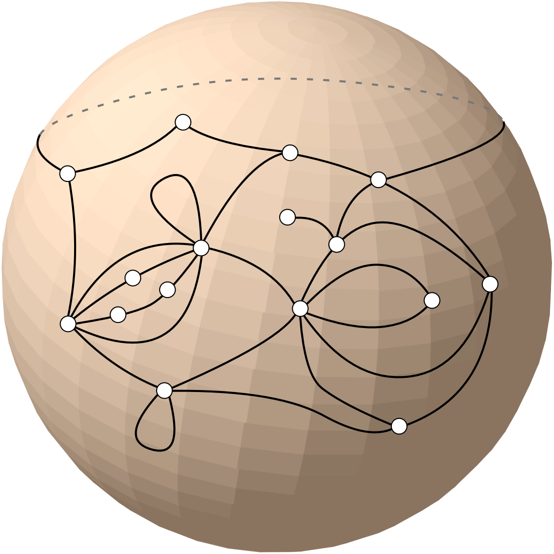
A bijection for nonorientable maps

Jérémie BETTINELLI

October 7, 2015

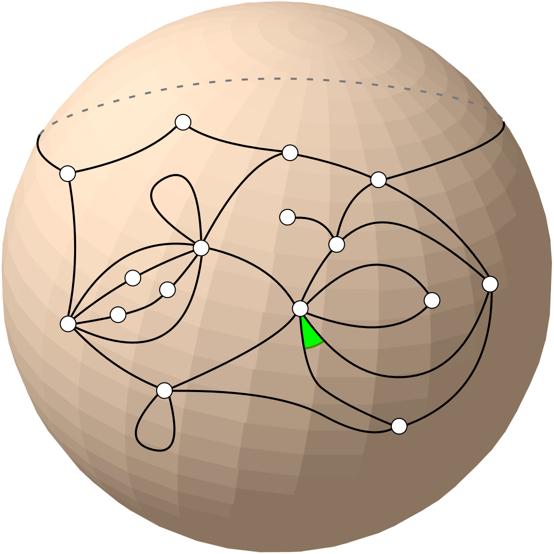
Plane maps



plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

Plane maps

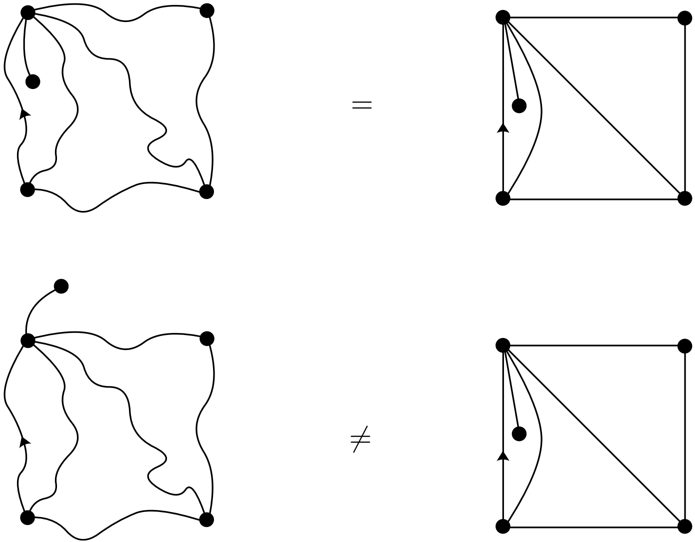


plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

root: distinguished corner

Edge deformation



Scaling limit: the Brownian map

- We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.

Theorem (Le Gall '11, Miermont '11)

Let $\mathfrak{q}_n$ be a uniform plane quadrangulation with n faces. The sequence $\left(V(\mathfrak{q}_n), (8n/9)^{-1/4}\, d_{\mathfrak{q}_n}\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space a.s. homeomorphic to the sphere, and called the Brownian map.

Definition (Convergence for the Gromov–Hausdorff topology)

A sequence $(\mathcal{X}_n)$ of compact metric spaces **converges in the sense of the Gromov–Hausdorff topology** toward a metric space $\mathcal{X}$ if there exist isometric embeddings $\varphi_n : \mathcal{X}_n \to \mathcal{Z}$ and $\varphi : \mathcal{X} \to \mathcal{Z}$ into a common metric space $\mathcal{Z}$ such that $\varphi_n(\mathcal{X}_n)$ converges toward $\varphi(\mathcal{X})$ in the sense of the Hausdorff topology.

Scaling limit: the Brownian map

- We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.

Theorem (B. & Jacob & Miermont '13)

Let $\mathfrak{m}_n$ be a uniform plane map with n edges. The sequence $\left(V(\mathfrak{m}_n),(8n/9)^{-1/4}\,d_{\mathfrak{m}_n}\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space a.s. homeomorphic to the sphere, and called the Brownian map.

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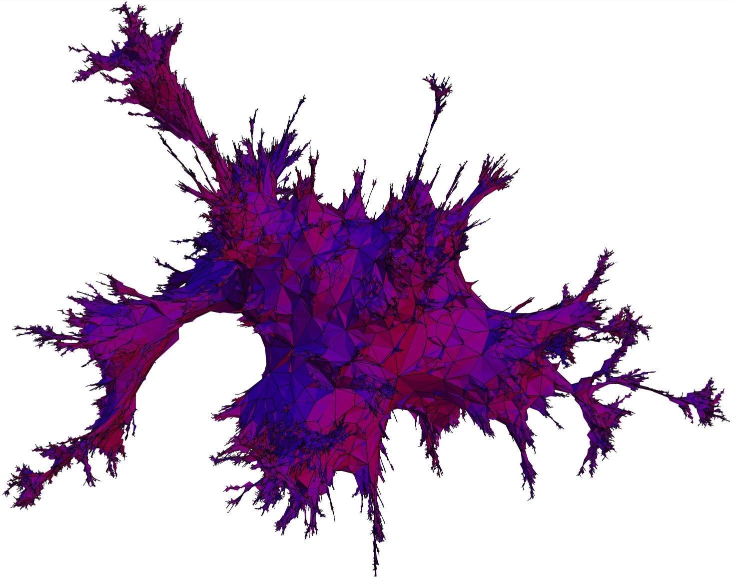
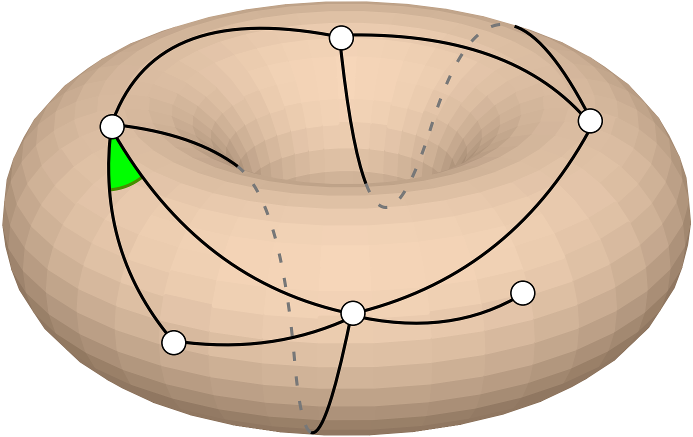
Uniform plane quadrangulation with 50 000 faces

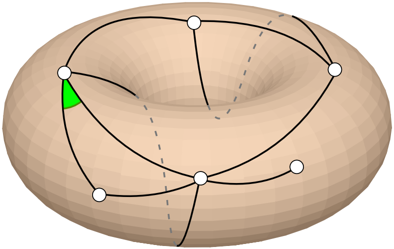


Genus g maps



genus g map: graph embedded in the surface of genus g, in such a way that the faces are homeomorphic to disks

Edge deformation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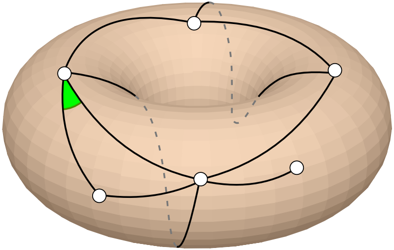


maps are defined up to direct homeomorphism of the underlying surface

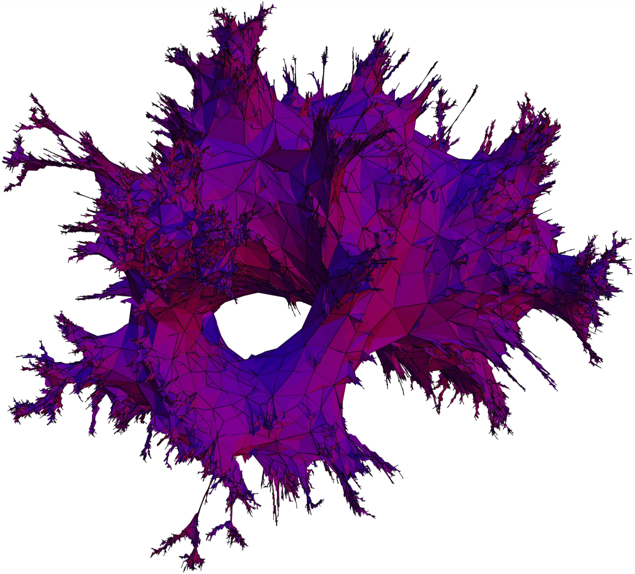
Scaling limit: the Brownian surface of genus g

✧ We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.

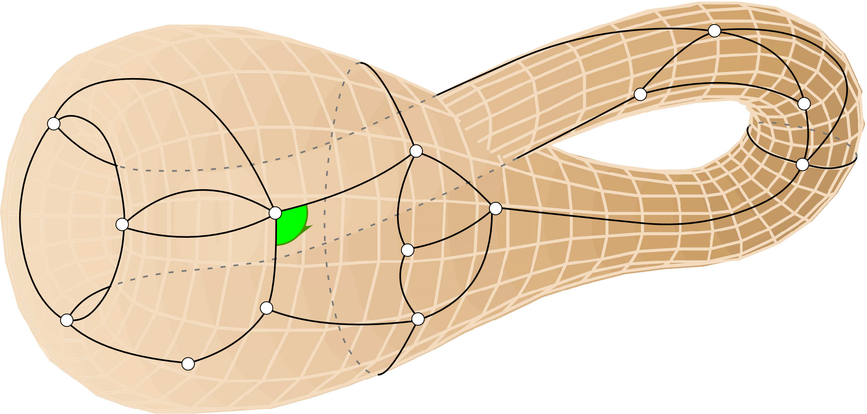
Theorem (B. & Miermont '15, in prep.)

Let $g \geq 1$ be fixed and $\mathfrak{q}_n$ be a uniform genus g quadrangulation with n faces. The sequence $\left(V(\mathfrak{q}_n), (8n/9)^{-1/4}\, d_{\mathfrak{q}_n}\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space a.s. homeomorphic to the surface of genus g, and called the Brownian surface of genus g.

Uniform genus 1 quadrangulation with 50 000 faces



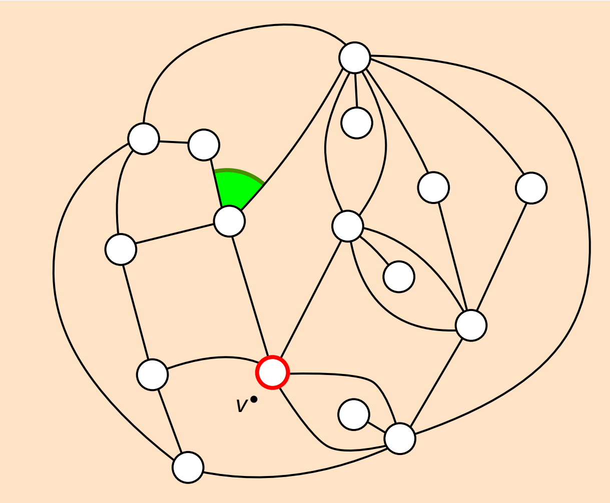
Nonorientable maps



root: distinguished corner given with a local orientation

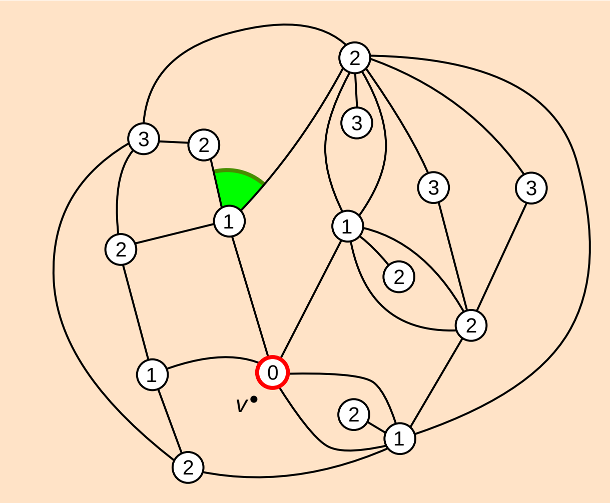
maps are defined up to homeomorphism of the underlying surface

