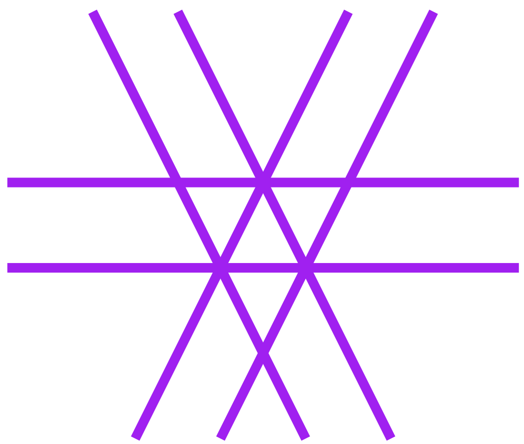
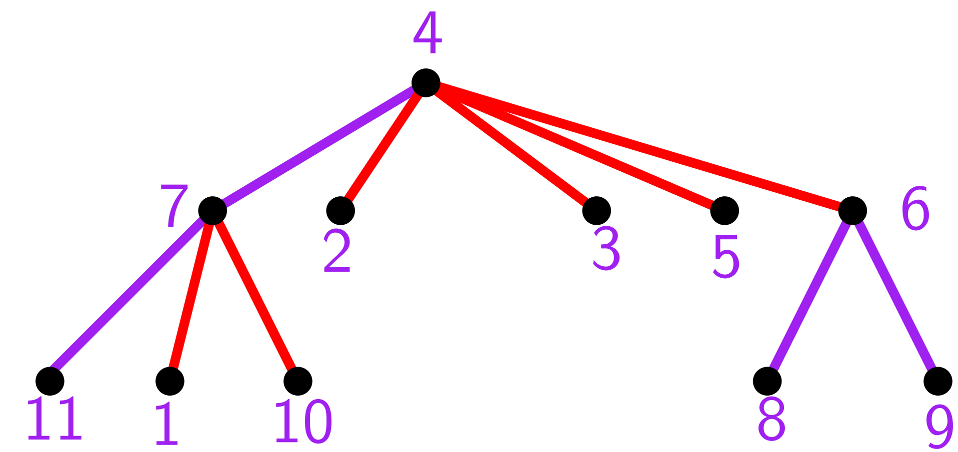
Hyperplane arrangements and trees





Anne Micheli (Paris 7)

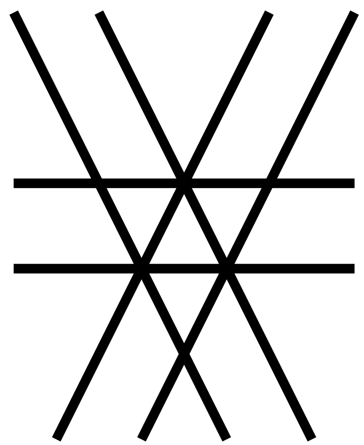
S. Corteel (Paris 7) and D. Forge (Paris Sud)

GT Combi du LIX

Introduction

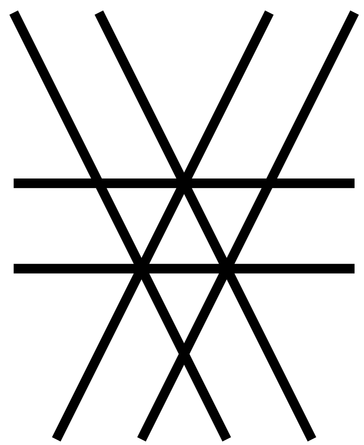
Hyperplane arrangements

$\mathcal{A}$: finite set of affine hyperplanes in $\mathbb{R}^n$



Hyperplane arrangements

$\mathcal{A}$: finite set of affine hyperplanes in $\mathbb{R}^n$



$x_i - x_j = 0$ or 1

$1 \leq i < j \leq 3$

Hyperplane arrangements

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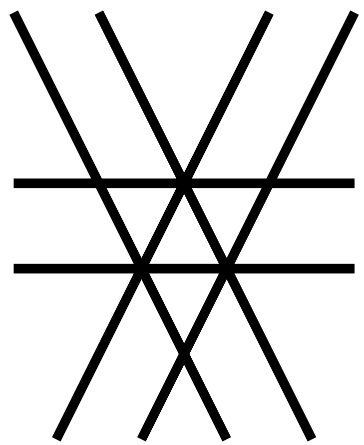

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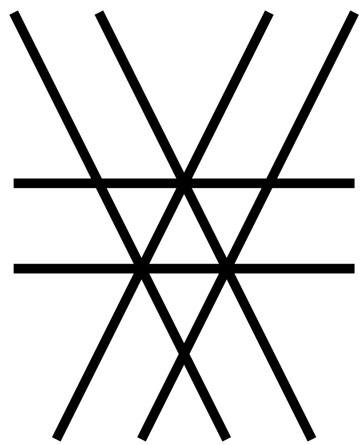
Regions: number of connected components obtained from $\mathbb{R}^n$ by removing $\mathcal{A}$

Hyperplane arrangements

$\mathcal{A}$: finite set of affine hyperplanes in $\mathbb{R}^n$



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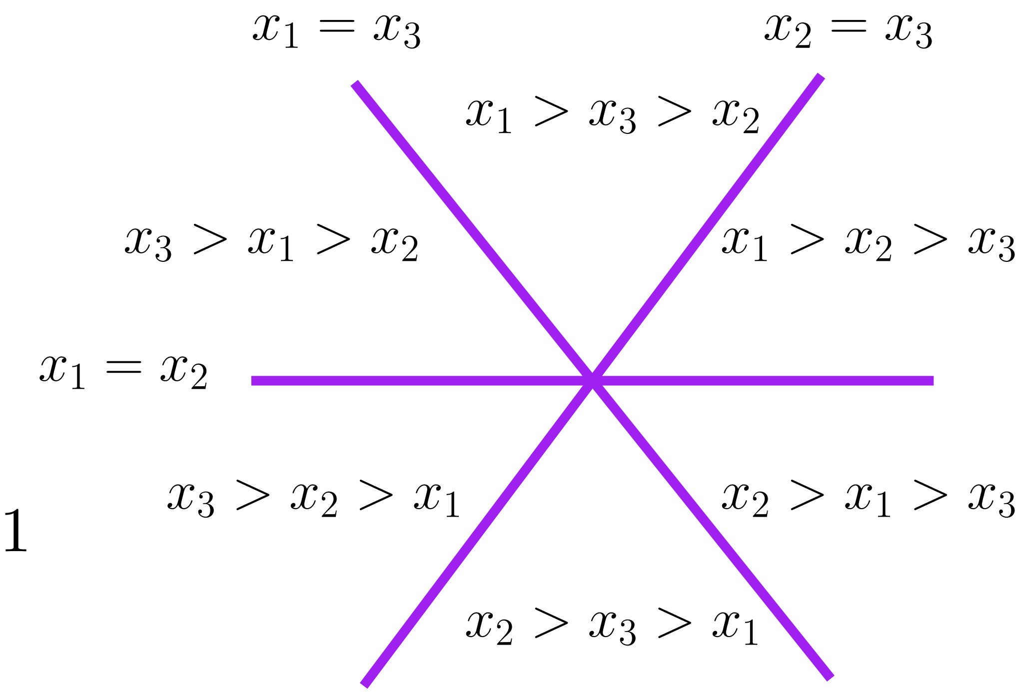
Regions: number of connected components obtained from $\mathbb{R}^n$ by removing $\mathcal{A}$

16 regions

Example of arrangements

Braid arrangement: $x_i - x_j = 0$

6 regions



Shi arrangement: $x_i - x_j = 0, 1$

Catalan arrangement: $x_i - x_j = -1, 0, 1$

Linial arrangement: $x_i - x_j = 1$

Example of arrangements

Braid arrangement: $x_i - x_j = 0$

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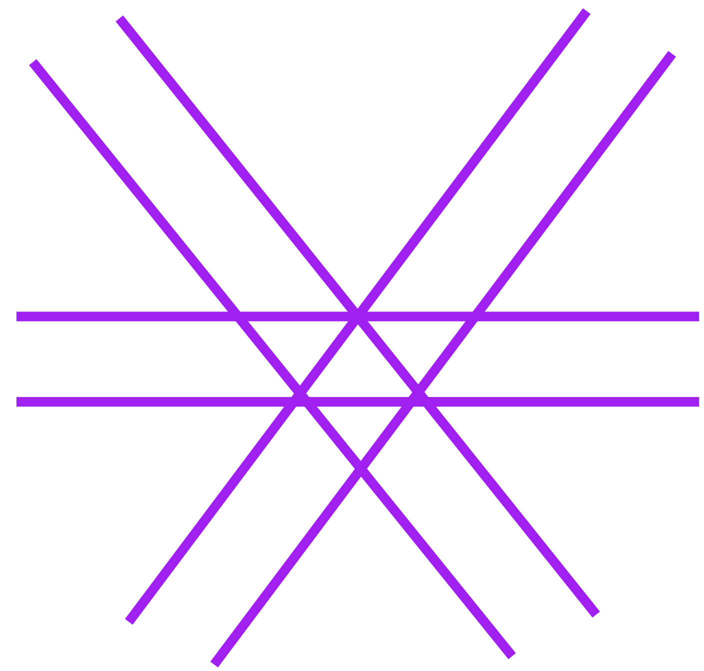

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Example of arrangements

Braid arrangement: $x_i - x_j = 0$

6 regions

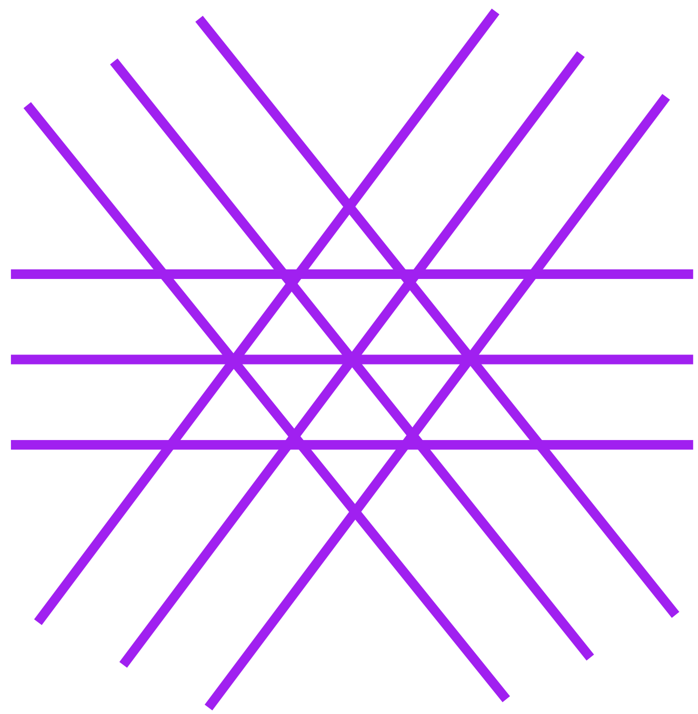
Shi arrangement: $x_i - x_j = 0, 1$

16 regions

Catalan arrangement: $x_i - x_j = -1, 0, 1$

30 regions

Linial arrangement: $x_i - x_j = 1$



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16 regions

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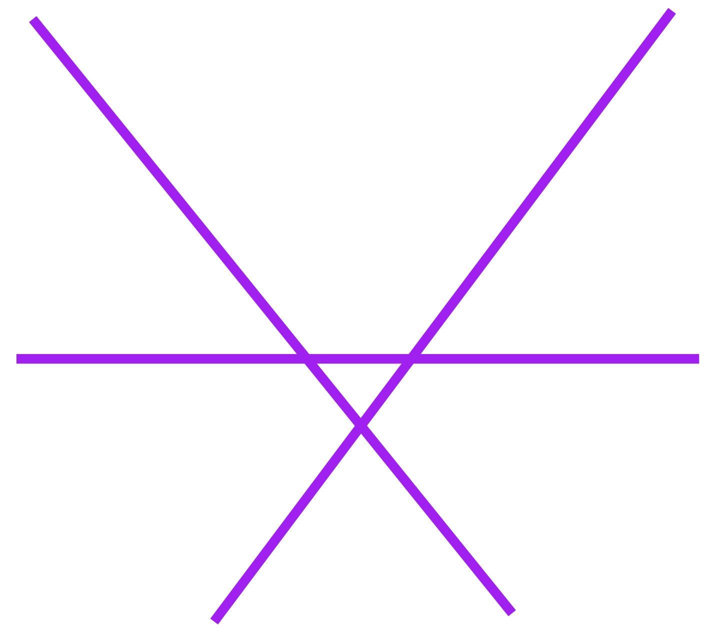
30 regions

Linial arrangement: $x_i - x_j = 1$

7 regions



Example of arrangements

Braid arrangement: $x_i - x_j = 0$ $n!$ regions

Shi arrangement: $x_i - x_j = 0, 1$ $(n+1)^{n-1}$ regions

Catalan arrangement: $x_i - x_j = -1, 0, 1$ $n! C_n$ regions

Linial arrangement: $x_i - x_j = 1$ $2^{-n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}$ regions

Lots of names, 90s Postnikov, Stanley, Athanasiadis, Linusson...

Example of arrangements

Braid arrangement: $x_i - x_j = 0$

Shi arrangement: $x_i - x_j = 0, 1$

Catalan arrangement: $x_i - x_j = -1, 0, 1$

Linial arrangement: $x_i - x_j = 1$

$$B(u_1, u_2, v_1, v_2) = \sum_T u_1^{LA(T)} u_2^{LD(T)} v_1^{RA(T)} v_2^{RD(T)}$$

Example of arrangements

Braid arrangement: $x_i - x_j = 0$

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[Gessel, Oberwolfach 14] $\frac{(1+u_1B)(1+v_2B)}{(1+v_1B)(1+u_2B)} = \exp(x(u_1 - u_2 - v_1 + v_2 + u_1v_2 - u_2v_1)B)$

Example of arrangements

Braid arrangement: $x_i - x_j = 0$ $\quad u_1 = v_1 = 1,\ u_2 = v_2 = 0$

Shi arrangement: $x_i - x_j = 0, 1$ $\quad u_1 = v_1 = u_2 = 1,\ v_2 = 0$

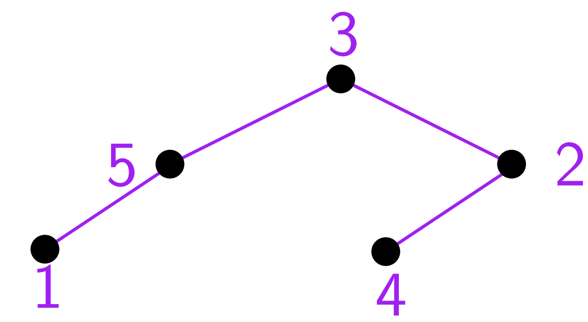
Catalan arrangement: $x_i - x_j = -1, 0, 1$ $\quad u_1 = v_1 = u_2 = v_2 = 1$

Linial arrangement: $x_i - x_j = 1$ $\quad u_2 = v_1 = 1,\ u_1 = v_2 = 0$

[Gessel, Oberwolfach 14] $\frac{(1+u_1B)(1+v_2B)}{(1+v_1B)(1+u_2B)} = \exp(x(u_1 - u_2 - v_1 + v_2 + u_1v_2 - u_2v_1)B)$

Arrangements and trees [Gessel]

labelled binary trees



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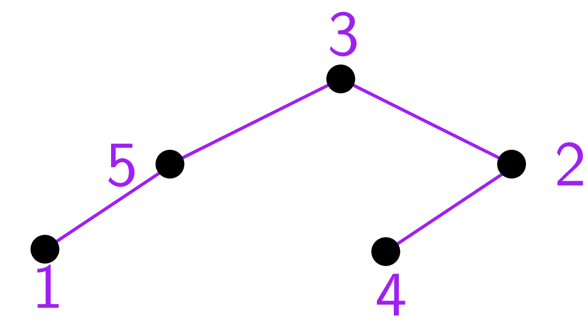


Braid arrangement: $x_i - x_j = 0$

left ascent, right ascent

Shi arrangement: $x_i - x_j = 0, 1$

No right descent

Catalan arrangement: $x_i - x_j = -1, 0, 1$

All possible

Linial arrangement: $x_i - x_j = 1$

left descent, right ascent

Arrangements and combinatorial objects

Braid arrangement: $x_i - x_j = 0$

13524 $\leftrightarrow x_1 > x_3 > x_5 > x_2 > x_4$

Permutations

Shi arrangement: $x_i - x_j = 0, 1$

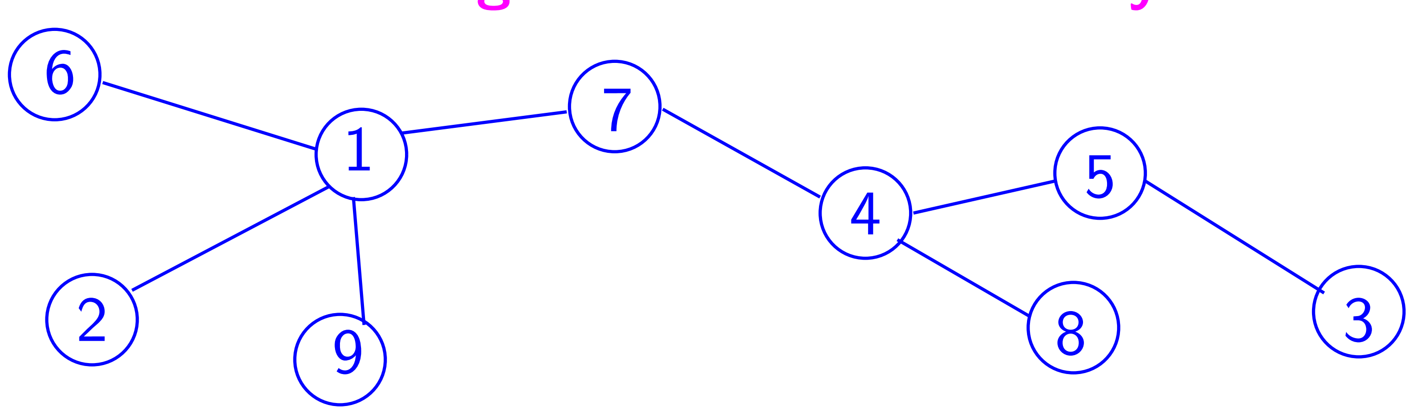
Parking functions [Athanasiadis and Linusson 99], Cayley forests

Catalan arrangement: $x_i - x_j = -1, 0, 1$

labelled binary trees

Linial arrangement: $x_i - x_j = 1$

Alternating trees and local binary search trees [Postnikov 97]



Deformation of Coxeter arrangements

[Postnikov and Stanley, 00]

$$x_i - x_j = g$$

$g \in [a, b]$ and 0 or 1 $\in [a, b]$

$f_n^{ab} = f_n$ number of regions

$$f(x) = \sum_n f_n \frac{x^n}{n!}$$

Theorem

- If $a + b = 0$ then $f = 1 + xf^{b+1}$
- Otherwise $f^{a+b} = \exp\left(xf^{1-a}\left(\frac{1-f^{a+b}}{1-f}\right)\right)$

Deformation of Coxeter arrangements

[Postnikov and Stanley, 00]

$$x_i - x_j = g$$

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Tools: NBC theorem and generating functions

No tree interpretation

$f_n^{ab} = f_n$ number of regions

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Theorem

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Our goal

Combinatorial interpretation!

Strategy:

- Gain graphs [Zavlavsky]
- No broken circuit (NBC)-trees
- Bijections with colored non-increasing or decreasing trees
- Generating functions [Gessel et al]
- Bijections (2,0)-decreasing $\leftrightarrow$ Rooted Cayley trees

Main result

Theorem [CFM 15]. There is a bijection between the regions of the arrangements $\mathcal{A}_n^{ab}$ and

- If $a \leq 0$ and $b \geq 0$, the $(a+b+1, -a)$-decreasing forests with n vertices
- If $a \leq 0$ and $b \geq 0$ or $a = 1$ and $b \geq 1$, the $(a+b-1, 1-a)$-non-increasing forests with n vertices

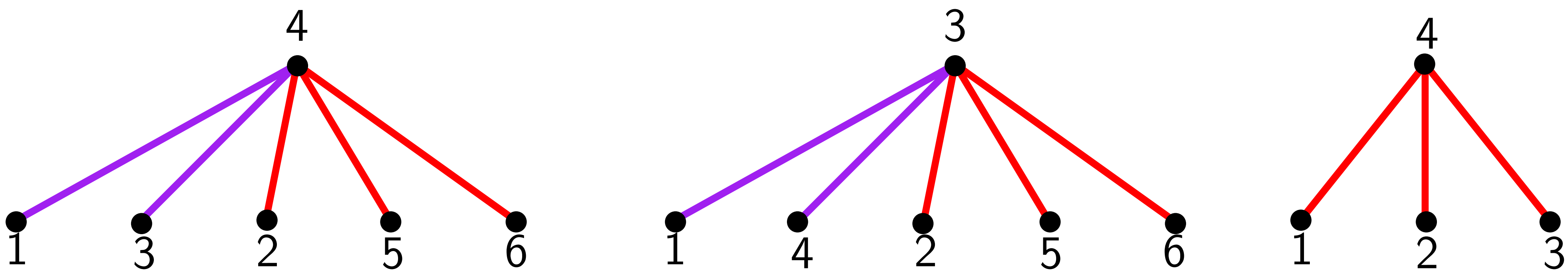
Generalization. Poincaré polynomial

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_{\text{forests}} q^{\#\text{edges(forests)}}$$

