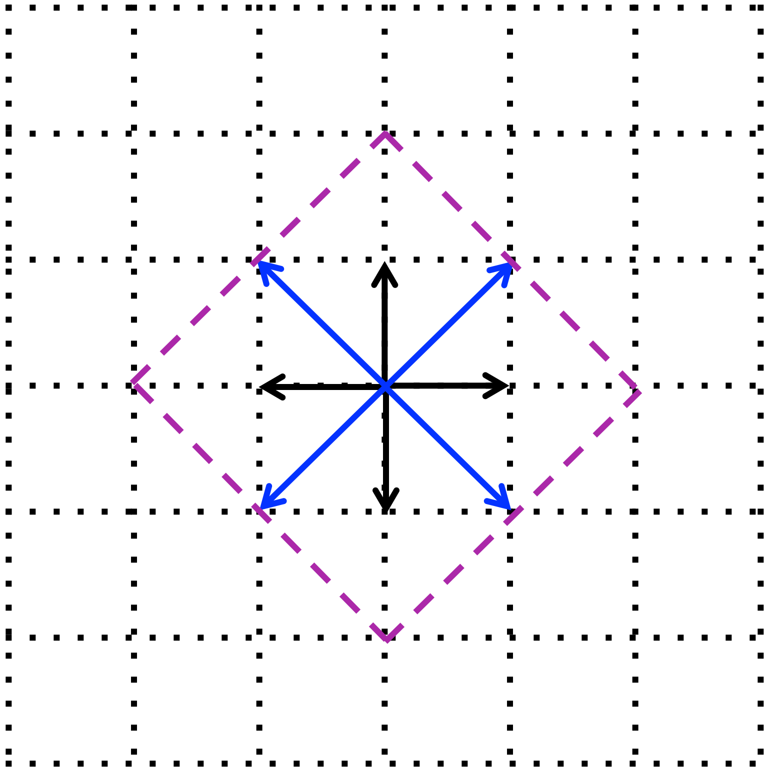
Lattice polygons with large diameter (i.e. many vertices)



$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

Lattice polygons with large diameter (i.e. many vertices)

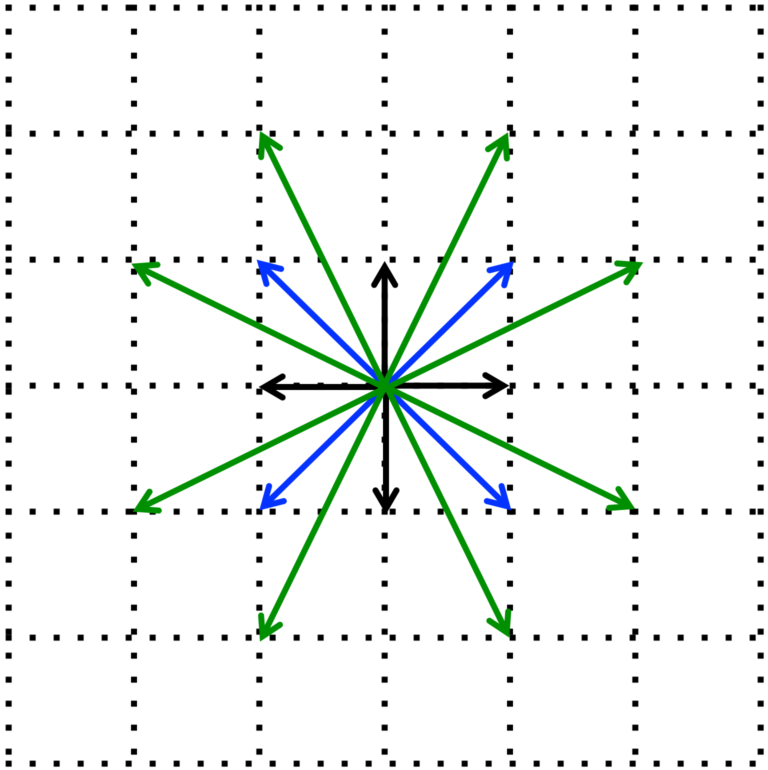


$||x||_1 \leq 2$

$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

Lattice polygons with large diameter (i.e. many vertices)