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



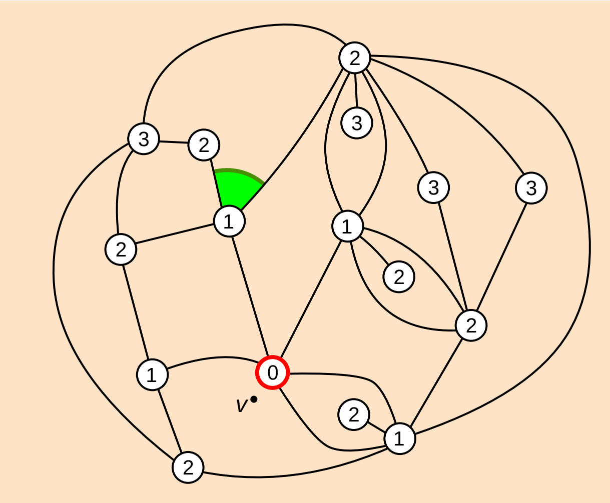
- Start with a pointed quadrangulation.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer

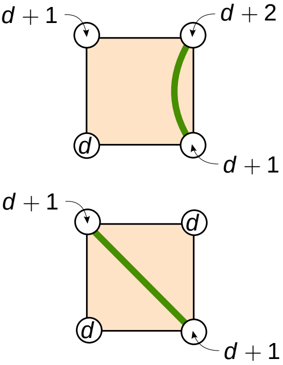


- ✧ Start with a pointed quadrangulation.
- ✧ Label the vertices with their distance to $v^\bullet$.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



- Apply the rule:



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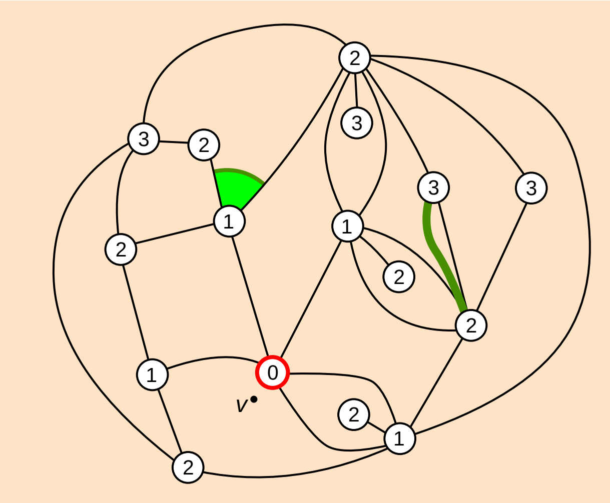

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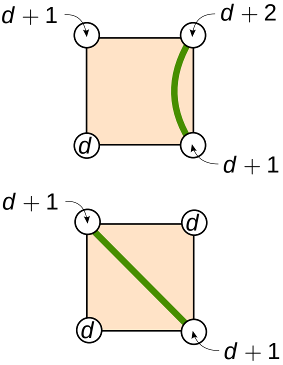

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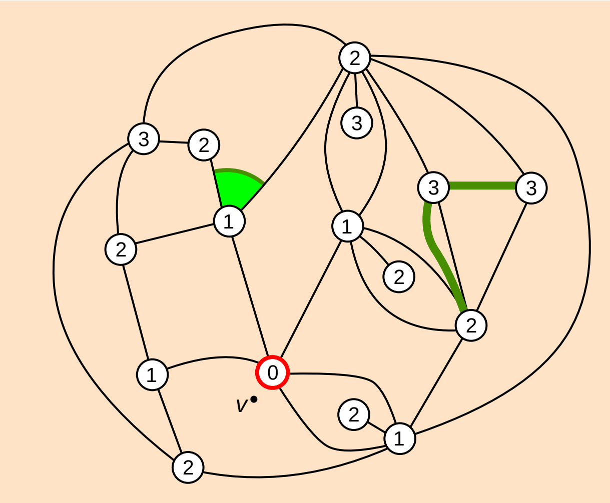
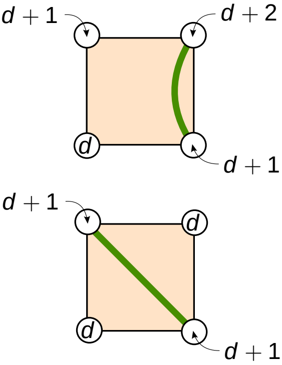
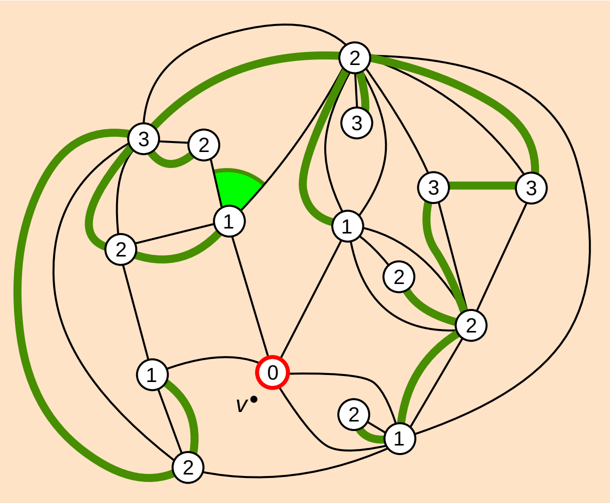


✧ Apply the rule:



Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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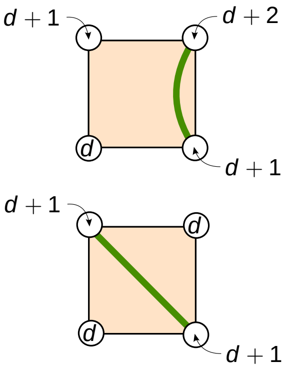

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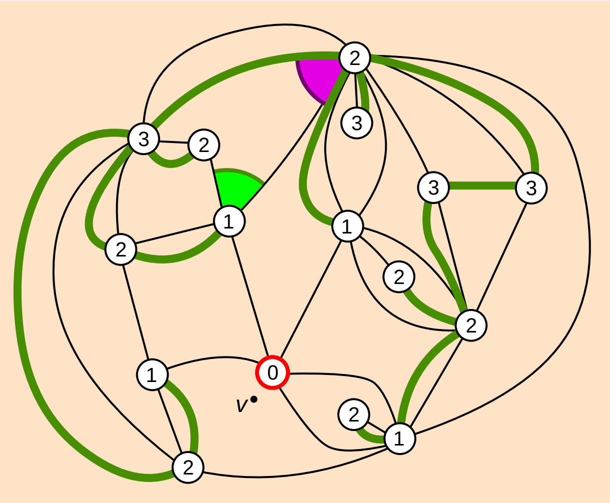
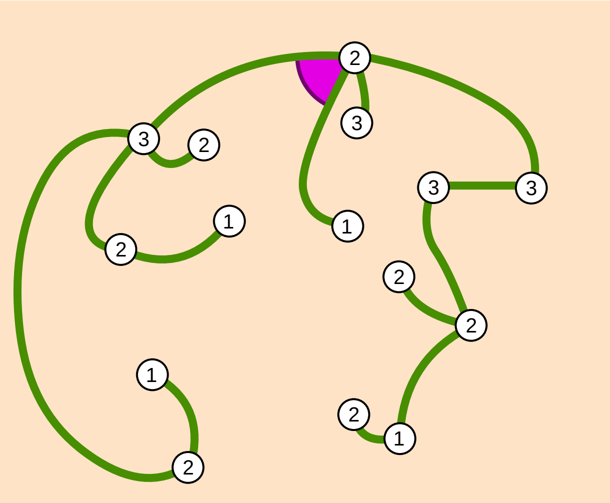


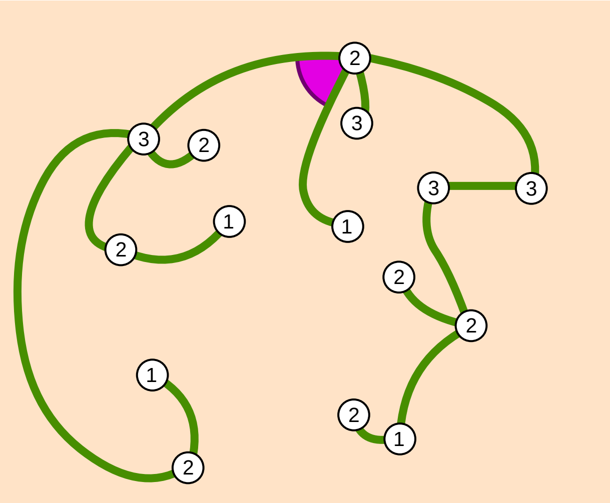
- ✧ Start with a pointed quadrangulation.
- ✧ Label the vertices with their distance to $v^\bullet$.
- ✧ Apply the rule.
- ✧ Root.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



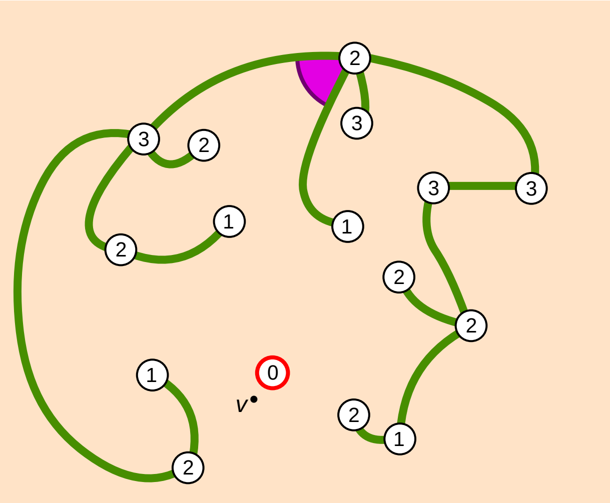
- ✧ Start with a pointed quadrangulation.
- ✧ Label the vertices with their distance to $v^{\bullet}$.
- ✧ Apply the rule.
- ✧ Root.
- ✧ Remove the initial edges and $v^{\bullet}$.

Inverse construction



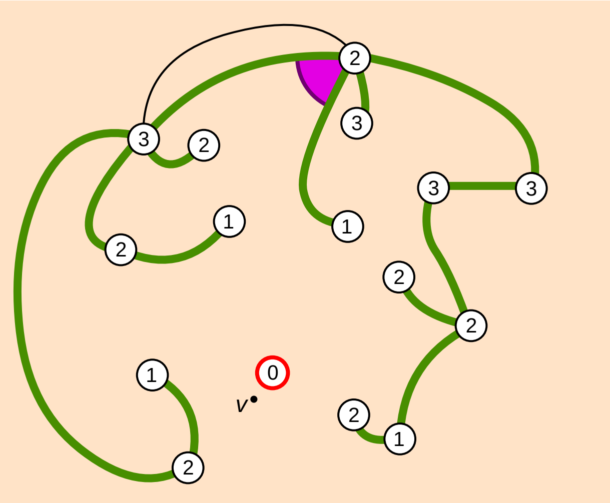
- Take a unicellular well-labeled map.

Inverse construction



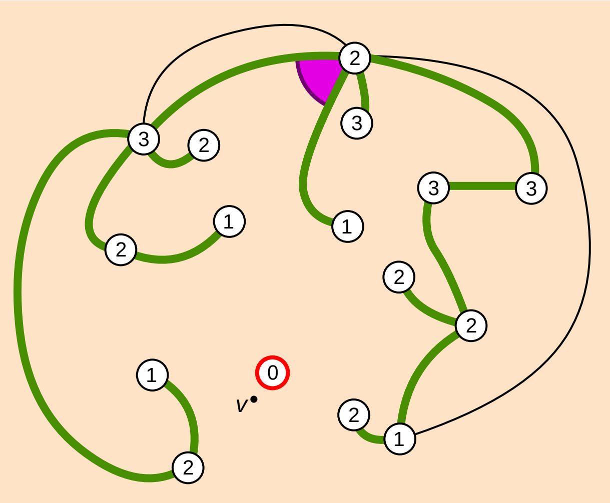
- Take a unicellular well-labeled map.
- Add a vertex $v^\bullet$ inside the unique face.

Inverse construction



- Take a unicellular well-labeled map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.

