Tamari Lattices for Parabolic Quotients of the Symmetric Group

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Catalan Objects

- **Catalan numbers**: $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$
- **Catalan objects**:
  - 231-avoiding permutations of $[n]$
  - triangulations of an $(n+2)$-gon
  - noncrossing set partitions of $[n]$
  - nonnesting set partitions of $[n]$
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The Symmetric Group $\mathfrak{S}_n$

- **symmetric group** $\mathfrak{S}_n$: group of permutations of $[n]$
- generators: $s_i = (i\ i{+}1), i \in [n-1]$
- $S = \{s_1, s_2, \ldots, s_{n-1}\}$
- **inversion set**: $\mathrm{inv}(w) = \big\{(i,j) \mid i < j, w_i > w_j\big\}$

Parabolic Quotients of $\mathfrak{S}_n$

- **(standard) parabolic subgroup**:
  subgroup $(\mathfrak{S}_n)_J$ generated by $J \subseteq S$
- **(standard) parabolic quotient**:
  $\mathfrak{S}_n^J = \{w \in \mathfrak{S}_n \mid \mathrm{inv}(w) \subsetneq \mathrm{inv}(ws) \text{ for all } s \in J\}$

Parabolic Quotients of $\mathfrak{S}_n$

- $J = S \setminus \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}$
- one-line notation for $w \in \mathfrak{S}_n^J$:
  $w_1 < \cdots < w_{i_1} | w_{i_1+1} < \cdots < w_{i_2} | \cdots | w_{i_k+1} < \cdots < w_n$

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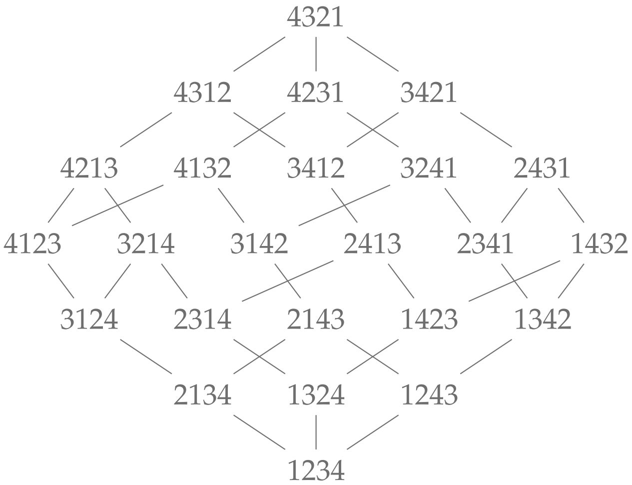
Weak Order

- **inversion set**: $\mathrm{inv}(w) = \{(i,j) \mid i < j, w_i > w_j\}$
- **weak order**: $u \leq_S v$ if and only if $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$ $\rightsquigarrow \mathrm{Weak}(\mathfrak{S}_n)$
- **longest element**: $w_o = n \cdots 21$

Example: Weak$(\mathfrak{S}_4)$



The Tamari Lattices

Theorem (Björner & Wachs, 1997)

For $n > 0$ the Tamari lattice $\mathcal{T}_n$ is isomorphic to the weak order on the 231-avoiding permutations of $\mathfrak{S}_n$, i.e. $\mathcal{T}_n \cong \mathit{Weak}\big(\mathfrak{S}_n(231)\big)$.

- $\mathcal{T}_n$ is a sublattice and a quotient lattice of $\mathrm{Weak}\big(\mathfrak{S}_n\big)$

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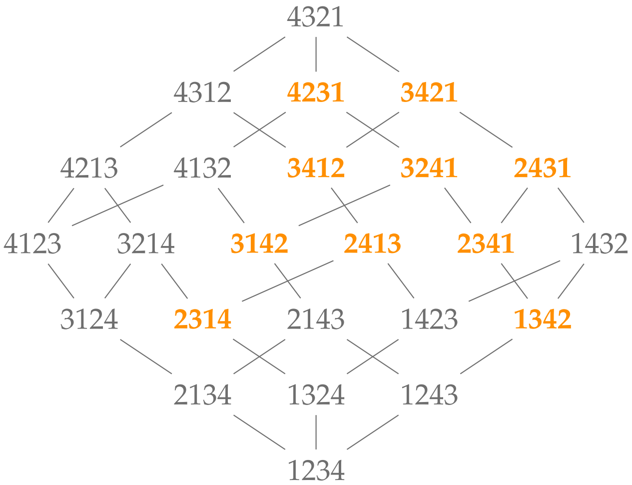

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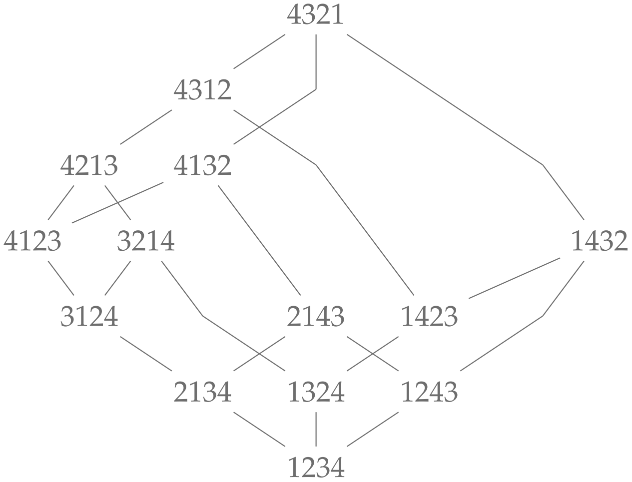

Parabolic Weak Order

- **parabolic weak order**: restrict $\mathrm{Weak}(\mathfrak{S}_n)$ to $\mathfrak{S}_n^J$
  $\rightsquigarrow \mathrm{Weak}(\mathfrak{S}_n^J)$
- $\mathrm{Weak}(\mathfrak{S}_n^J) \cong \mathrm{Weak}(e, w_o^J)$

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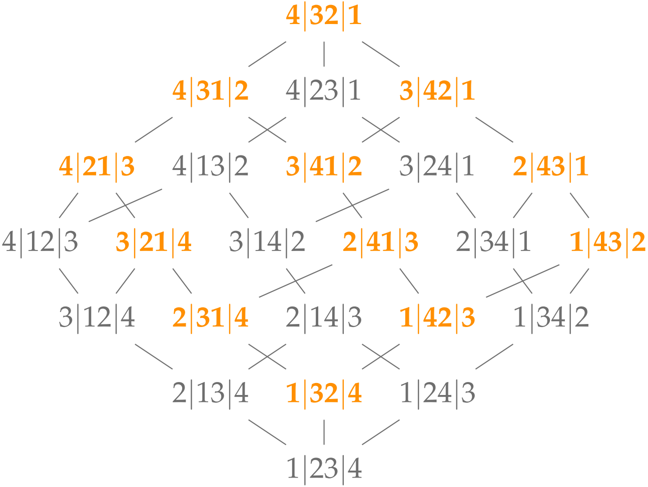
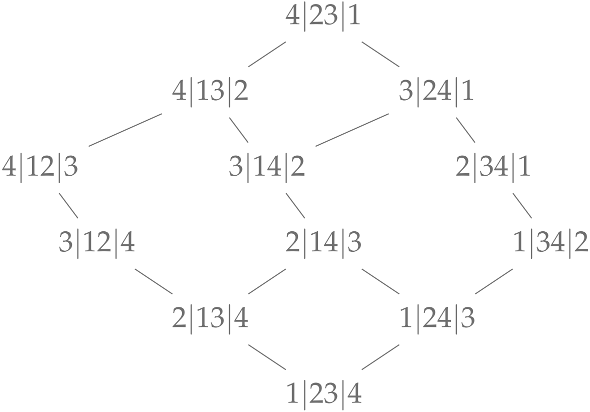


Example: Weak$\left(\mathfrak{S}_4^{\{s_2\}}\right)$



Parabolic Tamari Lattices

Theorem (& Williams, 2015)

For $n > 0$ and $J \subseteq S$, the poset Weak$\bigl(\mathfrak{S}_n^J(231)\bigr)$ is a lattice, the ***parabolic Tamari lattice*** $\mathcal{T}_n^J$.

- for any $w \in \mathfrak{S}_n^J$ there is a unique maximal $w' \in \mathfrak{S}_n^J(231)$ with $w' \leq_S w$

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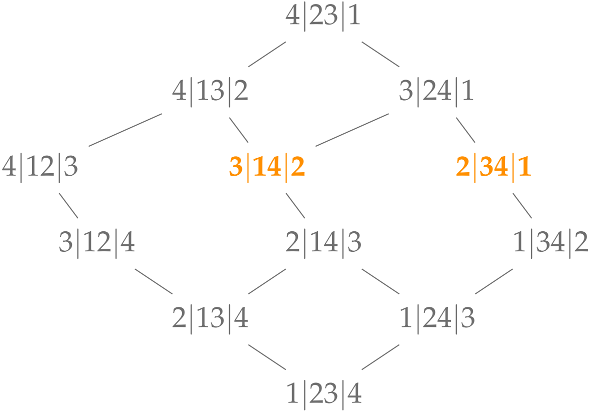

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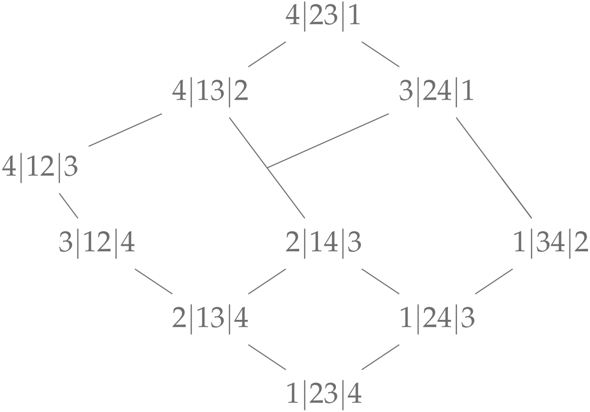

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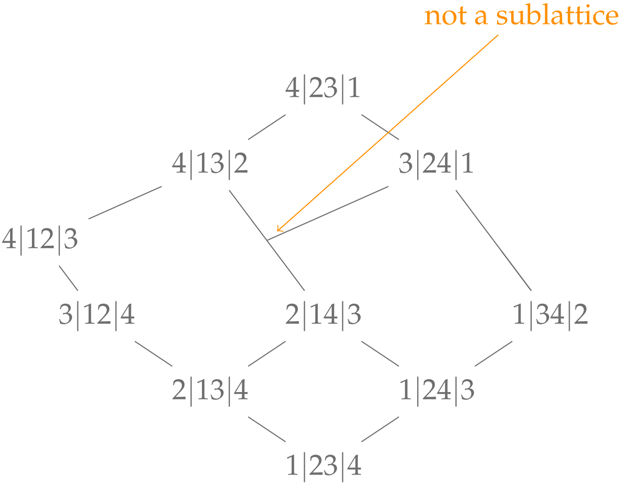

Outline

Noncrossing Partitions

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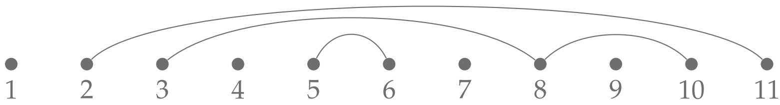

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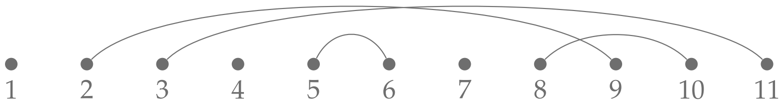

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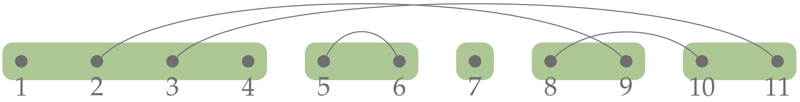

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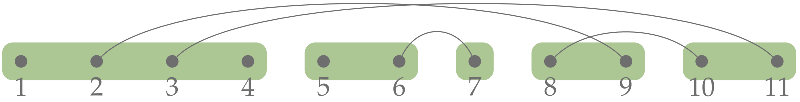

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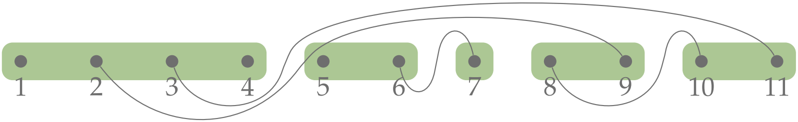

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For $n > 0$ and $J \subseteq S$, we have $\lvert NC_n^J \rvert = \lvert \mathfrak{S}_n^J(231) \rvert$.