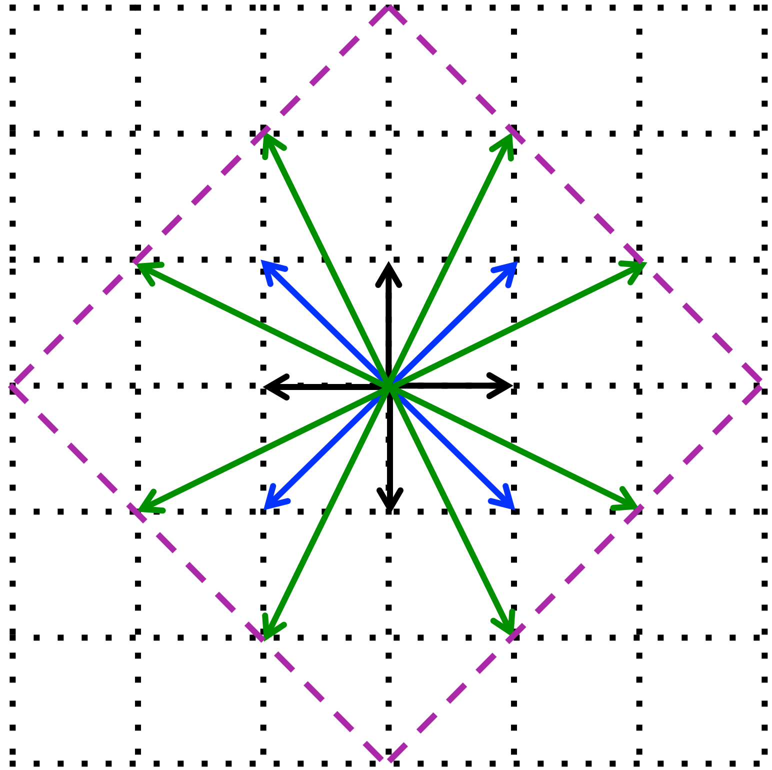


$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

Lattice polygons with large diameter (i.e. many vertices)



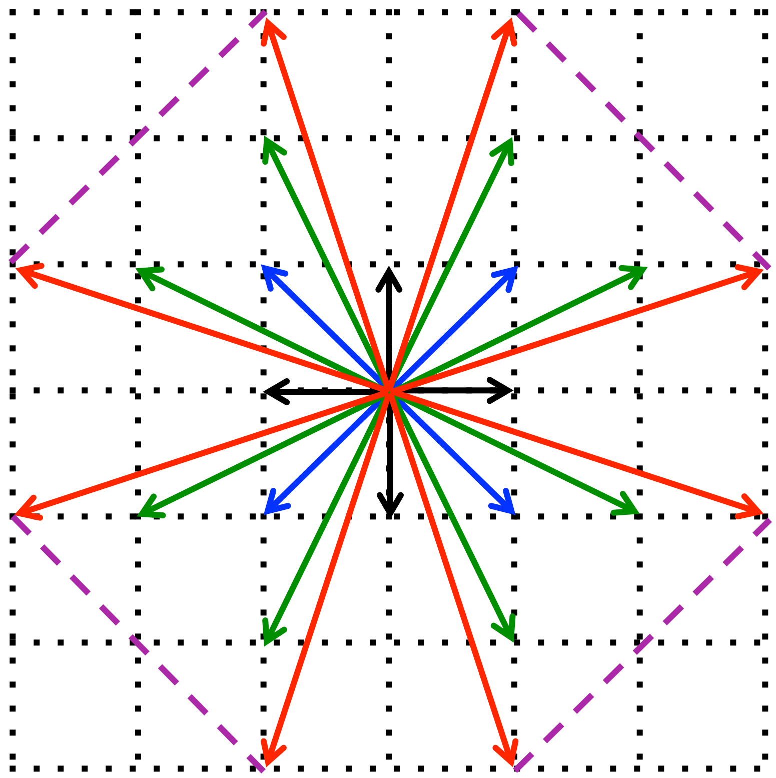
$||x||_1 \leq 3$

$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

$\delta(2,9) = 8$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$, $(\pm 1,\pm 2)$, $(\pm 2,\pm 1)$

Lattice polygons with large diameter (i.e. many vertices)