Inverse construction



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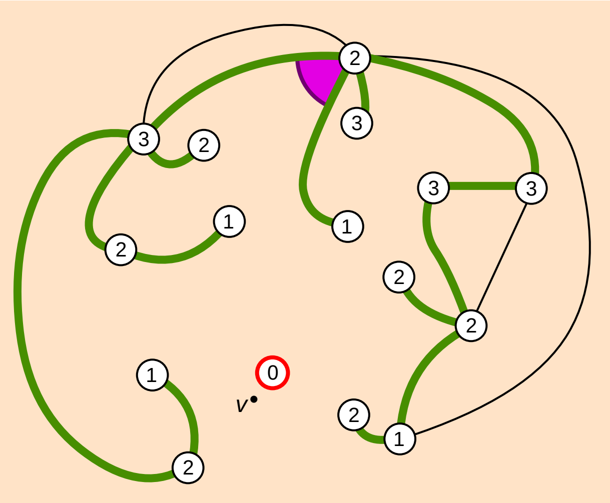

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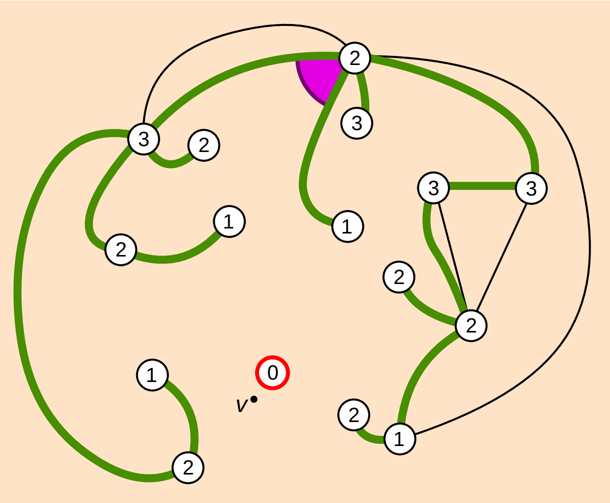

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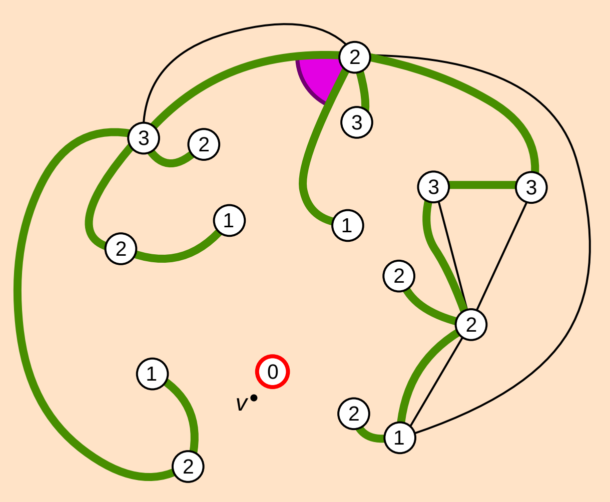

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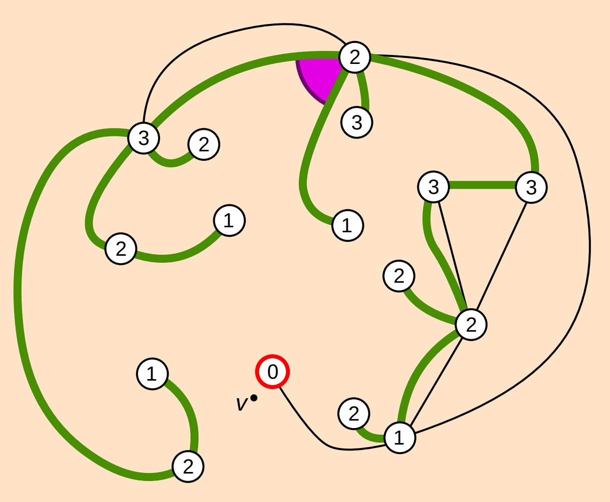

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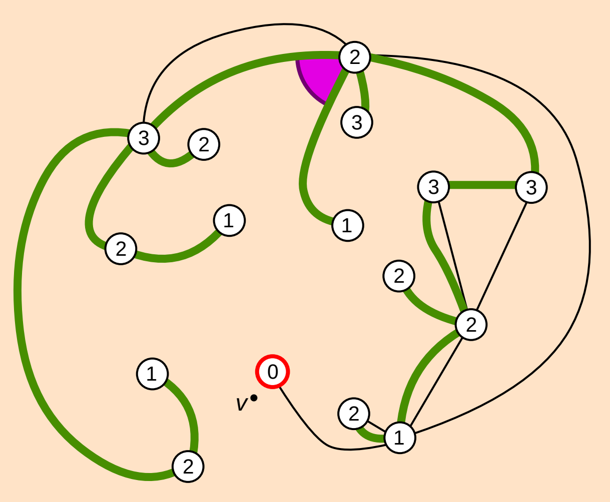

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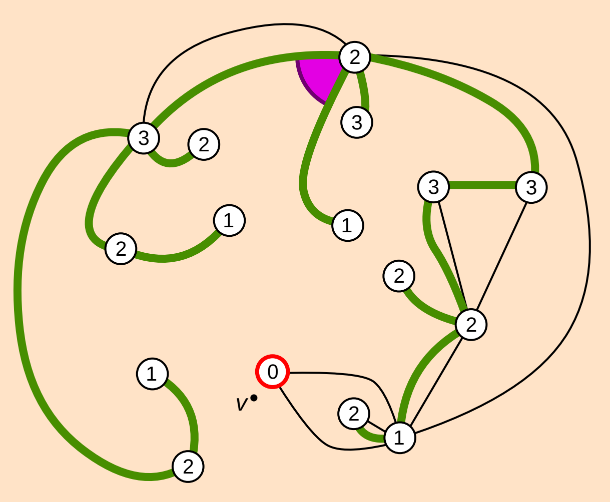

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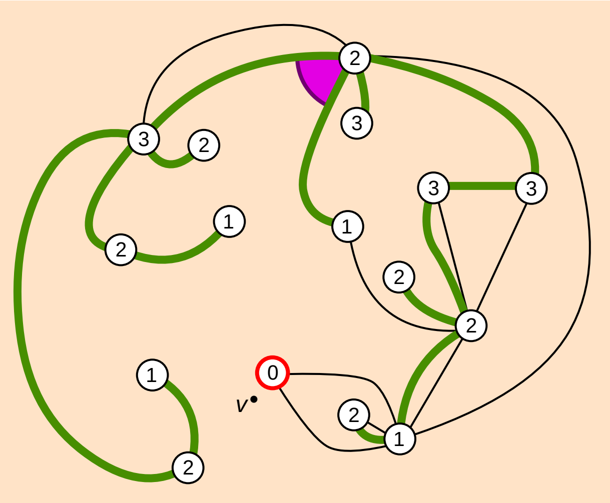

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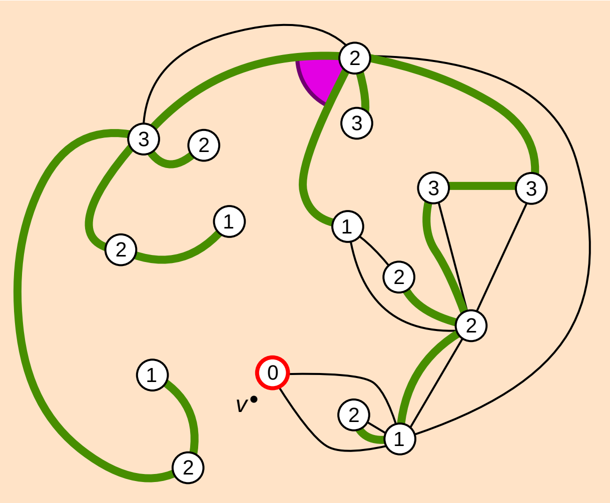

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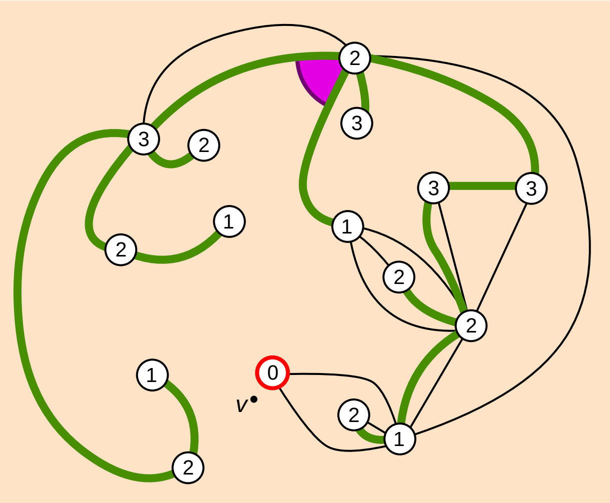

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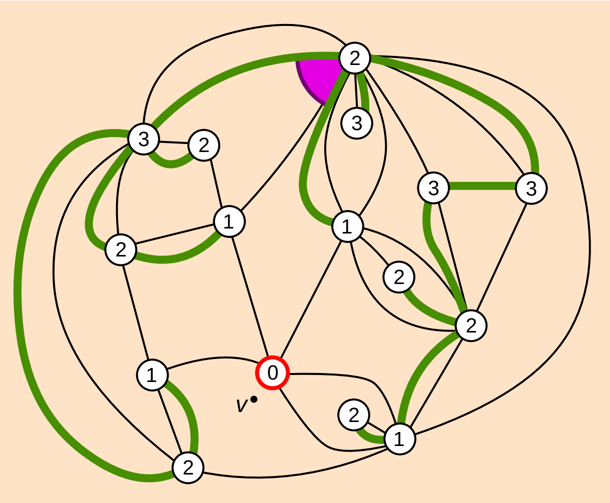

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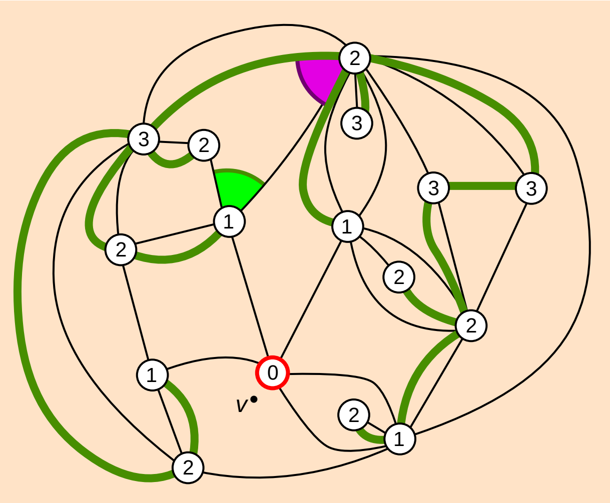
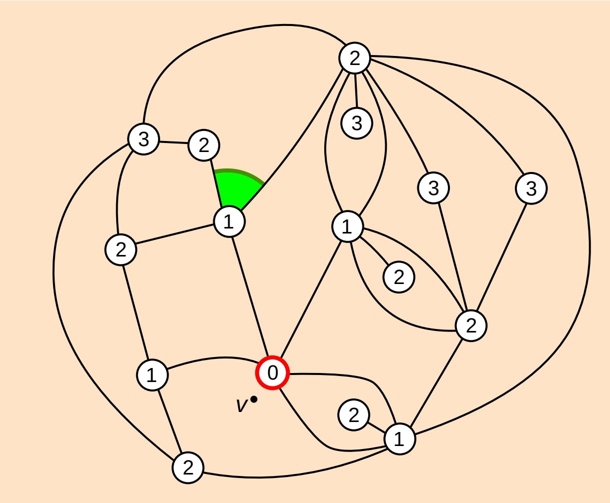


- Take a unicellular well-labeled map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.
- Root and remove the initial edges.

Inverse construction



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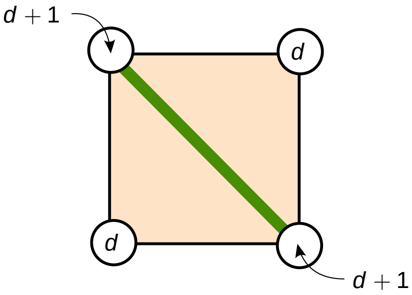
What could go wrong with nonorientable maps?