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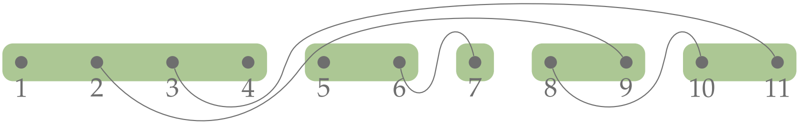

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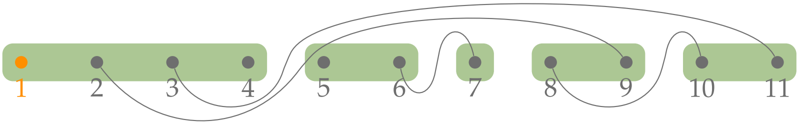

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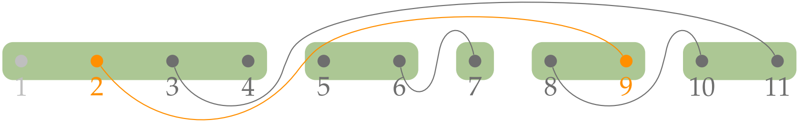

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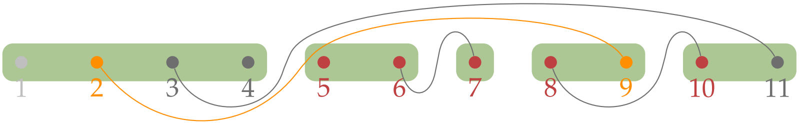

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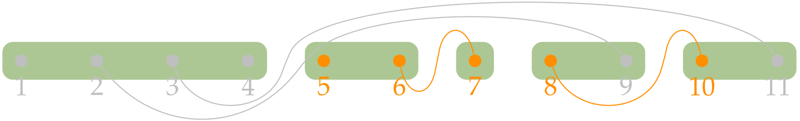

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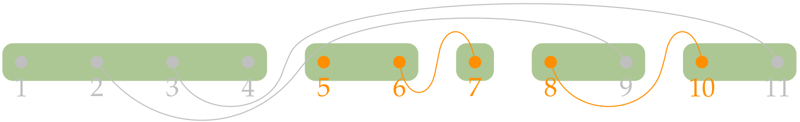

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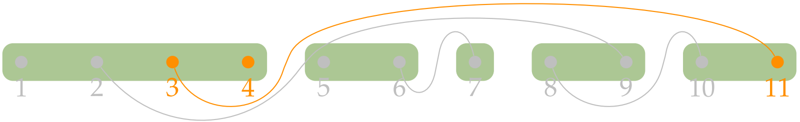

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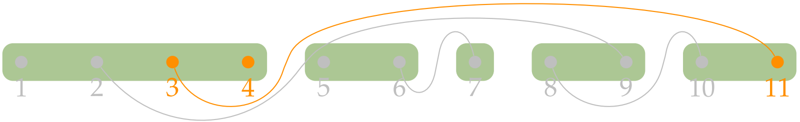

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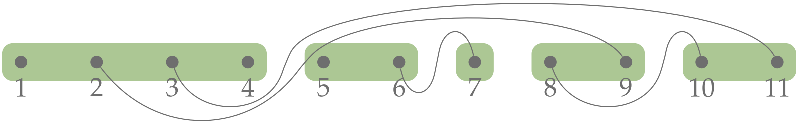

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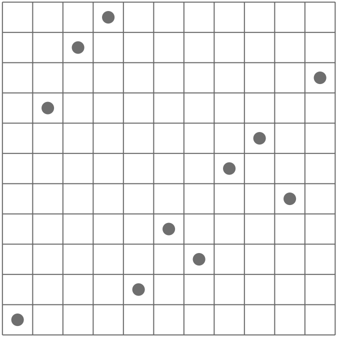

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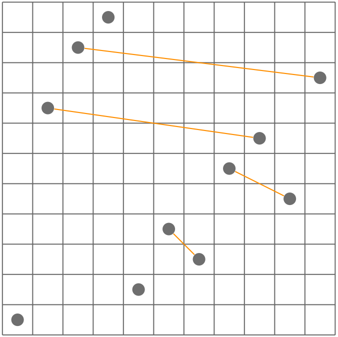

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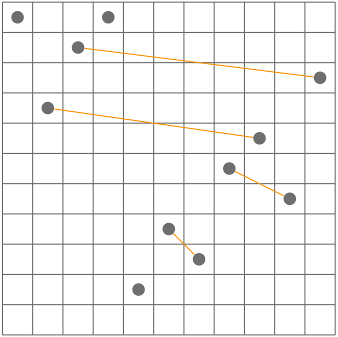

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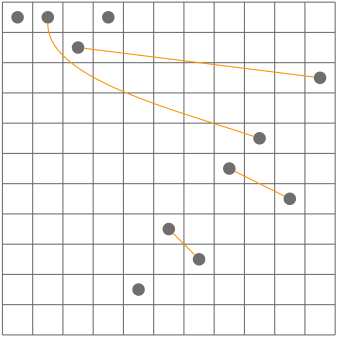

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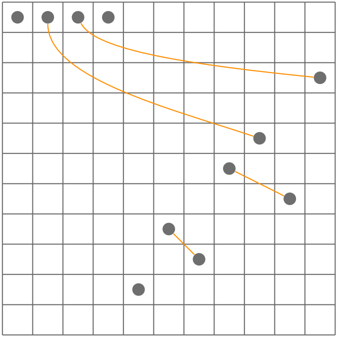

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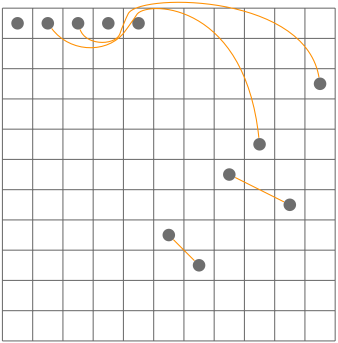

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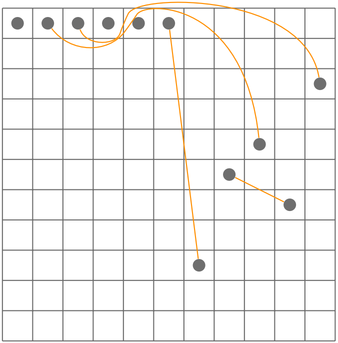

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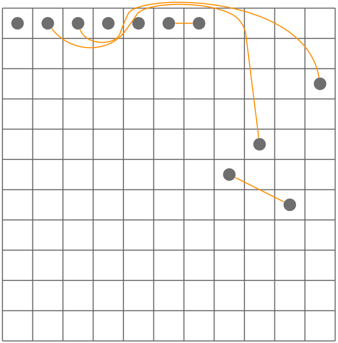

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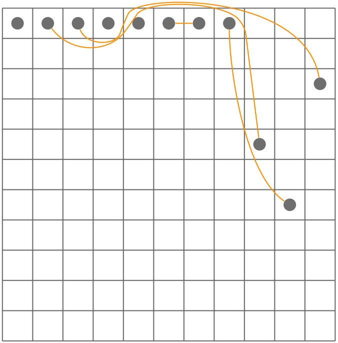

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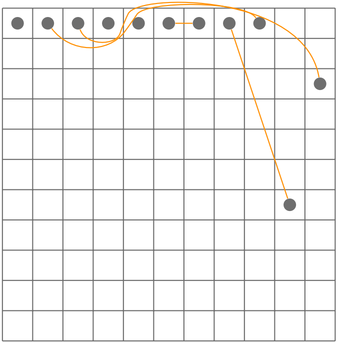

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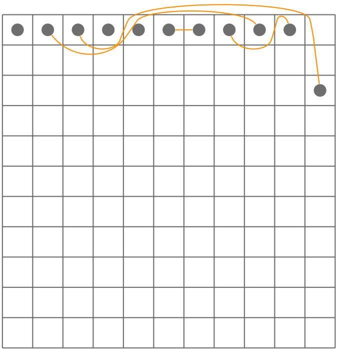

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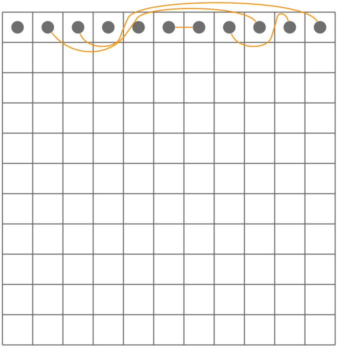

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- Reading recently gave an explicit bijection between noncrossing diagrams and permutations

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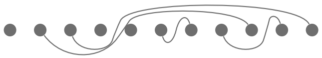

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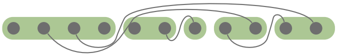
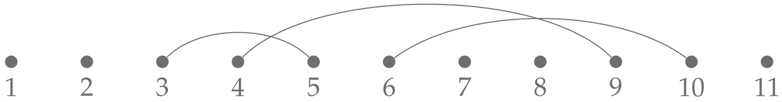


Outline

Nonnesting Partitions

- **nonnesting (set) partition**



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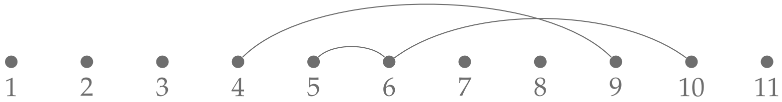
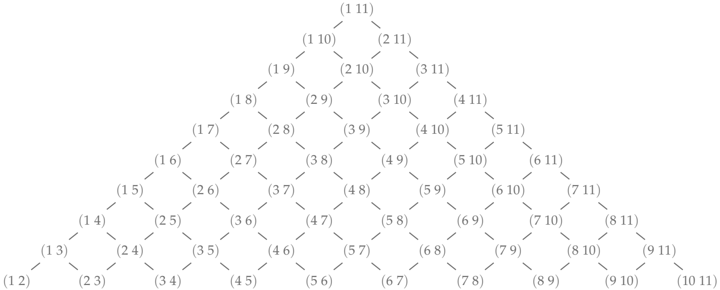
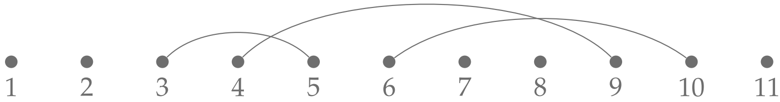


Nonnesting Partitions

- order ideals in the root poset





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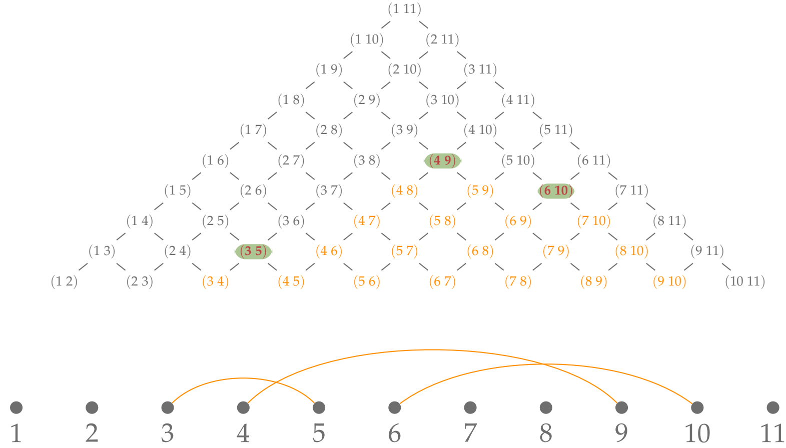

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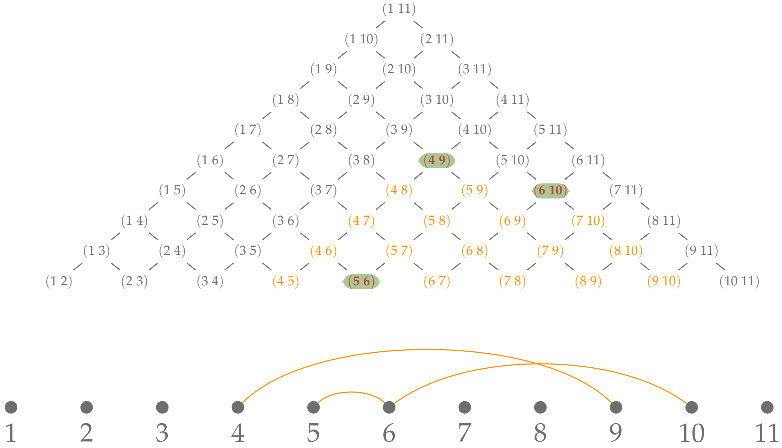