Labelled trees

(k, j)-decreasing/non-increasing trees

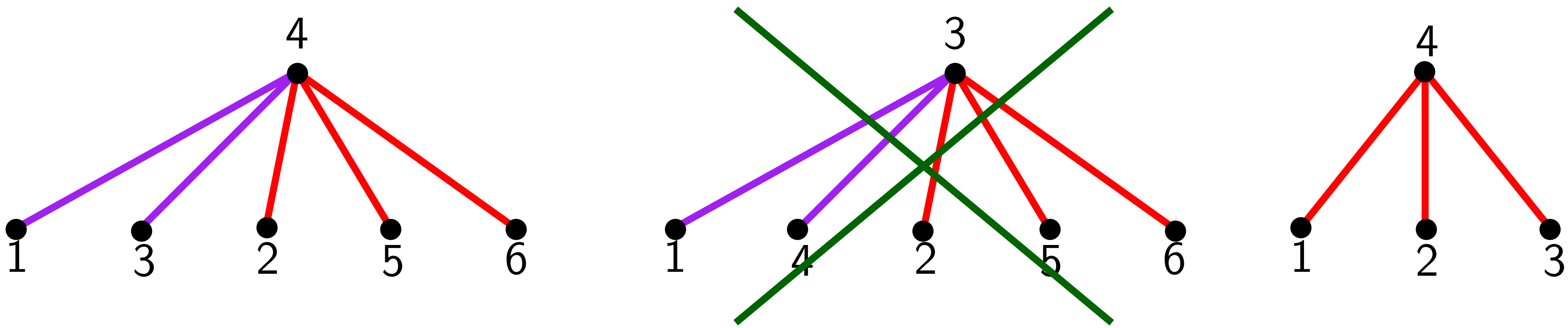
k-decreasing



ALL edges of the smallest appearing color MUST be decreasing

(k, j)-decreasing/non-increasing trees

k-decreasing



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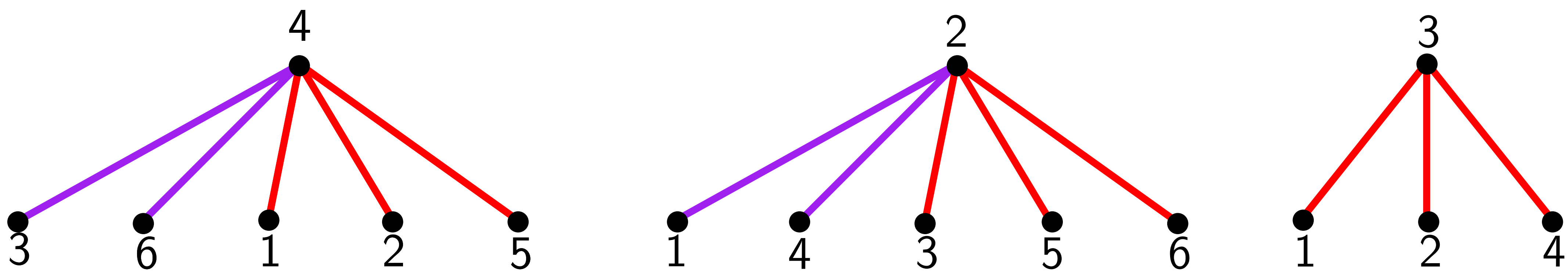
(k, j)-decreasing/non-increasing trees

k-non-increasing

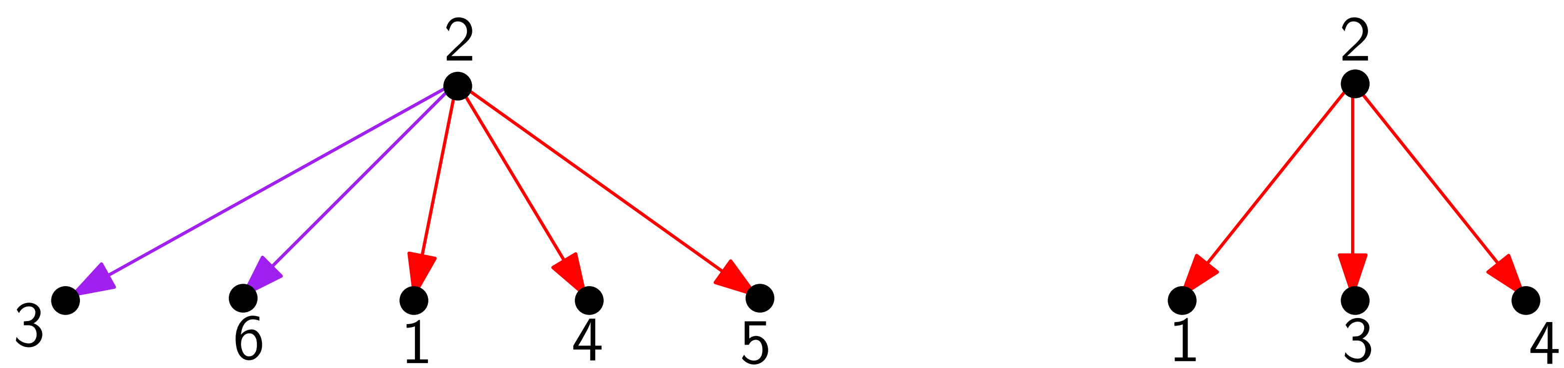


AT LEAST one edge of the smallest appearing color is decreasing

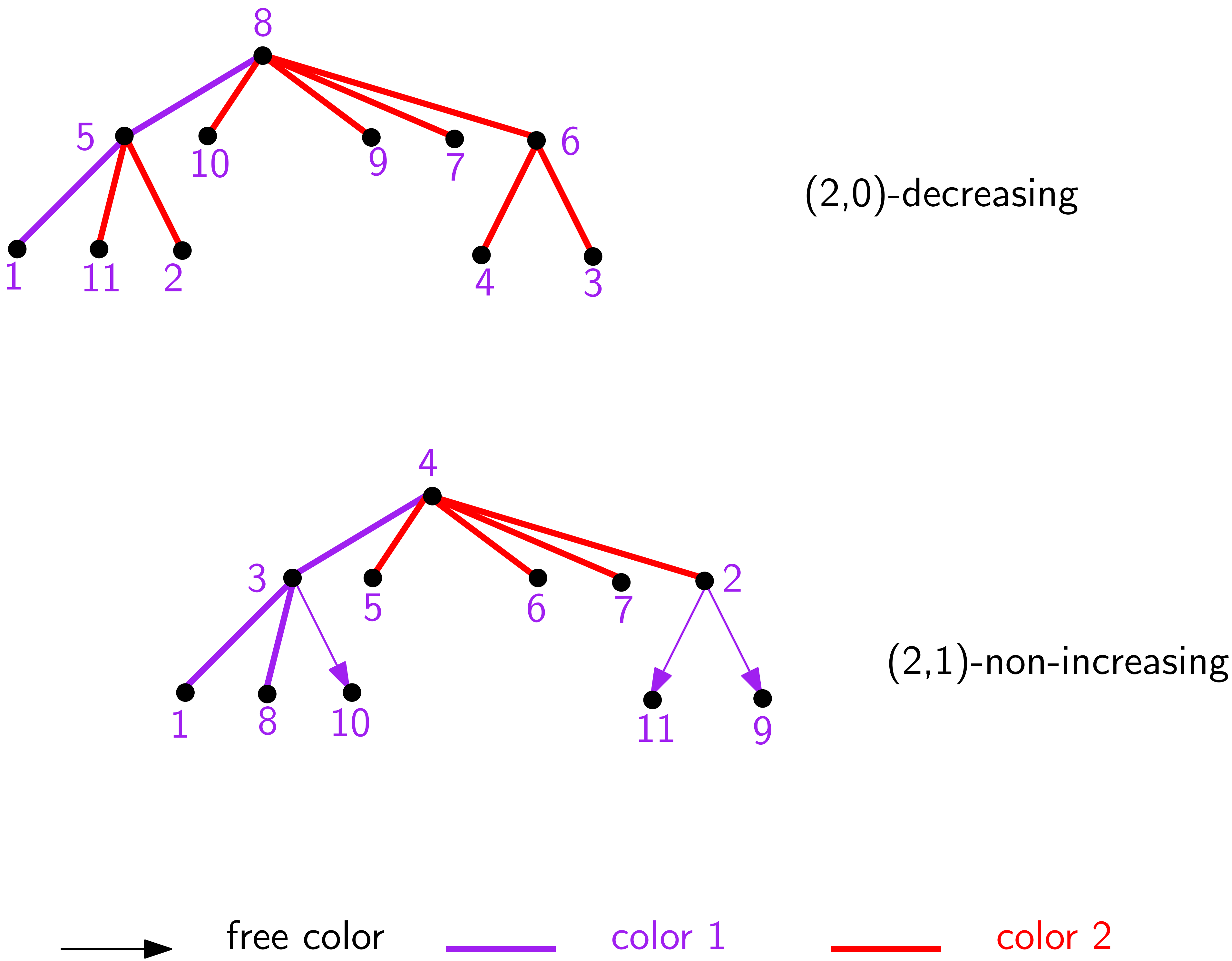
free color   color 1   color 2

(k, j)-decreasing/non-increasing trees

2 free colors



No conditions


8
5
10
9
7
6
1
11
2
4
3
(2,0)-decreasing
4
3
5
6
7
2
1
8
10
11
9
(2,1)-non-increasing
free color
color 1
color 2

Generating functions

Enumeration of $(k,0)$-decreasing trees

Decreasing forests $F_{n,k}$: n vertices, j internal vertices, $n-j$ leaves

$$F(x,y,t) = \sum_{n,j} F_{n,j} \frac{x^n}{n!} y^j t^{n-j}$$

$$F(x,y,t) = \frac{y-t}{y-t\exp(x(y-t))}$$

Decreasing trees $A(x,y,t) = \ln F(x,y,t)$

$(k,0)$-decreasing trees

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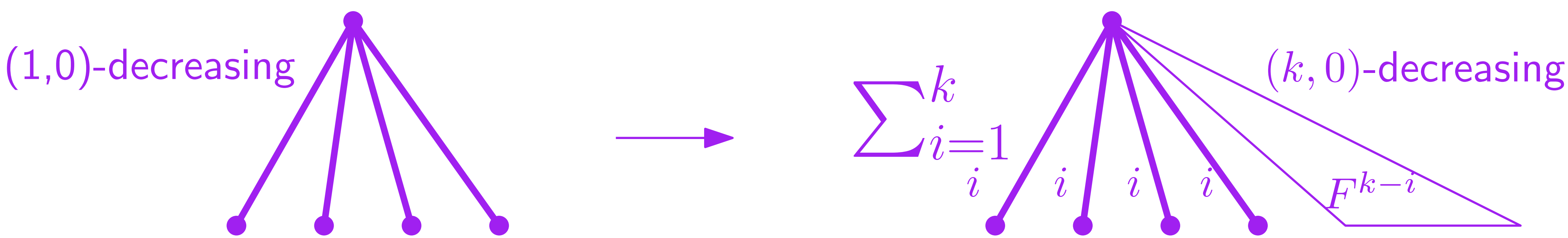

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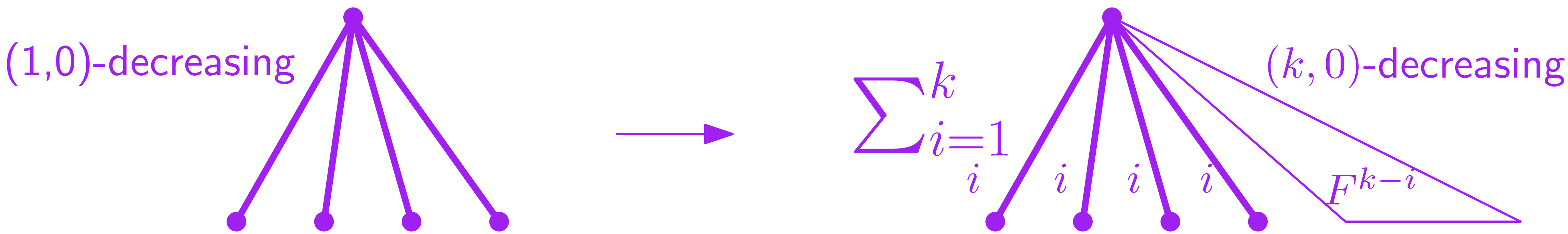

Lemma $F_k(x) = F(x, y + F_k + F_k^2 + \ldots + F_k^{k-1}, 1)$

Enumeration of $(k,0)$-decreasing trees

Decreasing forests $\quad F_{n,k}$: n vertices, j internal vertices, $n-j$ leaves

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Decreasing trees $A(x,y,t) = \ln F(x,y,t)$

$(k,0)$-decreasing trees $\qquad A_k(x) = \ln F_k(x)$

Proposition [CFM, 15] For $k \geq 2$, $F_k = F(x, y + F_k \frac{1-F_k^{k-1}}{1-F_k}, 1)$

$$(k-1)A_k = x\exp(A_k)(1+\exp(A_k)+\ldots+\exp((k-2)A_k))$$

Enumeration of $(k,0)$-decreasing trees

Decreasing forests $\quad F_{n,k}$: n vertices, j internal vertices, $n-j$ leaves

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Enumeration of $(k,0)$-decreasing trees

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$(k,0)$-decreasing trees

$$A_k(x) = \ln F_k(x)$$

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$$(k-1)A_k = x\exp(A_k)(1+\exp(A_k)+\ldots+\exp((k-2)A_k))$$

$k=2 : A_2 = x\exp(A_2)$

rooted Cayley trees!

Enumeration of (k, j)-non-increasing trees

Increasing forests

$$F(x, y, t) = \frac{y - t}{y - t\exp(x(y - t))}$$

Enumeration of (k,j)-non-increasing trees

Increasing forests

$$F(x,y,t) = \frac{y-t}{y-t\exp(x(y-t))}$$

(k,j)-non-increasing trees $\qquad \tilde{A}_k(x) = \ln \tilde{F}_k(x)$

Lemma $\quad \tilde{F}_k(x) = F(x, -1-\tilde{F}_k - \ldots - \tilde{F}_k^{k-1}, \tilde{F}_k^k)$

[Marked forests, Gessel 96]

Proposition [CFM, 15] For $k \geq 2$, $\tilde{F}_k^{k+1} = \exp\left(x\frac{1-\tilde{F}_k^{k+1}}{1-\tilde{F}_k}\right)$

$$(k+1)\tilde{A}_k = x(1+\exp(A_k)+\ldots+\exp(kA_k))$$

Link with the regions of the arrangements

Using [Postnikov and Stanley, 00], we get again

Theorem [CFM 15]. There is a bijection between the regions of the arrangements $\mathcal{A}_n^{ab}$ and

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Generalization. Poincaré polynomial

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_{\text{forests}} q^{\#\text{edges(forests)}}$$

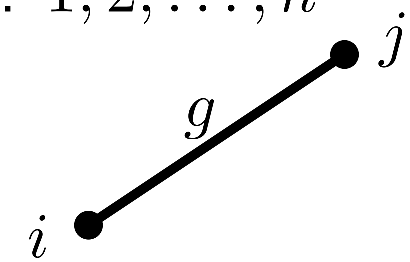
Gain graphs and NBC-trees

Gain graphs

[Zavlavsky]

Vertices: $1, 2, \ldots, n$

$$x_i - x_j = g$$
$$i < j$$



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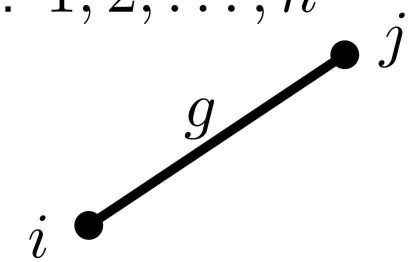

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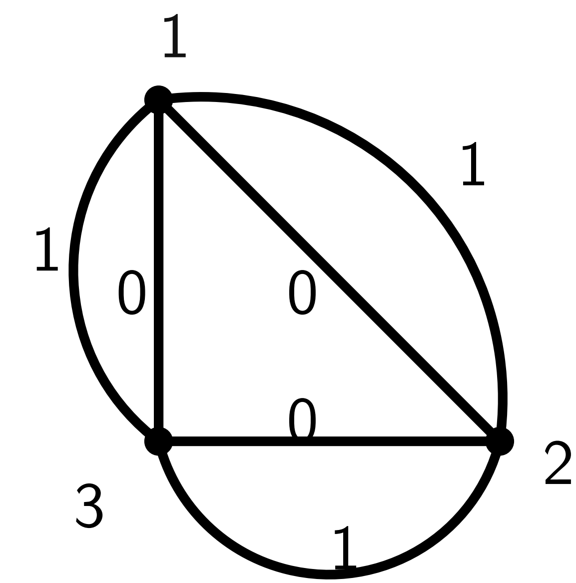

Gain graphs

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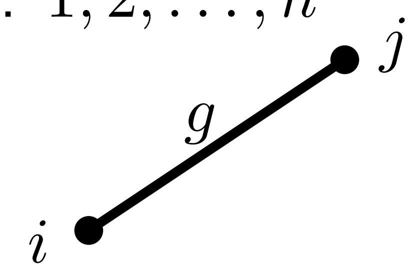
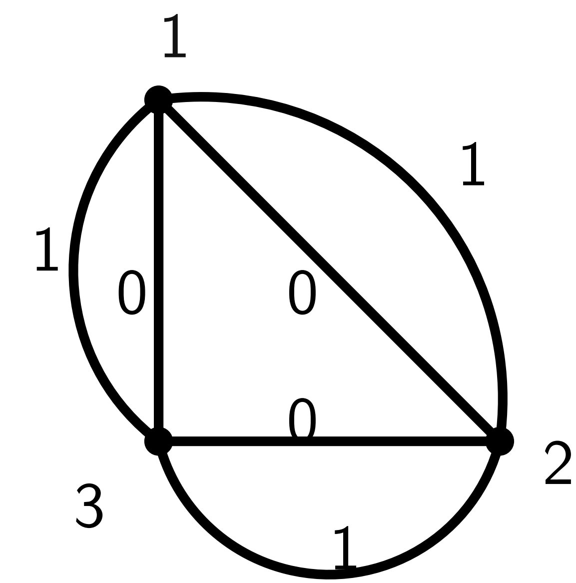
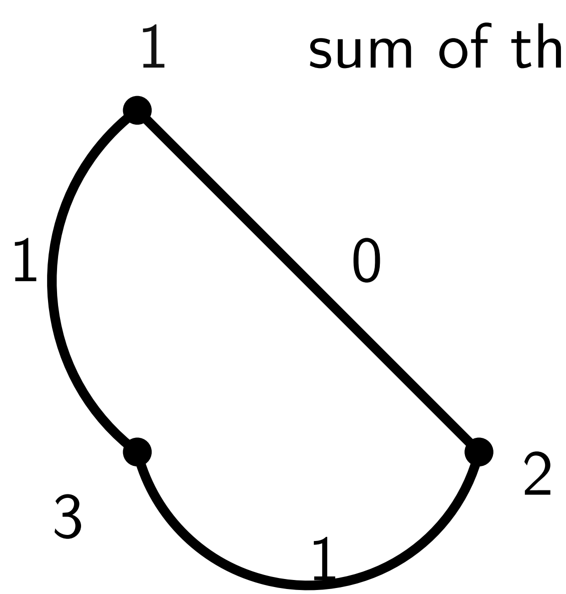
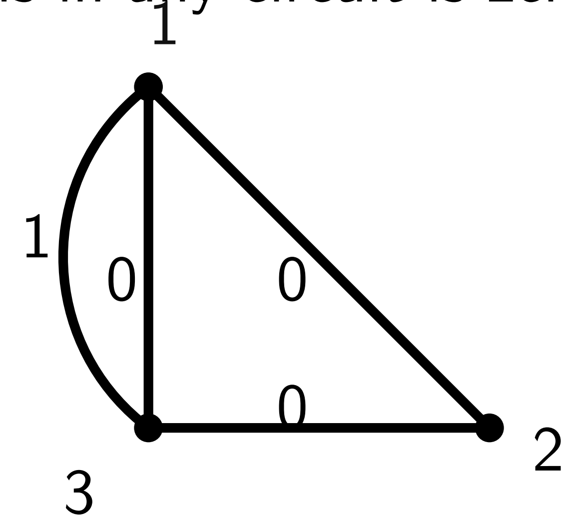


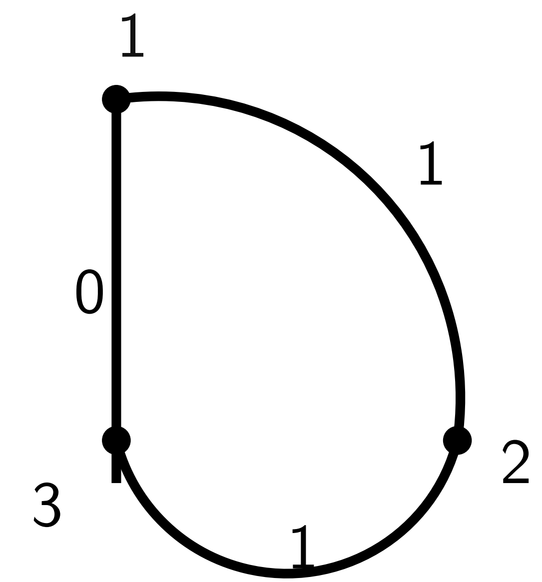


Balanced gain graph

sum of the gains in any circuit is zero







Gain graphs

[Zavlavsky]

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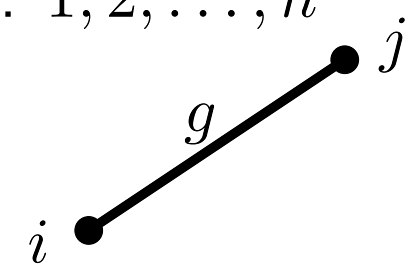
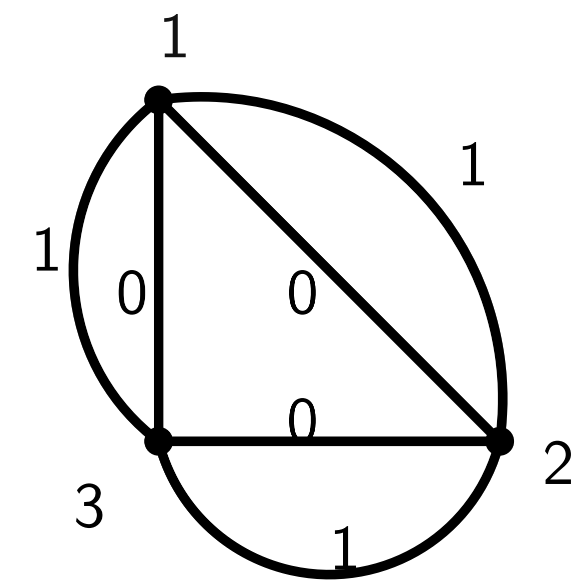
Vertices: $1, 2, \ldots, n$





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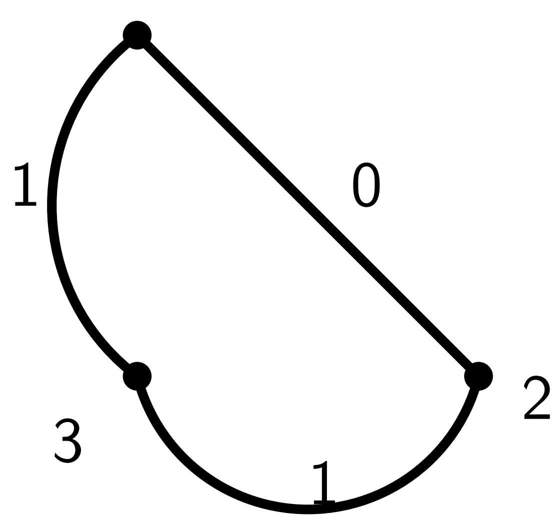
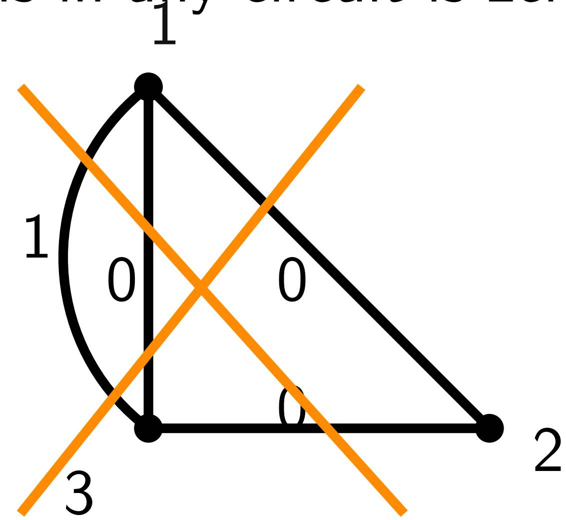
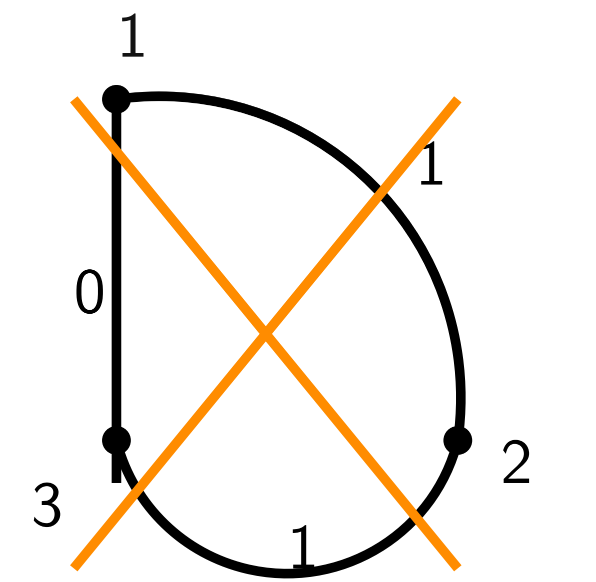