From quadrangulations to unicellular maps



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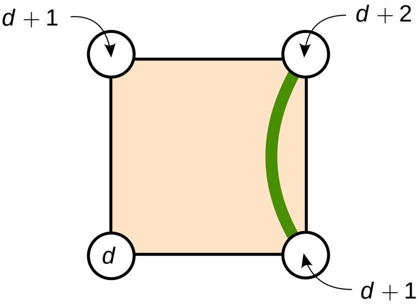
From quadrangulations to unicellular maps



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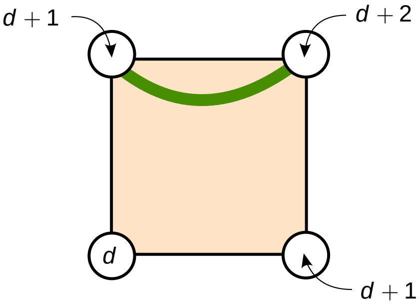

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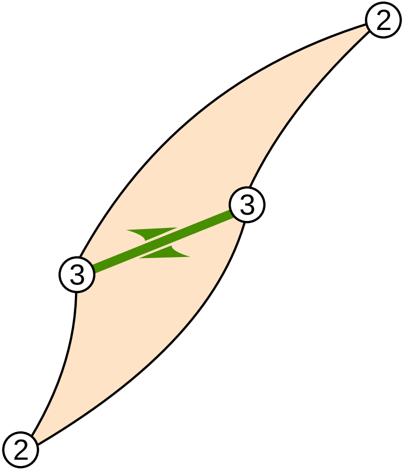

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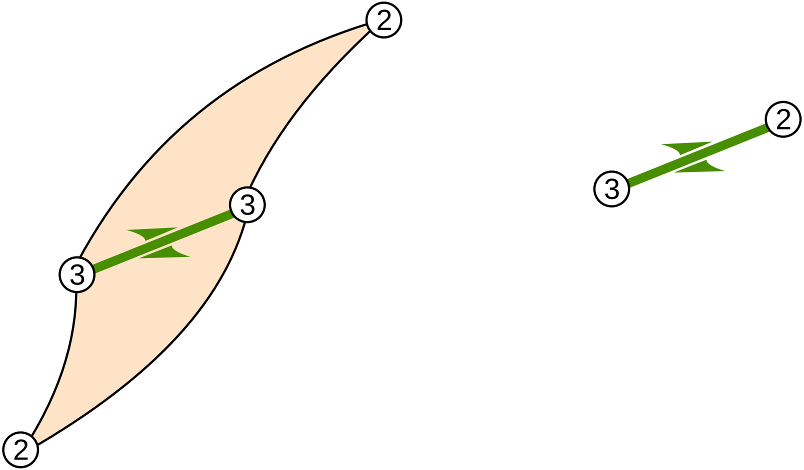

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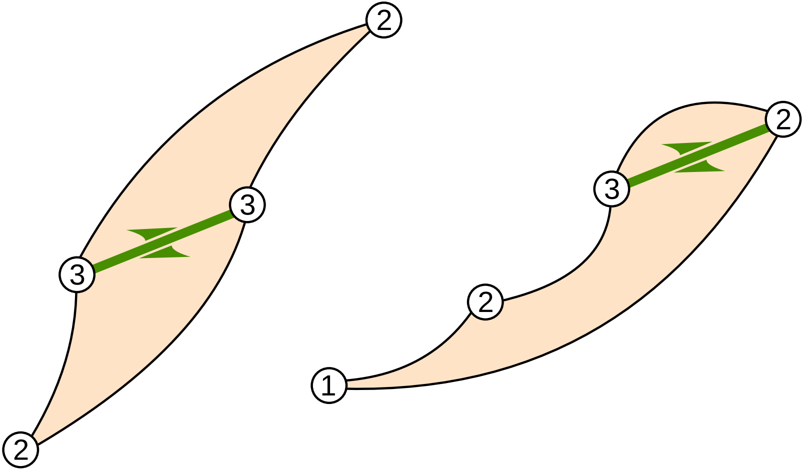

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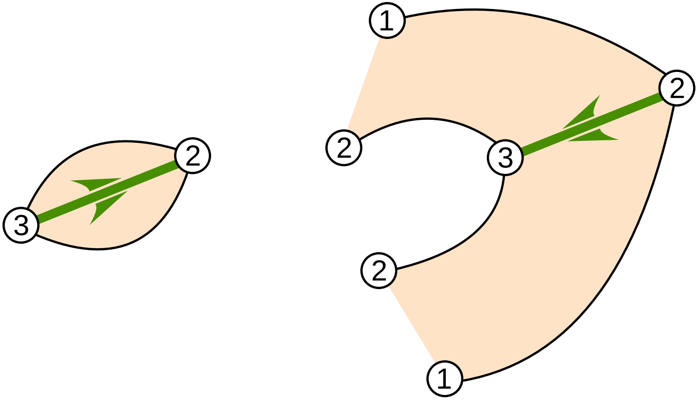
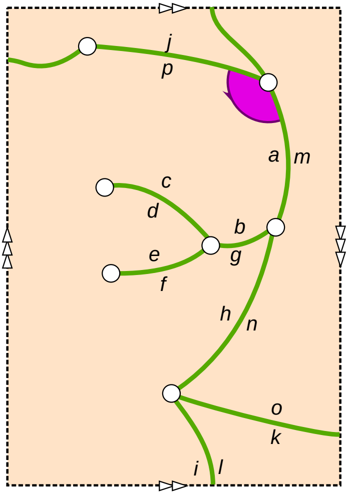
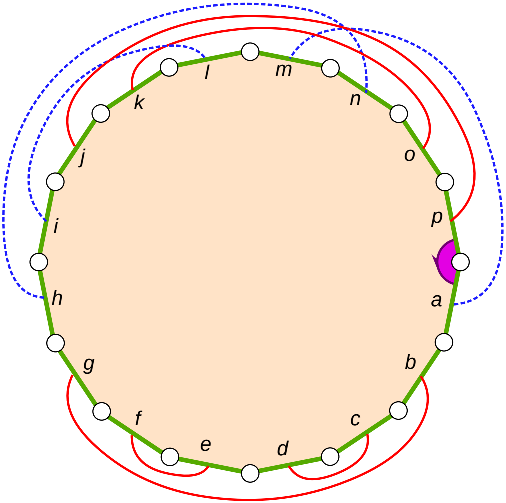
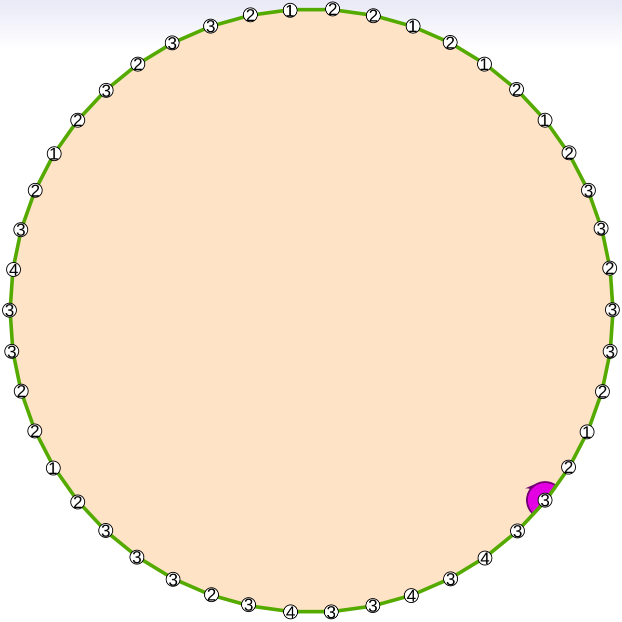


Unicellular maps seen as polygons with paired sides



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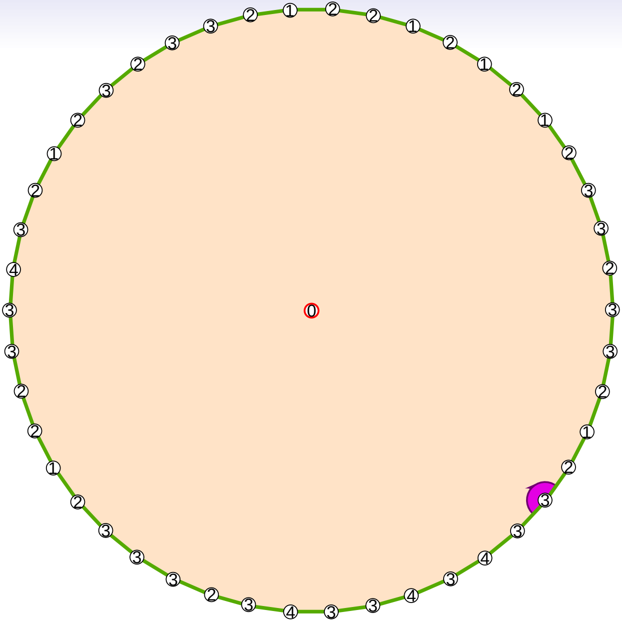