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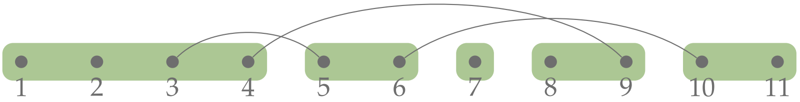

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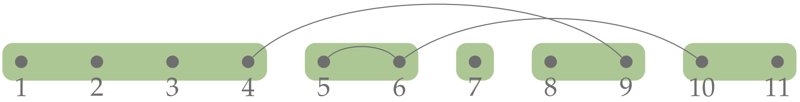

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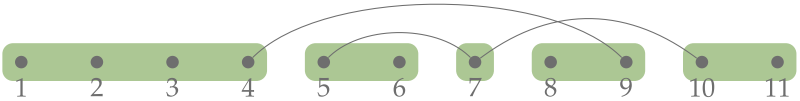

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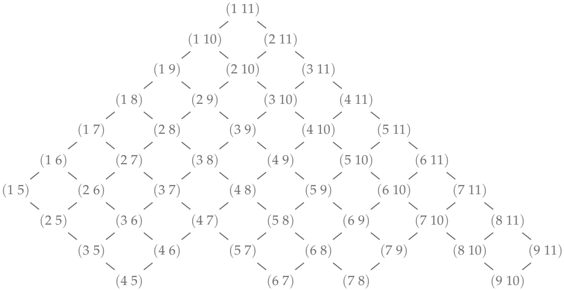

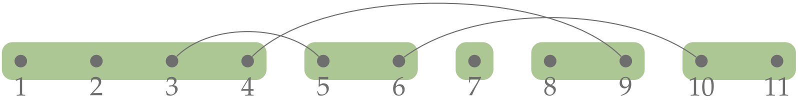

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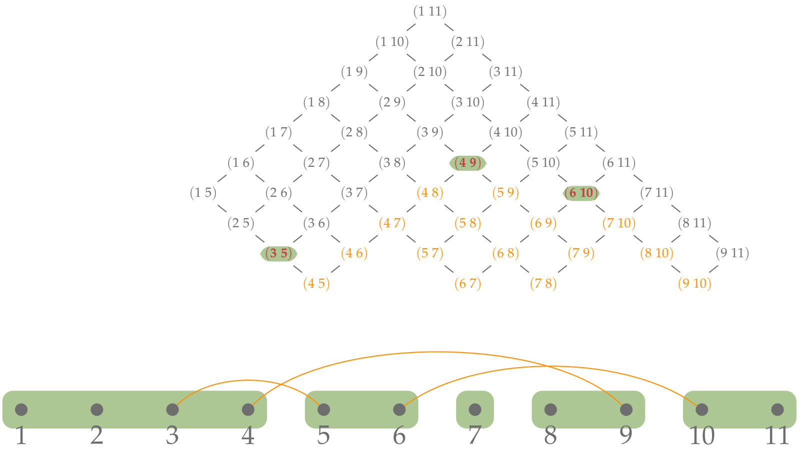

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Theorem (& Williams, 2015)

For $n > 0$ and $J \subseteq S$, we have $\lvert NN_n^J \rvert = \lvert NC_n^J \rvert$.

- associate bumps with minimal elements outside the order ideal

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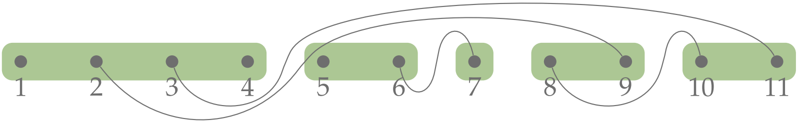

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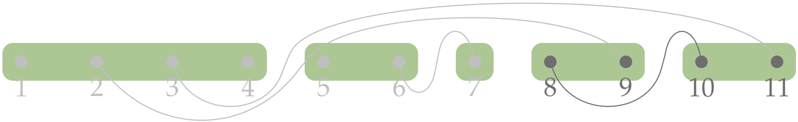

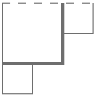

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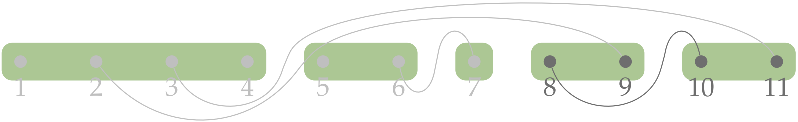

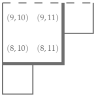

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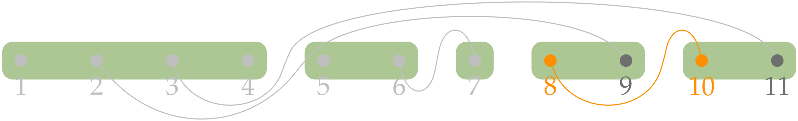

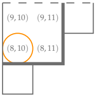

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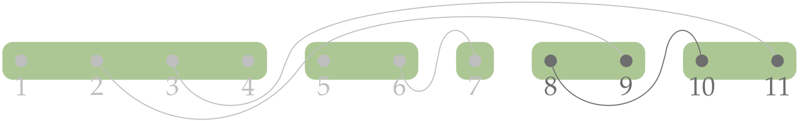

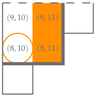

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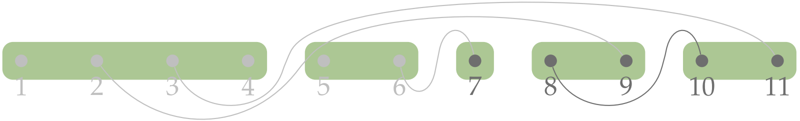

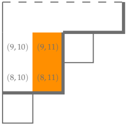

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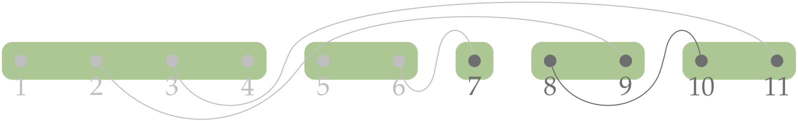

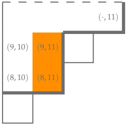

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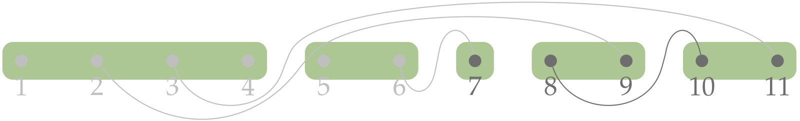

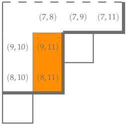

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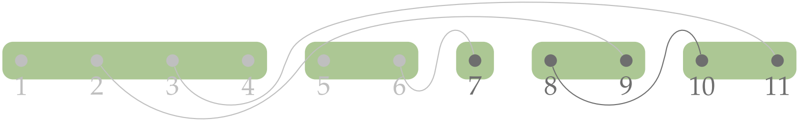

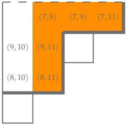

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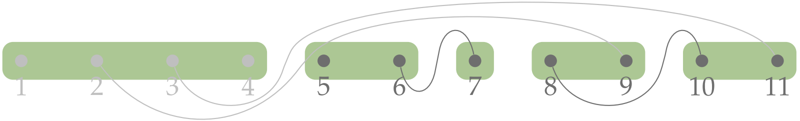

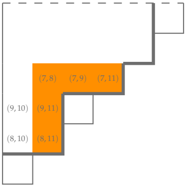

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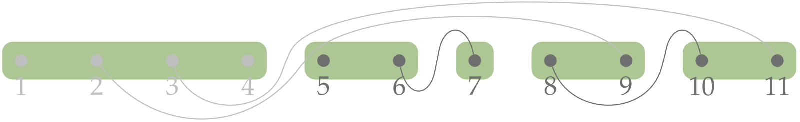

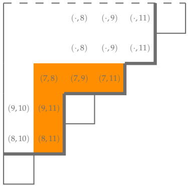

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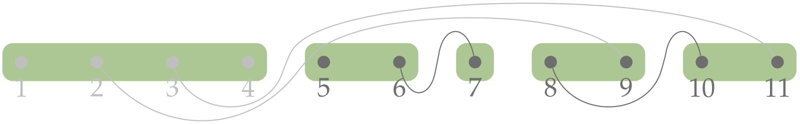

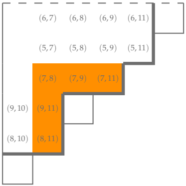

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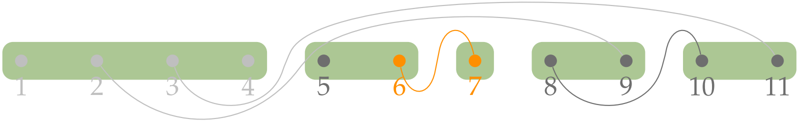

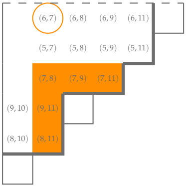

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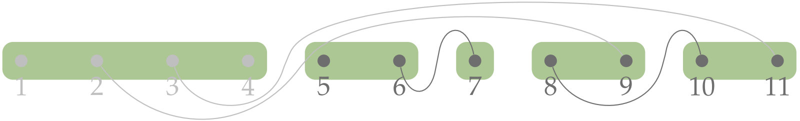

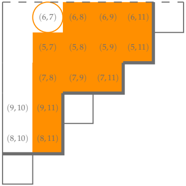

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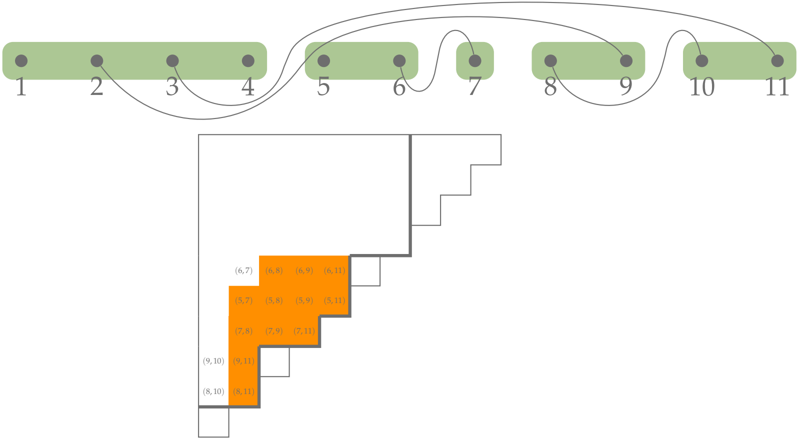

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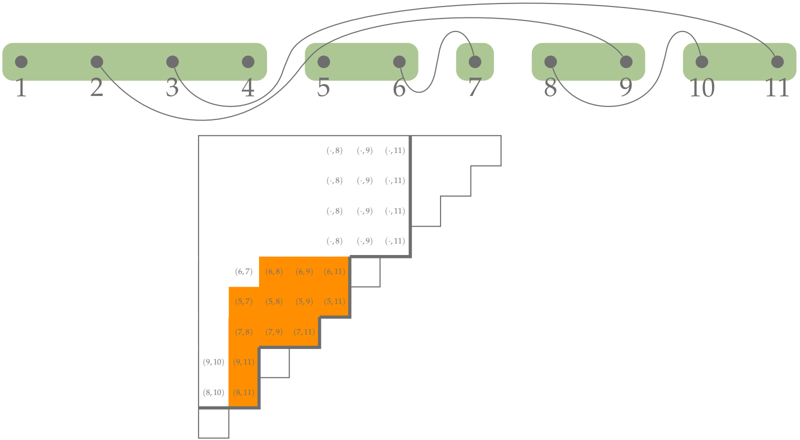

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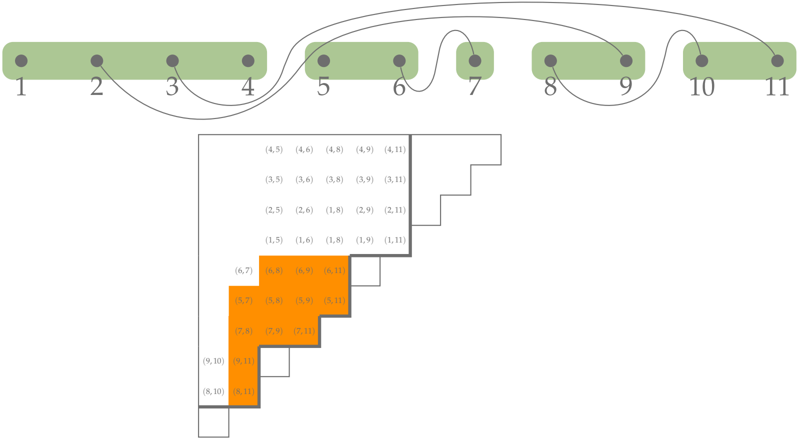