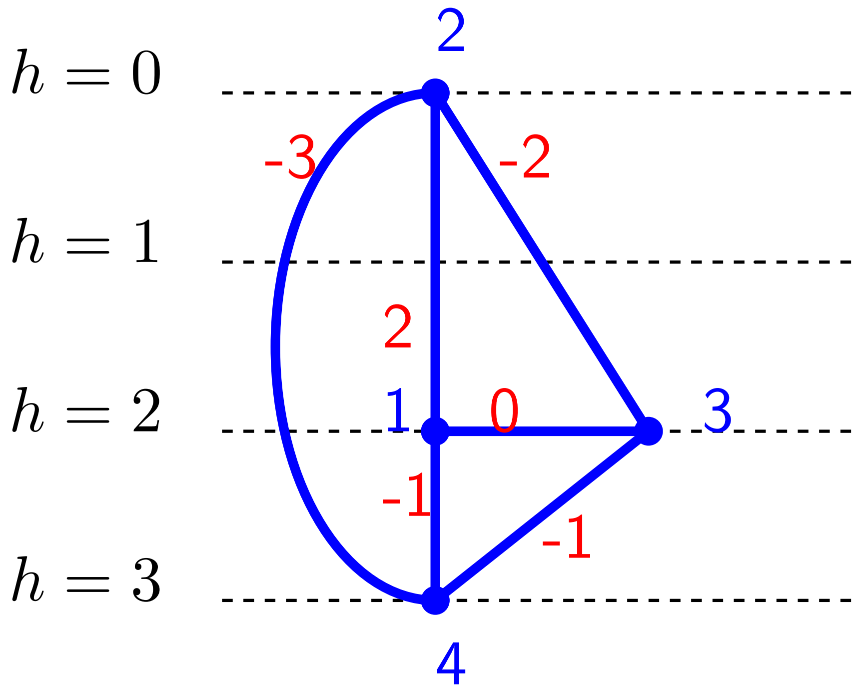




Height function

Height function ⇔ Balanced gain graph

Height function



$h(2) = 0,\ h(1) = h(3) = 2,\ h(4) = 1$

$x_i - x_j = g \quad \Leftrightarrow \quad h(i) - h(j) = g$

$g \in [-3, 2]$

No broken circuit

Order the edges of the gain graph

Broken circuit: balanced circuit without its minimal edge

NBC : set of edges with no broken circuit

Theorem [Whitney] There exists a bijection between

- the NBC-forests with n vertices and
- the regions of $\mathcal{A}_n$

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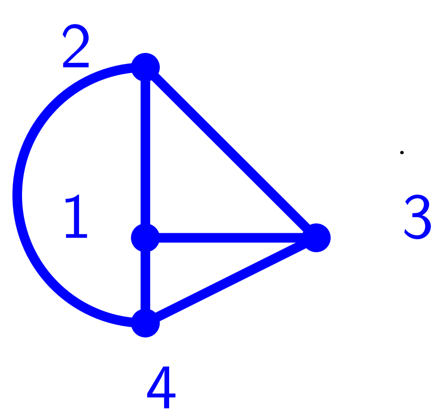

Order on vertices $2 < 3 < 1 < 4$

$\Rightarrow$ Order on edges $23 < 21 < 24 < 31 < 34 < 14$

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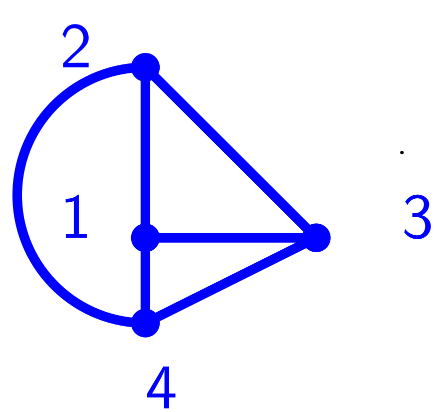

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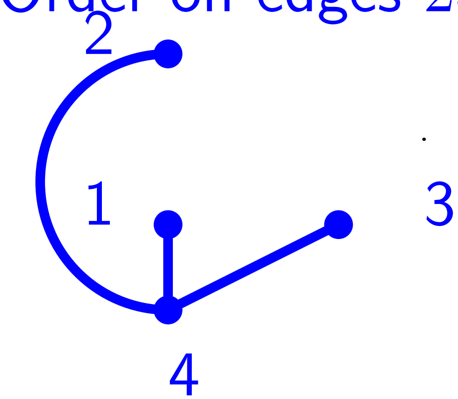

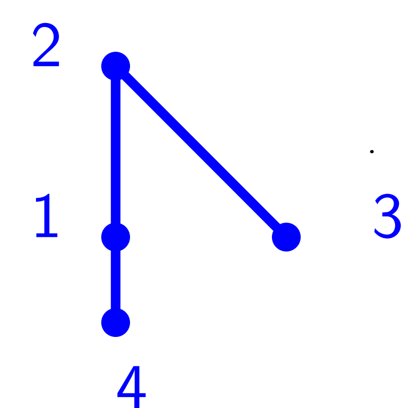

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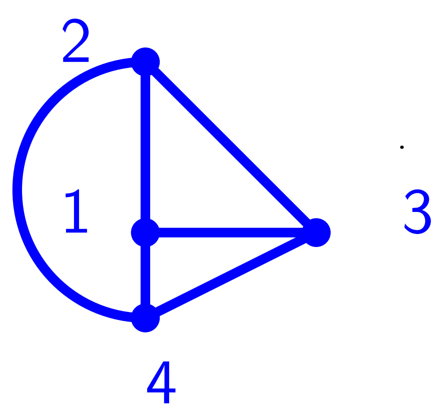

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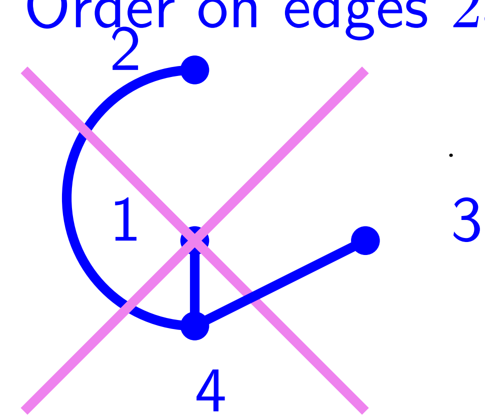
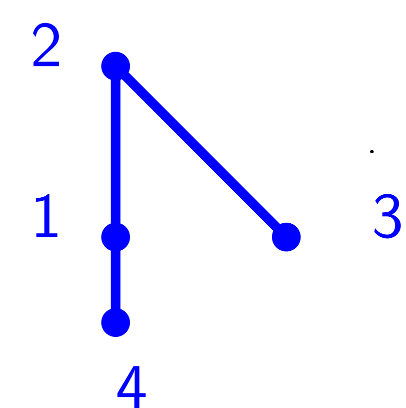
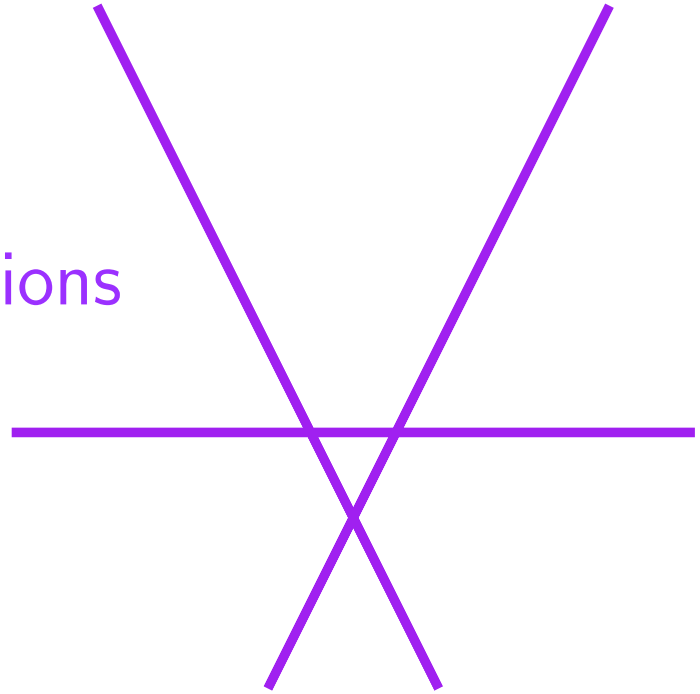


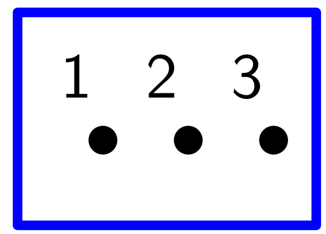


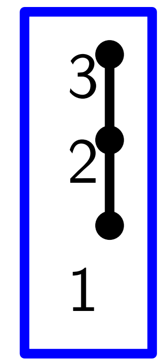
Example: linial

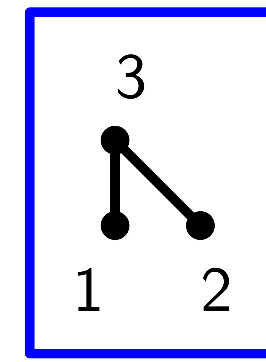
$$x_i - x_j = 1$$



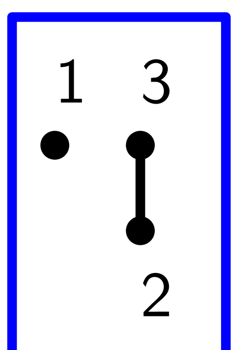
7 regions

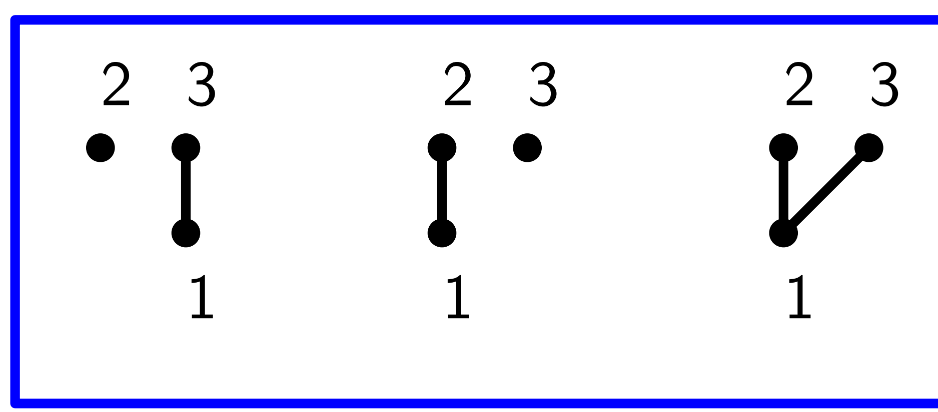






7 NBC forests





Bijection

NBC

trees

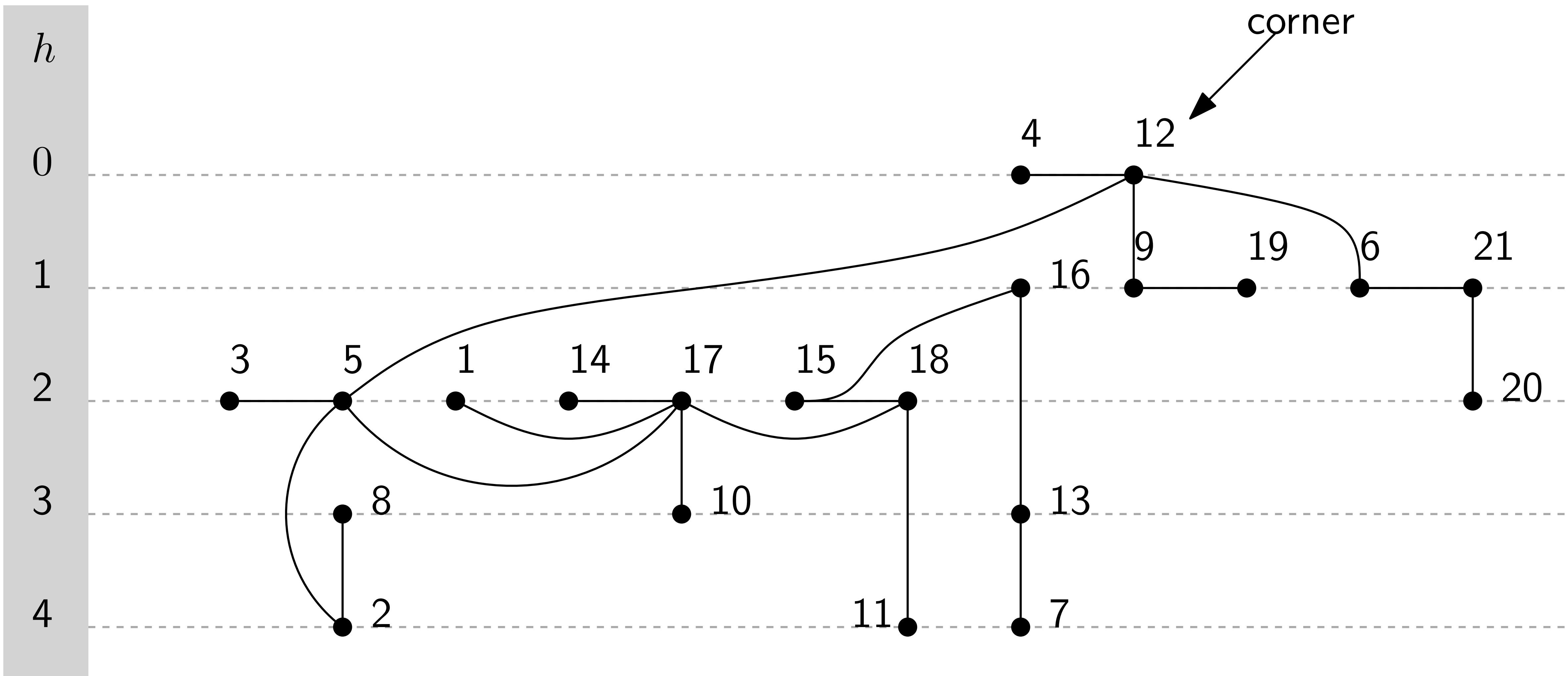
$\leftrightarrow$

Colored

trees

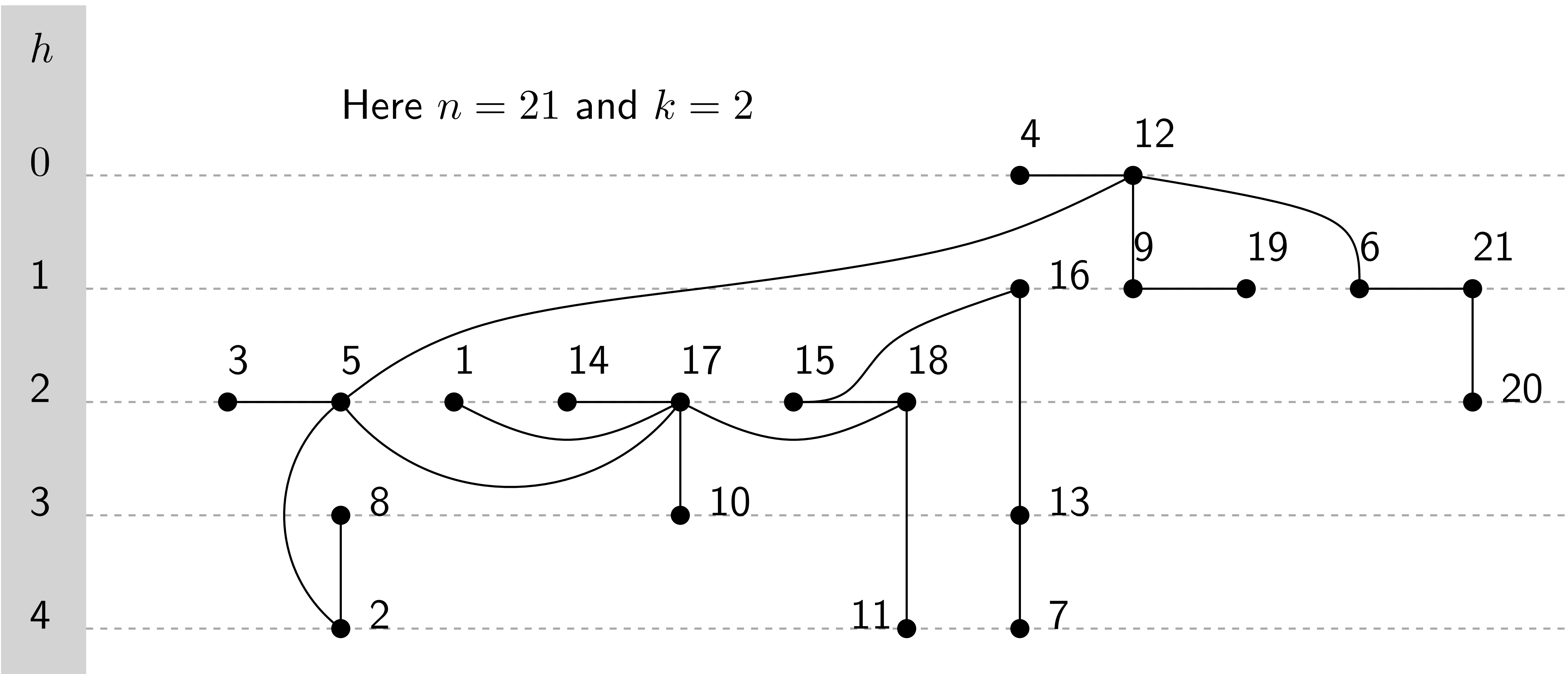
Bijection NBC-trees and decreasing trees

Corner: smallest for O_h



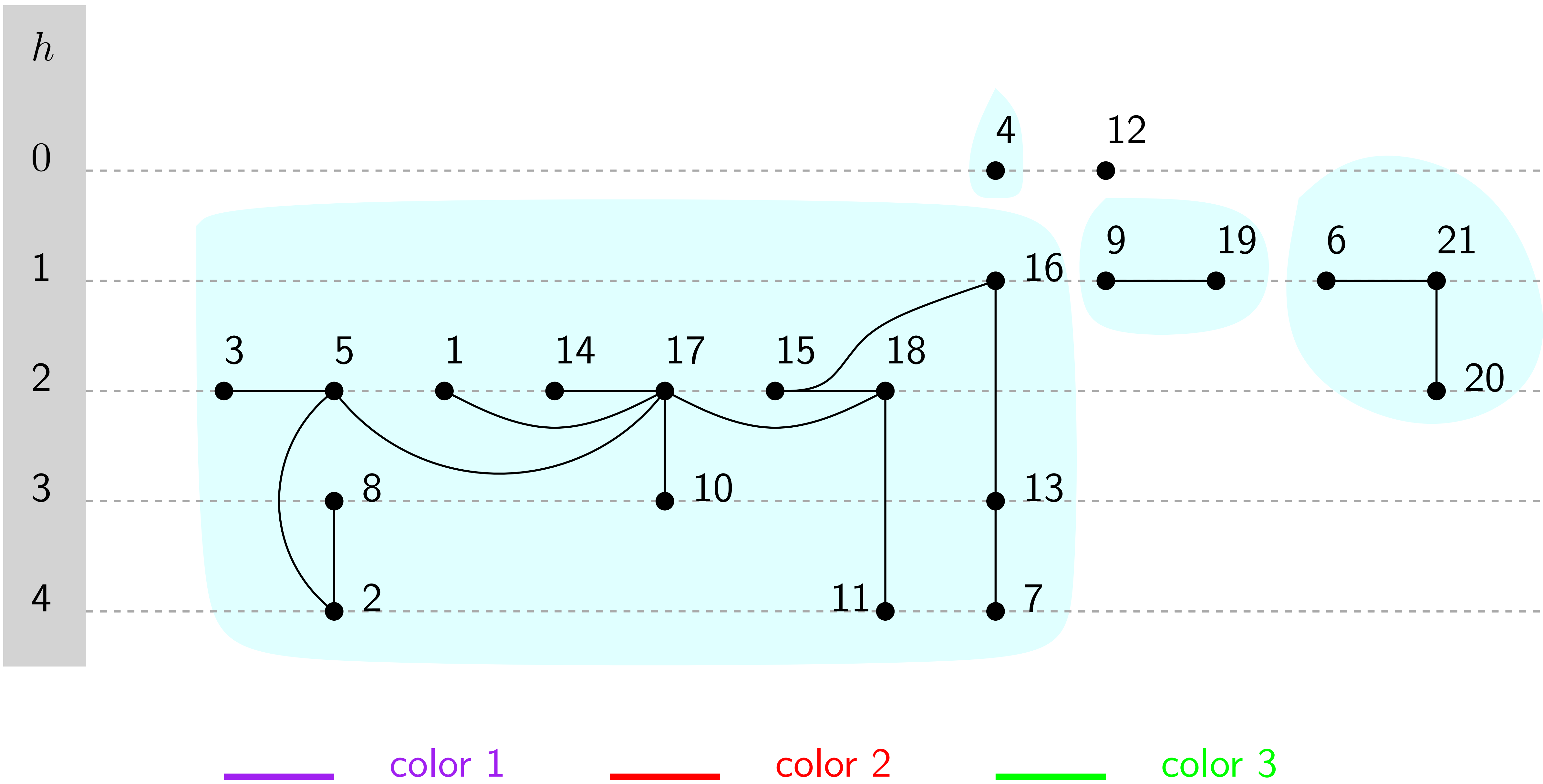
Bijection NBC-trees and decreasing trees

NBC-tree of $K_n^{[0,k]}$



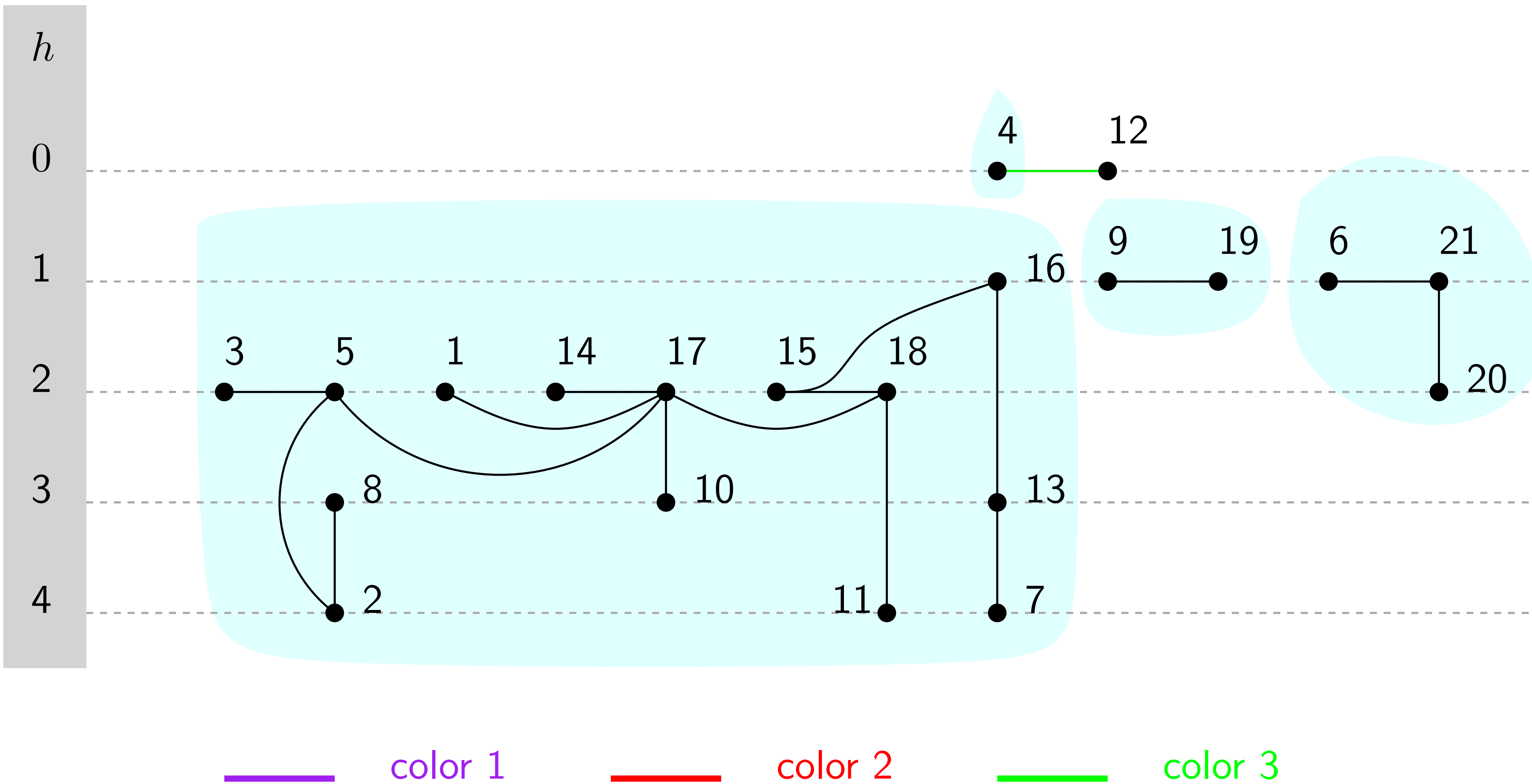
Bijection NBC-trees and decreasing trees

$$4 <_{O_h} 6 <_{O_h} 9 <_{O_h} 5$$



Bijection NBC-trees and decreasing trees

$$\text{label: } \begin{cases} k+1-h(r_i) & \text{if} \quad r_i < r \\ k+2-h(r_i) & \text{if} \quad r_i > r \end{cases}$$