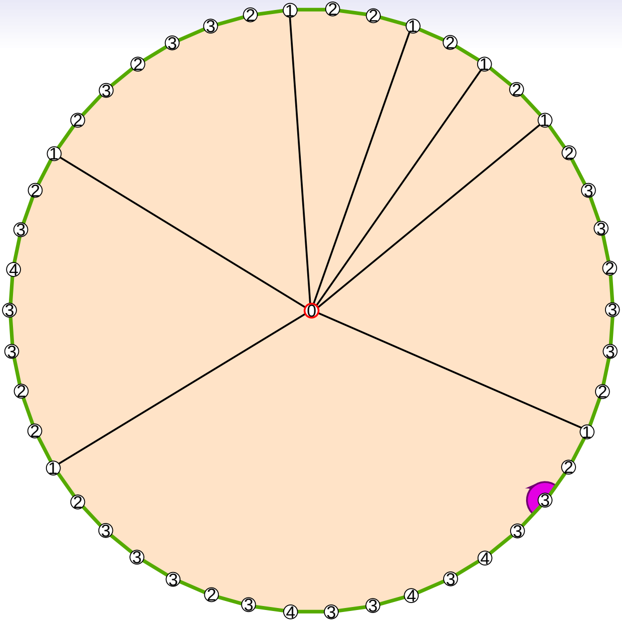
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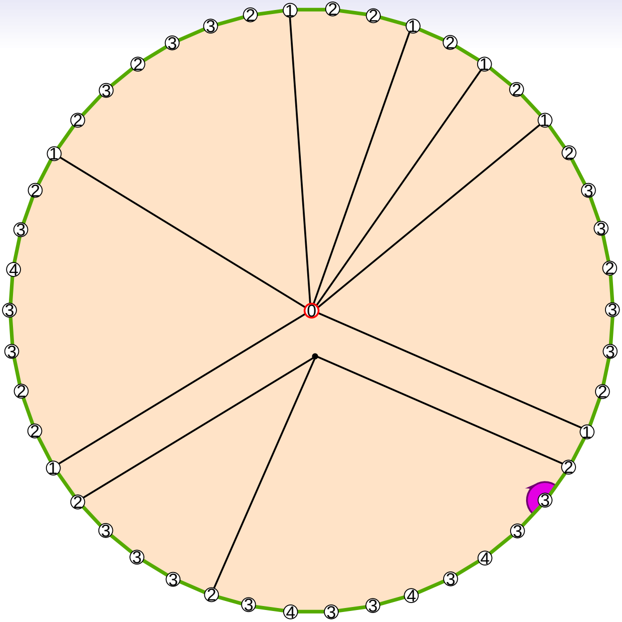
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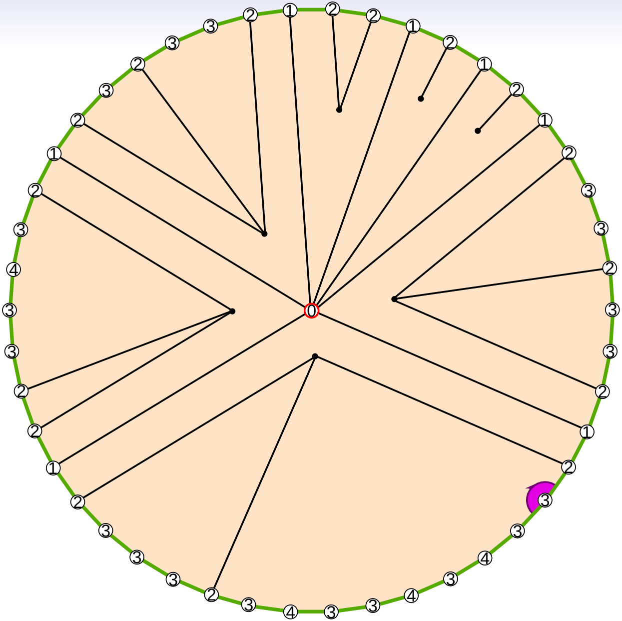
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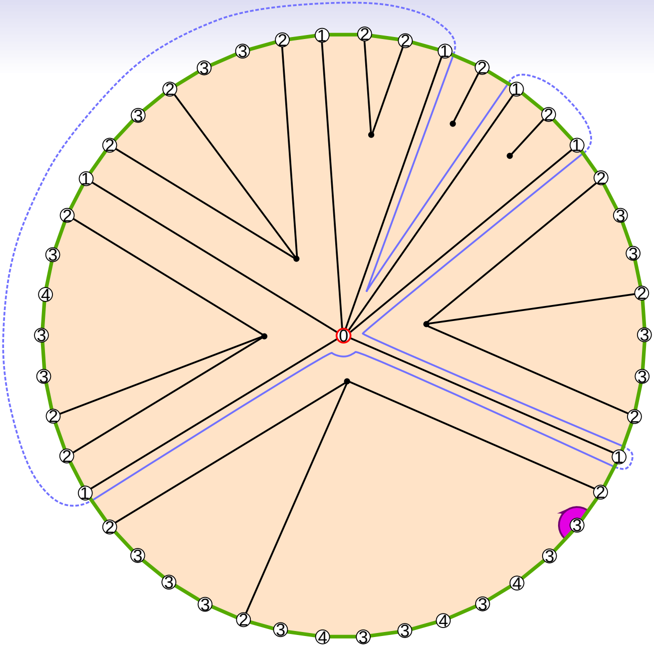
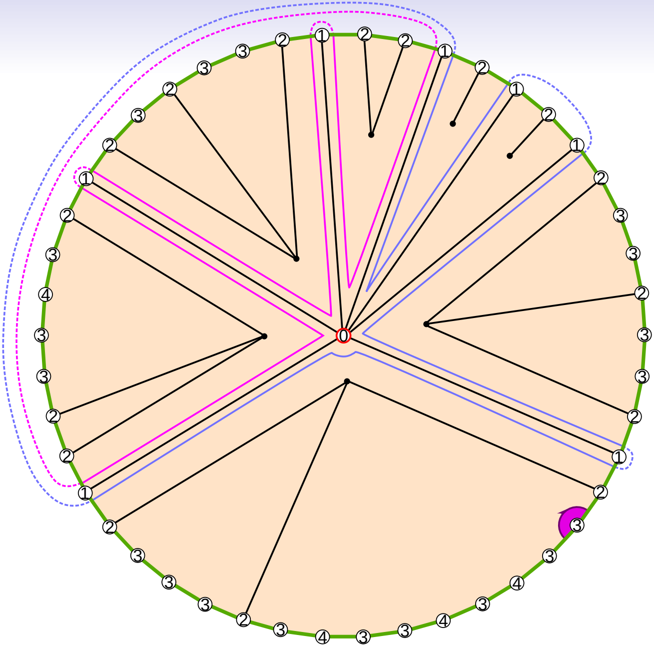
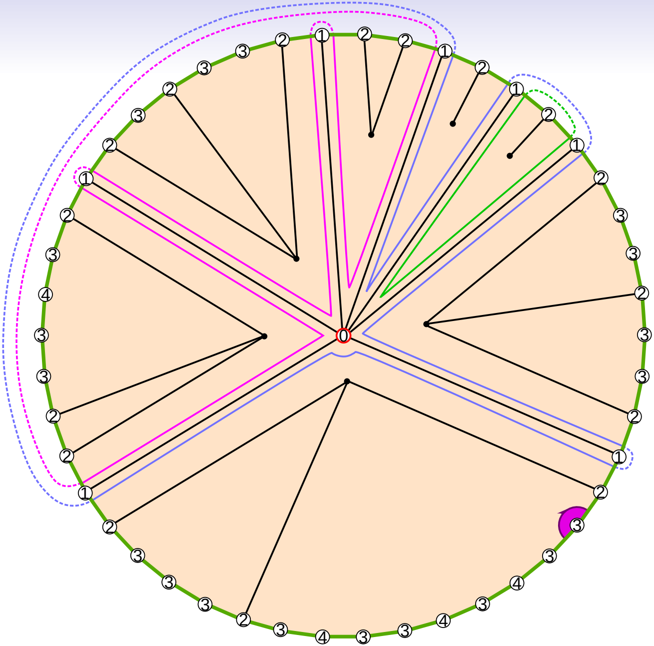
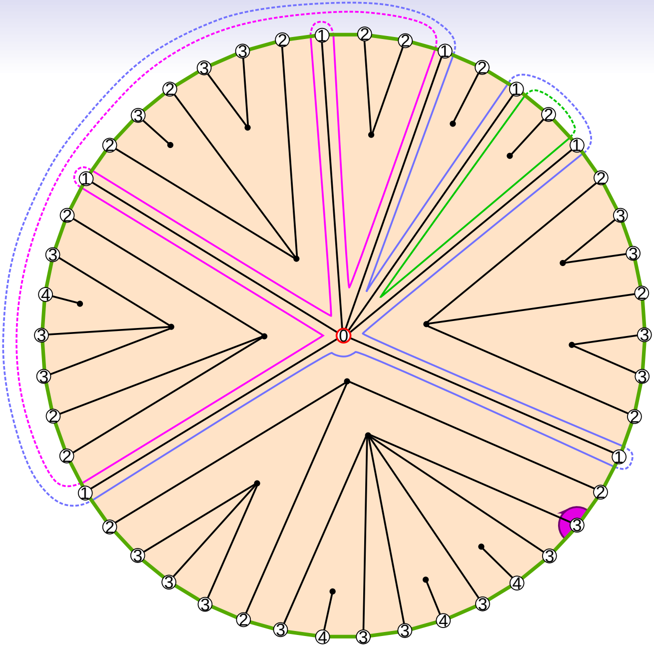
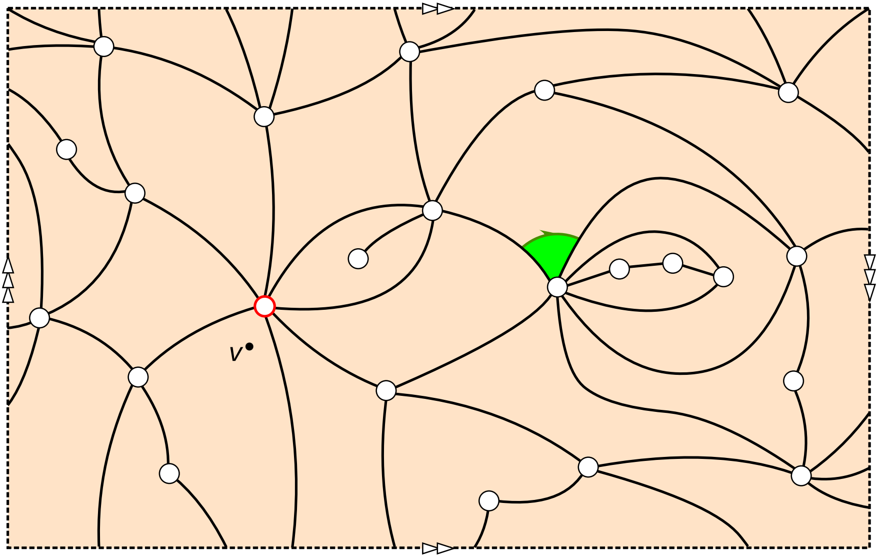








Chapuy–Dołęga bijection



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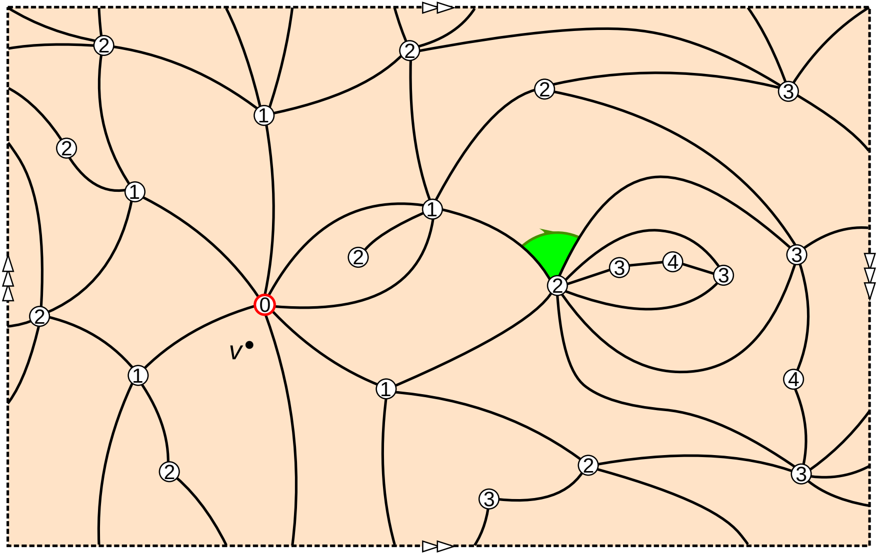

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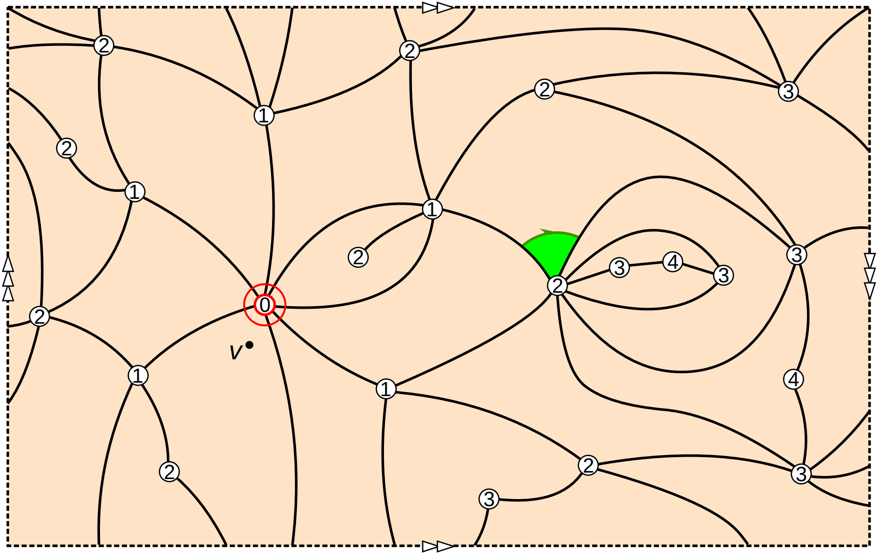

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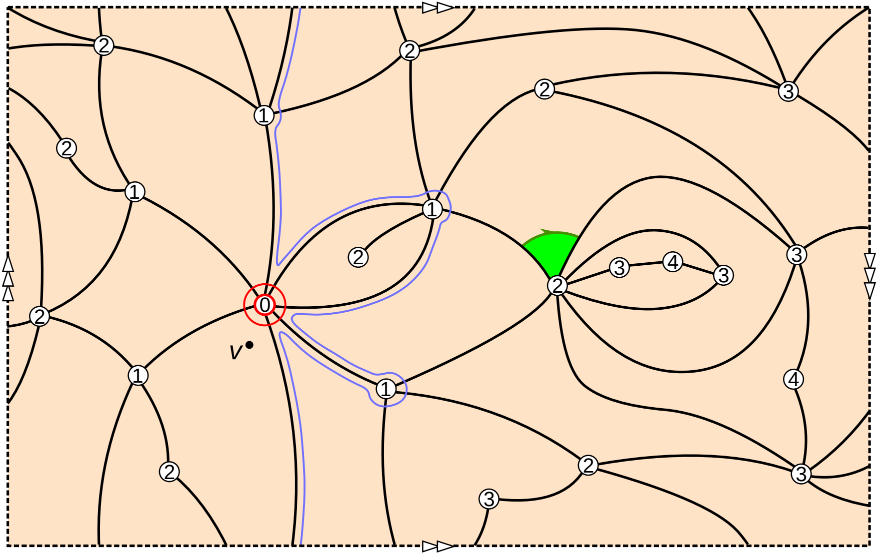

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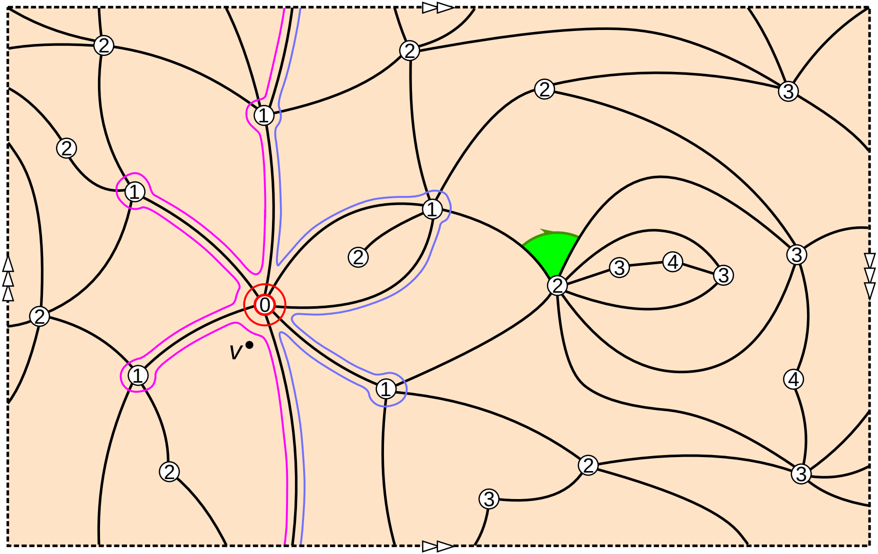