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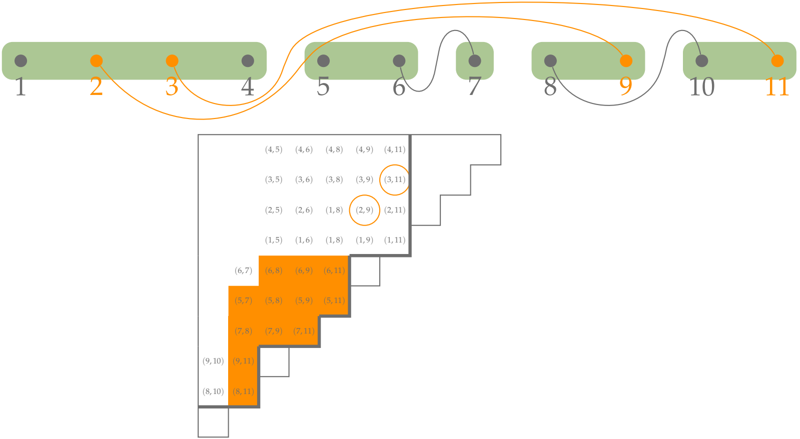

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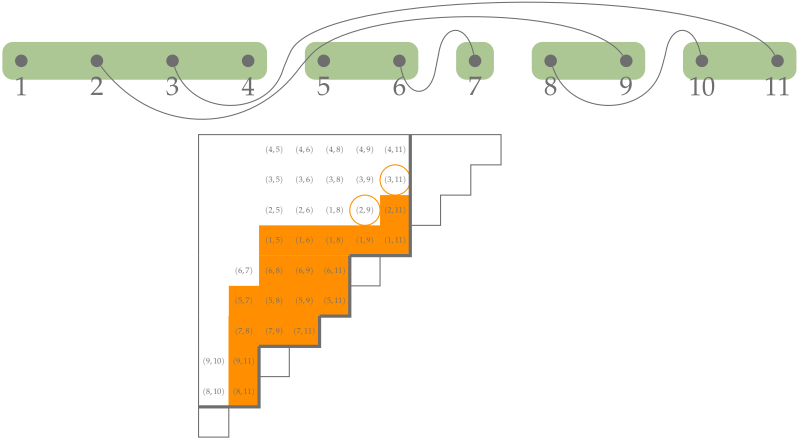

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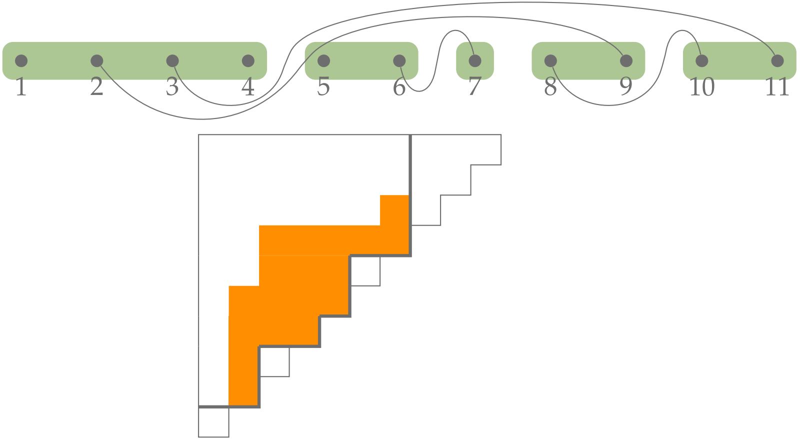

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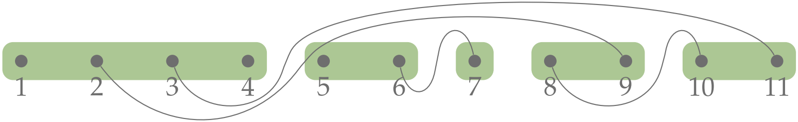
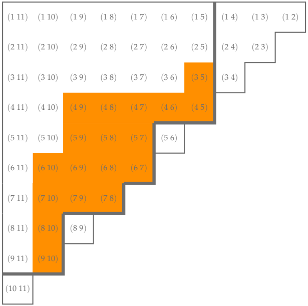
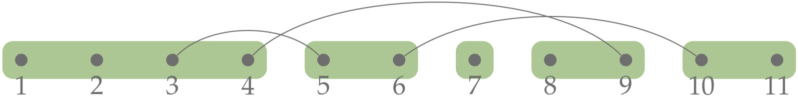






Outline

Finite Coxeter Groups

- **finite Coxeter group**: real reflection group
- S .. Coxeter generators; T .. reflections; ℓ_S .. Coxeter length
- **inversion set**: $\mathrm{inv}(w) = \{t \in T \mid \ell_S(tw) < \ell_S(w)\}$
- **longest element**: $w \in W$ with $\mathrm{inv}(w) = T$ $\rightsquigarrow w_o$
- **cover reflection**: $t \in \mathrm{inv}(w)$ such that $tw = ws$ for some $s \in S$ $\rightsquigarrow \mathrm{cov}(w)$
- **Coxeter element**: $c = s_{\pi_1} s_{\pi_2} \cdots s_{\pi_n}$ for $\pi \in \mathfrak{S}_n$
- ***c*-sorting word**: reduced word for w that is the lexicographically smallest subword of c^∞ $\rightsquigarrow \mathbf{w}(\mathbf{c})$

Parabolic Quotients

- **(standard) parabolic subgroup**:
  subgroup W_J generated by $J \subseteq S$
- **(standard) parabolic quotient**:
  $$W^J = \{w \in W \mid \mathrm{inv}(w) \subsetneq \mathrm{inv}(ws) \text{ for all } s \in J\}$$

Outline

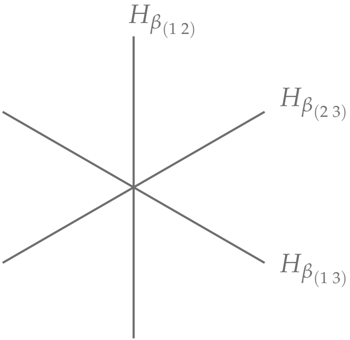
Root Systems

- reflection t $\longleftrightarrow$ reflection hyperplane H_t
- roots $\pm\beta_t$ $\longleftrightarrow$ normal vectors to H_t
- $\beta_t = \sum_{s\in S} c_s\beta_s$, where all c_s have same sign

$W = \mathfrak{S}_3$

$$\beta_{(1\,2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\beta_{(2\,3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$



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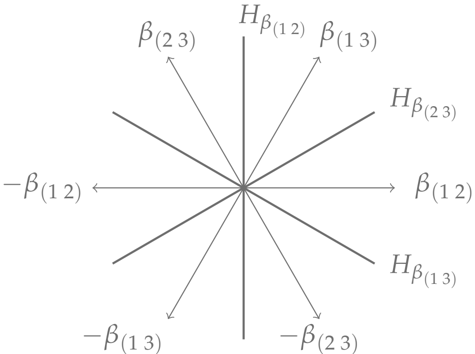

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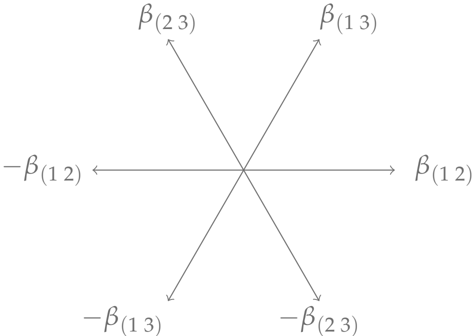

Root Order

- fix $\mathbf{w} = a_1 a_2 \cdots a_k \in W$
- **root order**: $\beta_{t_1} < \beta_{t_2} < \cdots < \beta_{t_k}$, where $t_i = a_1 a_2 \cdots a_i \cdots a_2 a_1$ $\rightsquigarrow \mathbf{Inv}(\mathbf{w})$

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- **root order**: $\beta_{t_1} < \beta_{t_2} < \cdots < \beta_{t_k}$, where $t_i = a_1 a_2 \cdots a_i \cdots a_2 a_1$ $\rightsquigarrow \mathbf{Inv}(\mathbf{w})$

$$W = \mathfrak{S}_4, J = \{s_2\}, \mathbf{w_o^J} = s_1 s_2 s_3 s_2 s_1$$

$$s_1 \quad < \quad s_1 s_2 s_1 \quad < \quad s_1 s_2 s_3 s_2 s_1$$

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$$W = \mathfrak{S}_4, J = \{s_2\}, \mathbf{w_o^J} = s_1 s_2 s_3 s_2 s_1$$

$$(1\,2) \quad < \quad (1\,3) \quad < \quad (1\,4) \quad < \quad (3\,4) \quad < \quad (2\,4)$$

Parabolic Aligned Elements

- (W^J, c)**-aligned element**: whenever $\alpha < a\alpha + b\beta < \beta$ in $\mathbf{Inv}\big(\mathbf{w_o^J}(\mathbf{c})\big)$, then $t_{a\alpha+b\beta} \in \mathrm{cov}(w)$ implies $t_\alpha \in \mathrm{inv}(w)$ $\rightsquigarrow \mathrm{Align}\big(W^J, c\big)$

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$$(1\ 2) < (1\ 3) < (1\ 4) < (3\ 4) < (2\ 4)$$

$w \in \mathfrak{S}_4^{\{s_2\}}$	$\mathrm{cov}(w)$	$\mathrm{inv}(w)$	aligned?
$1\|23\|4$	$\emptyset$	$\emptyset$	yes
$2\|13\|4$	$(1\ 2)$	$(1\ 2)$	yes
$1\|24\|3$	$(3\ 4)$	$(3\ 4)$	yes
$3\|12\|4$	$(1\ 3)$	$(1\ 2), (1\ 3)$	yes
$2\|14\|3$	$(1\ 2), (3\ 4)$	$(1\ 2), (3\ 4)$	yes
$1\|34\|2$	$(2\ 4)$	$(3\ 4), (2\ 4)$	yes

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$$(1\,2) < (1\,3) < (1\,4) < (3\,4) < (2\,4)$$

$w \in \mathfrak{S}_4^{\{s_2\}}$	$\mathrm{cov}(w)$	$\mathrm{inv}(w)$	aligned?
$4\|12\|3$	$(1\,4)$	$(1\,2), (1\,3), (1\,4)$	yes
$3\|14\|2$	$(1\,4)$	$(1\,2), (1\,4), (3\,4)$	no
$2\|34\|1$	$(1\,4)$	$(1\,4), (2\,4), (3\,4)$	no
$4\|13\|2$	$(1\,3), (3\,4)$	$(1\,2), (1\,3), (1\,4), (3\,4)$	yes
$3\|24\|1$	$(1\,2), (2\,4)$	$(1\,2), (1\,4), (3\,4), (2\,4)$	yes
$4\|23\|1$	$(1\,3), (2\,4)$	$(1\,2), (1\,3), (1\,4), (3\,4), (2\,4)$	yes

Parabolic Aligned Elements

- **(W^J, c)-aligned element**: whenever $\alpha < a\alpha + b\beta < \beta$ in $\mathbf{Inv}\big(\mathbf{w_o^J(c)}\big)$, then $t_{a\alpha+b\beta} \in \mathrm{cov}(w)$ implies $t_\alpha \in \mathrm{inv}(w)$ $\rightsquigarrow \mathrm{Align}\big(W^J, c\big)$

Lemma (Mühle & Williams, 2015)

Let $W = \mathfrak{S}_n$, let $c = s_1 s_2 \cdots s_{n-1}$, and choose $J \subseteq S$. An element $w \in \mathfrak{S}_n^J$ is (W^J, c)-aligned if and only if it is J-231-avoiding.

Parabolic Aligned Elements

- **weak order**: $u \leq_S v$ if and only if $\mathrm{inv}(u) \subseteq \mathrm{inv}(v)$ $\rightsquigarrow \mathrm{Weak}(W)$

Conjecture (& Williams, 2015)

For any finite Coxeter group W, any Coxeter element $c \in W$, and any $J \subseteq S$, the poset $\mathrm{Weak}\big(\mathrm{Align}(W^J, c)\big)$ is a lattice. Moreover, it is a lattice quotient of $\mathrm{Weak}(W^J)$.