$||x||_1 \leq 4$

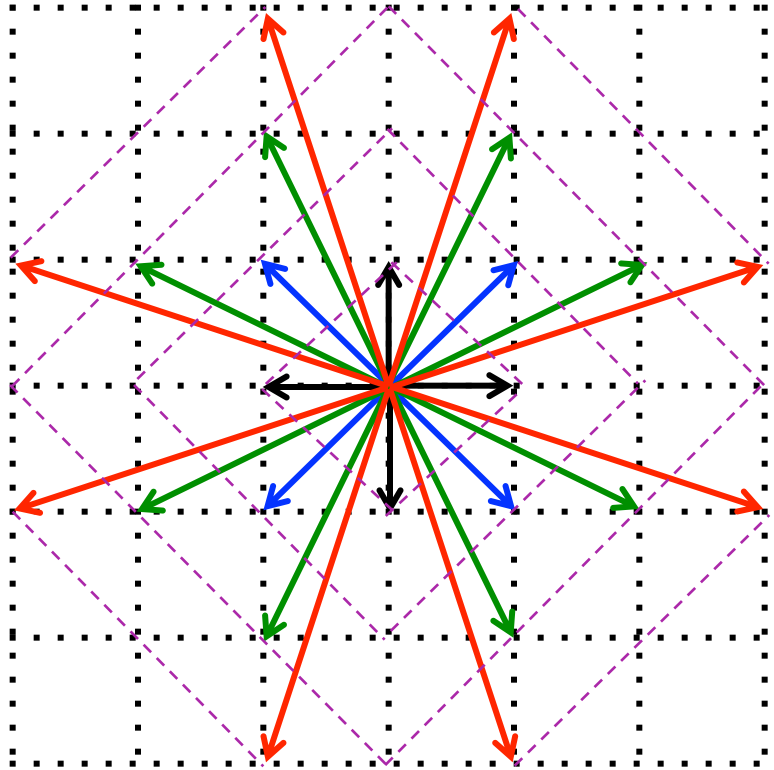
$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

$\delta(2,17) = 12$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$, $(\pm1,\pm3)$, $(\pm3,\pm1)$

Lattice polygons with large diameter (i.e. many vertices)



$||x||_1 \leq p$

$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

φ(*p*) : ***Euler totient function*** counting positive integers less or equal to *p* relatively prime with *p*

φ(1) = φ(2) = 1, φ(3) = φ(4) = 2,…

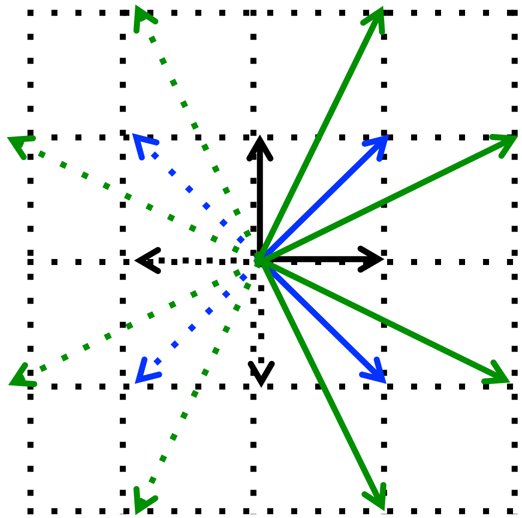
Lattice polygons with large diameter (i.e. many vertices)

δ(2,***k***)		***k***								
		1	2	3	4	5	6	7	8	9
	p	1		2						3
	v	4	6	8	8	10	12	12	14	16
	δ	2	3	4	4	5	6	6	7	8

$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

φ(***p***) : ***Euler totient function*** counting positive integers less or equal to ***p*** relatively prime with ***p***
φ(1) = φ(2) = 1, φ(3) = φ(4) = 2,…

Lattice polytopes with large diameter and many vertices



$||x||_1 \leq p$

$E_q(d,p)$: zonotope generated by $\{x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0\}$

($E_q(d,p)$: Minkowski sum of segments)

Ex. $E_1(2,p)$ has diameter $\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $E_1(2,2)$ generated by (1,0), (0,1), (1,1), (1,-1) (fits, *up to translation*, in 3x3 grid)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Lattice polytopes with large diameter and many vertices

$E_q(d,p)$: zonotope generated by $\{x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0\}$

- large symmetry group : ***permutation and sign flips***
- $\delta(E_1(2,p)) = \delta(2,k)$ for infinitely many k

Lattice polytopes with large diameter and many vertices

$E_q(d,p)$: zonotope generated by $\{x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0\}$

- large symmetry group : ***permutation and sign flips***

- $\delta(E_1(2,p)) = \delta(2,k)$ for infinitely many k

- $E_1(d,2)$: d^2 generators $g^1, g^2, \ldots$ ($d!\,2^d$ vertices)