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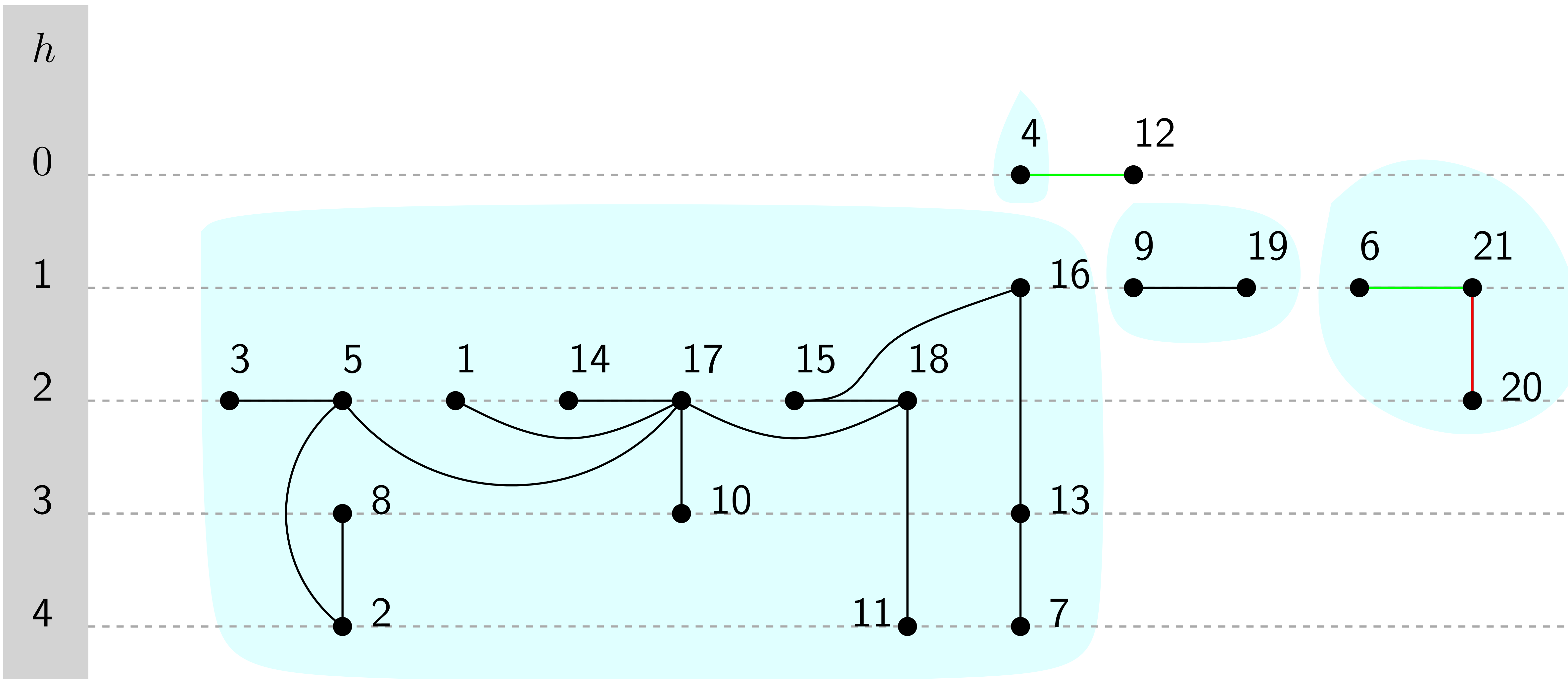

color 1 color 2 color 3

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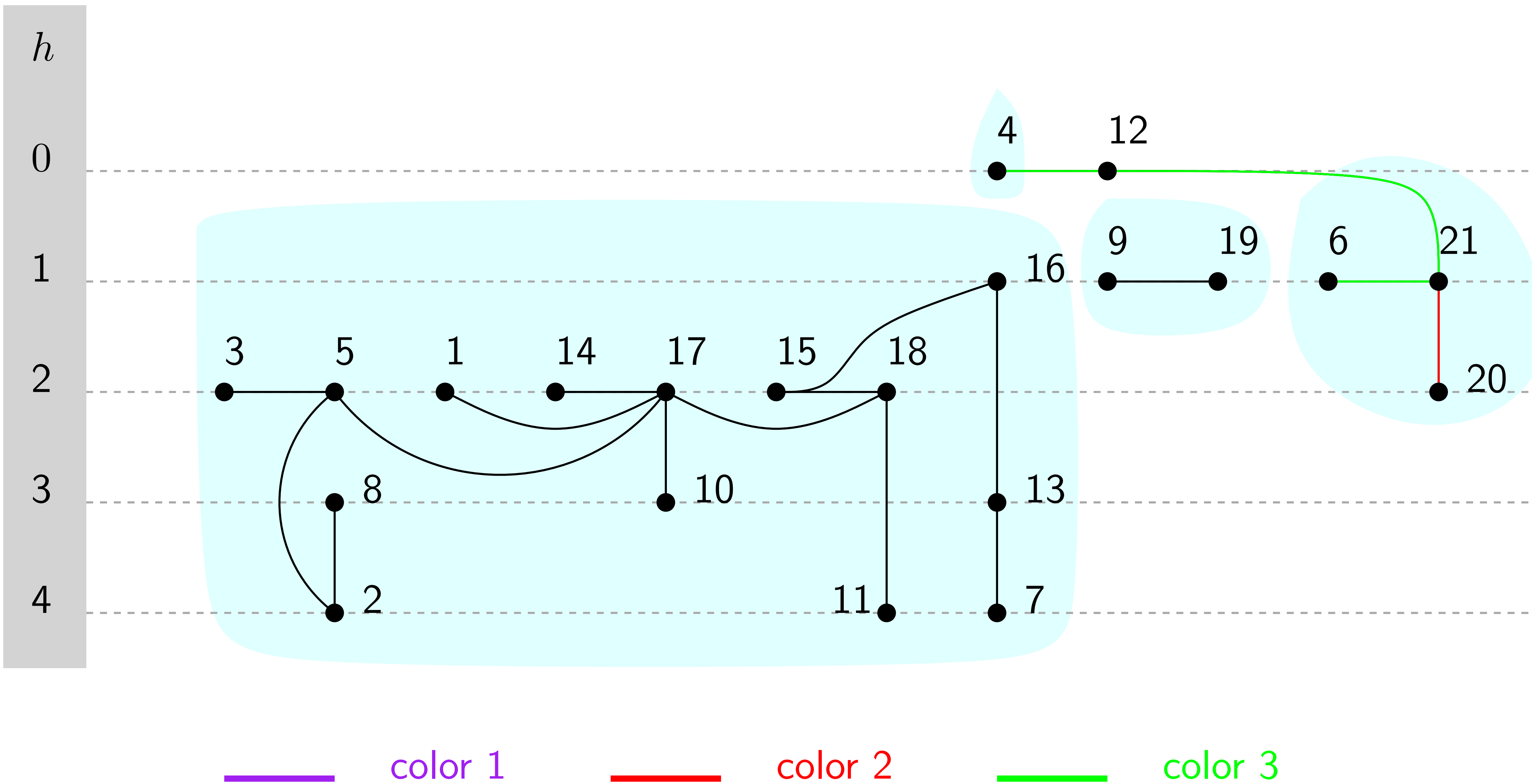

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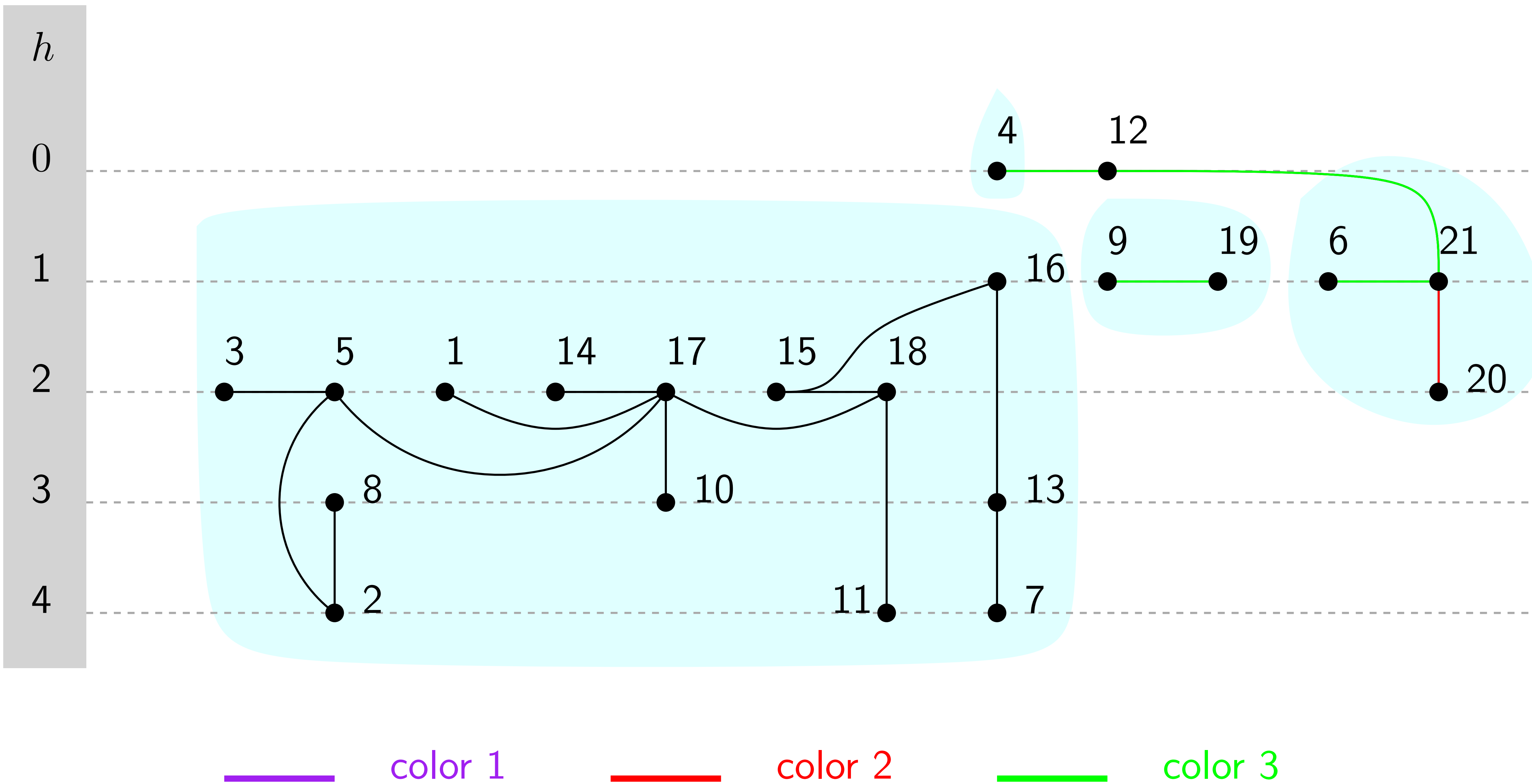

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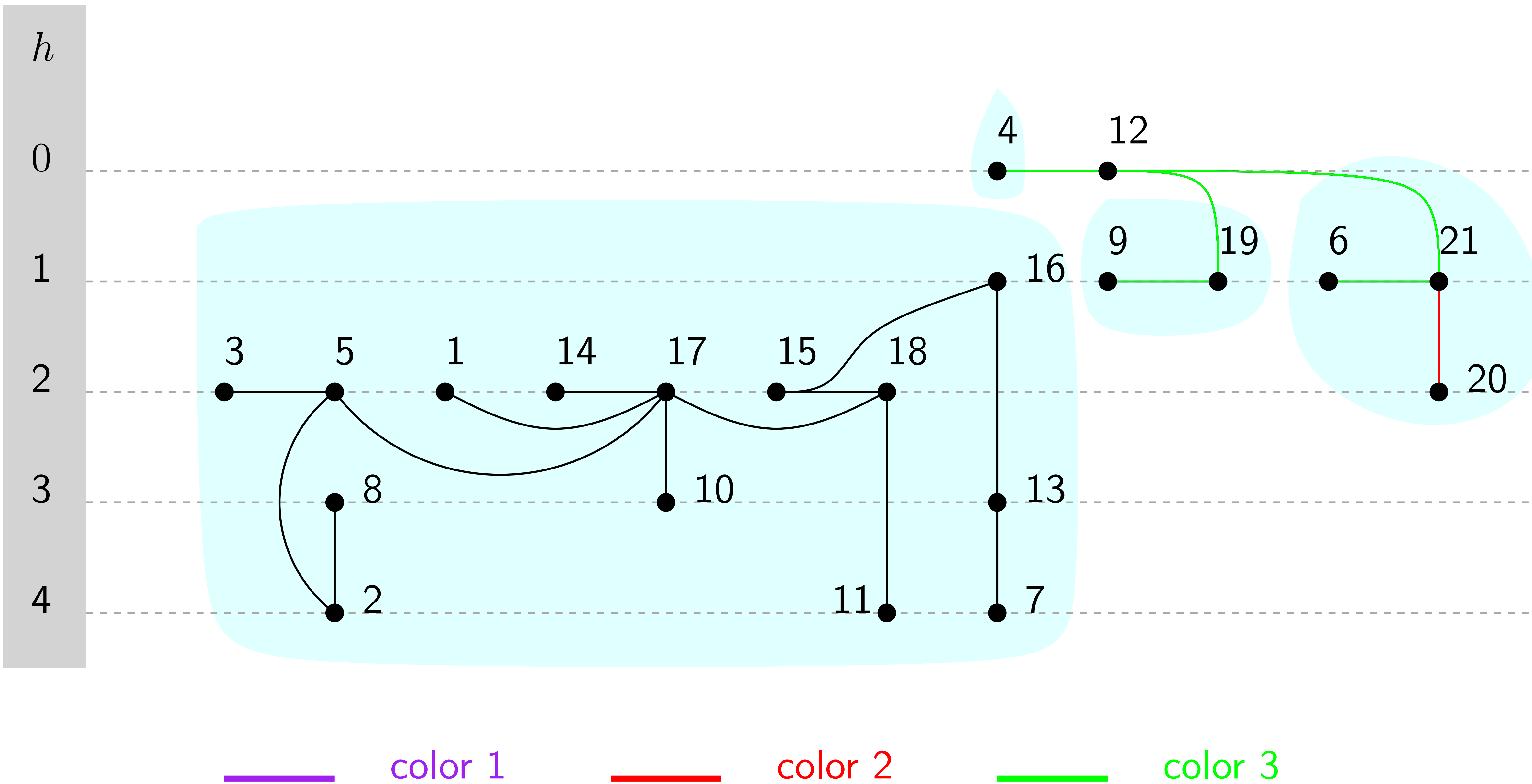

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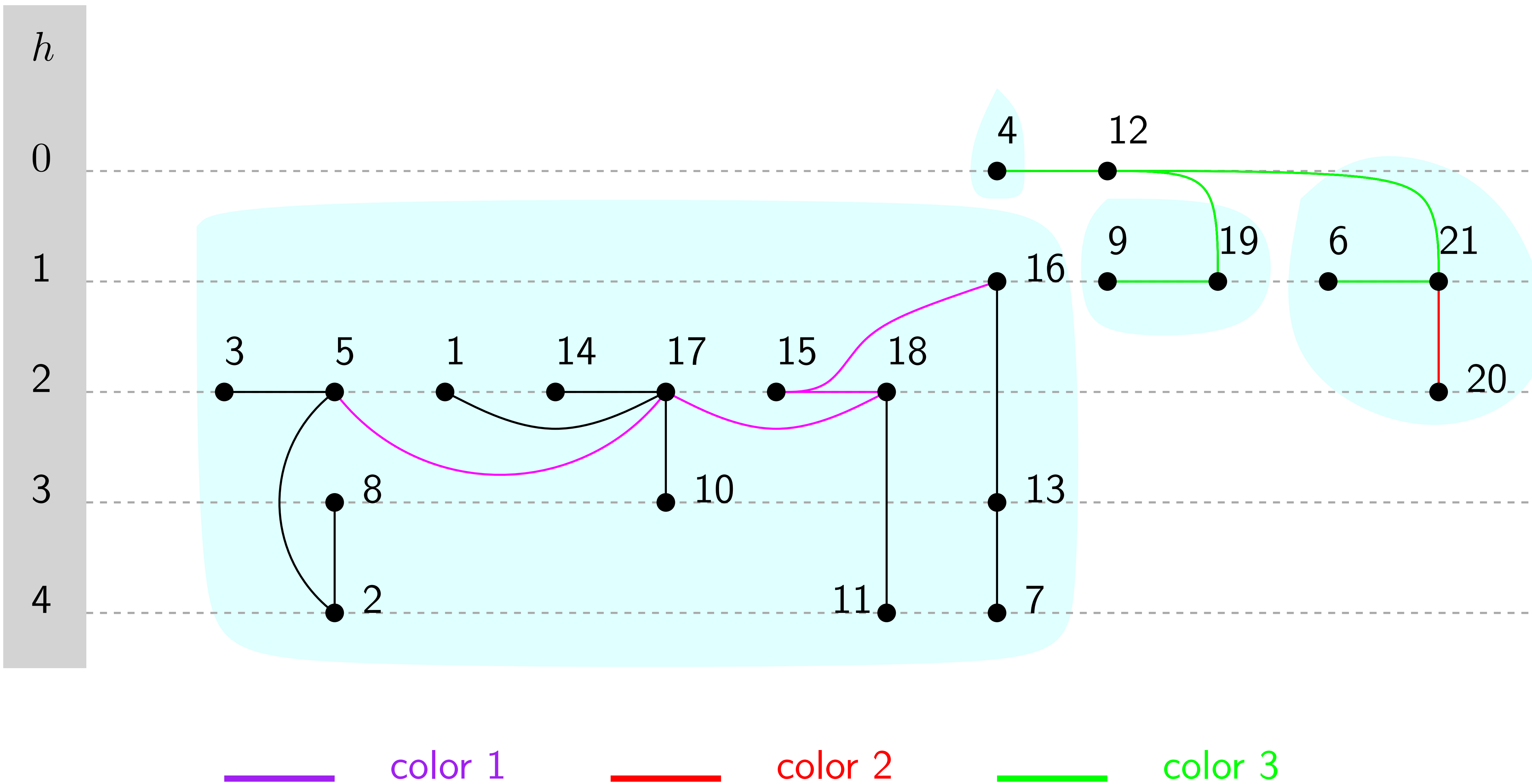

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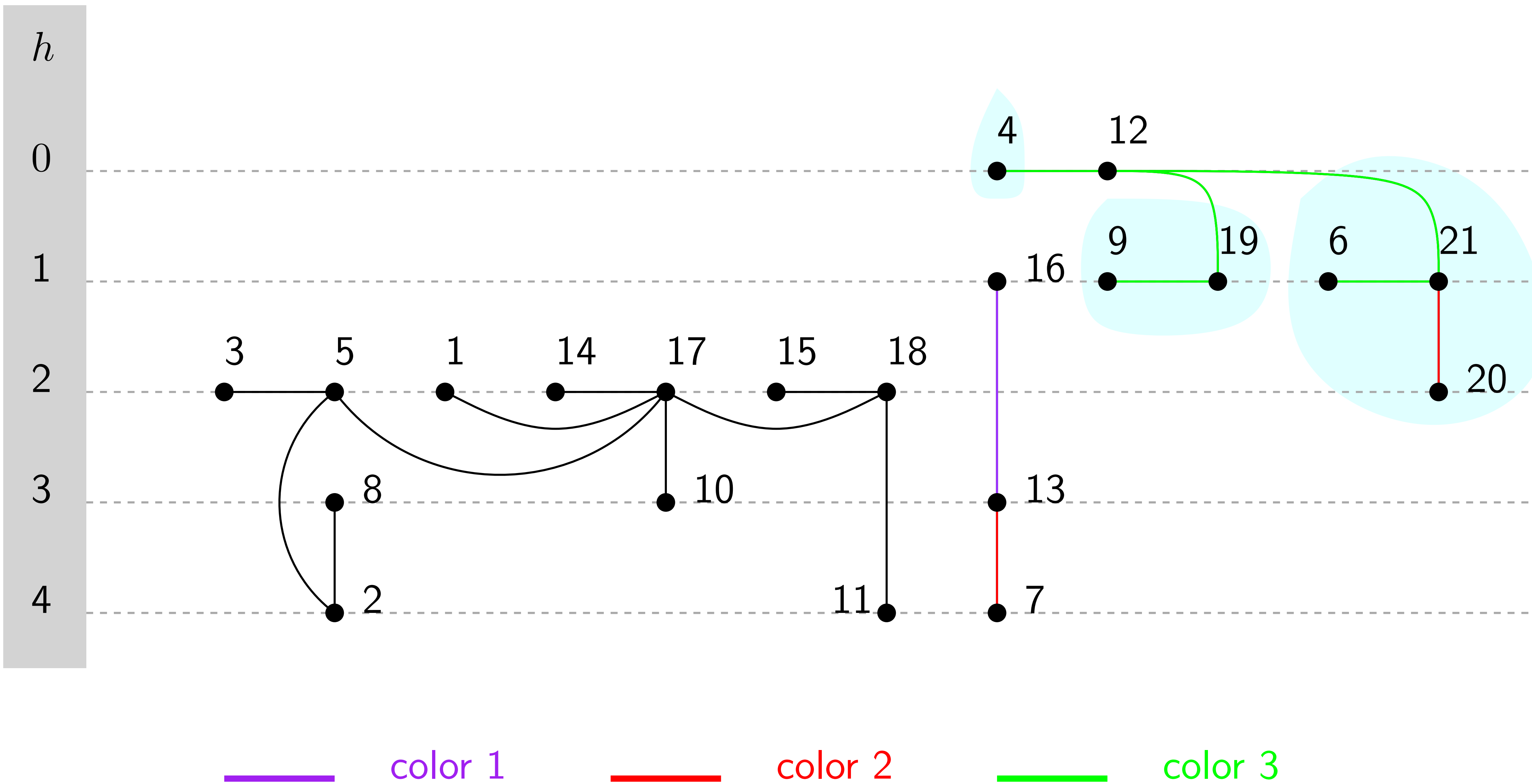

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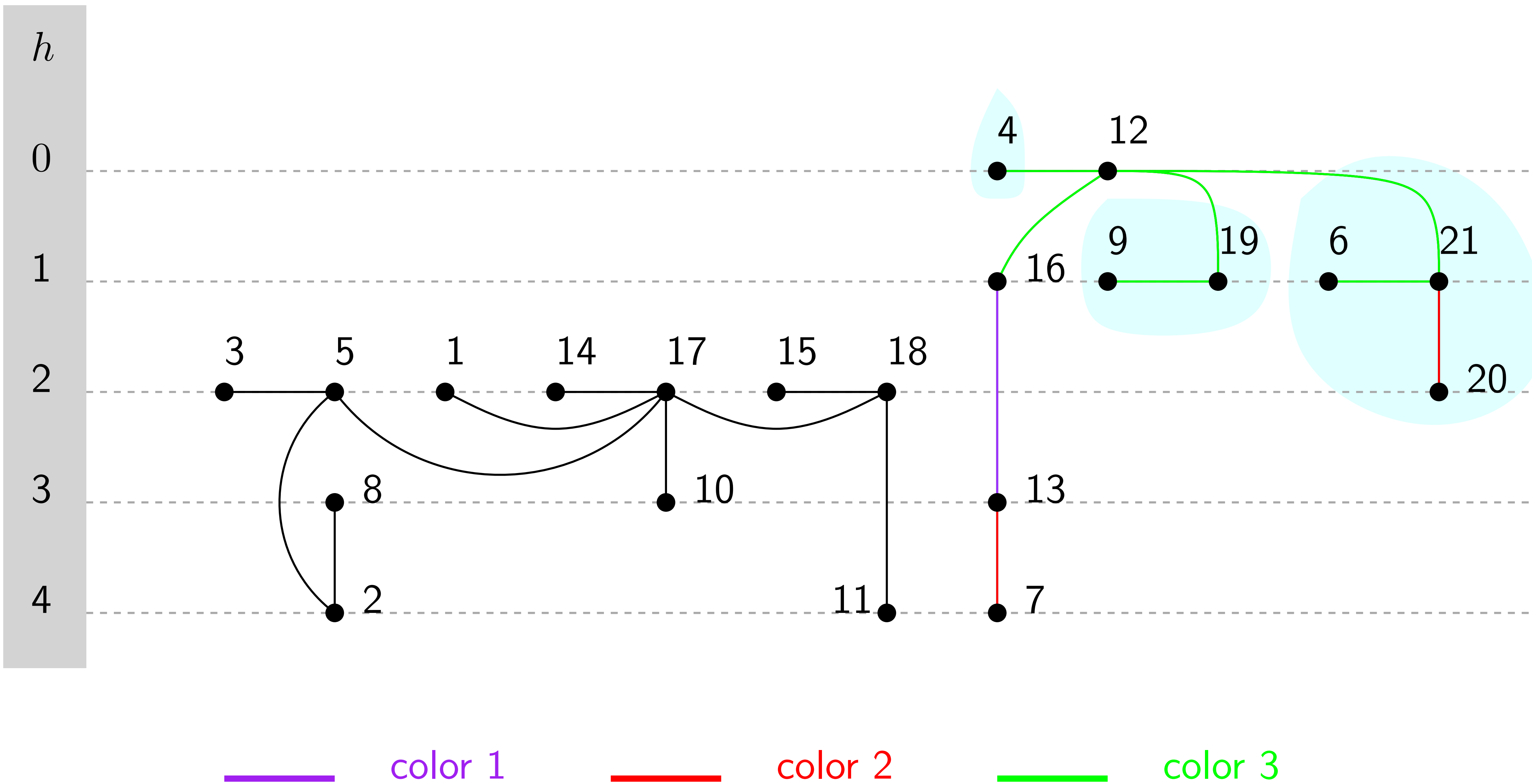