Outline

Parabolic Noncrossing Partitions

- **(W^J, c)-noncrossing partitions**: $\mathrm{cov}(w)$ for $w \in \mathrm{Align}(W^J, c)$ $\rightsquigarrow NC(W^J)$

Outline

Parabolic Nonnesting Partitions

- **parabolic root poset**: order filter of root poset induced by $S \setminus J$
- **(W^J)-nonnesting partitions**: order ideals in parabolic root poset $\rightsquigarrow NN\big(W^J\big)$

Outline

Subword Complexes

- **subword complex** $\rightsquigarrow \mathcal{S}(Q,w)$
- **flip order**: $F \lessdot G$ if and only if $F \setminus \{i\} = G \setminus \{j\}$ for $i < j$ $\rightsquigarrow \operatorname{Flip}\bigl(\mathcal{S}(Q,w)\bigr)$
- $Q = s_1s_2s_3s_1s_2s_3s_1s_2s_1$, $w = 4321$
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- $(1,2,3)$

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**
s_1	s_2	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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- $(1,2,3)$, $(2,3,4)$

1	2	3	4	**5**	**6**	**7**	**8**	**9**
$\mathbf{s_1}$	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	3	4	5	**6**	**7**	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	**3**	4	5	6	**7**	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	**3**	**4**	5	6	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_1}$

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1	**2**	**3**	**4**	**5**	6	**7**	8	9
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	s_2	s_1

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1	**2**	**3**	4	**5**	6	**7**	**8**	9
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	s_1

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1	2	**3**	4	**5**	**6**	**7**	**8**	9
$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_1

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1	2	**3**	**4**	**5**	**6**	**7**	**8**	9
s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	s_1

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1	**2**	**3**	**4**	**5**	**6**	**7**	8	9
s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_1

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1	**2**	**3**	**4**	**5**	**6**	7	8	**9**
s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	$\mathbf{s_1}$

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1	**2**	3	**4**	**5**	**6**	7	**8**	**9**
s_1	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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1	**2**	3	**4**	5	**6**	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

Subword Complexes

- **parabolic subword complex**: $Q = cw_o(c), w = w_o^J$
  $\rightsquigarrow SW(W^J, c)$

Subword Complexes

- **parabolic subword complex**: $Q = cw_o(c), w = w_o^J$, where $W = \mathfrak{S}_n$, $c = s_1 s_2 \cdots s_{n-1}$, and $w_o = s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} s_1 \cdots s_1$

$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1$, $w = 4|23|1$
-

1	2	3	4	5	6	7	8	9
s_1	s_2	s_3	s_1	s_2	s_3	s_1	s_2	s_1

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$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1$, $w = 4|23|1$
- $(1, 2, 3, 7)$

1	2	3	**4**	**5**	**6**	7	**8**	**9**
s_1	s_2	s_3	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1s_2s_3s_1s_2s_3s_1s_2s_1$, $w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$

1	2	3	4	**5**	**6**	7	**8**	**9**
$\mathbf{s_1}$	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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$\rightsquigarrow \mathcal{S}_n^J$

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- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$

1	**2**	3	4	5	**6**	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1$, $w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$

1	**2**	**3**	4	5	6	7	**8**	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	s_3	s_1	$\mathbf{s_2}$	$\mathbf{s_1}$

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$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1$, $w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$

1	**2**	**3**	4	**5**	6	7	8	**9**
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_3	s_1	s_2	$\mathbf{s_1}$

Subword Complexes

- **parabolic subword complex**: $Q = cw_o(c), w = w_o^J$, where $W = \mathfrak{S}_n$, $c = s_1s_2\cdots s_{n-1}$, and $w_o = s_1s_2\cdots s_{n-1}s_1s_2\cdots s_{n-2}s_1\cdots s_1$

$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1s_2s_3s_1s_2s_3s_1s_2s_1$, $w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$

1	**2**	**3**	4	**5**	6	**7**	8	9
$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_3	$\mathbf{s_1}$	s_2	s_1

Subword Complexes

- **parabolic subword complex**: $Q = cw_o(c), w = w_o^J$, where $W = \mathfrak{S}_n$, $c = s_1 s_2 \cdots s_{n-1}$, and $w_o = s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} s_1 \cdots s_1$

$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1$, $w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$, $(2,4,8,9)$

1	2	**3**	4	**5**	**6**	**7**	8	9
$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_1

Subword Complexes

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- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$, $(2,4,8,9)$, $(2,4,7,8)$

1	2	**3**	4	**5**	**6**	7	8	**9**
$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	$\mathbf{s_1}$

Subword Complexes

- **parabolic subword complex**: $Q = cw_o(c), w = w_o^J$, where $W = \mathfrak{S}_n$, $c = s_1s_2\cdots s_{n-1}$, and $w_o = s_1s_2\cdots s_{n-1}s_1s_2\cdots s_{n-2}s_1\cdots s_1$

$\rightsquigarrow \mathcal{S}_n^J$

- $Q = s_1s_2s_3s_1s_2s_3s_1s_2s_1$, $w = 4|23|1$
- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$, $(2,4,8,9)$, $(2,4,7,8)$, $(1,2,7,8)$

1	2	**3**	**4**	**5**	**6**	7	8	**9**
s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	s_1	s_2	$\mathbf{s_1}$

Subword Complexes

- **parabolic subword complex**: $Q = cw_o(c), w = w_o^J$, where $W = \mathfrak{S}_n$, $c = s_1 s_2 \cdots s_{n-1}$, and $w_o = s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} s_1 \cdots s_1$

$\rightsquigarrow \mathcal{S}_n^J$

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- $(1,2,3,7)$, $(2,3,4,7)$, $(3,4,5,7)$, $(4,5,6,7)$, $(4,6,7,8)$, $(4,6,8,9)$, $(2,4,8,9)$, $(2,4,7,8)$, $(1,2,7,8)$, $(1,2,8,9)$

1	2	**3**	**4**	**5**	**6**	**7**	8	9
s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_1}$	s_2	s_1

Subword Complexes

Theorem (Pilaud & Stump, 2015)

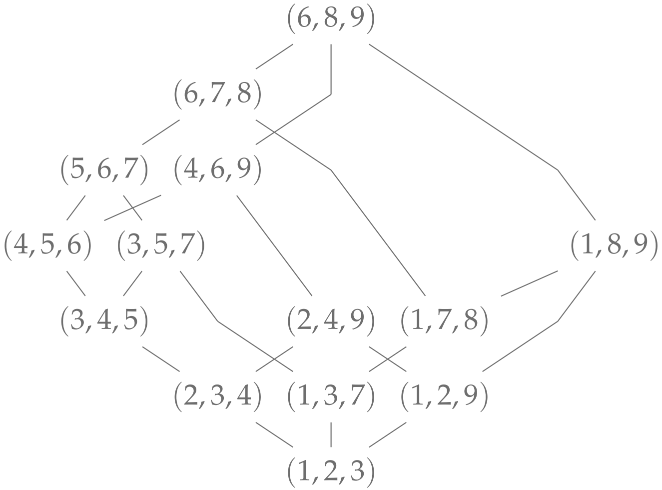
For any finite Coxeter group W, and any Coxeter element $c \in W$, we have $\mathit{Weak}\big(\mathit{Align}(W^{\emptyset}, c)\big) \cong \mathit{Flip}\big(\mathcal{SW}(W^{\emptyset}, c)\big)$.

Subword Complexes

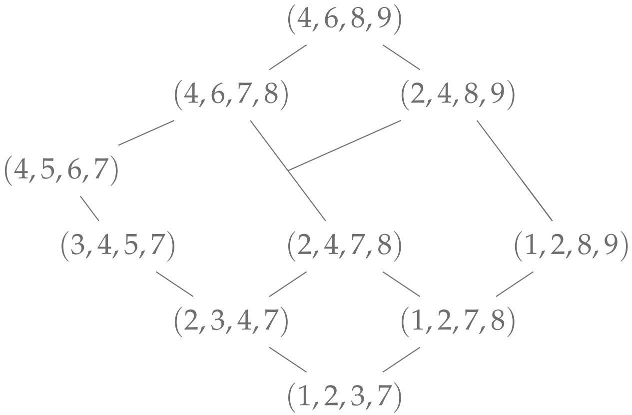
Conjecture (🌬 & Williams, 2015)

For any finite Coxeter group W, any Coxeter element $c \in W$, and any $J \subseteq S$, we have $Weak\big(Align(W^J, c)\big) \cong Flip\big(SW(W^J, c)\big)$.

Example: Flip$(\mathcal{S}_4^{\emptyset})$



Example: $\mathrm{Flip}\big(\mathcal{S}_4^{\{s_2\}}\big)$



A Bijection

Theorem (Serrano & Stump, 2011; & Williams, 2015)

For $n > 0$ and $J \subseteq S$, we have $\lvert \mathcal{S}_n^J \rvert = \lvert NN_n^J \rvert$.

- Edelman-Greene insertion on positions of subword
- slight modification of the recording tableau

A Bijection

Theorem (Serrano & Stump, 2011; & Williams, 2015)

For $n > 0$ and $J \subseteq S$, we have $\left|\mathcal{S}_n^J\right| = \left|NN_n^J\right|$.

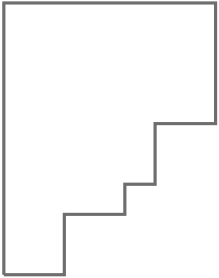
- Edelman-Greene insertion on positions of subword
- slight modification of the recording tableau

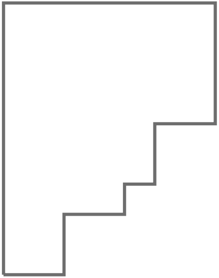
Example: $n = 11, J = \{s_1, s_2, s_3, s_5, s_8, s_{10}\}$

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	11	12	**13**
s_1	s_2	s_3	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	$\mathbf{s_{10}}$	s_1	s_2	$\mathbf{s_3}$
14	**15**	**16**	**17**	**18**	**19**	20	21	**22**	**23**	**24**	**25**	**26**
$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	s_{10}	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$
27	28	29	30	31	**32**	33	34	**35**	36	**37**	**38**	**39**
$\mathbf{s_7}$	s_8	s_9	s_1	s_2	$\mathbf{s_3}$	s_4	s_5	$\mathbf{s_6}$	s_7	$\mathbf{s_8}$	$\mathbf{s_1}$	$\mathbf{s_2}$
40	**41**	**42**	**43**	**44**	45	46	**47**	**48**	**49**	**50**	**51**	**52**
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_1}$	$\mathbf{s_2}$
53	**54**	**55**	**56**	**57**	**58**	**59**	**60**	61	**62**	63	**64**	65
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_1

Example: $n = 11, J = \{s_1, s_2, s_3, s_5, s_8, s_{10}\}$

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	11	12	**13**
s_1	s_2	s_3	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	$\mathbf{s_{10}}$	s_1	s_2	$\mathbf{s_3}$
14	**15**	**16**	**17**	**18**	**19**	20	21	**22**	**23**	**24**	**25**	**26**
$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	s_{10}	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$
27	28	29	30	31	**32**	33	34	**35**	36	**37**	**38**	**39**
$\mathbf{s_7}$	s_8	s_9	s_1	s_2	$\mathbf{s_3}$	s_4	s_5	$\mathbf{s_6}$	s_7	$\mathbf{s_8}$	$\mathbf{s_1}$	$\mathbf{s_2}$
40	**41**	**42**	**43**	**44**	45	46	**47**	**48**	**49**	**50**	**51**	**52**
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_1}$	$\mathbf{s_2}$
53	**54**	**55**	**56**	**57**	**58**	**59**	**60**	61	**62**	63	**64**	65
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_1





Example: $n = 11, J = \{s_1, s_2, s_3, s_5, s_8, s_{10}\}$

1	2	3	**4**	**5**	**6**	**7**	**8**	**9**	**10**	11	12	**13**
s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_1	s_2	s_3
14	**15**	**16**	**17**	**18**	**19**	20	21	**22**	**23**	**24**	**25**	**26**
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_1	s_2	s_3	s_4	s_5	s_6
27	28	29	30	31	**32**	33	34	**35**	36	**37**	**38**	**39**
s_7	s_8	s_9	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_1	s_2
40	**41**	**42**	**43**	**44**	45	46	**47**	**48**	**49**	**50**	**51**	**52**
s_3	s_4	s_5	s_6	s_7	s_1	s_2	s_3	s_4	s_5	s_6	s_1	s_2
53	**54**	**55**	**56**	**57**	**58**	**59**	**60**	61	**62**	63	**64**	65
s_3	s_4	s_5	s_1	s_2	s_3	s_4	s_1	s_2	s_3	s_1	s_2	s_1

s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					

4	5	6	7	8	9	10
13	14	15	16	17	18	19
22	23	24	25	26	27	37
32	35	40	41	42	43	44
38	39	48	49	50		
47	52	53	54	55		
51	57	58	59			
56	62					
60	64					