$\Rightarrow \delta(E_1(d,2)) = d^2$ (no two generators are parallel)

- $E_1(d,2)$ in $\{0,1,\ldots,k\}^d$ with $k = 2d - 1$

Lattice polytopes with large diameter and many vertices

$E_q(d,p)$: zonotope generated by $\{x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \geq 0\}$

- large symmetry group : ***permutation and sign flips***
- $\delta(E_1(2,p)) = \delta(2,k)$ for infinitely many k
- $E_1(d,2)$: d^2 generators $g^1, g^2, \ldots$ ($d!\,2^d$ vertices)

$\Rightarrow \delta(E_1(d,2)) = d^2$ (no two generators are parallel)

- $E_1(d,2)$ in $\{0,1,\ldots,k\}^d$ with $k = 2d - 1$

$\Rightarrow \delta(d,k) \geq d^2$ with $k = 2d - 1$

$\Rightarrow \delta(d,k) \geq (k+1)d/2$ for infinitely many (d,k)

$\Rightarrow \delta(d,k) \geq kd/2 - k^2/4$ for any (d,k)

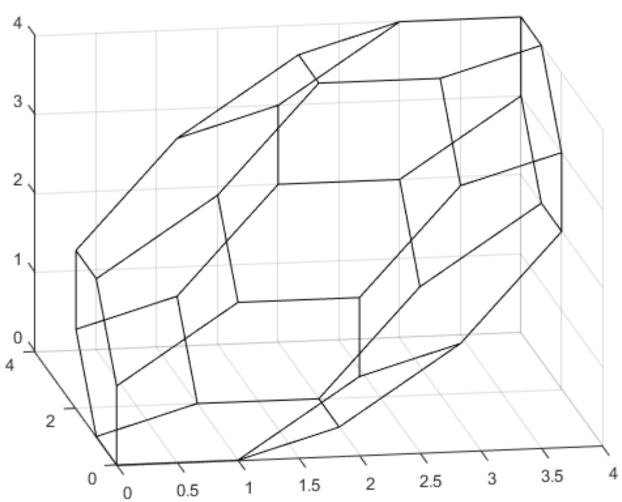
- $kd/2 - k^2/4 \leq \delta(d,k) \leq kd - \lceil d/2 \rceil$

Lattice polytopes with large diameter and many vertices

$E_q(d,p)$: zonotope generated by $\{x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0\}$

$E_q(d,p)^+ = E_q(d,p) \cap \mathbb{Z}^d_+$

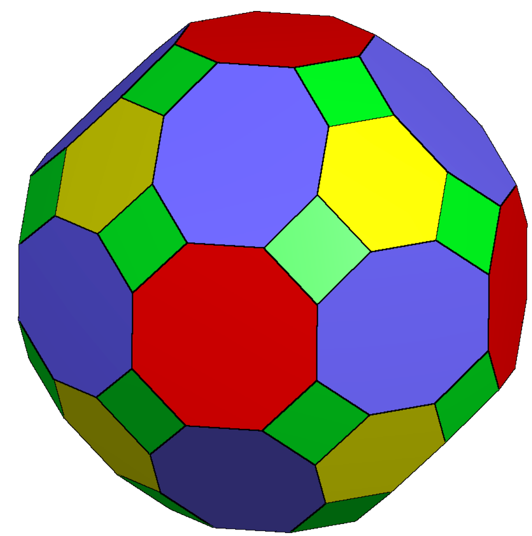
- $|E_\infty(d,p)^+| \leq \mathbf{m}(d,p) \leq |E_\infty(d,p)| \quad \Rightarrow \quad |E_\infty(d,1)^+| \leq \mathbf{m}(d) \leq |E_\infty(d,1)|$



$E_\infty(3,1)^+$

$\mathbf{32} \leq \mathbf{m}(3) \leq \mathbf{96}$

$\mathbf{370} \leq \mathbf{m}(4) \leq \mathbf{5376}$



$E_\infty(3,1)$: ***truncated small rhombicuboctahedron***

Lattice polytopes with large diameter and many vertices

$E_\infty(d,p)^+$		p			
		1	2	3	4
d	2	6 (3,2)	10 (5,5)	18 (9,14)	26 (13,26)
	3	**32** (7,4)	212 (19,19)	1418 (49,76)	4916 (91,184)
	4	**370** (15,8)	27778 (65,65)	(225,344)	(529,1064)