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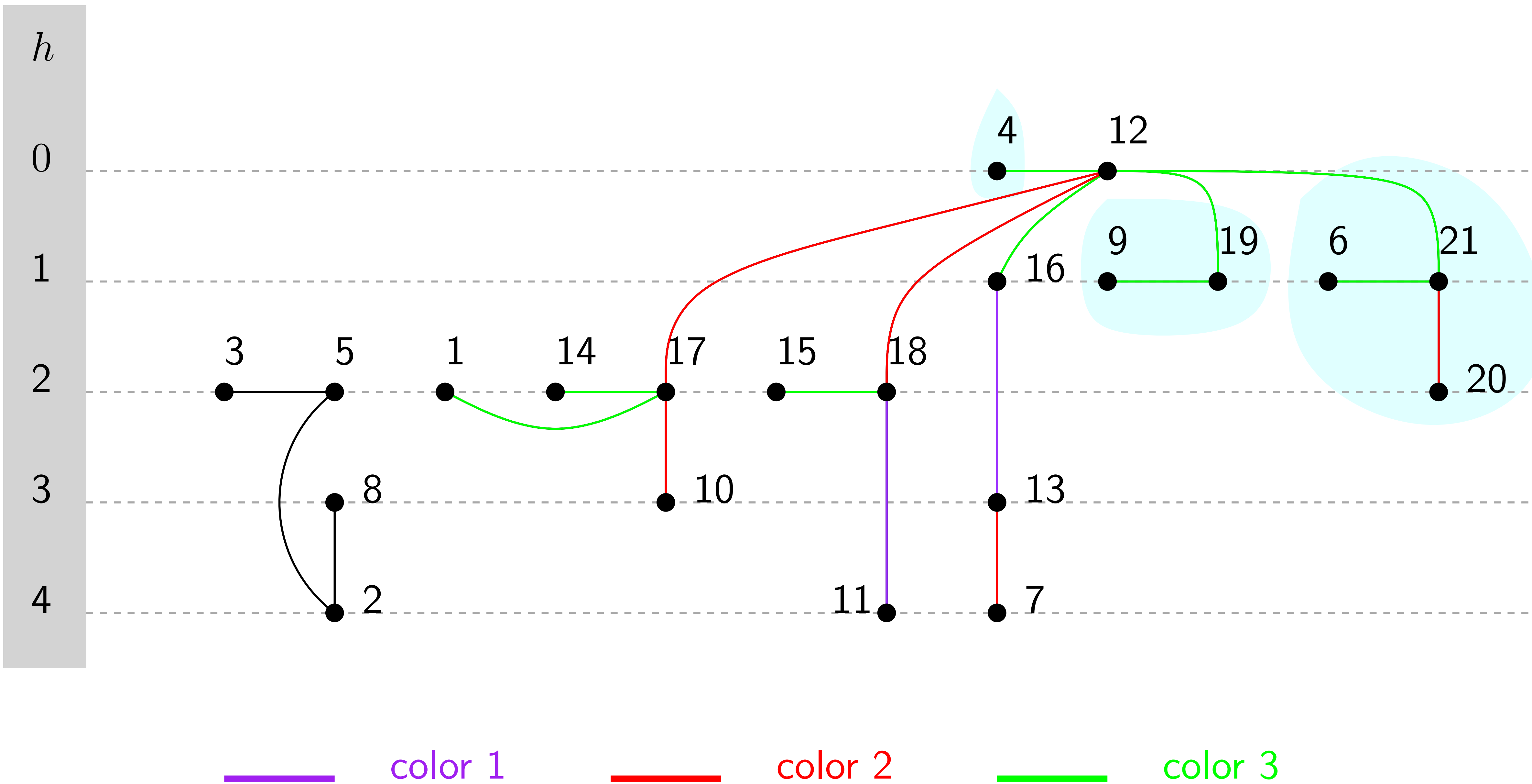

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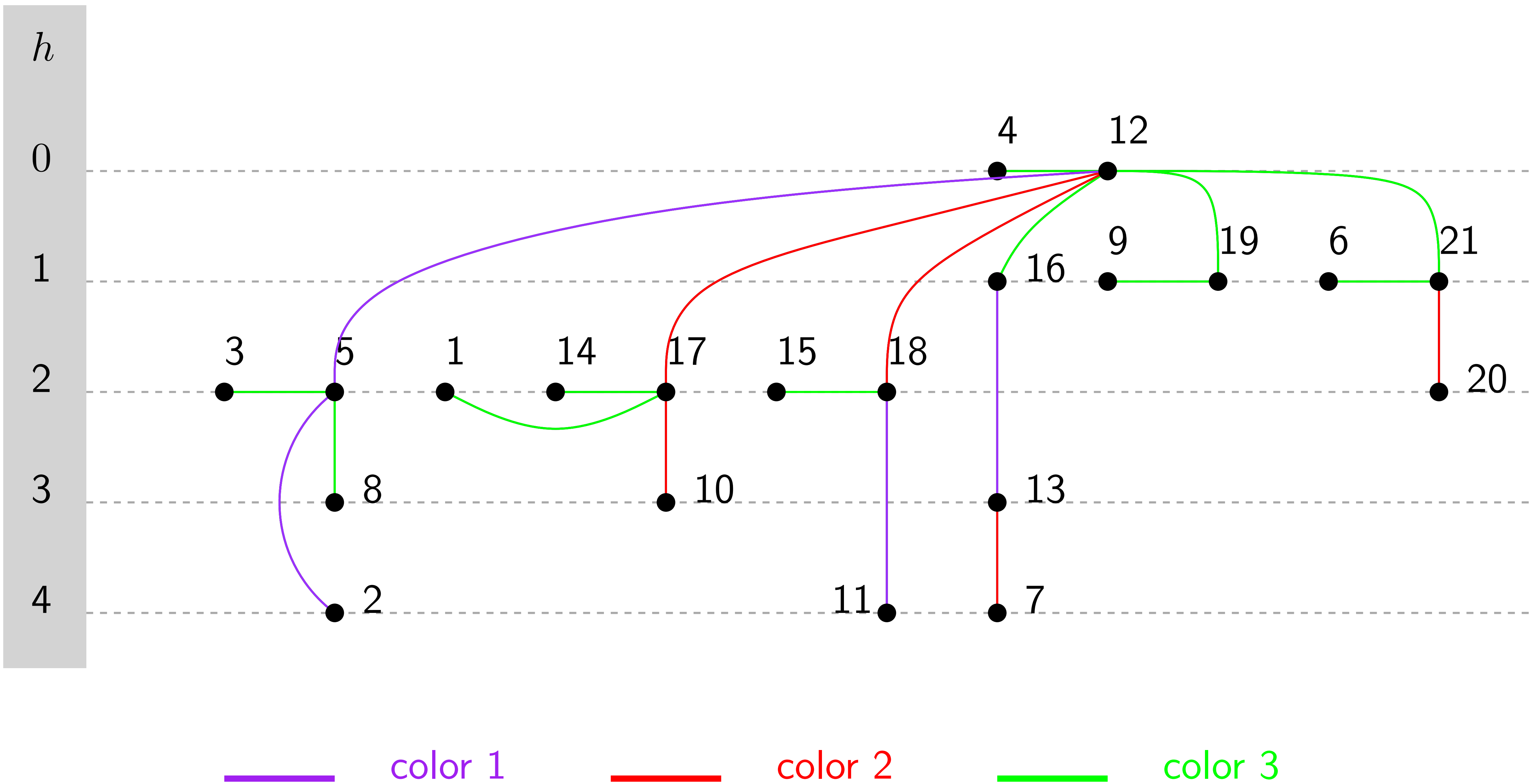
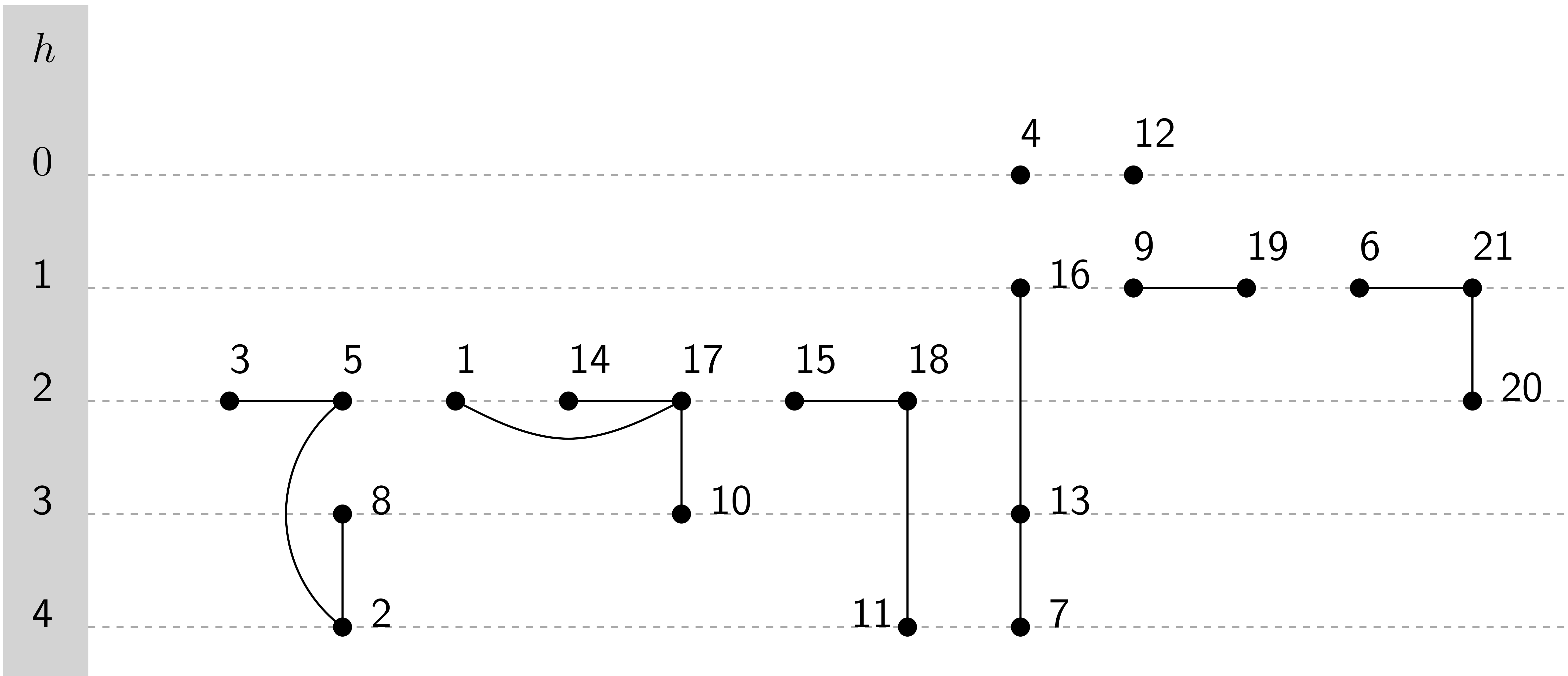


Bijection NBC-trees and decreasing trees

$$5 >_{O_h} 17 >_{O_h} 18 >_{O_h} 16 >_{O_h} 19 >_{O_h} 21 >_{O_h} 4$$



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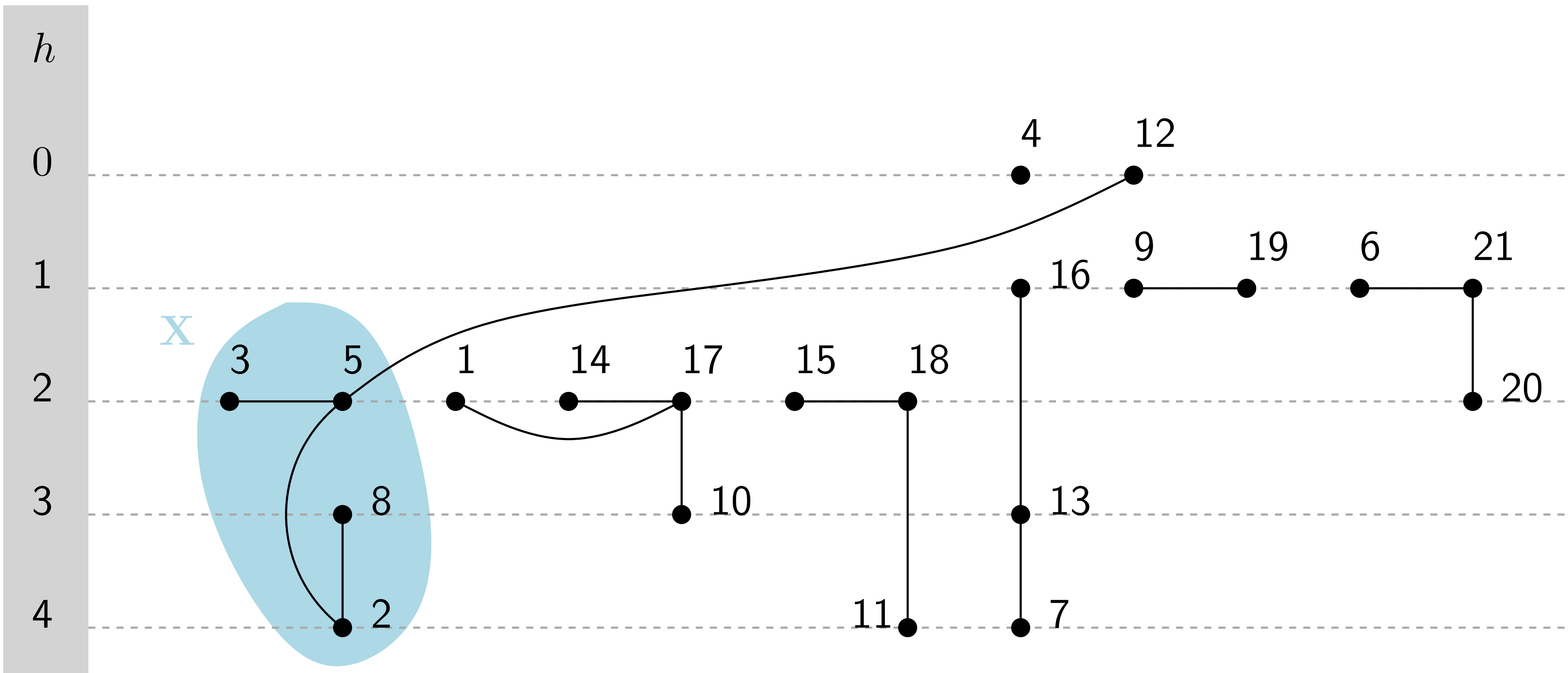

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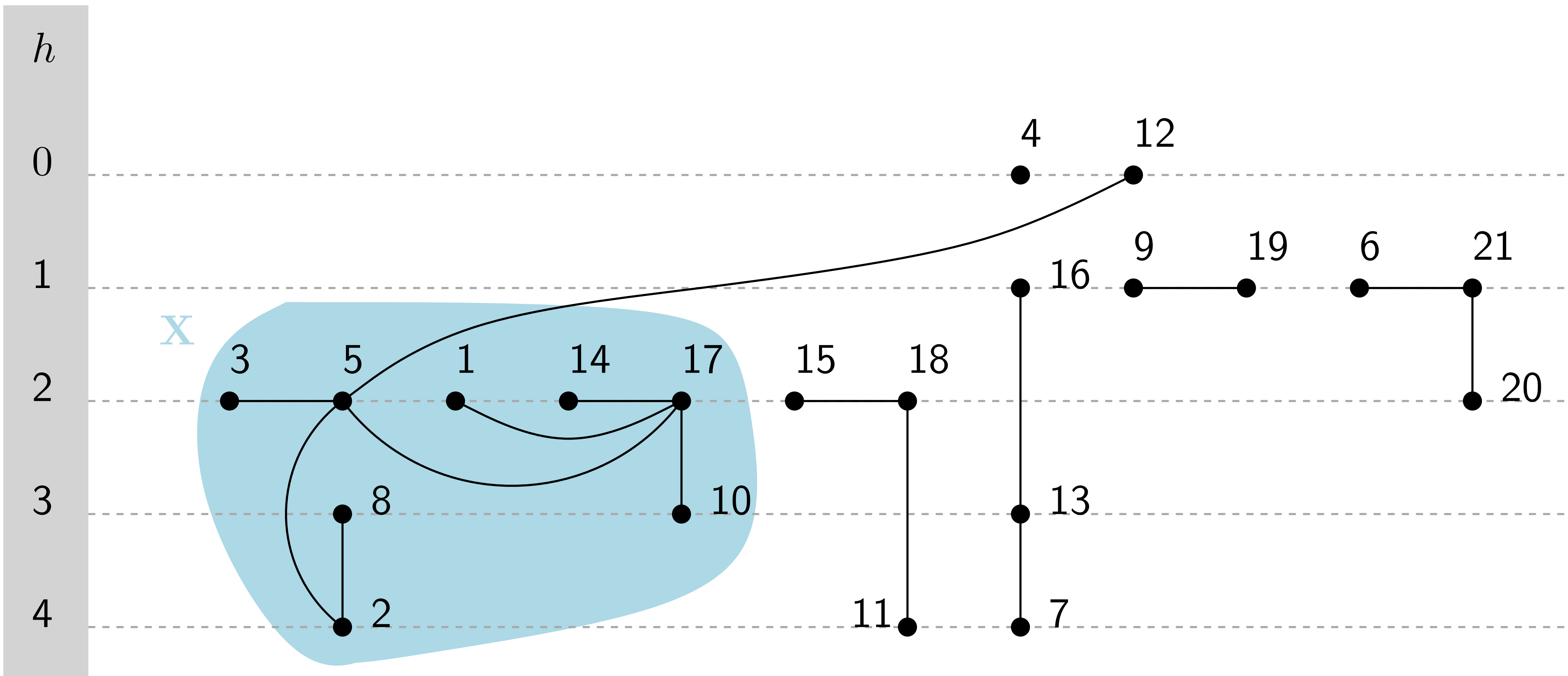

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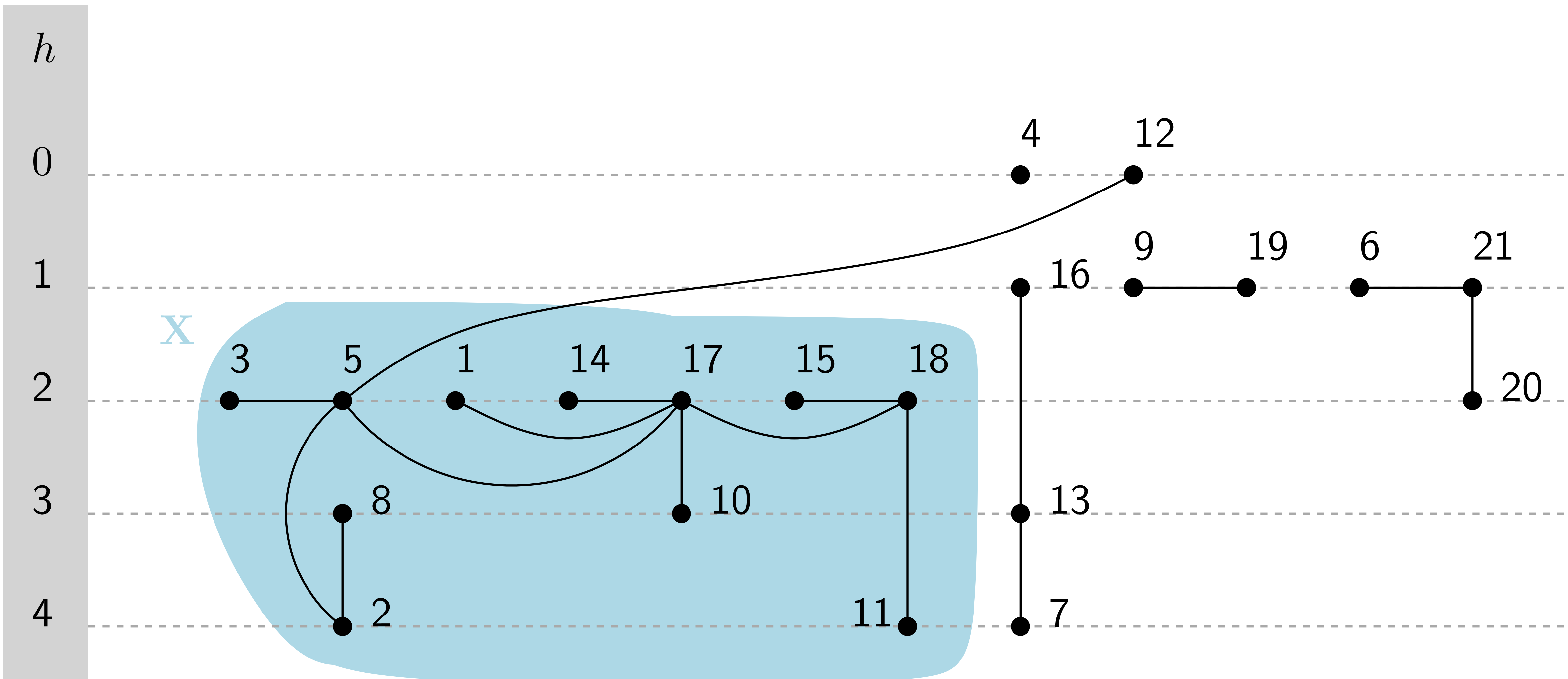