Example: $n = 11, J = \{s_1, s_2, s_3, s_5, s_8, s_{10}\}$

1	1	1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	2	2	**2**
s_1	s_2	s_3	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	$\mathbf{s_{10}}$	s_1	s_2	$\mathbf{s_3}$
2	**2**	**2**	**2**	**2**	**2**	2	3	**3**	**3**	**3**	**3**	**3**
$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	$\mathbf{s_8}$	$\mathbf{s_9}$	s_{10}	s_1	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$
3	3	3	4	4	**4**	4	4	**4**	4	**4**	**5**	**5**
$\mathbf{s_7}$	s_8	s_9	s_1	s_2	$\mathbf{s_3}$	s_4	s_5	$\mathbf{s_6}$	s_7	$\mathbf{s_8}$	$\mathbf{s_1}$	$\mathbf{s_2}$
5	**5**	**5**	**5**	**5**	6	6	**6**	**6**	**6**	**6**	**7**	**7**
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_7}$	s_1	s_2	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_6}$	$\mathbf{s_1}$	$\mathbf{s_2}$
7	**7**	**7**	**8**	**8**	**8**	**8**	**9**	9	**9**	10	**10**	11
$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_5}$	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$	$\mathbf{s_4}$	$\mathbf{s_1}$	s_2	$\mathbf{s_3}$	s_1	$\mathbf{s_2}$	s_1

s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	4
4	4	5	5	5	5	5
5	5	6	6	6		
6	7	7	7	7		
7	8	8	8			
8	9					
9	10					

Example: $n = 11, J = \{s_1, s_2, s_3, s_5, s_8, s_{10}\}$

1 s_1	1 s_2	1 s_3	**1 s_4**	**1 s_5**	**1 s_6**	**1 s_7**	**1 s_8**	**1 s_9**	**1 s_{10}**	2 s_1	2 s_2	**2 s_3**
2 s_4	**2 s_5**	**2 s_6**	**2 s_7**	**2 s_8**	**2 s_9**	2 s_{10}	3 s_1	**3 s_2**	**3 s_3**	**3 s_4**	**3 s_5**	**3 s_6**
3 s_7	3 s_8	3 s_9	4 s_1	4 s_2	**4 s_3**	4 s_4	4 s_5	**4 s_6**	4 s_7	**4 s_8**	**5 s_1**	**5 s_2**
5 s_3	**5 s_4**	**5 s_5**	**5 s_6**	**5 s_7**	6 s_1	6 s_2	**6 s_3**	**6 s_4**	**6 s_5**	**6 s_6**	**7 s_1**	**7 s_2**
7 s_3	**7 s_4**	**7 s_5**	**8 s_1**	**8 s_2**	**8 s_3**	**8 s_4**	**9 s_1**	9 s_2	**9 s_3**	10 s_1	**10 s_2**	11 s_1

s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	1
0	0	1	1	1	1	1
0	0	1	1	1		
0	1	1	1	1		
0	1	1	1			
0	1					
0	1					

Example: $n = 11, J = \{s_1, s_2, s_3, s_5, s_8, s_{10}\}$

1 s_1	1 s_2	1 s_3	**1 s_4**	**1 s_5**	**1 s_6**	**1 s_7**	**1 s_8**	**1 s_9**	**1 s_{10}**	2 s_1	2 s_2	**2 s_3**
2 s_4	**2 s_5**	**2 s_6**	**2 s_7**	**2 s_8**	**2 s_9**	2 s_{10}	3 s_1	**3 s_2**	**3 s_3**	**3 s_4**	**3 s_5**	**3 s_6**
3 s_7	3 s_8	3 s_9	4 s_1	4 s_2	**4 s_3**	4 s_4	4 s_5	**4 s_6**	4 s_7	**4 s_8**	**5 s_1**	**5 s_2**
5 s_3	**5 s_4**	**5 s_5**	**5 s_6**	**5 s_7**	6 s_1	6 s_2	**6 s_3**	**6 s_4**	**6 s_5**	**6 s_6**	**7 s_1**	**7 s_2**
7 s_3	**7 s_4**	**7 s_5**	**8 s_1**	**8 s_2**	**8 s_3**	**8 s_4**	**9 s_1**	9 s_2	**9 s_3**	10 s_1	**10 s_2**	11 s_1

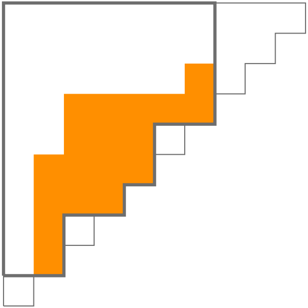
s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	1
0	0	1	1	1	1	1
0	0	1	1	1		
0	1	1	1	1		
0	1	1	1			
0	1					
0	1					

Example: $n = 11, J = \{s_1, s_2, s_3, s_5, s_8, s_{10}\}$

1 s_1	1 s_2	1 s_3	**1 s_4**	**1 s_5**	**1 s_6**	**1 s_7**	**1 s_8**	**1 s_9**	**1 s_{10}**	2 s_1	2 s_2	**2 s_3**
2 s_4	**2 s_5**	**2 s_6**	**2 s_7**	**2 s_8**	**2 s_9**	2 s_{10}	3 s_1	**3 s_2**	**3 s_3**	**3 s_4**	**3 s_5**	**3 s_6**
3 s_7	3 s_8	3 s_9	4 s_1	4 s_2	**4 s_3**	4 s_4	4 s_5	**4 s_6**	4 s_7	**4 s_8**	**5 s_1**	**5 s_2**
5 s_3	**5 s_4**	**5 s_5**	**5 s_6**	**5 s_7**	6 s_1	6 s_2	**6 s_3**	**6 s_4**	**6 s_5**	**6 s_6**	**7 s_1**	**7 s_2**
7 s_3	**7 s_4**	**7 s_5**	**8 s_1**	**8 s_2**	**8 s_3**	**8 s_4**	**9 s_1**	9 s_2	**9 s_3**	10 s_1	**10 s_2**	11 s_1

s_1	s_2	s_3	s_4	s_5	s_6	s_7
s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_3	s_4	s_5	s_6	s_7	s_8	s_9
s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_5	s_6	s_7	s_8	s_9		
s_6	s_7	s_8	s_9	s_{10}		
s_7	s_8	s_9	s_{10}			
s_8	s_9					
s_9	s_{10}					



Outline

Parabolic Catalan Numbers

- four families of **parabolic Catalan objects**...
- ... parametrized by a Coxeter group and a Coxeter element

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Theorem (& Williams, 2015)

For $n > 0$ and $J \subseteq S$ we have

$$\left|\mathfrak{S}_n^J(231)\right| = \left|NC_n^J\right| = \left|NN_n^J\right| = \left|\mathcal{S}_n^J\right|.$$

Parabolic Catalan Numbers

- four families of **parabolic Catalan objects**...
- ... parametrized by a Coxeter group and a Coxeter element

Theorem (& Williams, 2015)

For $W = \mathfrak{S}_n$, $c = s_1 s_2 \cdots s_{n-1}$, and $J \subseteq S$ we have

$$\bigl|Align\bigl(W^J, c\bigr)\bigr| = \bigl|NC\bigl(W^J, c\bigr)\bigr| = \bigl|NN\bigl(W^J\bigr)\bigr| = \bigl|SW\bigl(W^J, c\bigr)\bigr|.$$

Parabolic Catalan Numbers

- does this work for all types?

Parabolic Catalan Numbers

- does this work for all types? No!

- let $W = D_4, c = s_3s_2s_1s_4, c' = s_2s_3s_4s_1, J = \{s_1, s_2\}$
- we have:
  - $\bigl|\mathrm{Align}\bigl(W^J, c\bigr)\bigr| = 21$
  - $\bigl|\mathrm{Align}\bigl(W^J, c'\bigr)\bigr| = \bigl|NN\bigl(W^J\bigr)\bigr| = 22$

Parabolic Catalan Numbers

- does this work for all types? But possibly for the coincidental types!

Conjecture (& Williams, 2015)

For $W\in\{A_n,B_n,I_2(k),H_3\}$, $J\subseteq S$, and $c\in W$ a Coxeter element, we have

$$\lvert Align(W^J,c)\rvert = \lvert NC(W^J,c)\rvert = \lvert NN(W^J)\rvert = \lvert SW(W^J,c)\rvert.$$

Outline

Parabolic Aligned Elements

- **(W^J, c)-aligned element**: whenever $\alpha < a\alpha + b\beta < \beta$ in $\mathbf{Inv}\big(\mathbf{w}_{\mathbf{o}}^{\mathbf{J}}(\mathbf{c})\big)$, then $t_{a\alpha+b\beta} \in \text{cov}(w)$ implies $t_\alpha \in \text{inv}(w)$ $\rightsquigarrow \text{Align}\big(W^J, c\big)$

w-Aligned Elements

- **w-aligned element**: whenever $\alpha < a\alpha + b\beta < \beta$ in $\mathbf{Inv}(\mathbf{w})$, then $t_{a\alpha+b\beta} \in \text{cov}(w)$ implies $t_\alpha \in \text{inv}(w)$

  $\rightsquigarrow \text{Align}(W, \mathbf{w})$

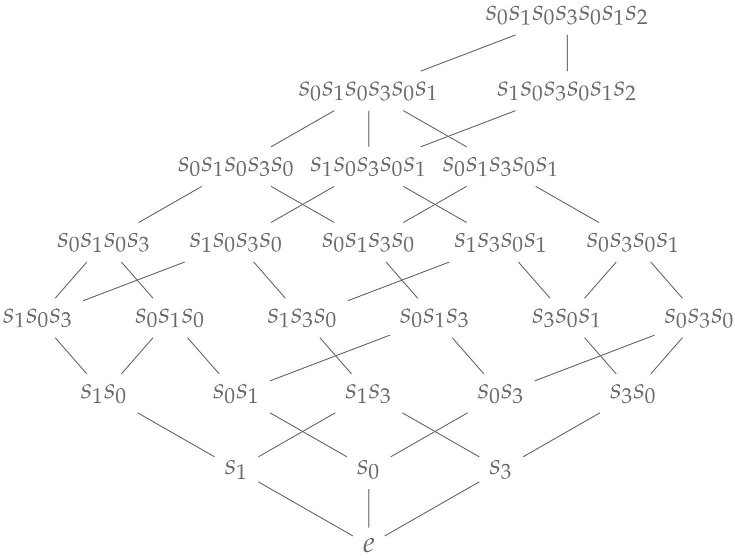
$\mathbf{w}$-Aligned Elements

- **w-aligned element**: whenever $\alpha < a\alpha + b\beta < \beta$ in $\mathbf{Inv}(\mathbf{w})$, then $t_{a\alpha+b\beta} \in \mathrm{cov}(w)$ implies $t_\alpha \in \mathrm{inv}(w)$ $\rightsquigarrow \mathrm{Align}(W, \mathbf{w})$
- works for any reduced word of any element in any Coxeter group

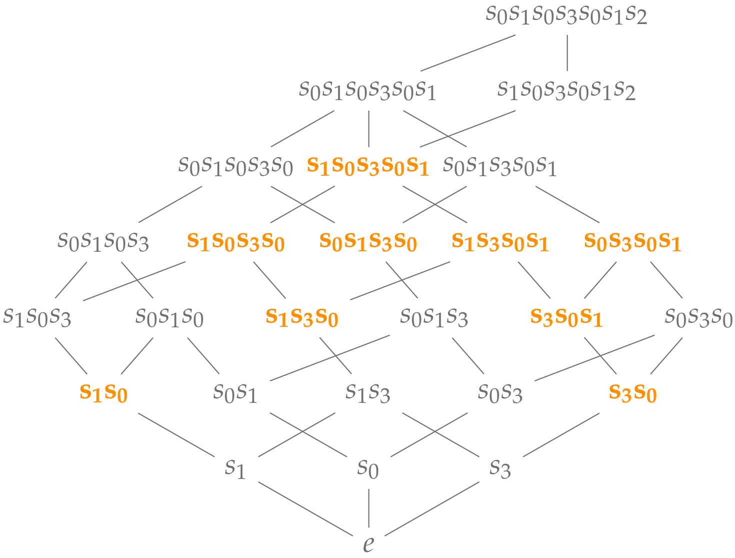
$\mathbf{w}$-Aligned Elements

- **w-aligned element**: whenever $\alpha < a\alpha + b\beta < \beta$ in $\mathbf{Inv}(\mathbf{w})$, then $t_{a\alpha+b\beta} \in \mathrm{cov}(w)$ implies $t_\alpha \in \mathrm{inv}(w)$ $\rightsquigarrow \mathrm{Align}(W, \mathbf{w})$
- works for any reduced word of any element in any Coxeter group
- in general, $\mathrm{Weak}(\mathrm{Align}(W, \mathbf{w}))$ is not a lattice
  - take $W = \tilde{A}_3$ and $\mathbf{w} = s_0s_1s_0s_3s_0s_1s_2$
- not even in finite type
- no counterexamples for rank ≤ 3 known