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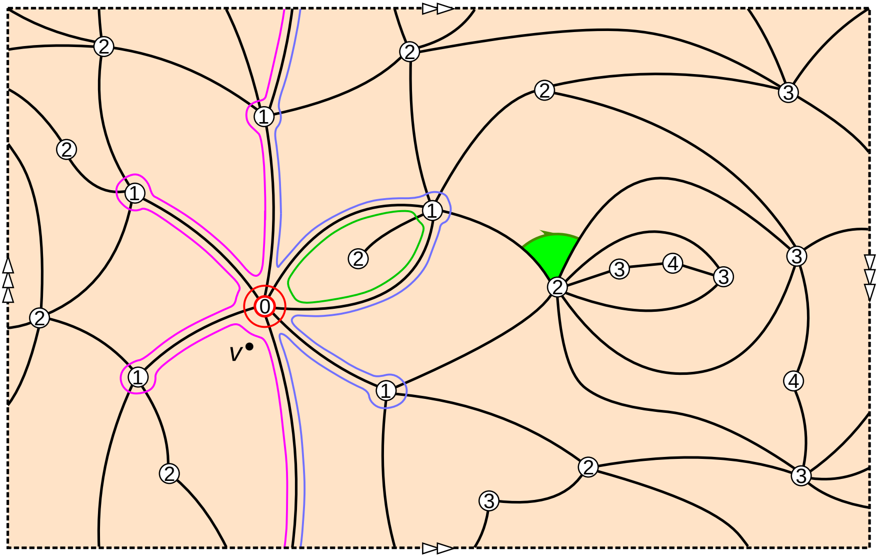

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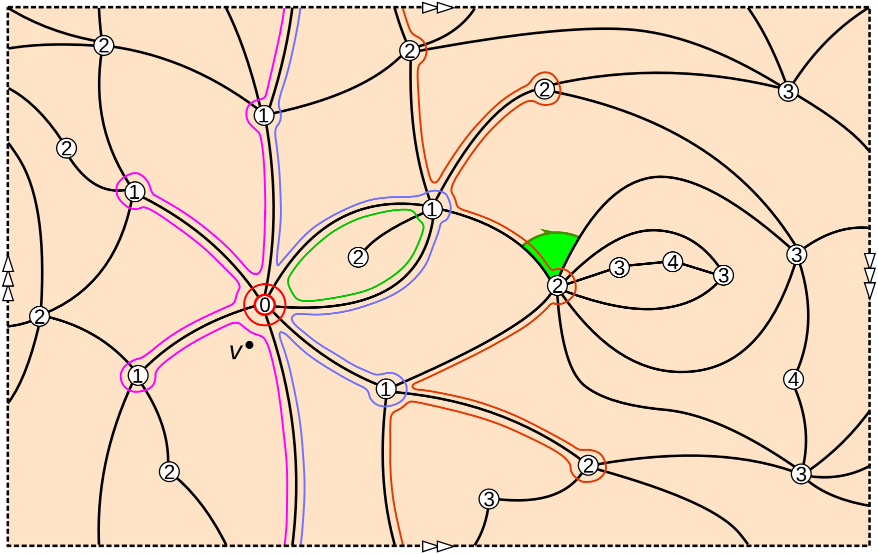

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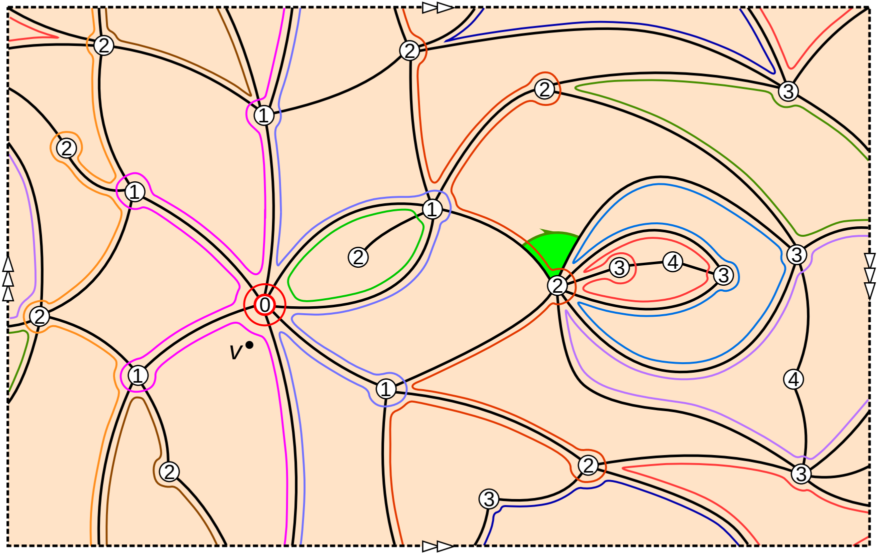

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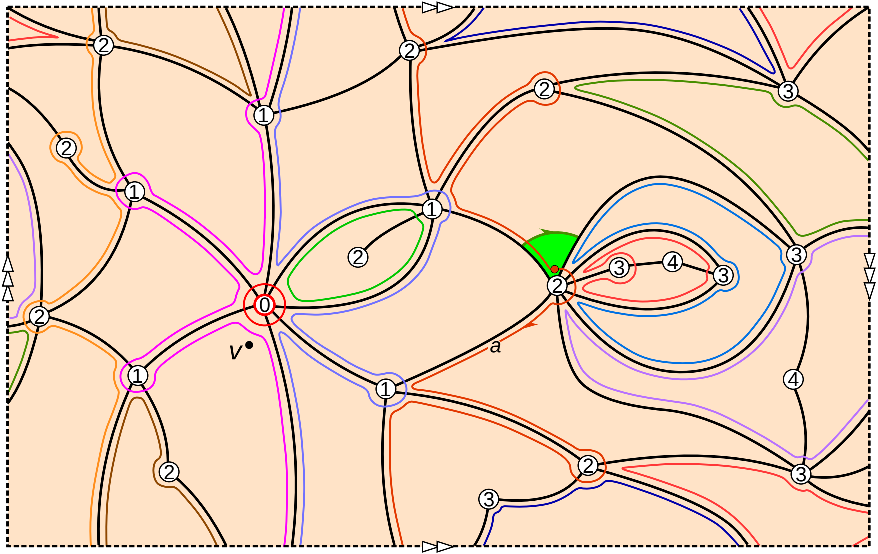

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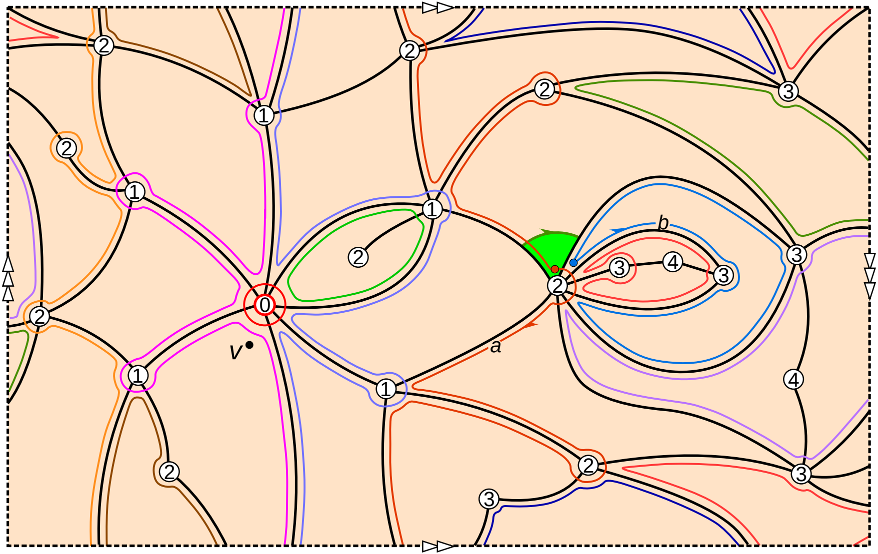

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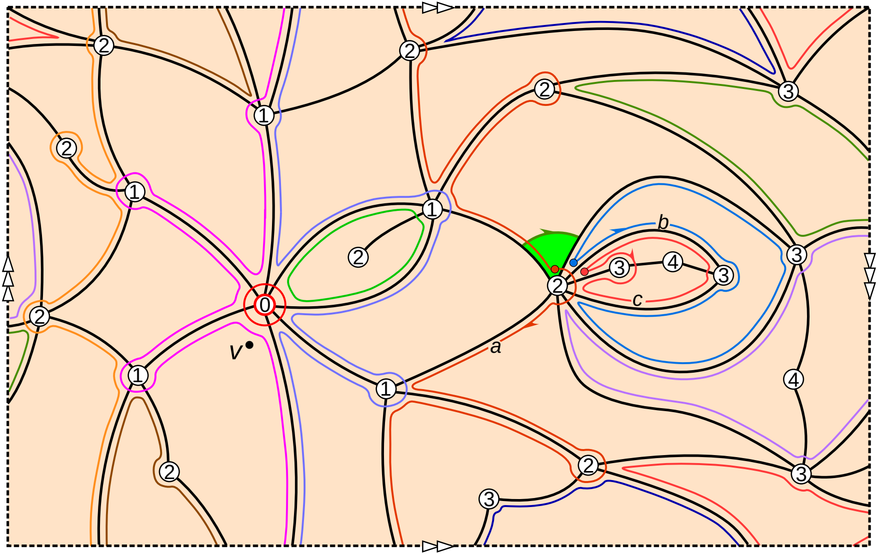

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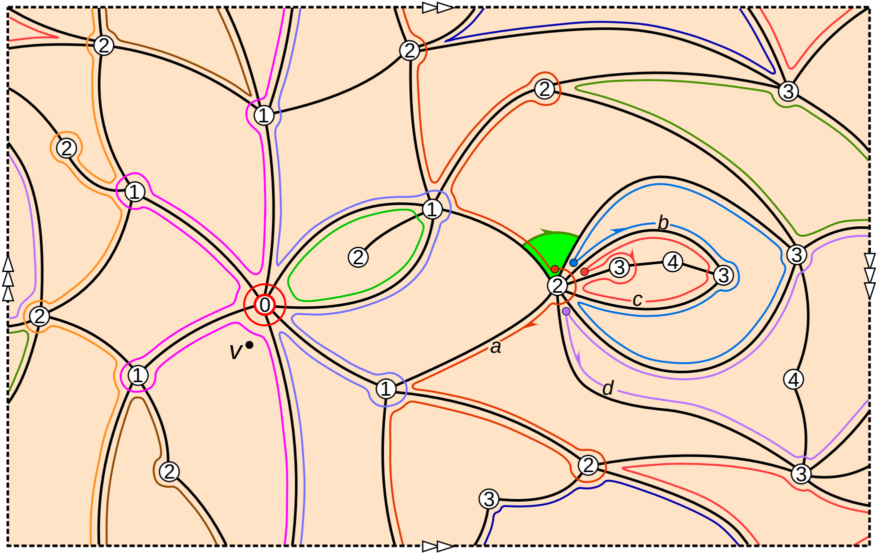

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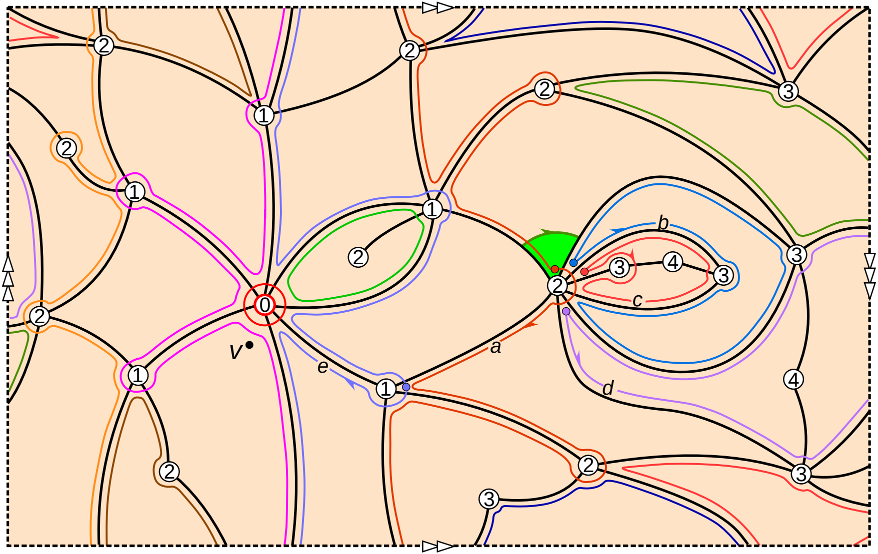

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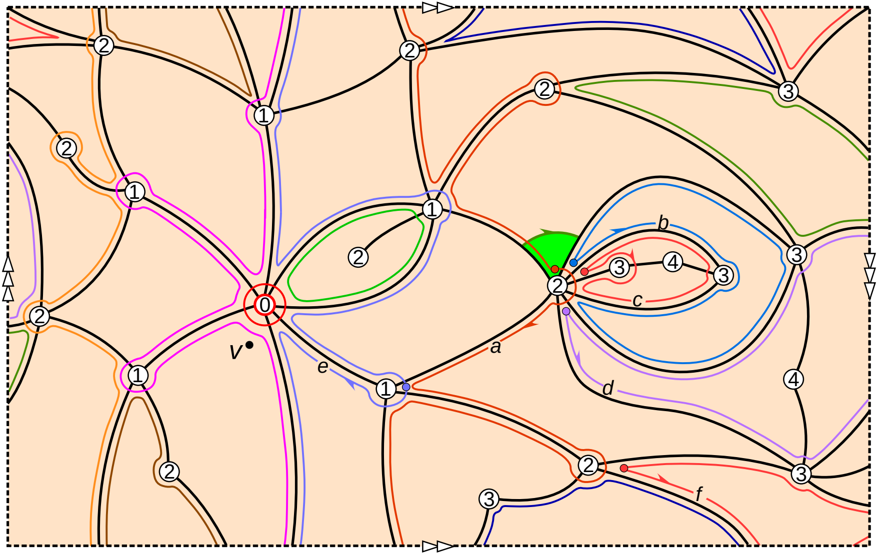