$|E_\infty(d,p)^+|$ (δ,k)

Lattice polytopes with large diameter and many vertices

$E_\infty(d,p)$		p			
		1	2	3	4
d	2	8 (4,3)	16 (8,9)	32 (16,27)	48 (24,51)
	3	**96** (13,9)	1248 (49,57)	10940 (145,249)	43680 (289,633)
	4	**5376** (40,27)	(272,321)	(1120,1923)	(2928,6459)

$| E_\infty(d,p) | \, (\delta,k)$

Lattice polytopes with large diameter and many vertices

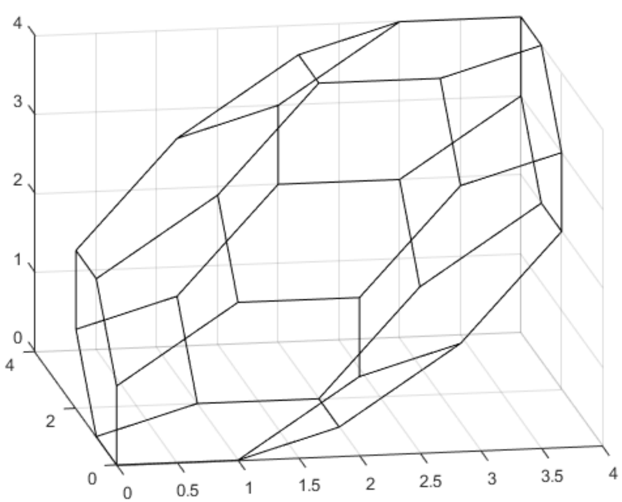
$E_q(d,p)$: zonotope generated by $\{x \in \mathbb{Z}^d : \|x\|_q \leq p,\ \gcd(x)=1,\ x \succeq 0\}$

$E_q(d,p)^+ = E_q(d,p) \cap \mathbb{Z}^d_+$

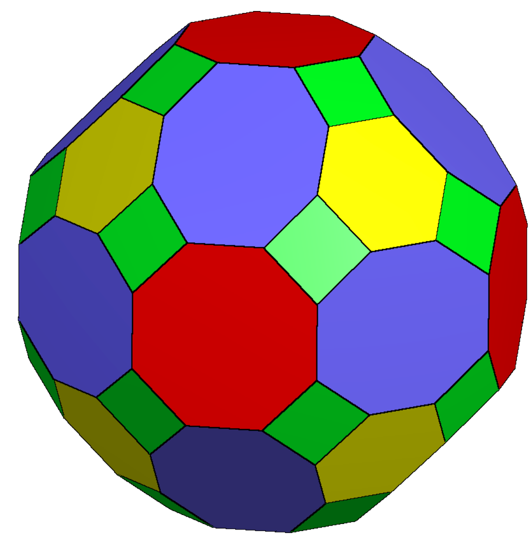
- $|\, E_\infty(d,1)^+ \,| \leq \mathbf{m}(d) \leq |\, E_\infty(d,1) \,|$

32 $\leq \mathbf{m}(3) \leq$ **96**

370 $\leq \mathbf{m}(4) \leq$ **5376**



$E_\infty(3,1)^+$



$E_\infty(3,1)$: ***truncated small rhombicuboctahedron***

Lattice polytopes with large diameter and many vertices

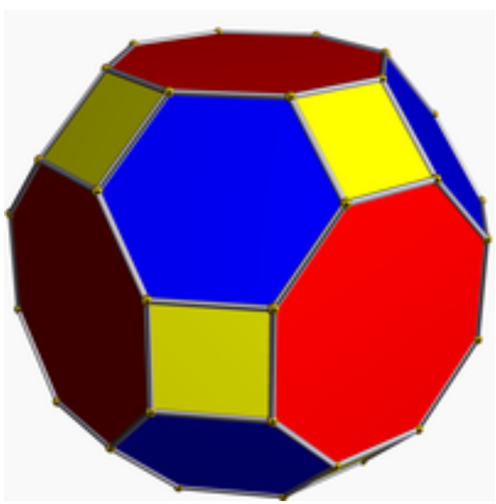
$E_q(d,p)$: zonotope generated by $\{x \in \mathbf{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0\}$