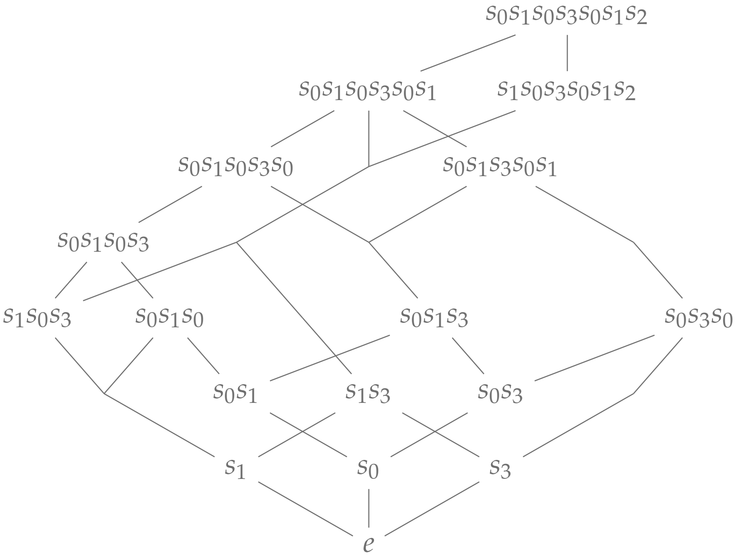
Weak$(e, s_0s_1s_0s_3s_0s_1s_2)$



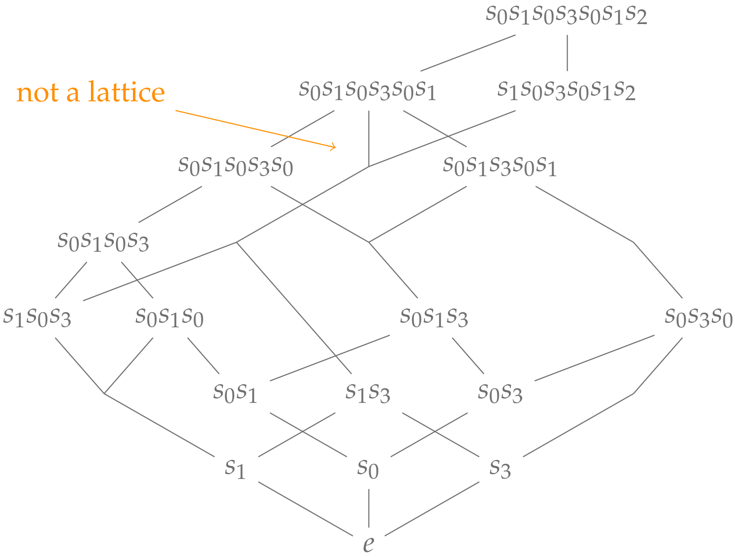
Weak$(e, s_0s_1s_0s_3s_0s_1s_2)$



$\mathrm{Weak}\big(\mathrm{Align}\big(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2\big)\big)$

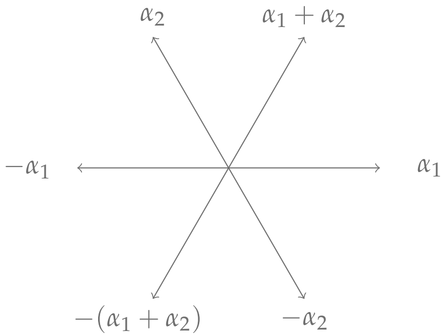


Weak$\left(\text{Align}\left(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2\right)\right)$

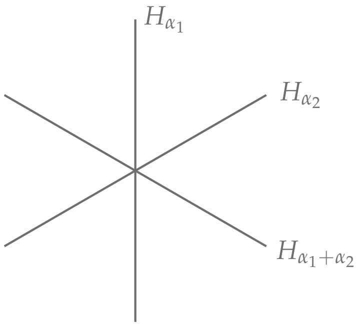


A Geometric Interpretation

- **shards**



- **shards**



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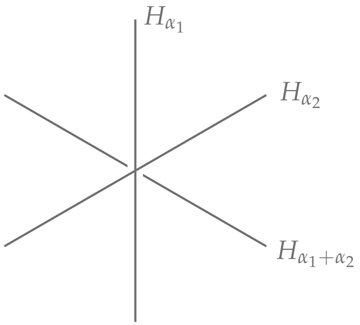
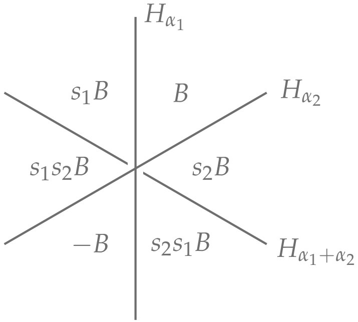


- **w-selected shards**



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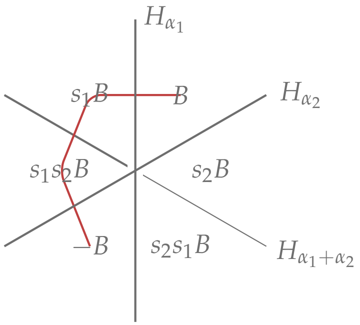
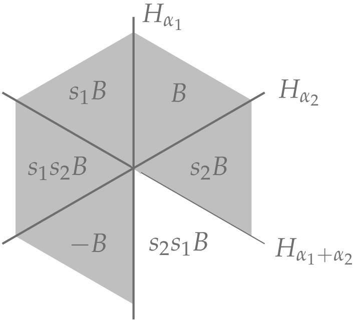
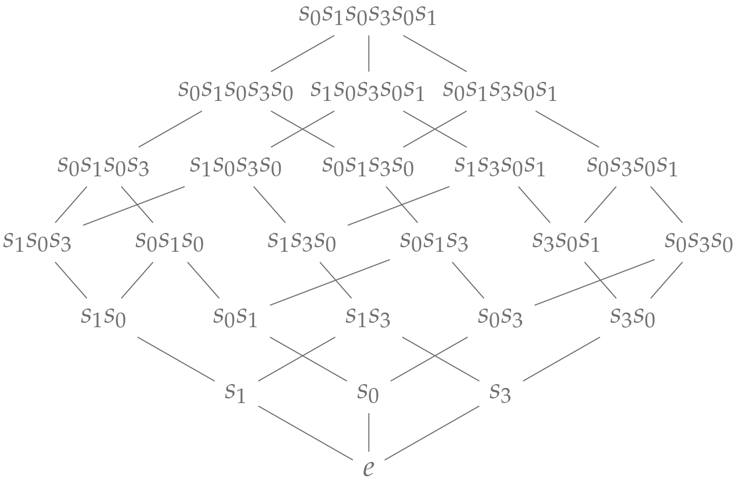


A Geometric Interpretation

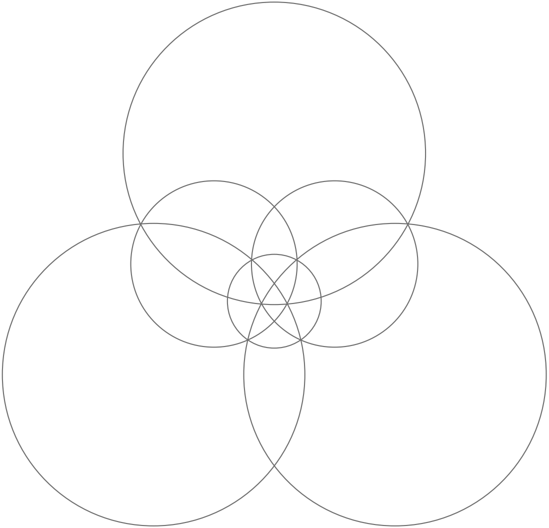
- **w-selected shards**
- **w-sortable elements**: all lower shards are $\mathbf{w}$-selected $\rightsquigarrow \mathrm{Sort}(W, \mathbf{w})$