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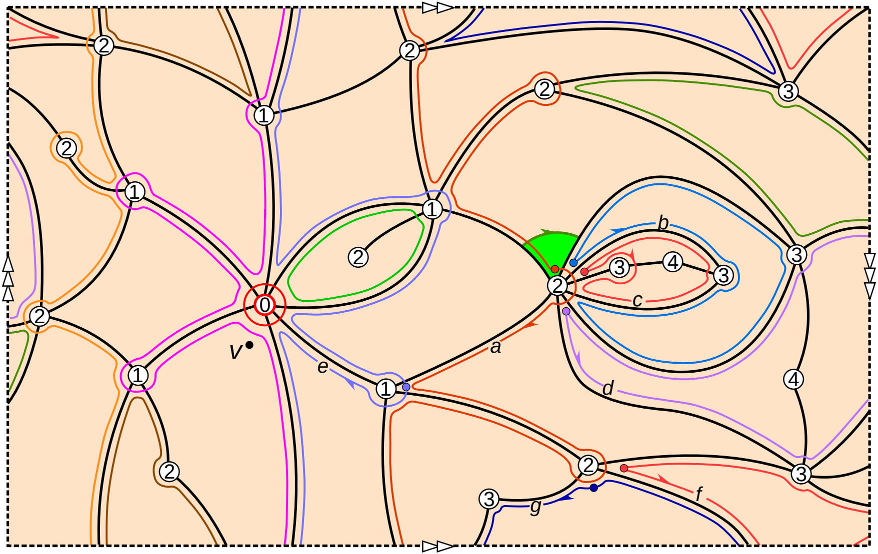

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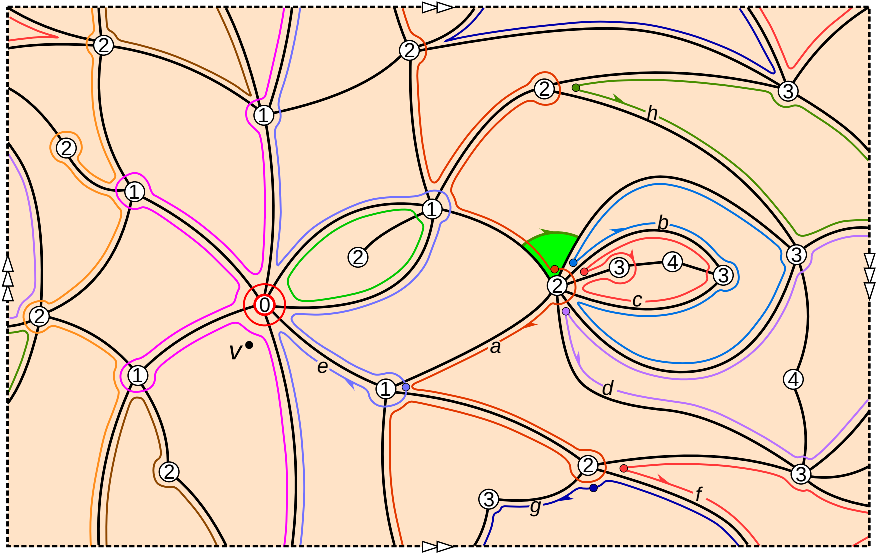

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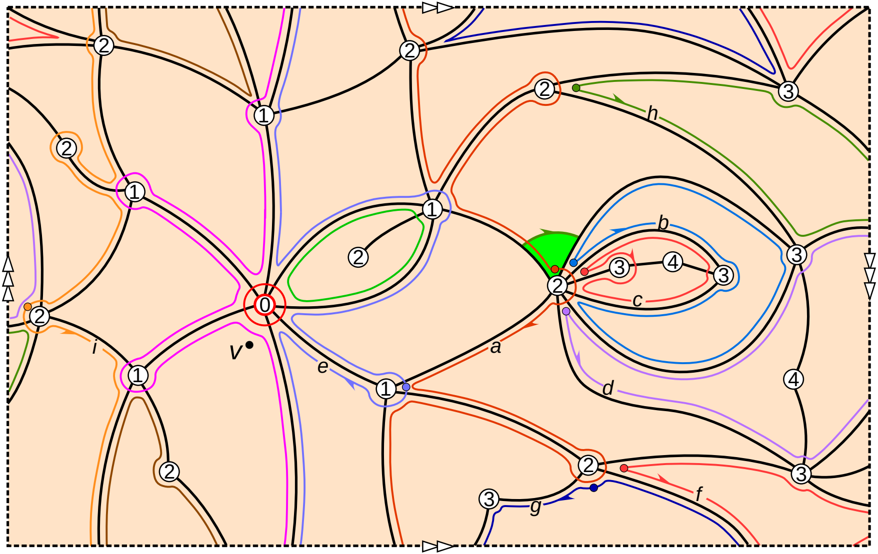

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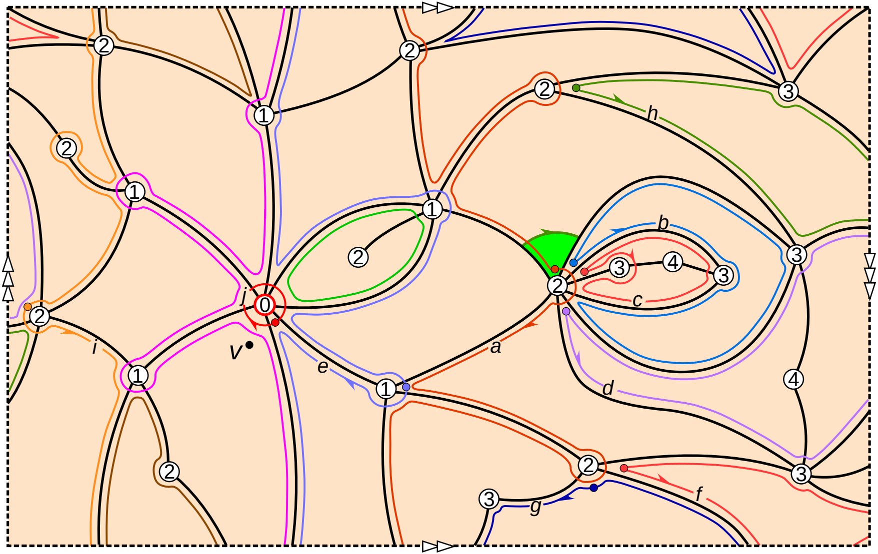

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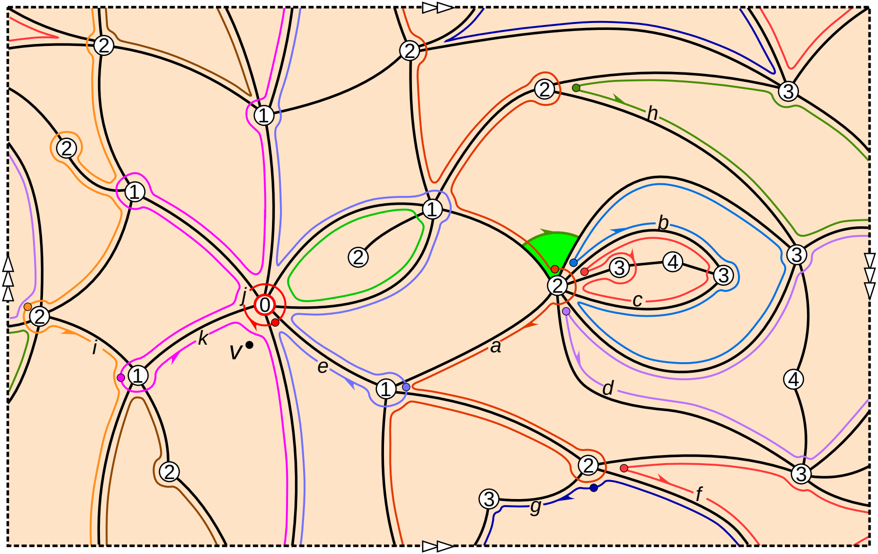

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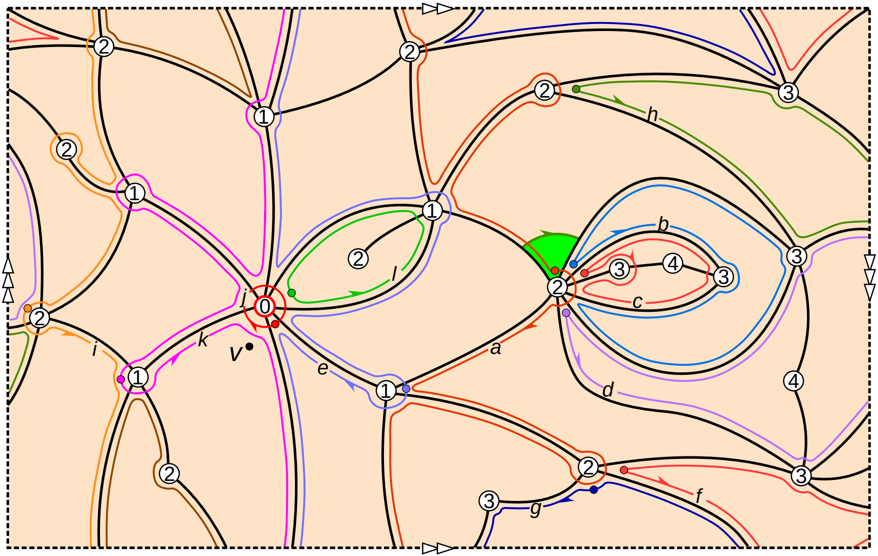

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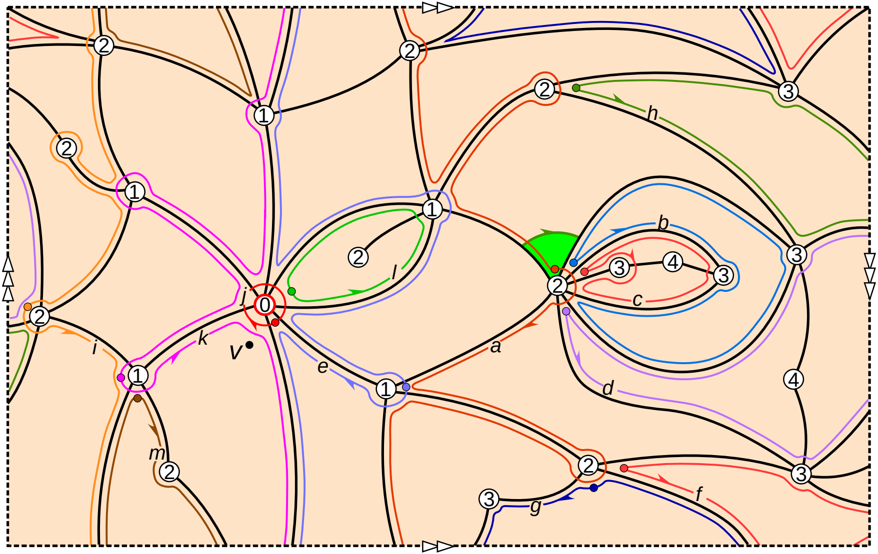

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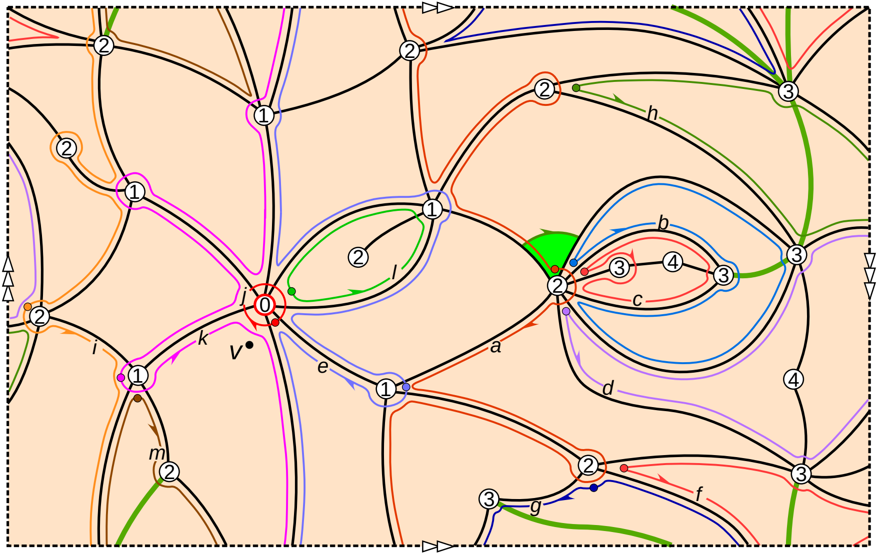