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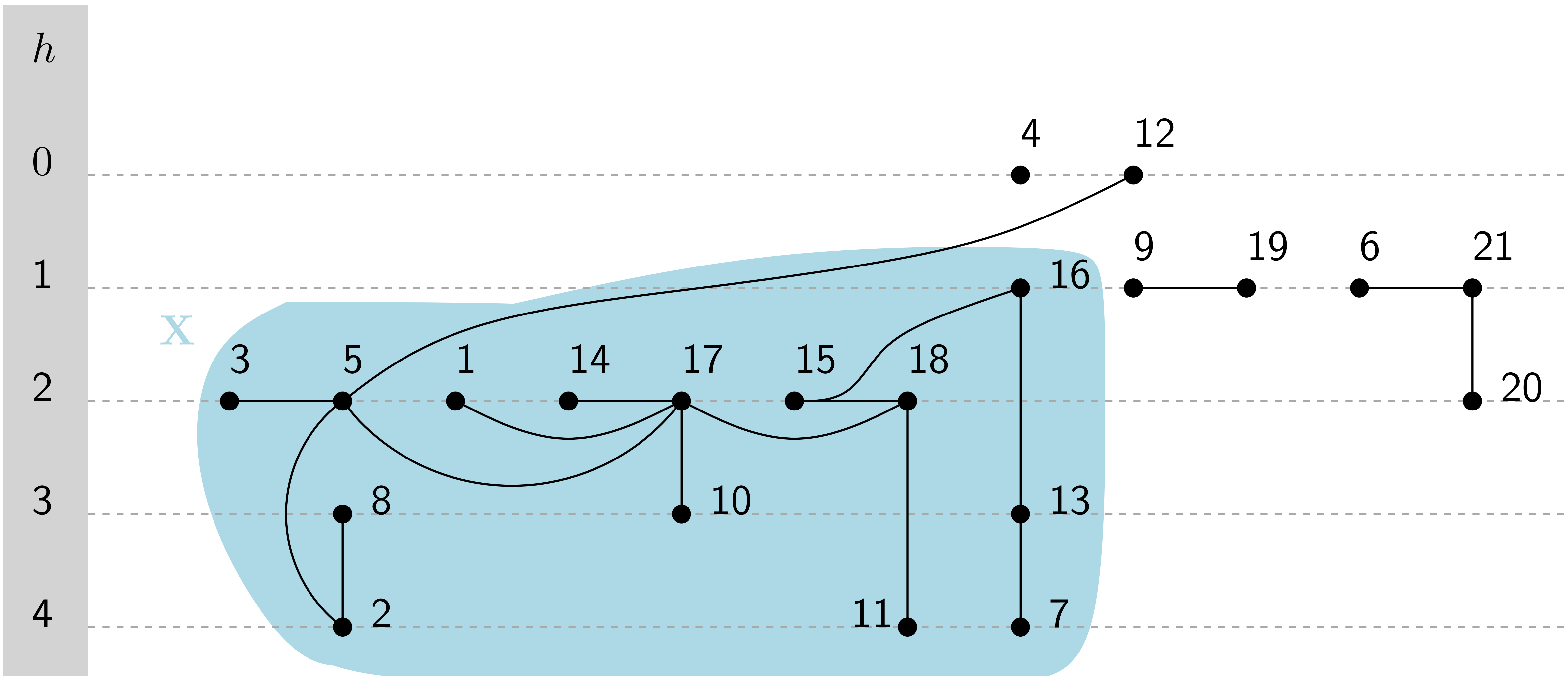

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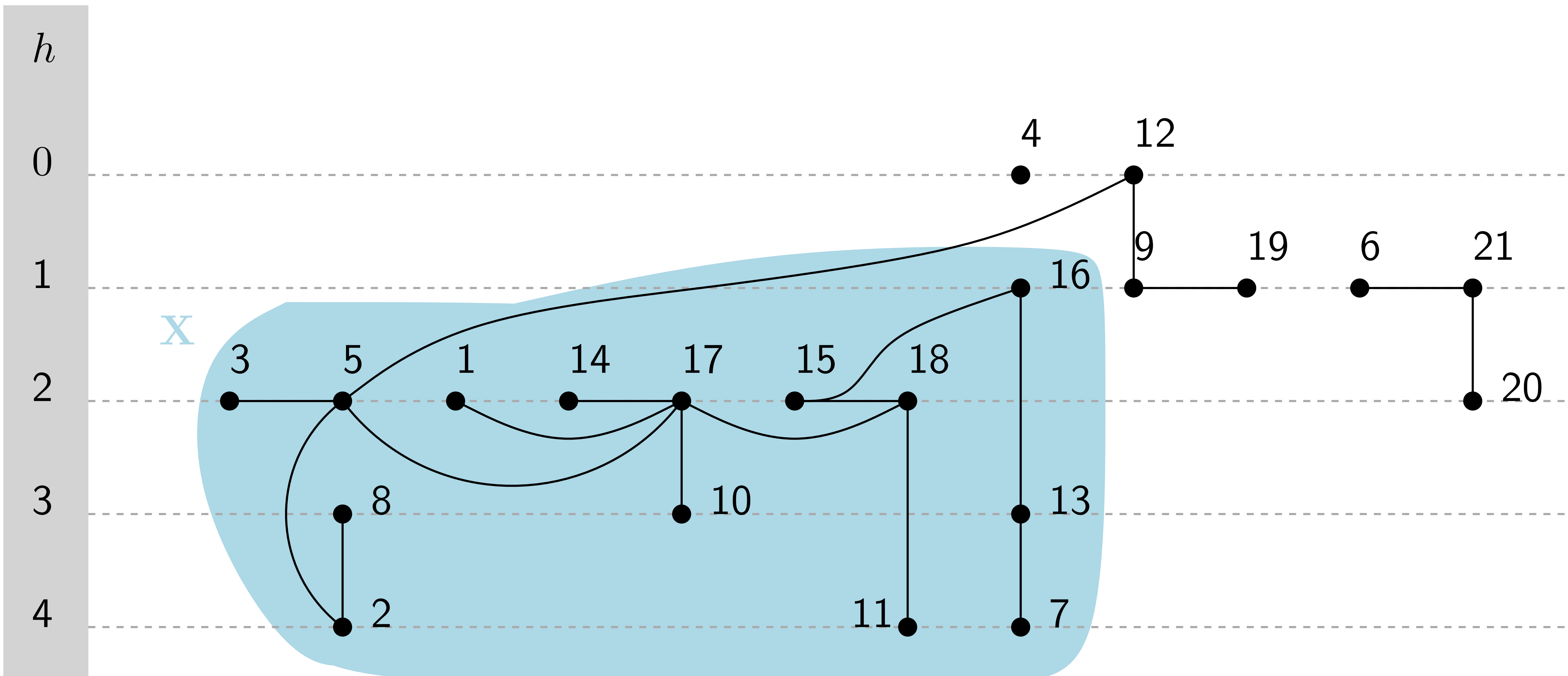

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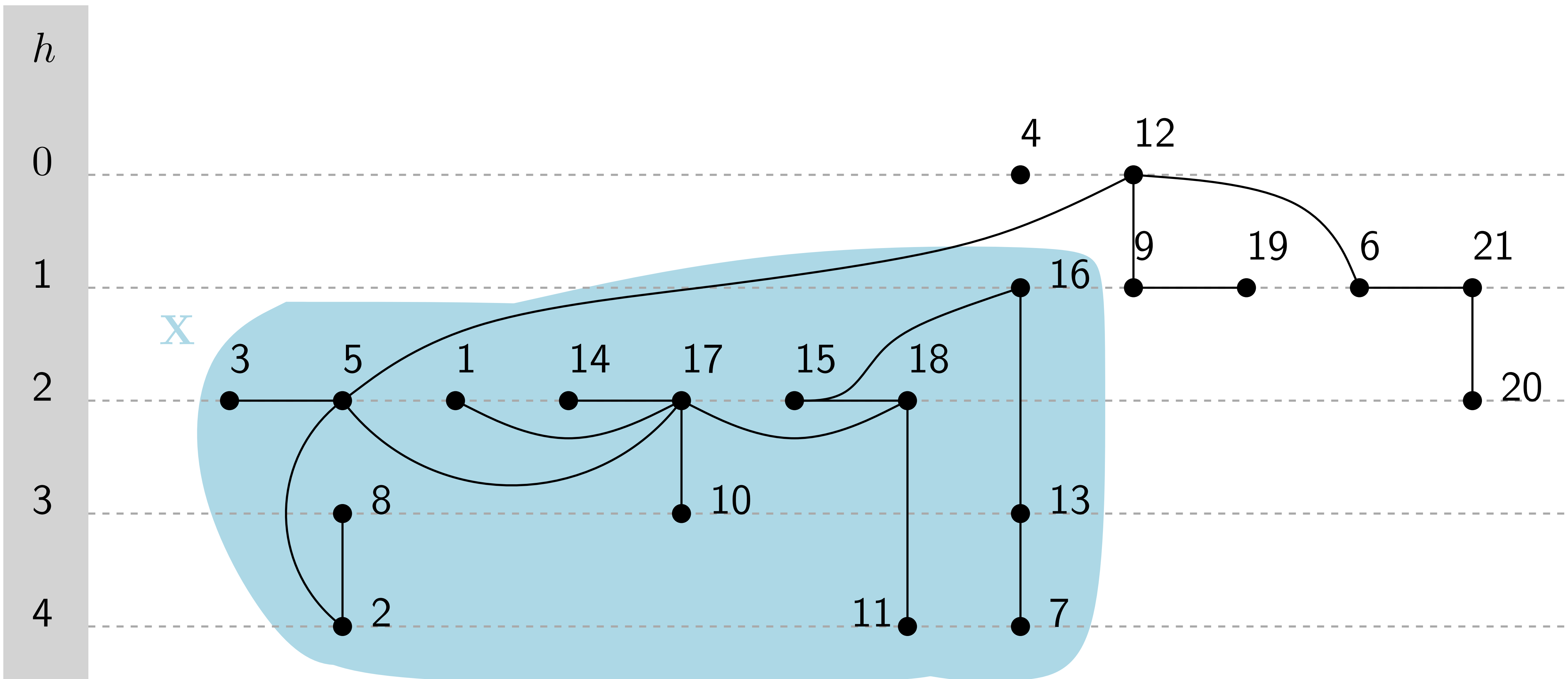

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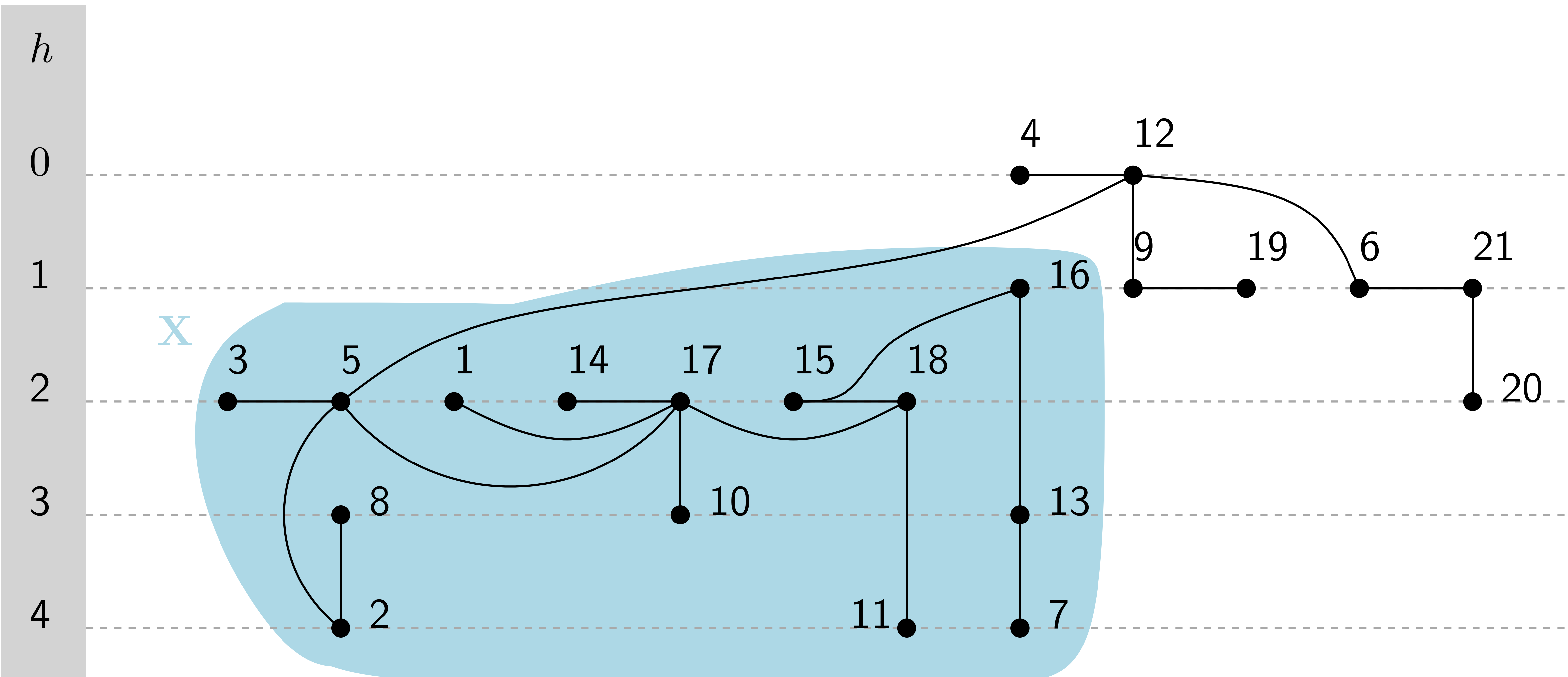
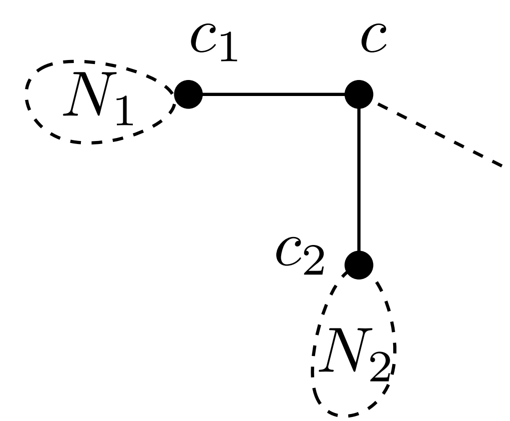


Bijection NBC-trees and decreasing trees

NBC-tree in $K_n^{[-k_2,k_1+k_2]}$



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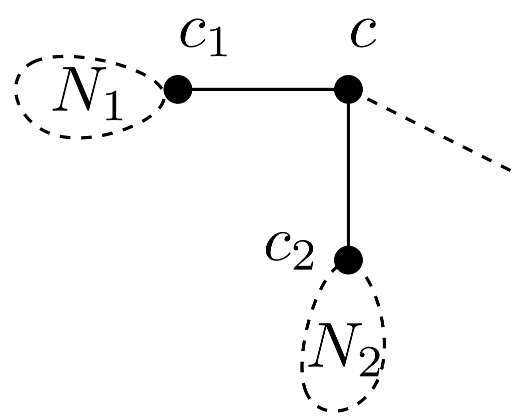
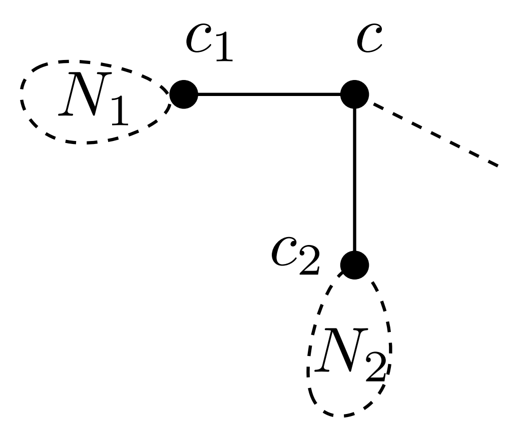


If gain $\in [-k_2, k_2 - 1]$

Bijection NBC-trees and decreasing trees

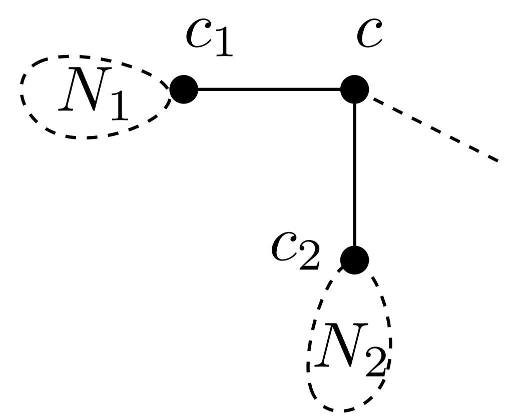
NBC-tree in $K_n^{[-k_2,k_1+k_2]}$



If gain $\in [-k_2, k_2 - 1] \;\Rightarrow\; c_i$ corner of N_i

Bijection NBC-trees and decreasing trees

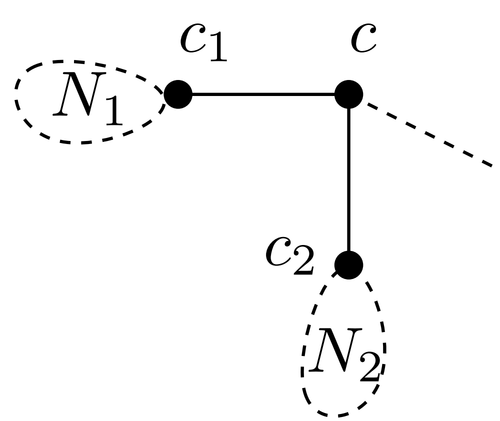
NBC-tree in $K_n^{[-k_2,k_1+k_2]}$



If gain $\in [-k_2, k_2-1] \Rightarrow c_i$ corner of $N_i \Rightarrow (cc_i)$ edge in the decreasing tree of color

Bijection NBC-trees and decreasing trees

NBC-tree in $K_n^{[-k_2,k_1+k_2]}$



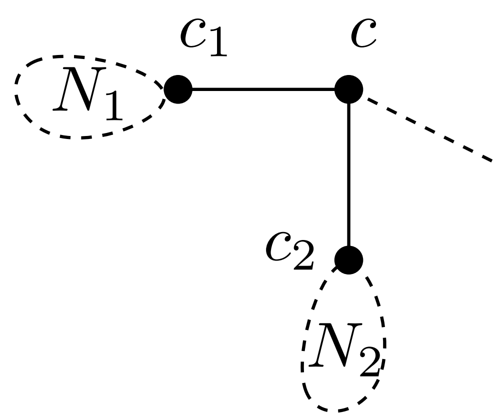
If gain $\in [-k_2, k_2-1] \;\Rightarrow\; c_i$ corner of $N_i \;\Rightarrow\; (cc_i)$ edge in the decreasing tree of color

$k_1+k_2-h(c_i)$ if $c_i < c$ $\qquad$ ($h(c_i) =$ gain ≥ 0)

$k_1+k_2+1-h(c_i)$ if $c_i > c$ $\qquad$ ($h(c_i) = -$gain > 0)

Bijection NBC-trees and decreasing trees

NBC-tree in $K_n^{[-k_2,k_1+k_2]}$



If gain $\in [-k_2, k_2-1]$ $\Rightarrow$ c_i corner of N_i $\Rightarrow$ (cc_i) edge in the decreasing tree of color

$k_1+k_2-h(c_i)$ if $c_i < c$ $\qquad$ $(h(c_i) =$ gain $\geq 0)$

$k_1+k_2+1-h(c_i)$ if $c_i > c$ $\qquad$ $(h(c_i) = -$gain $> 0)$

If gain $\in [k_2, k_1+k_2]$ $\qquad$ apply the preceding bijection with the colors in $[1, k_1+1]$

$\longleftrightarrow$ $\qquad$ (k_1+1, k_2)- decreasing trees

General results

Theorem [CFM 15]. There is a bijection between the regions of the arrangements $\mathcal{A}_n^{ab}$ and

- If $a \leq 0$ and $b \geq 0$, the $(a+b+1, -a)$-decreasing forests with n vertices
- If $a \leq 0$ and $b \geq 0$ or $a = 1$ and $b \geq 1$, the $(a+b-1, 1-a)$-non-increasing forests with n vertices

General results

Regions of the arrangements $\mathcal{A}_n^{ab}$

Theorem [CFM 15]. There is a bijection between the regions of the arrangements $\mathcal{A}_n^{ab}$ and

- If $a \leq 0$ and $b \geq 0$, the $(a+b+1, -a)$-decreasing forests with n vertices
- If $a \leq 0$ and $b \geq 0$ or $a = 1$ and $b \geq 1$, the $(a+b-1, 1-a)$-non-increasing forests with n vertices

D. Forge (2015)

Other order O'_h

$i <_{O'_h} j$ if $h(i) < h(j)$ or $h(i) = h(j)$ and $i < j$

Enumerative consequences

Theorem [CFM 15] There exists a bijection between (k, j)-decreasing trees with n vertices and $(k-2, j+1)$-non-increasing trees with n vertices

[Conjectured first thanks to Sage and F. Chapoton]