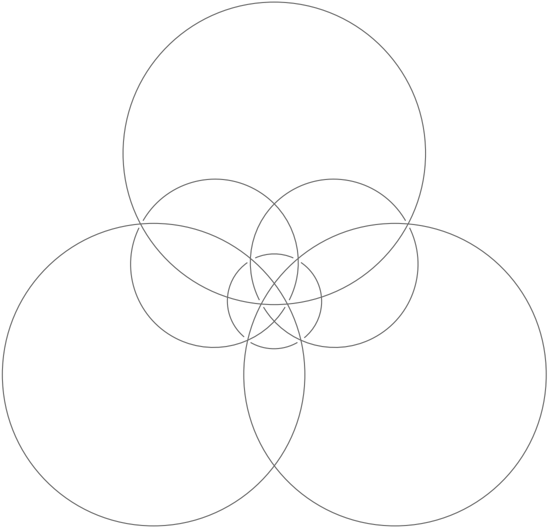
Weak$(e, s_0s_1s_0s_3s_0s_1)$



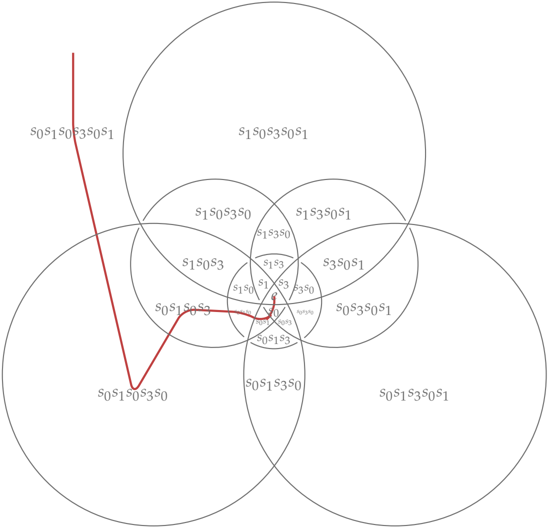
Sort$(\tilde{A}_3, s_0s_1s_0s_3s_0s_1)$



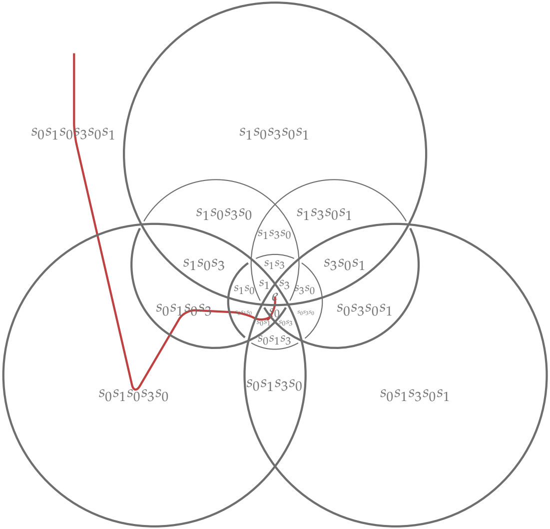
Sort$(\tilde{A}_3, s_0s_1s_0s_3s_0s_1)$



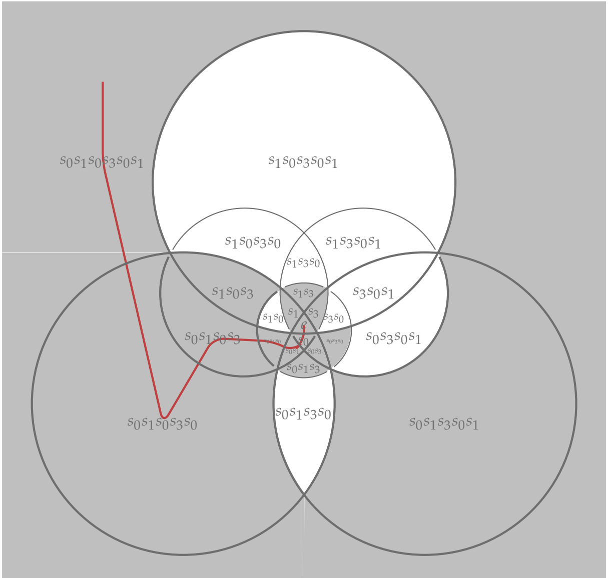
Sort$(\tilde{A}_3, s_0s_1s_0s_3s_0s_1)$



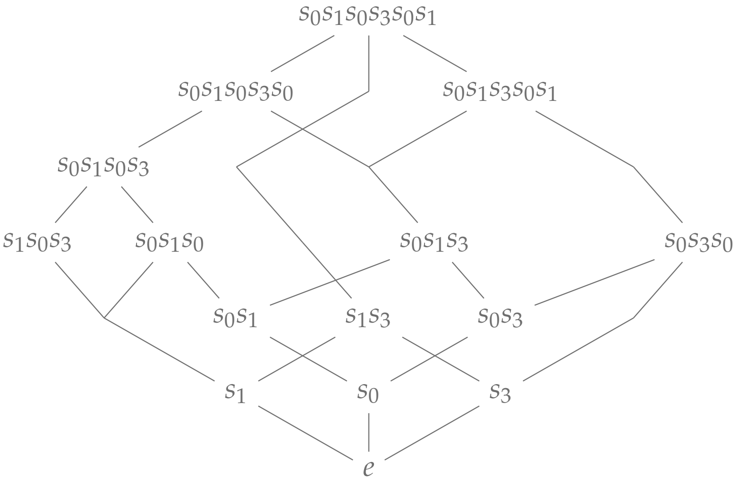
Sort$(\tilde{A}_3, s_0s_1s_0s_3s_0s_1)$



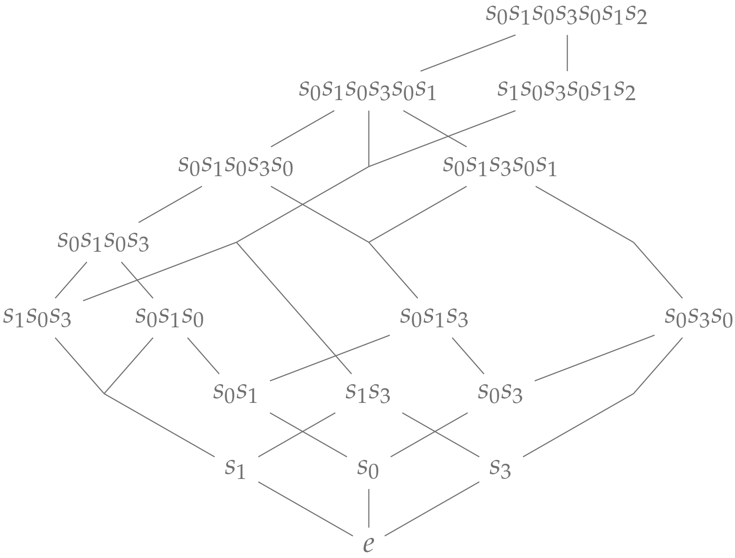
$\mathrm{Sort}(\tilde{A}_3, s_0s_1s_0s_3s_0s_1)$



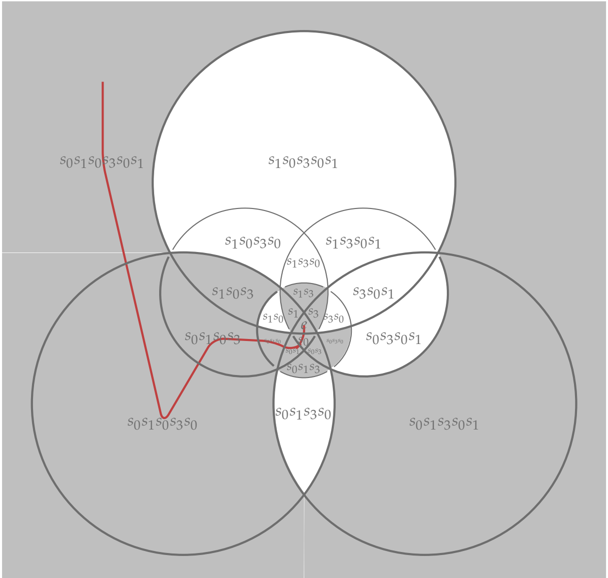
Weak$\left(\text{Align}\left(\tilde{A}_3, s_0s_1s_0s_3s_0s_1\right)\right)$



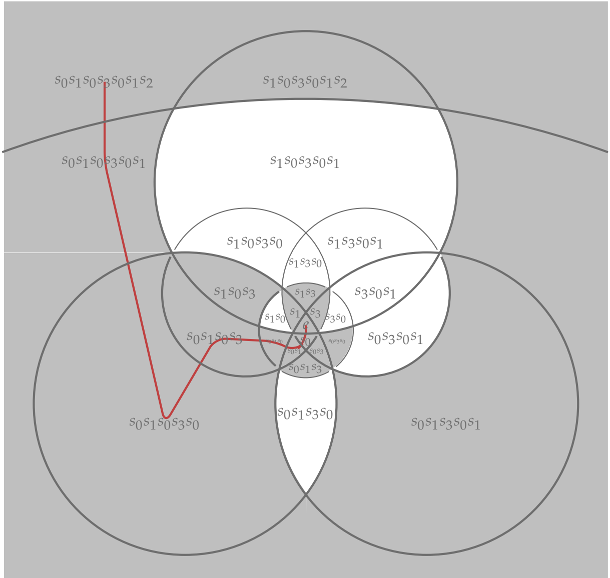
Weak$(\text{Align}(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2))$



$\mathrm{Sort}(\tilde{A}_3, s_0s_1s_0s_3s_0s_1)$



$\mathrm{Sort}(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2)$



A Geometric Interpretation

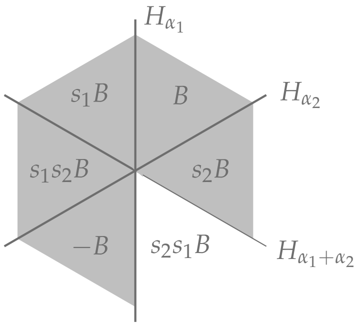
- **w-sortable elements**: all lower shards are $\mathbf{w}$-selected $\rightsquigarrow \mathrm{Sort}(W, \mathbf{w})$

Theorem (& Williams, 2015)

For any Coxeter group W, and any reduced word $\mathbf{w}$, we have $\mathit{Align}(W, \mathbf{w}) = \mathit{Sort}(W, \mathbf{w})$.

A Lattice-Theoretic Extension

- **shard-inversion set**: inversions that correspond to selected shards $\rightsquigarrow \mathrm{sinv}(w)$
- **w-selected**: minimal regions with given shard-inversion set $\rightsquigarrow \mathrm{Select}(W, \mathbf{w})$



$$\mathrm{sinv}(e) = \emptyset$$
$$\mathrm{sinv}(s_1) = \{s_1\}$$
$$\mathrm{sinv}(s_2) = \{s_2\}$$
$$\mathrm{sinv}(s_1s_2) = \{s_1, s_1s_2s_1\}$$
$$\mathrm{sinv}(s_2s_1) = \{s_2\}$$
$$\mathrm{sinv}(s_1s_2s_1) = \{s_1, s_1s_2s_1, s_2\}$$

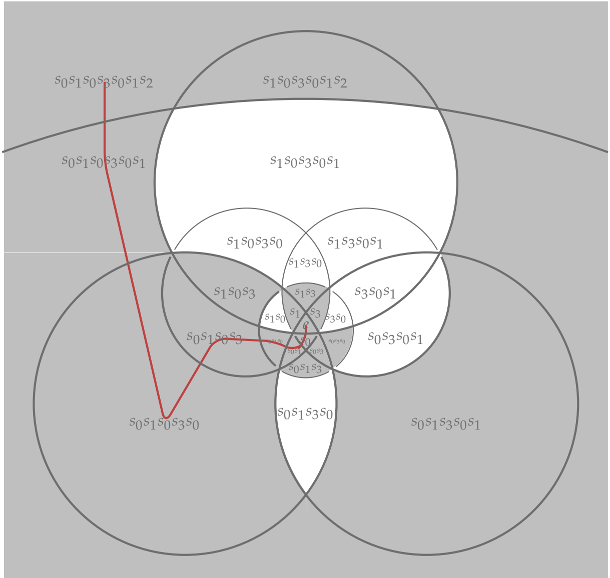
A Lattice-Theoretic Extension

- **shard-inversion set**: inversions that correspond to selected shards $\rightsquigarrow \mathrm{sinv}(w)$
- **w-selected**: minimal regions with given shard-inversion set $\rightsquigarrow \mathrm{Select}(W, \mathbf{w})$

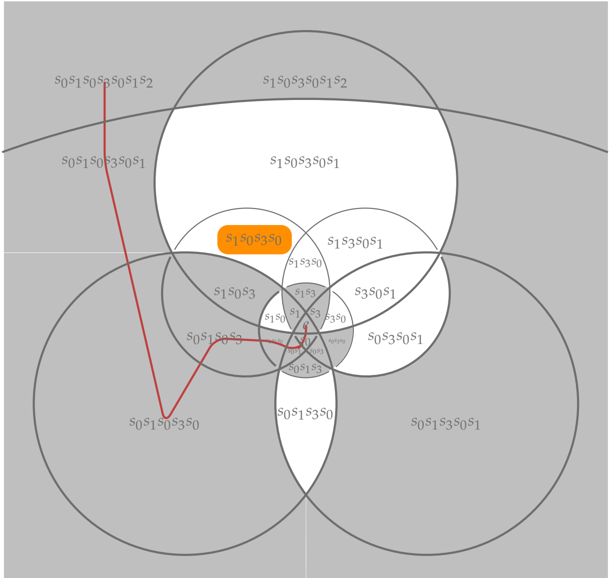
Theorem (& Williams, 2015)

For any Coxeter group W, and any reduced word $\mathbf{w}$, the poset $\mathit{Weak}(\mathit{Select}(W, \mathbf{w}))$ is a lattice. Moreover, it is a lattice completion of $\mathit{Weak}(\mathit{Sort}(W, \mathbf{w}))$.

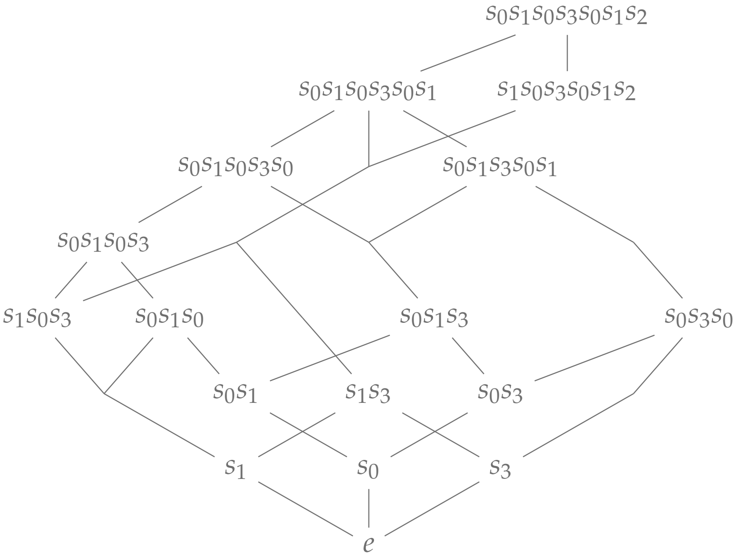
Sort$(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2)$



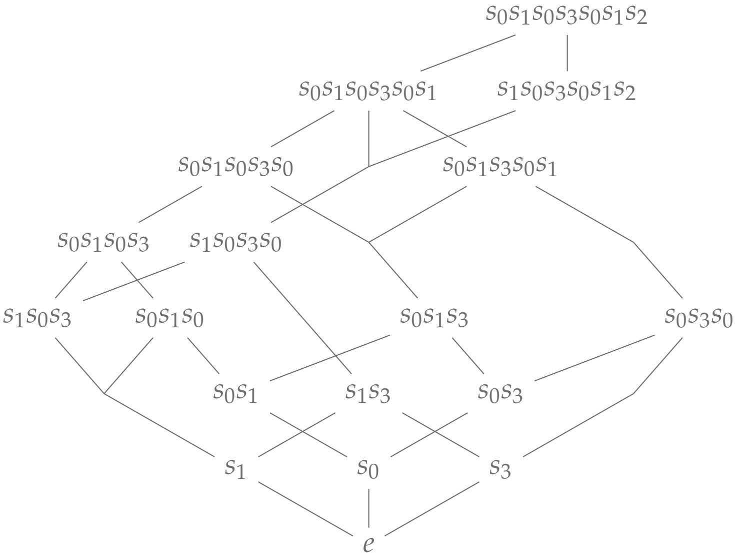
$\mathrm{Sort}\left(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2\right)$



$\mathrm{Weak}\big(\mathrm{Align}(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2)\big)$



Weak$\left(\text{Select}\left(\tilde{A}_3, s_0s_1s_0s_3s_0s_1s_2\right)\right)$



Thank You.