$E_q(d,p)^+ = E_q(d,p) \cap \mathbf{Z}^d_+$

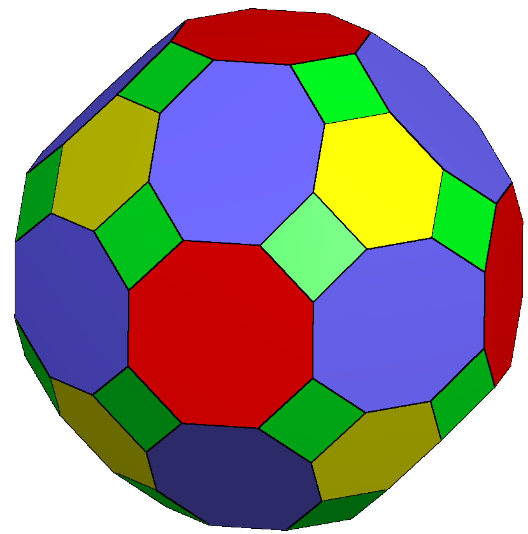
- $| E_\infty(d,1)^+ | \leq \mathbf{m}(d) \leq | E_\infty(d,1) |$

$48 \leq \mathbf{m}(3) \leq 96$

$672 \leq \mathbf{m}(4) \leq 5376$



$\mathbf{M}(3,5,2)$ ***truncated cuboctahedron***
(great rhombicuboctahedron)



$E_\infty(3,1)$: ***truncated small rhombicuboctahedron***

Lattice polytopes with large diameter and many vertices

$\mathbf{M}(d,r,s)$ = conv(**WS**) where

W : {0,1}-valued d x $s2^d$ matrix
whose $s2^d$ columns consist of s copies of the 2^d vectors in $\{0,1\}^d$

S : uniform matroid of rank r and order $s2^d$ (matroid having all r-subsets of $\{1,2,\ldots,s2^d\}$ as bases); that is, the columns of **S** consist of all $s2^d$ dimensional {0,1}-vectors with support r

$\mathbf{M}(d,r,s \geq r)$:	$\{0,1,\ldots,s\}^d$ cube	$\lvert \mathbf{M}(d,r,s \geq r) \rvert = 2^d$
$\mathbf{M}(d,2,1)$:	truncated $\{0,1,2\}^d$ cube	$\lvert \mathbf{M}(d, 2,1) \rvert = d2^{d-1}$
$\mathbf{M}(d,s+1,s \geq 2)$:	truncated $\{0,1,\ldots,s\}^d$ cube	$\lvert \mathbf{M}(d, 2,1) \rvert = d2^d$

Lattice polytopes with large diameter and many vertices

M(3,*r*,*s*)		*r*					
		1	2	3	4	5	6
s	1	8 (3,1)	12 (3,2)	24 (5,3)	14 (4,4)	24 (5,4)	12 (3,4)
	2	8 (3,1)	8 (3,2)	24 (6,3)	12 (3,4)	**48 (9,5)**	24 (5,6)
	3	8 (3,1)	8 (3,2)	8 (3,3)	24 (6,4)	24 (6,5)	12 (3,6)
	4	8 (3,1)	8 (3,2)	8 (3,3)	8 (3,4)	24 (6,5)	24 (6,6)

| M(3,*r*,*s*) | (δ,*k*)

Lattice polytopes with large diameter and many vertices

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

$\mathbf{m}(d)$: maximum number of vertices of conv(**WS**) over matroid **S** of order n, and $\{0,1\}$-valued d x n matrix **W**

- $kd/2 - k^2/4 \leq \delta(d,k) \leq kd - \lceil d/2 \rceil$ using $E_1(d,2)$
- $48 \leq \mathbf{m}(3) \leq 96$ using $\mathbf{M}(3,5,2)$ and $E_\infty(3,1)$
- $672 \leq \mathbf{m}(4) \leq 5376$ using $\mathbf{M}(4,9,2)$ and $E_\infty(4,1)$
- $2d! \leq \mathbf{m}(d) \leq 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$ using $E_\infty(d,1)^+$ and $E_\infty(d,1)$

Lattice polytopes with large diameter and many vertices

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

Conjecture

- $\delta(d,k)$ is achieved, up to translation, by a zonotope
- $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$
- $\delta(d,k) = \delta(E_1(d,p))$ for $k = k(E_1(d,p))$ and is uniquely achieved

  in particular, $\delta(d,2d-1) = d^2$ is uniquely achieved by $E_1(d,2)$

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✓ ***thank you***