The case of Shi

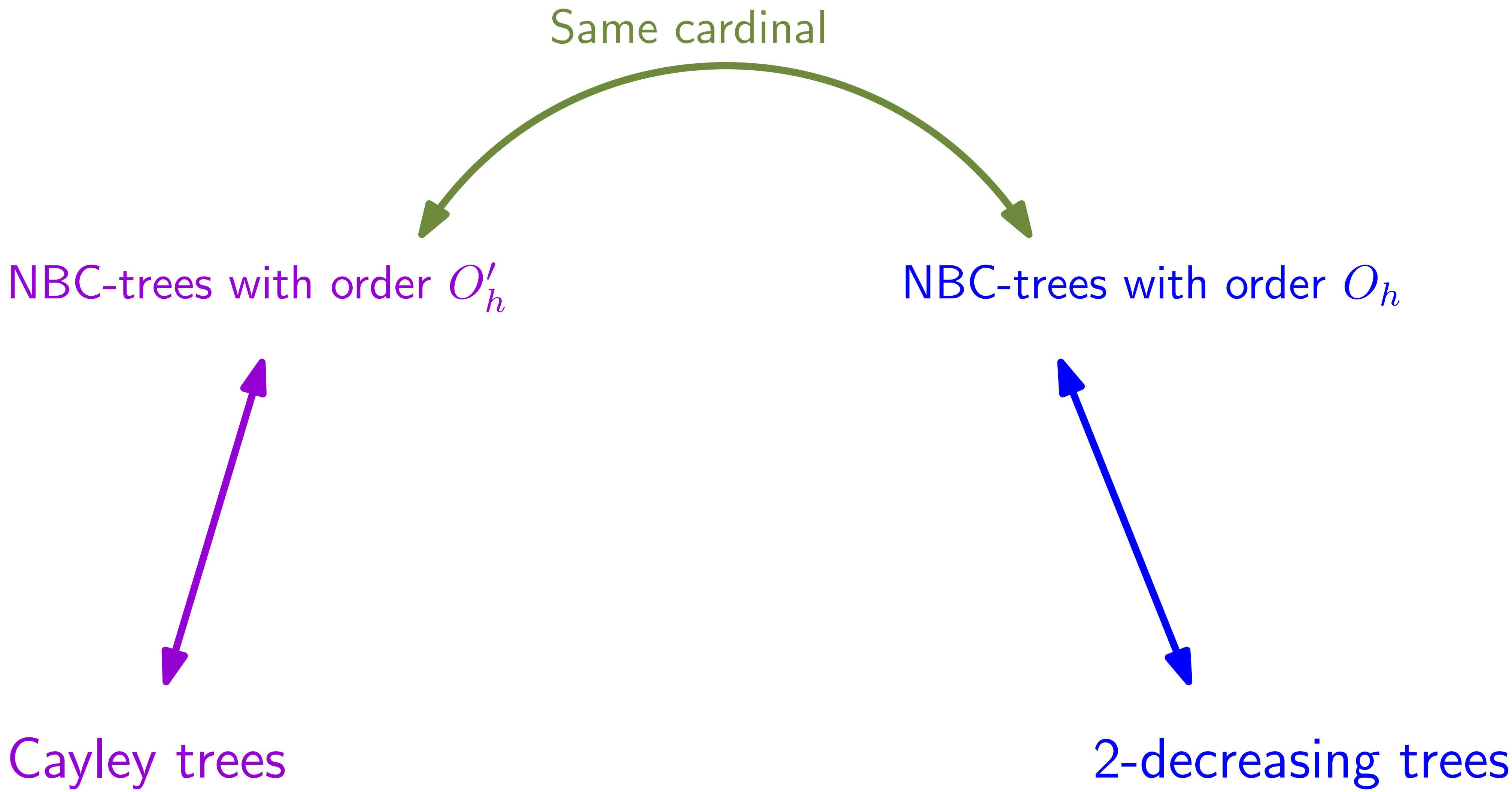
NBC-trees with order O'_h

NBC-trees with order O_h

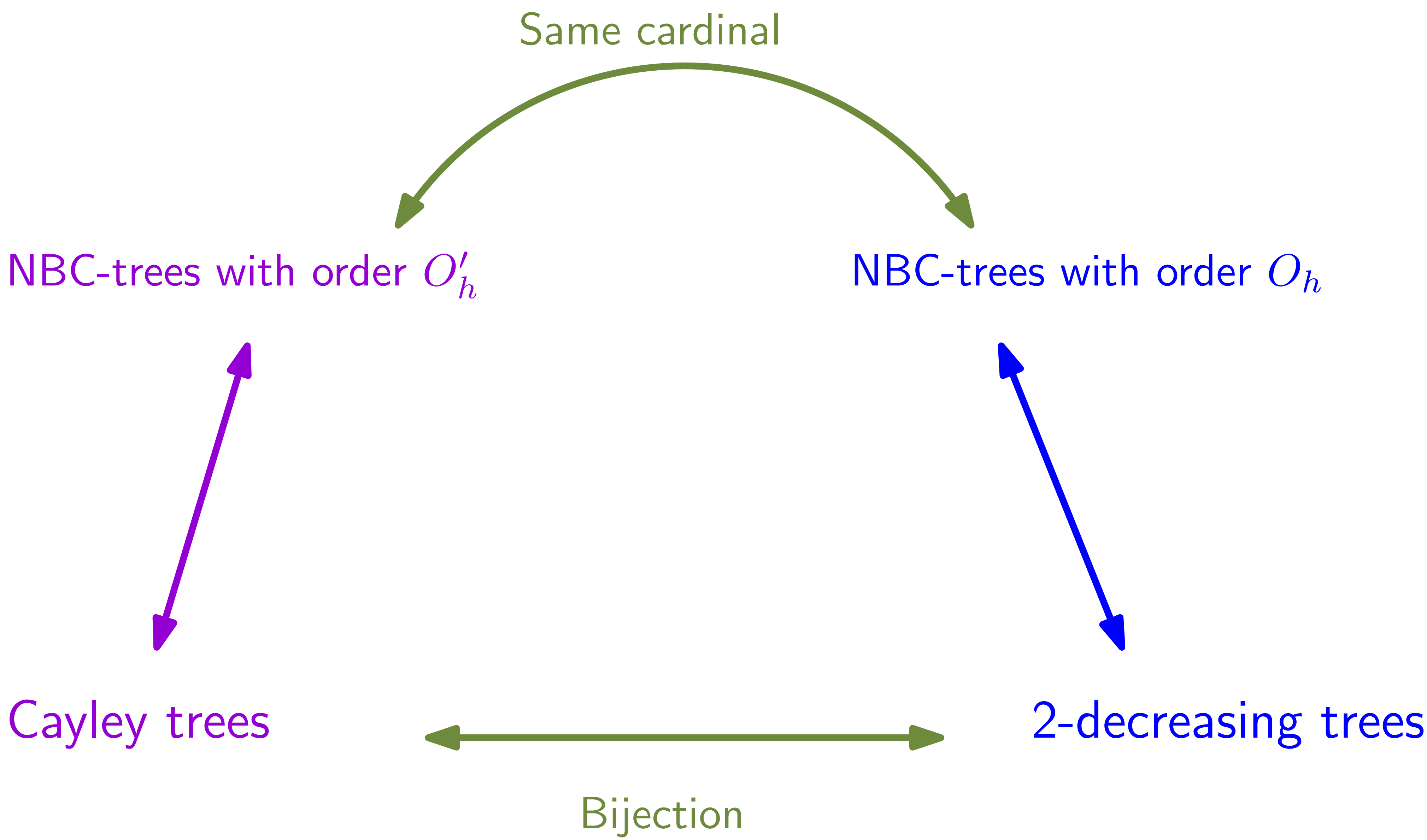
Cayley trees

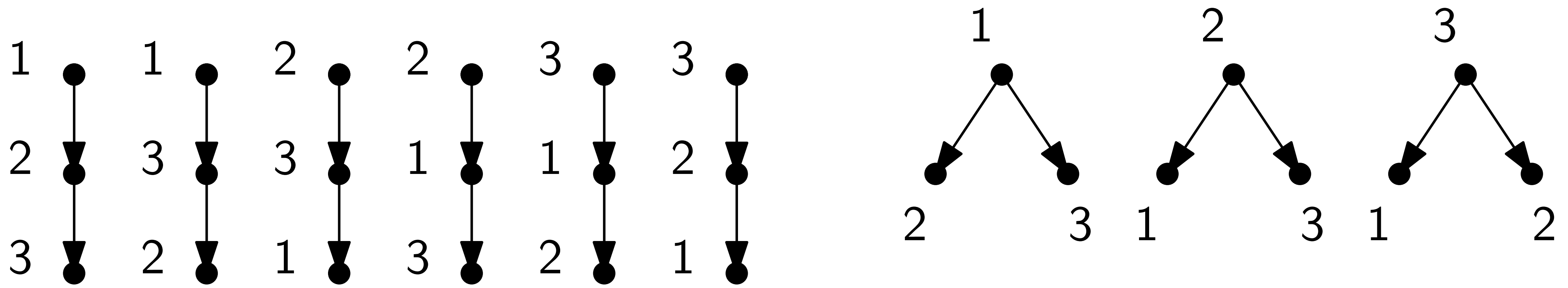
2-decreasing trees

The case of Shi

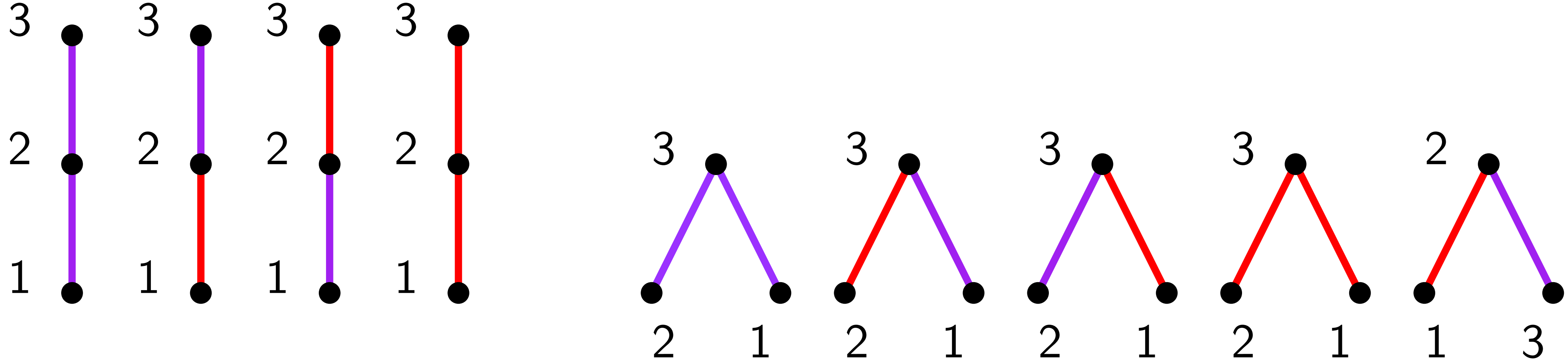


The case of Shi

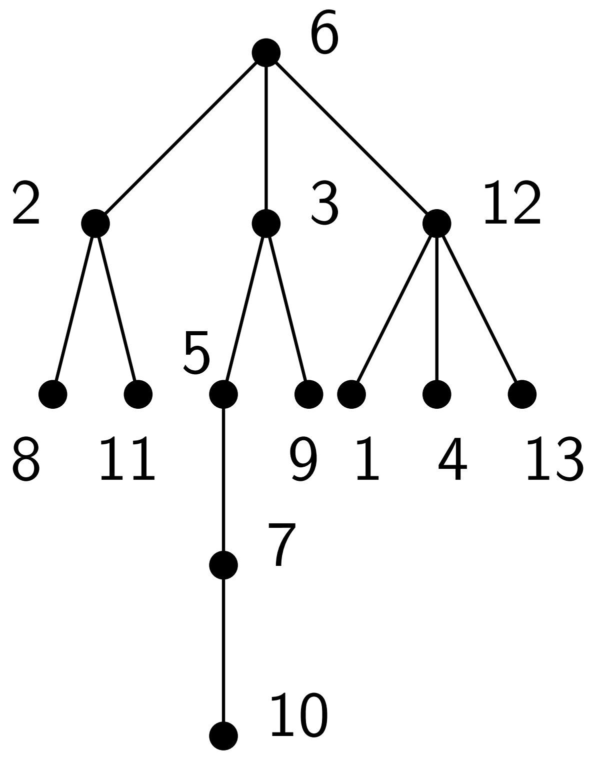




Bijection Cayley $\leftrightarrow$ (2,0)-decreasing



Cayley trees ↔ 2-decreasing trees



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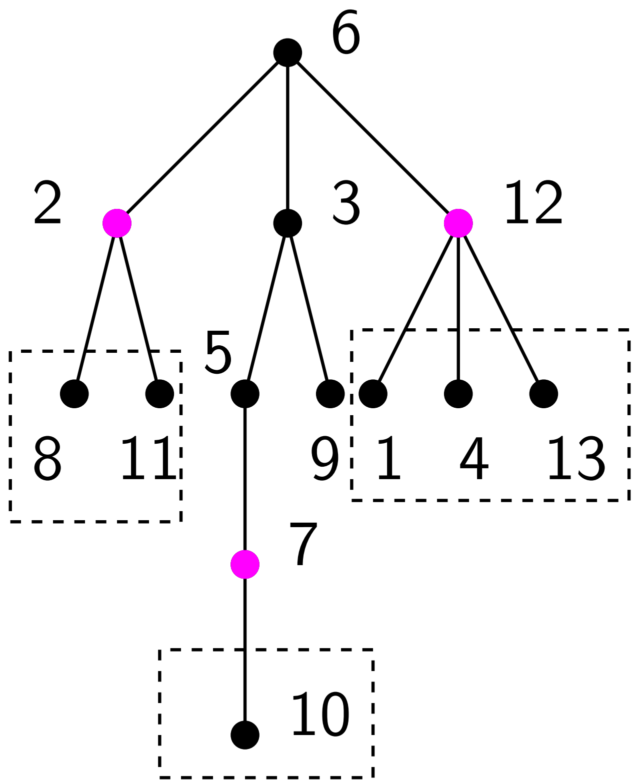
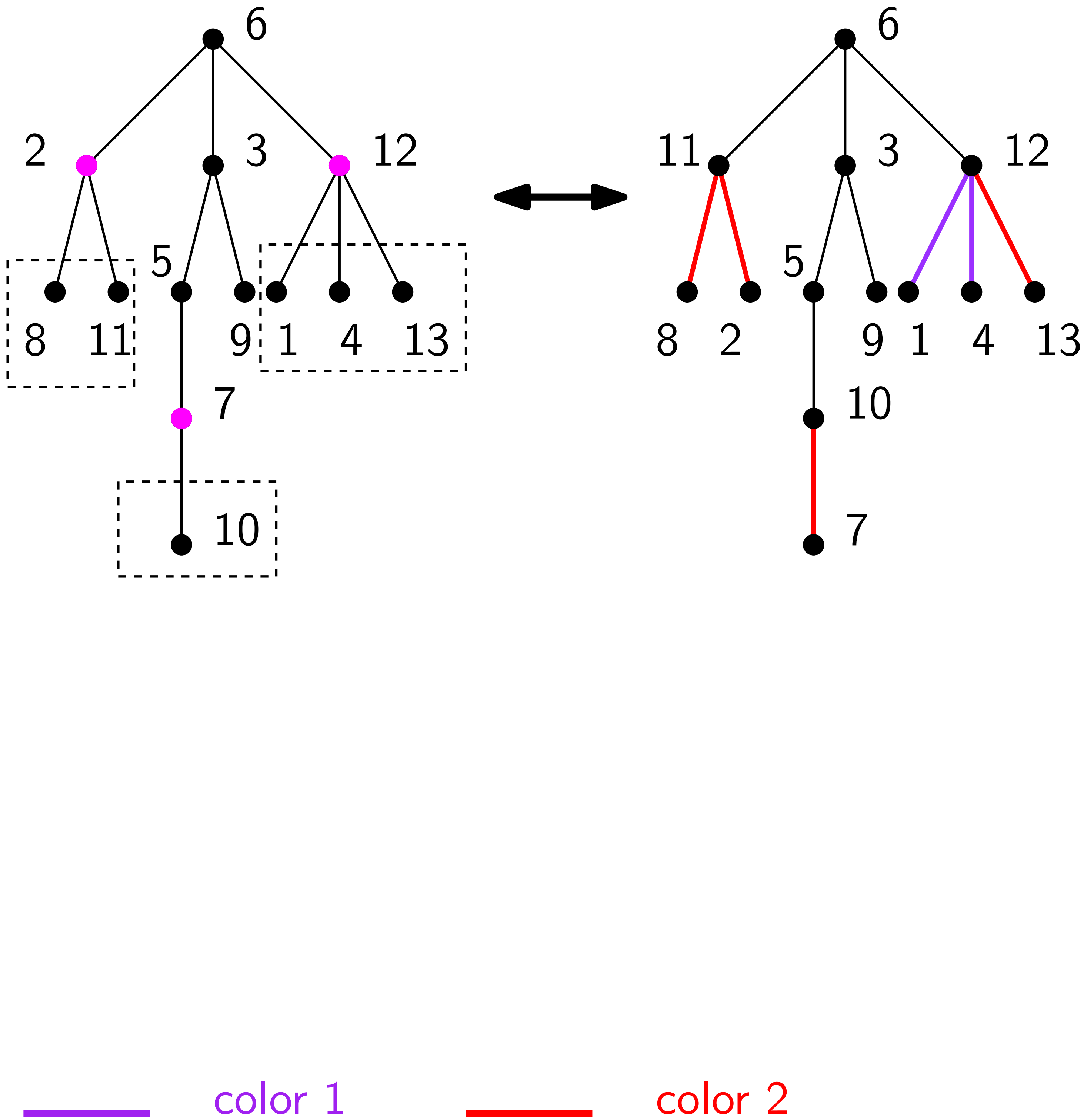


Cayley trees ↔ 2-decreasing trees



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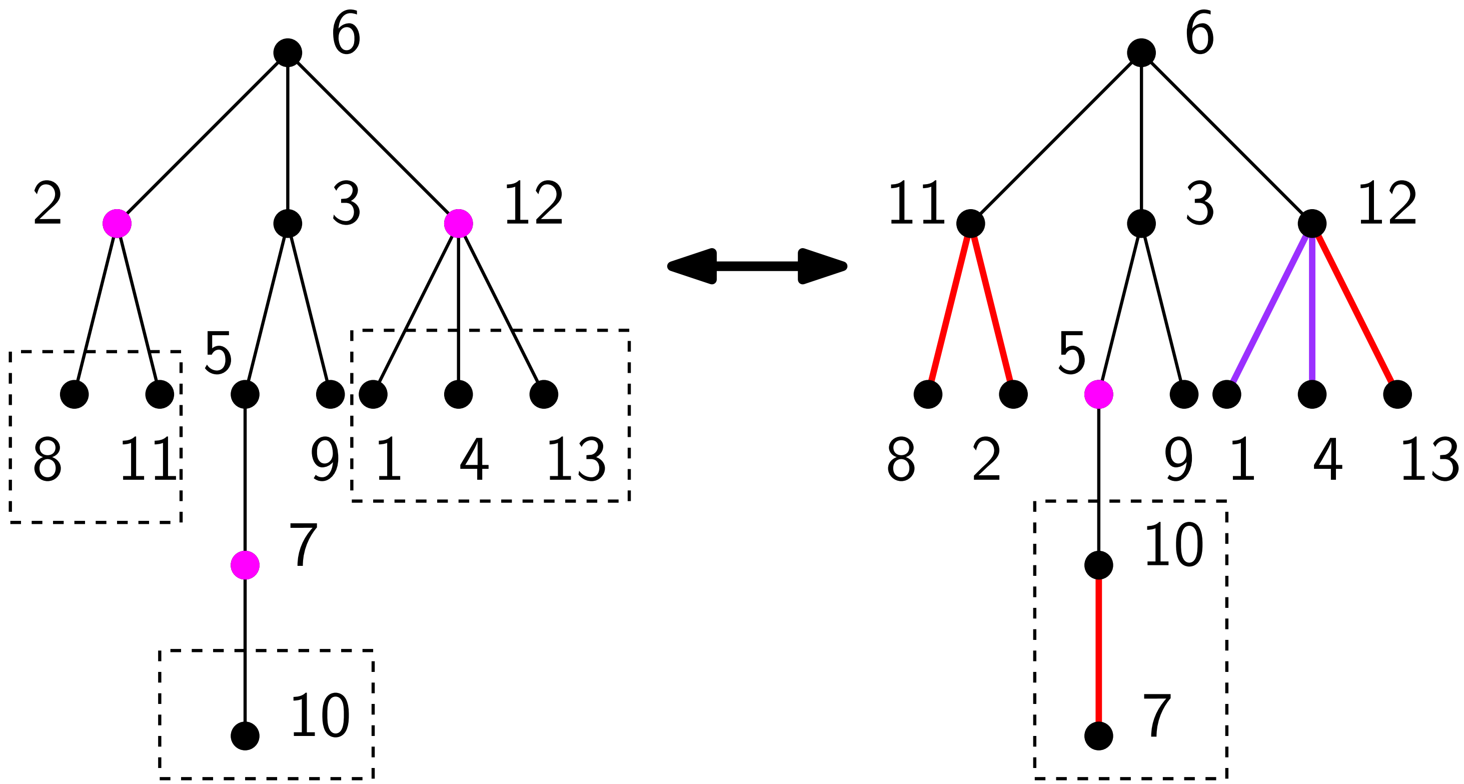

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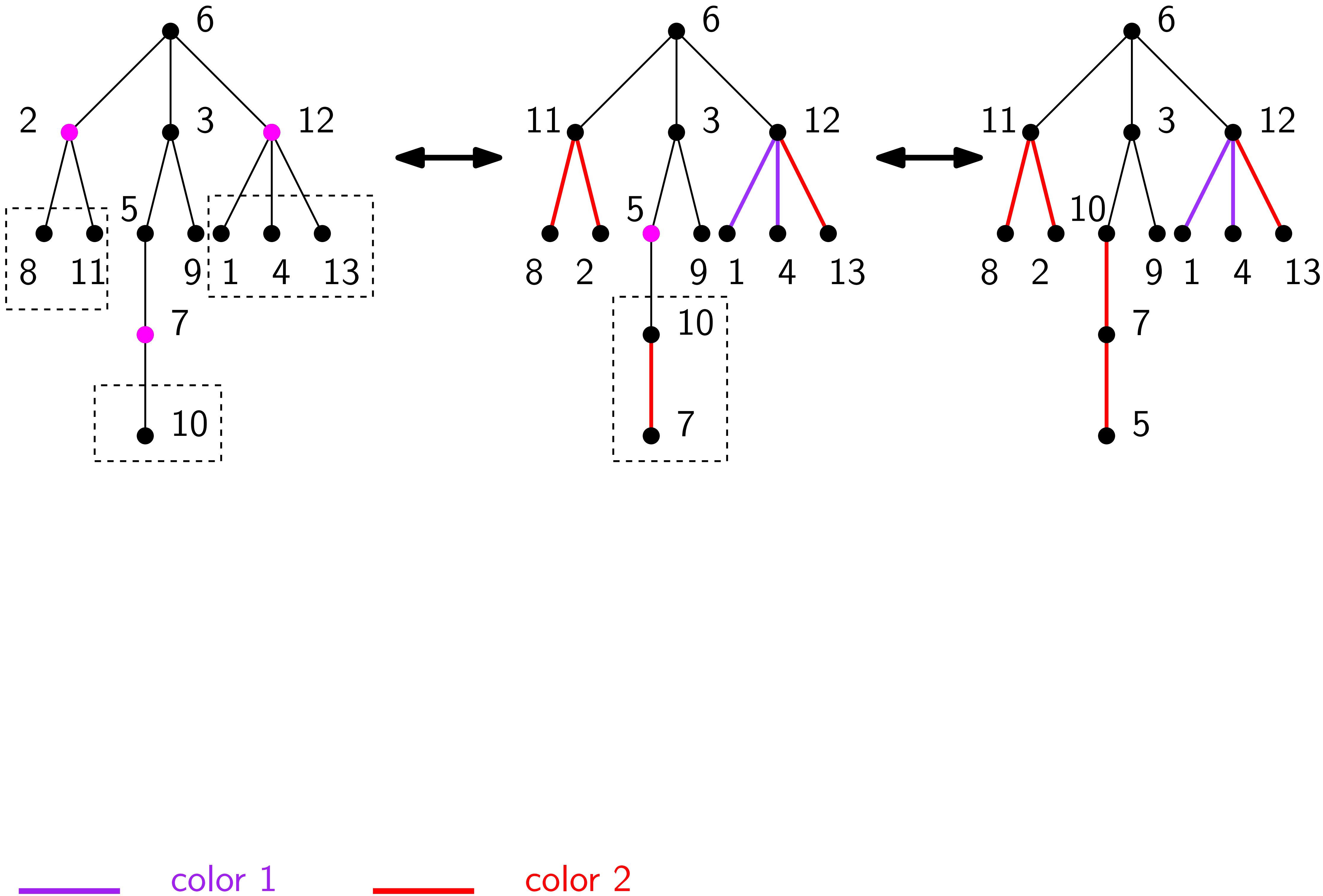

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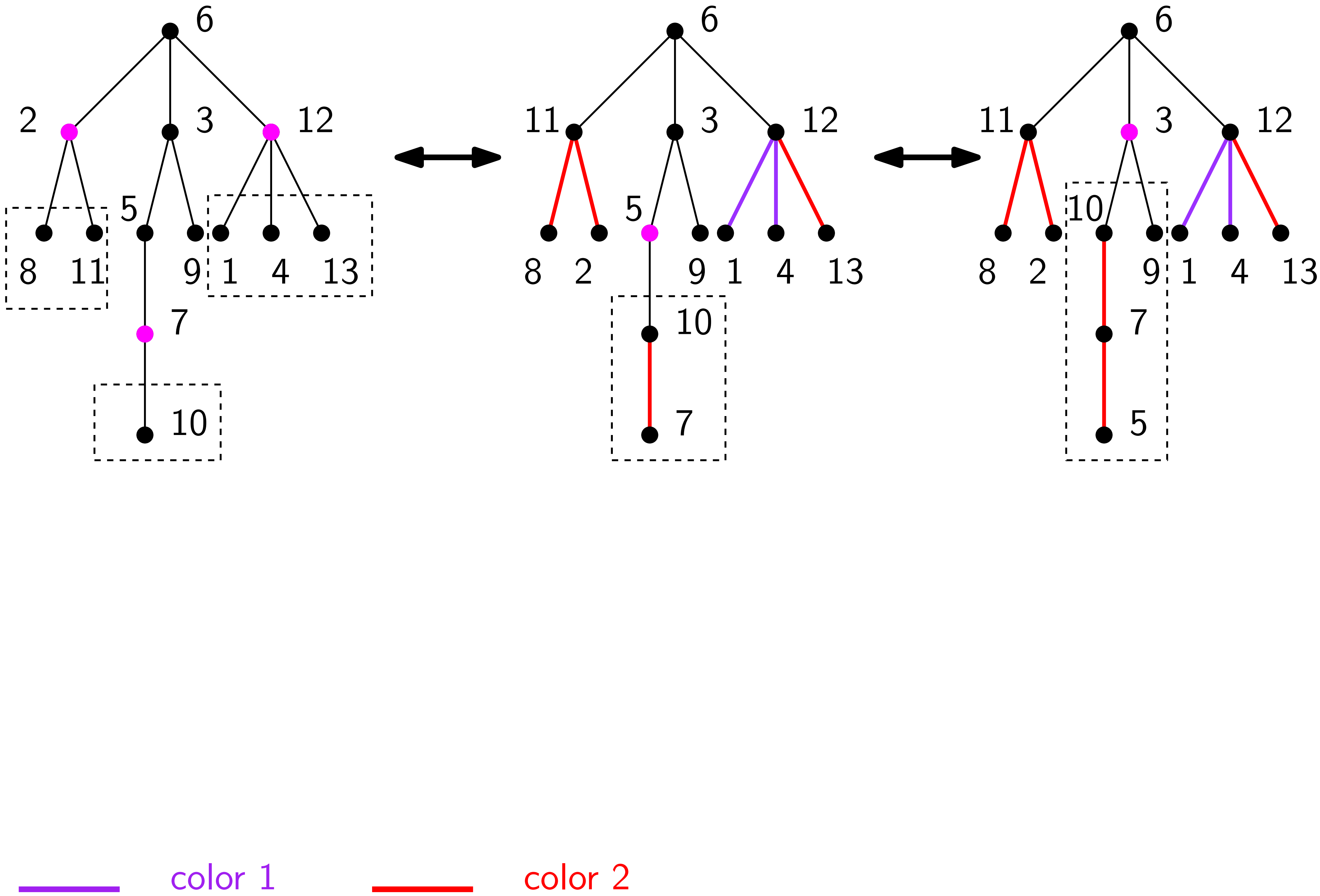
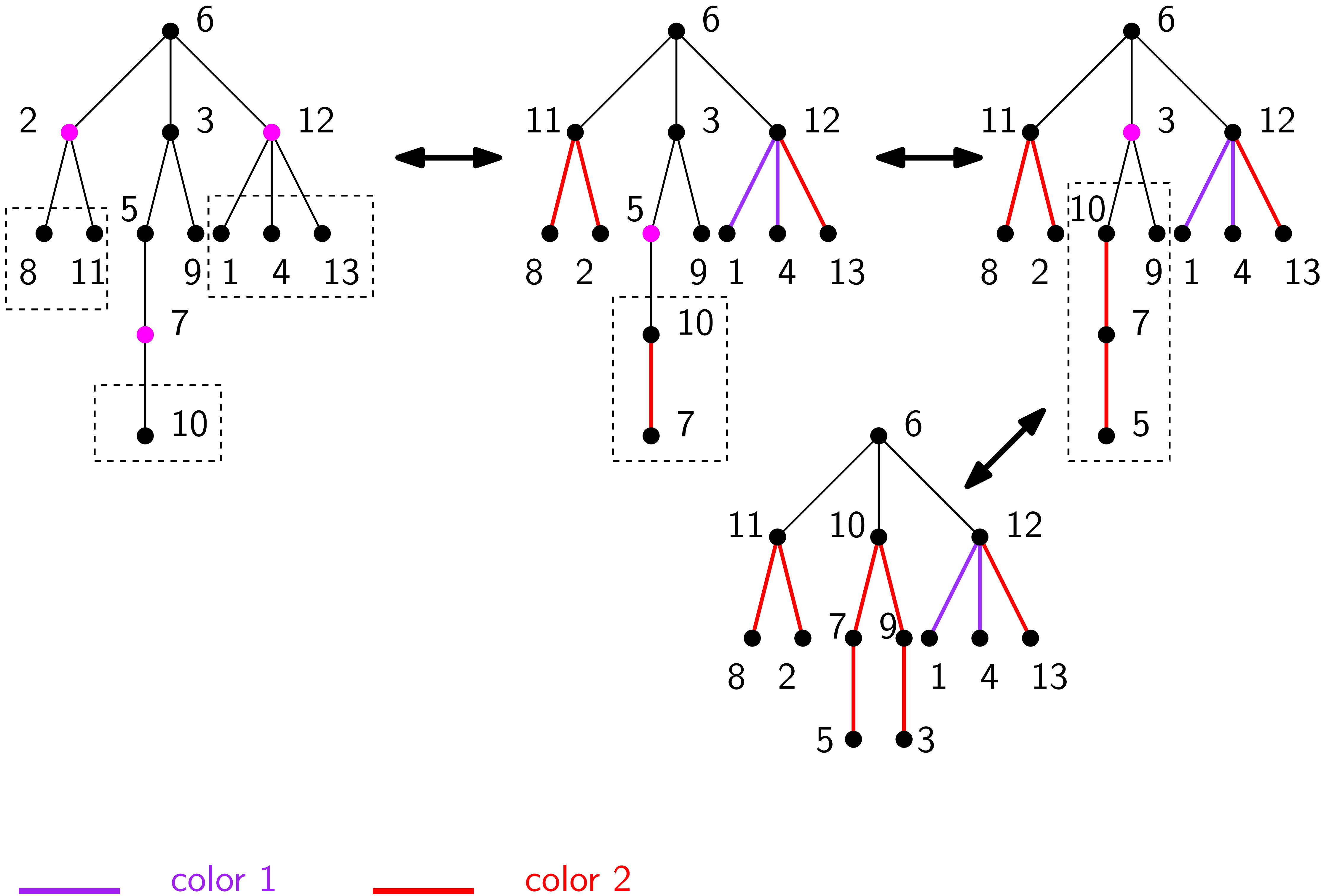


Cayley trees $\leftrightarrow$ 2-decreasing trees



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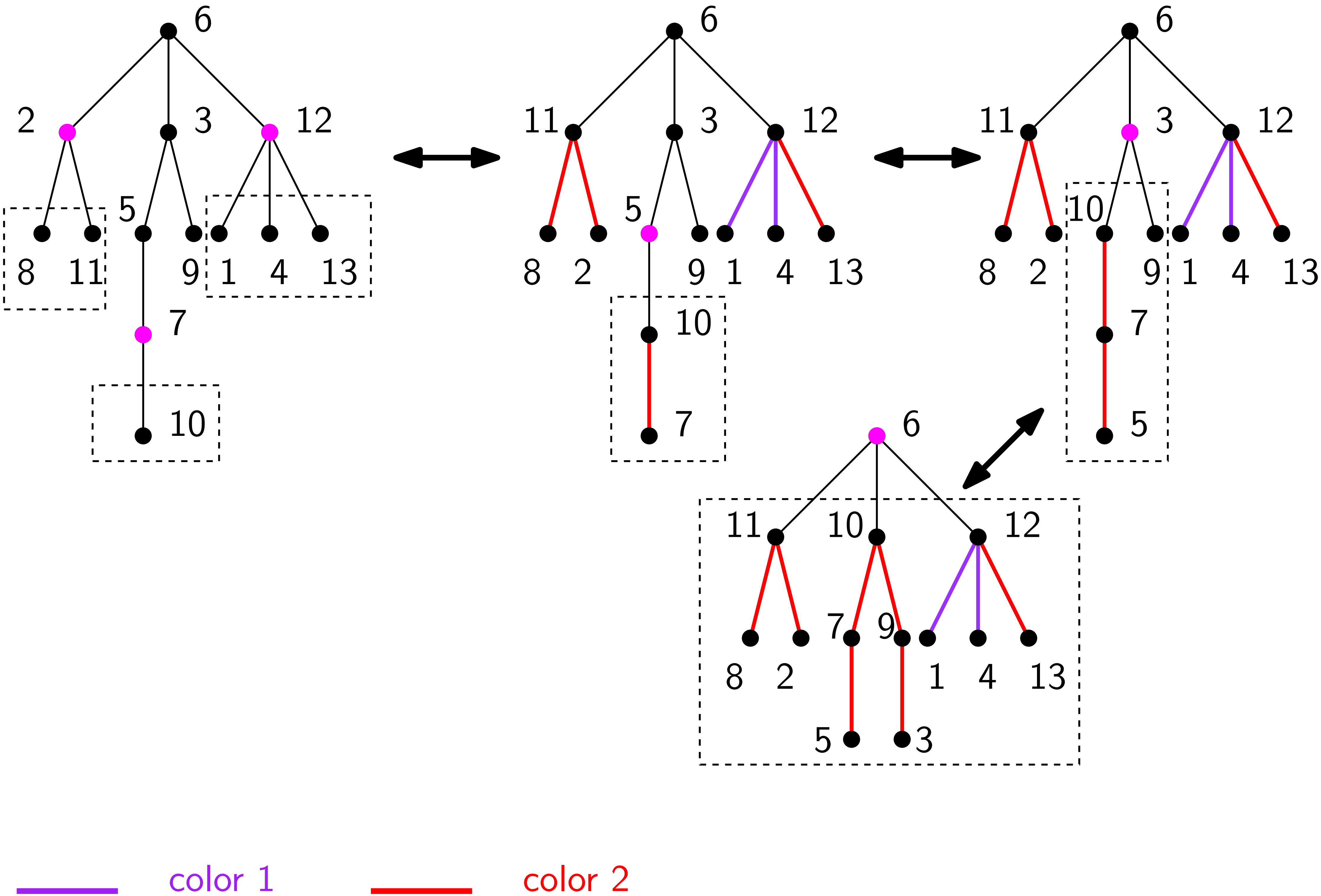

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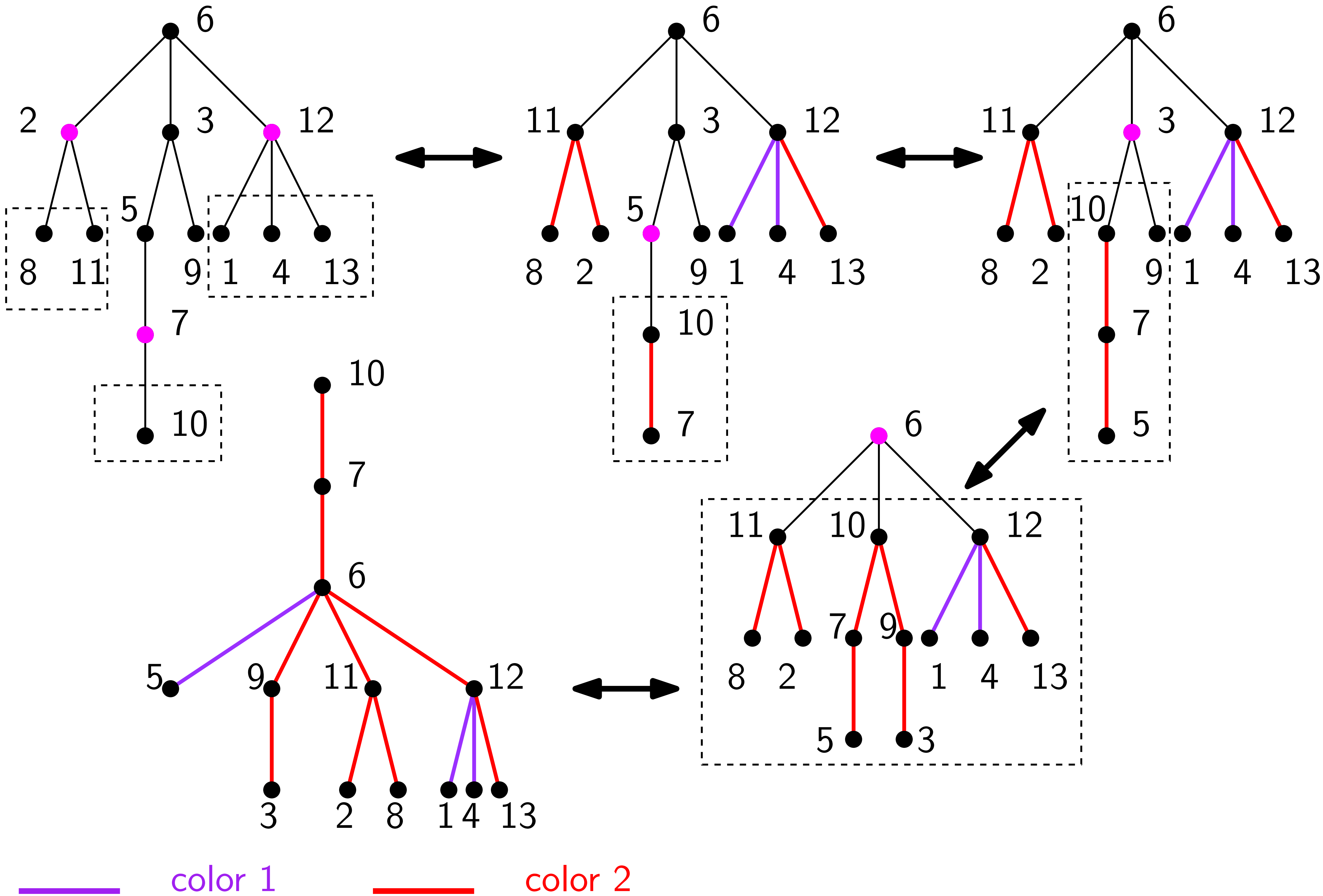


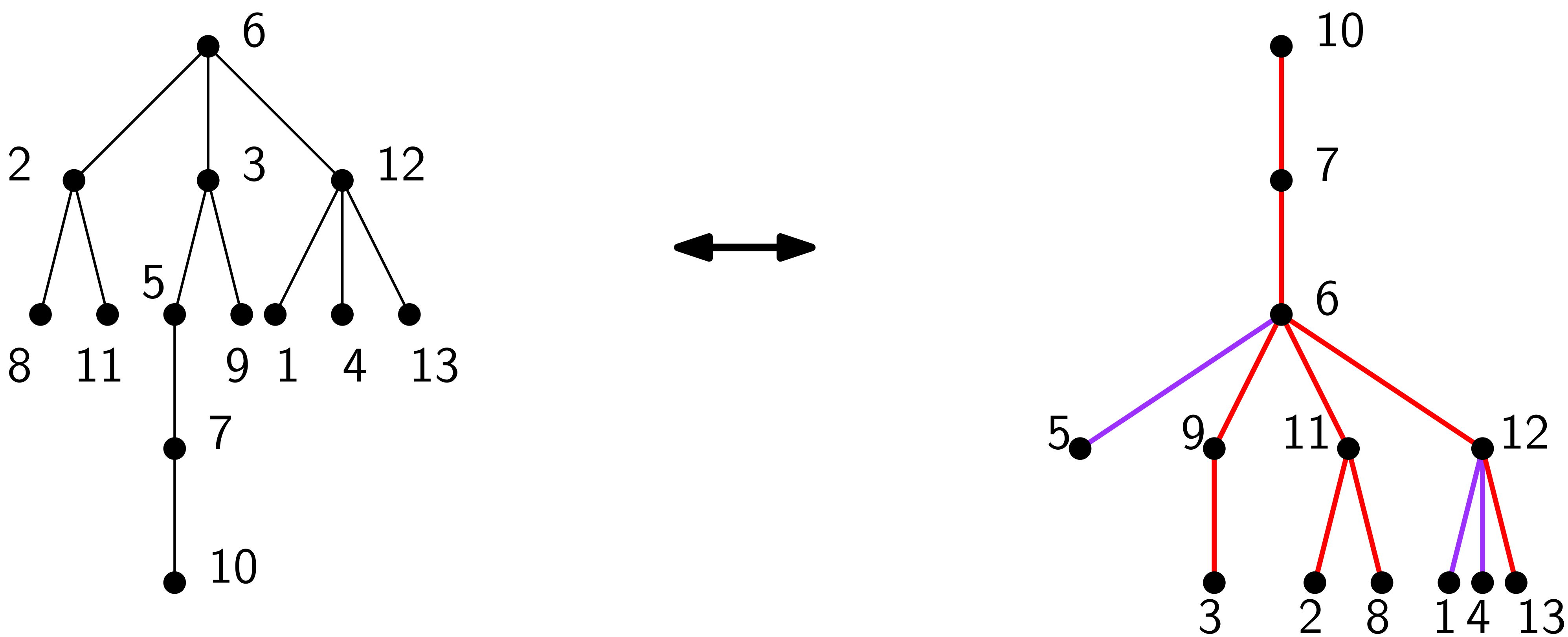
Cayley trees ↔ (2,0)-decreasing trees

Lemma. Root of the Cayley tree

↔ vertex at the end of the 2-decreasing path of the (2,0)-decreasing tree.

Lemma. Increasing vertices in the Cayley trees

↔ vertices who have a decreasing 2-edge in the (2,0)-decreasing tree.



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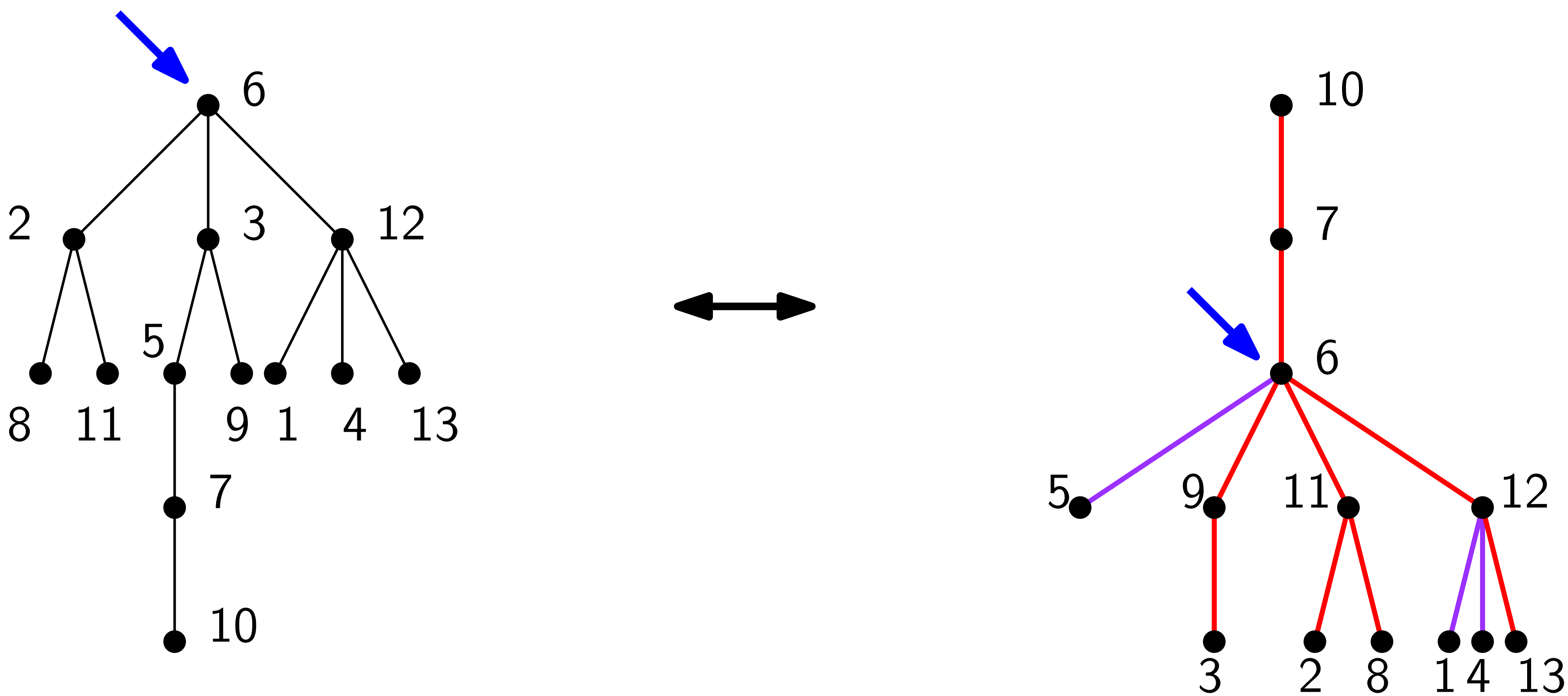

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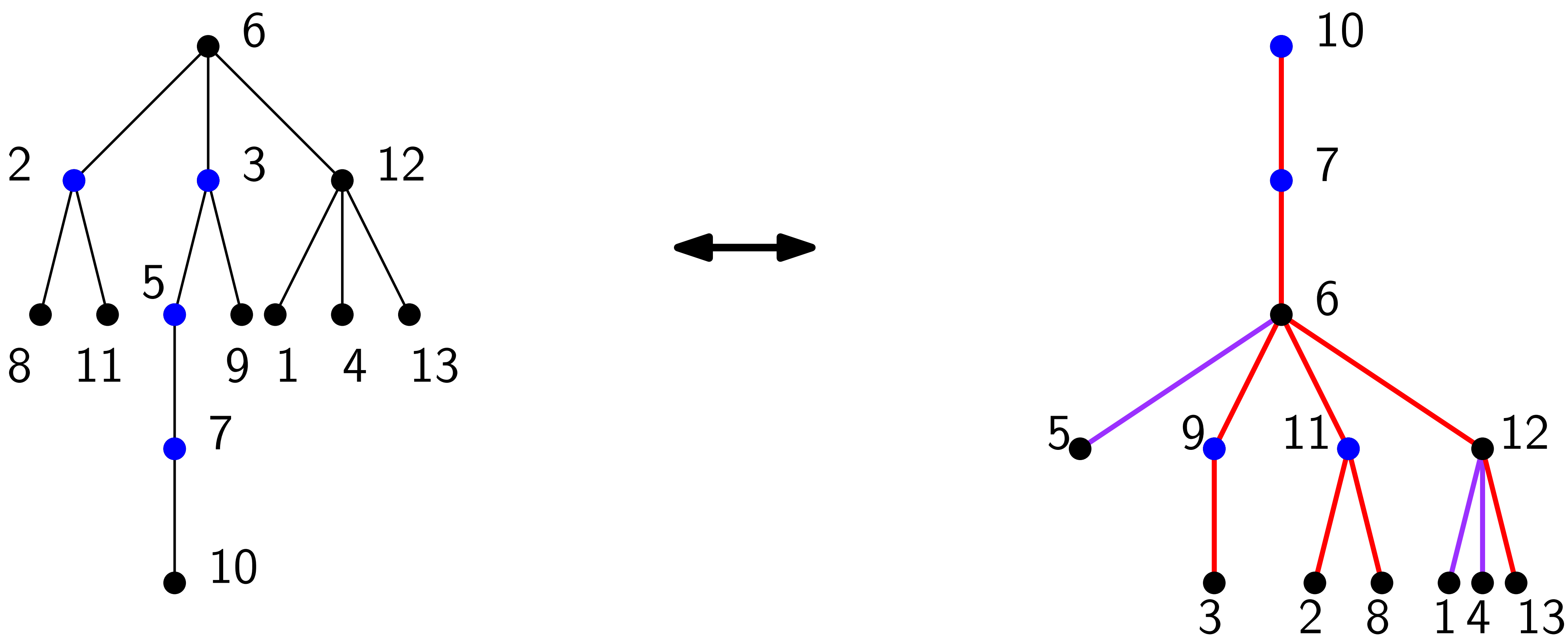

Generalization

Theorem [CFM 15] There exists a bijection between
(k, j)-decreasing trees with n vertices and
$(k-2, j+1)$-non-increasing trees with n vertices

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vertices with a decreasing k-edge $\leftrightarrow$
increasing vertices on the first free color

Ongoing projects

- Simple bijections? (2,0)-decreasing trees $\leftrightarrow$ rooted Cayley trees

  (1,1)-decreasing $\leftrightarrow$ labelled binary trees....

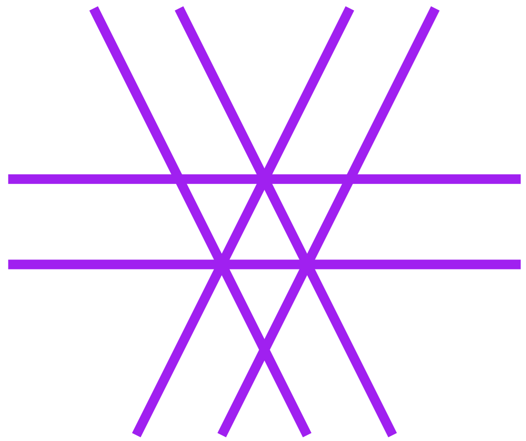
- link with parking functions

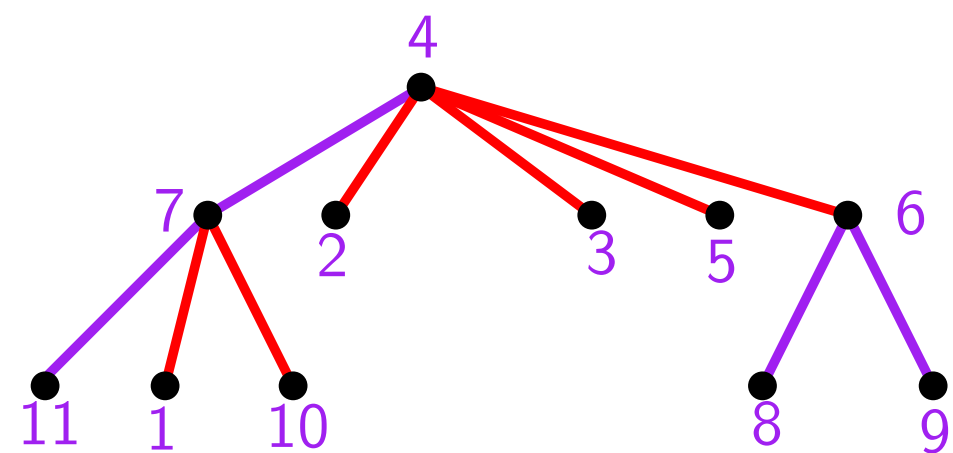
  regions of the Shi arrangement $\leftrightarrow$ parking functions [Athanasiadis & Linusson 99]

- Other Coxeter arrangements $x_i \pm x_j = g$ [Athanasiadis 99]

- Ish arrangement [Armstrong et al, 10]

- Counting bounded regions





Merci!