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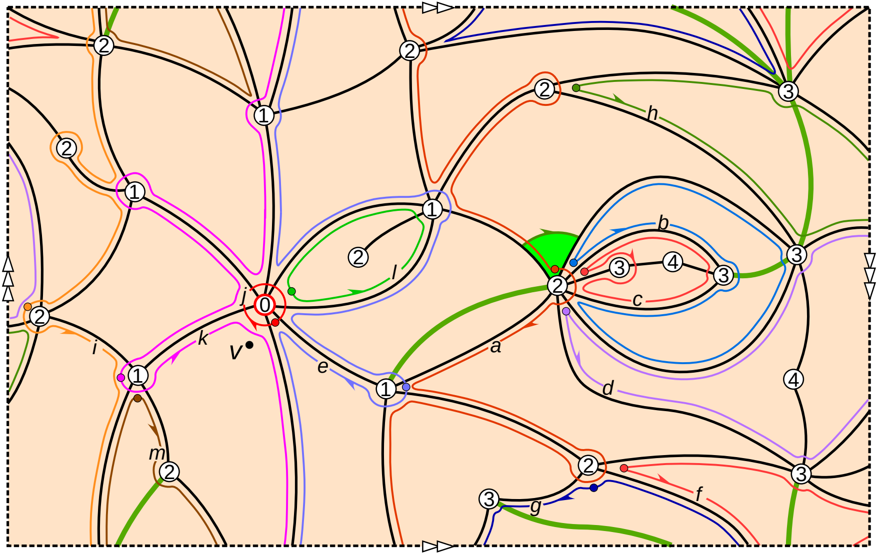

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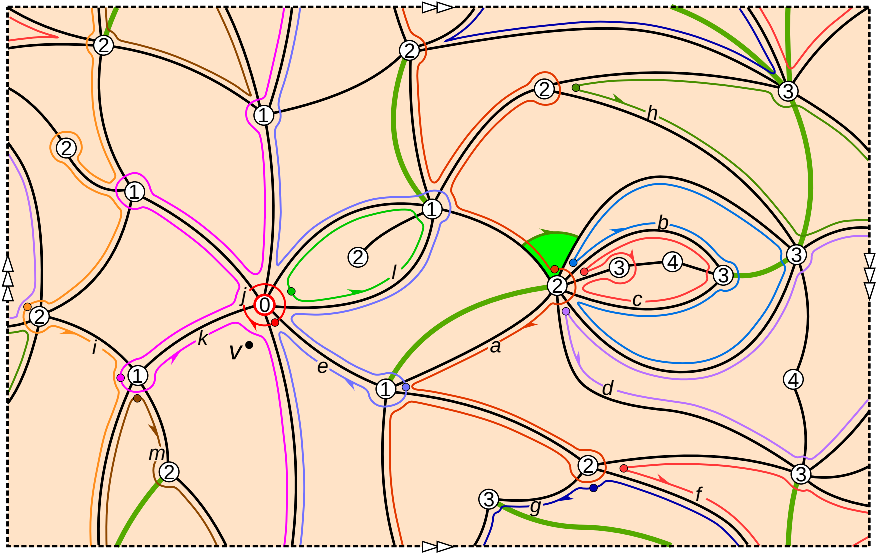

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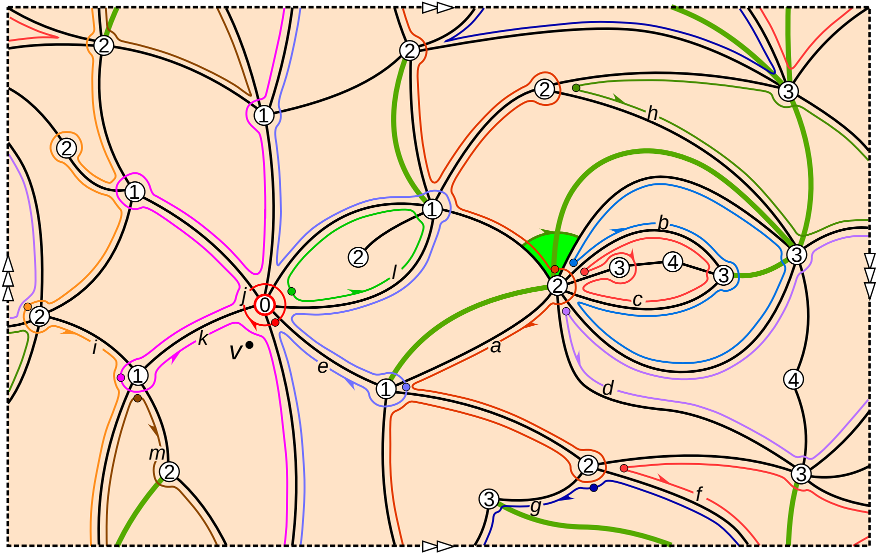

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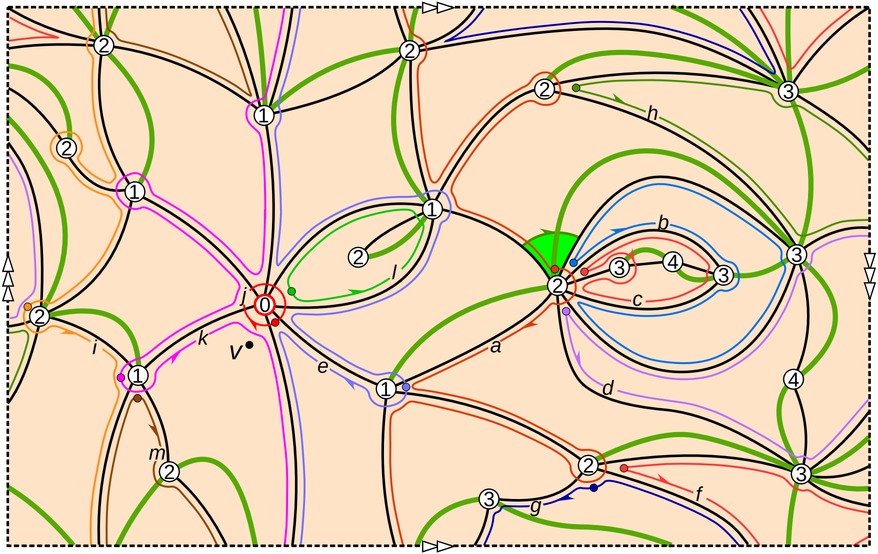

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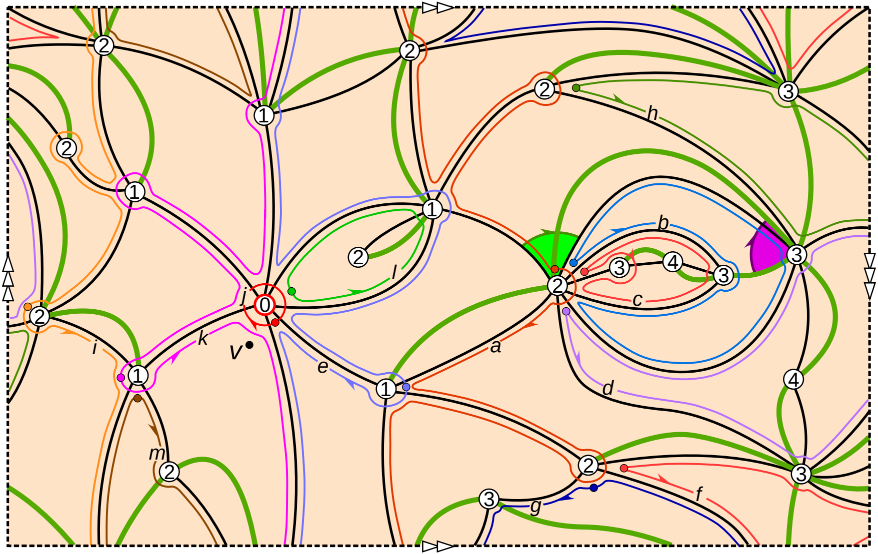

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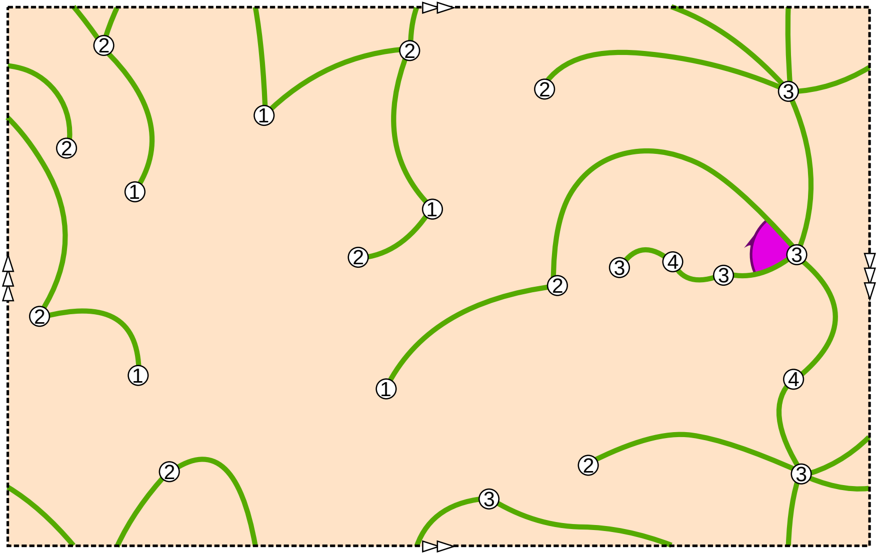

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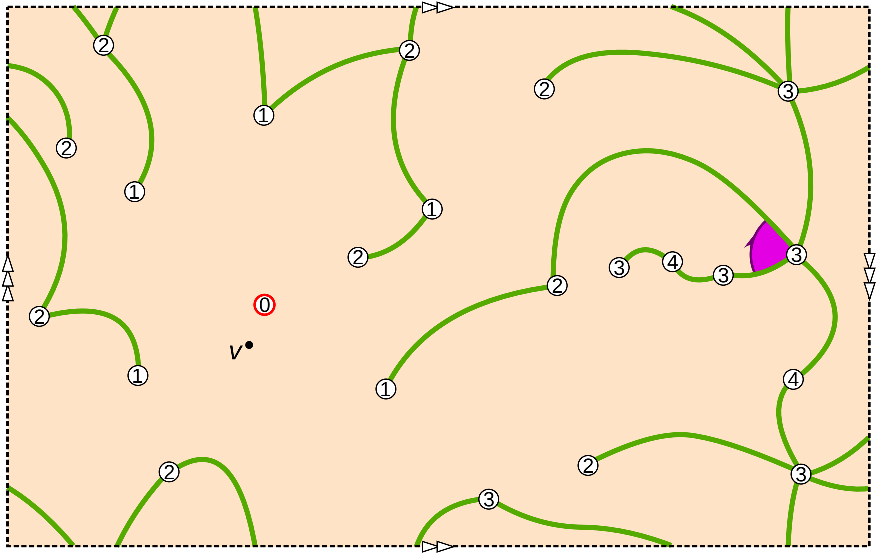

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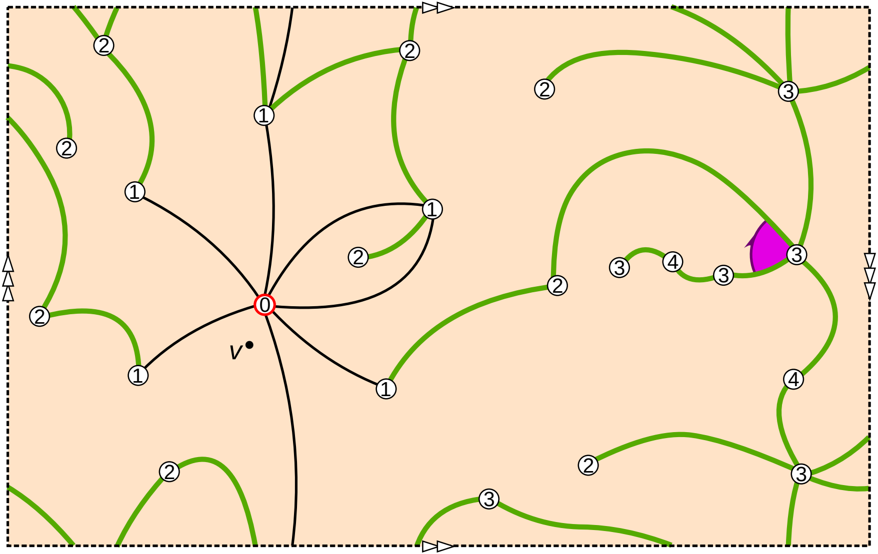

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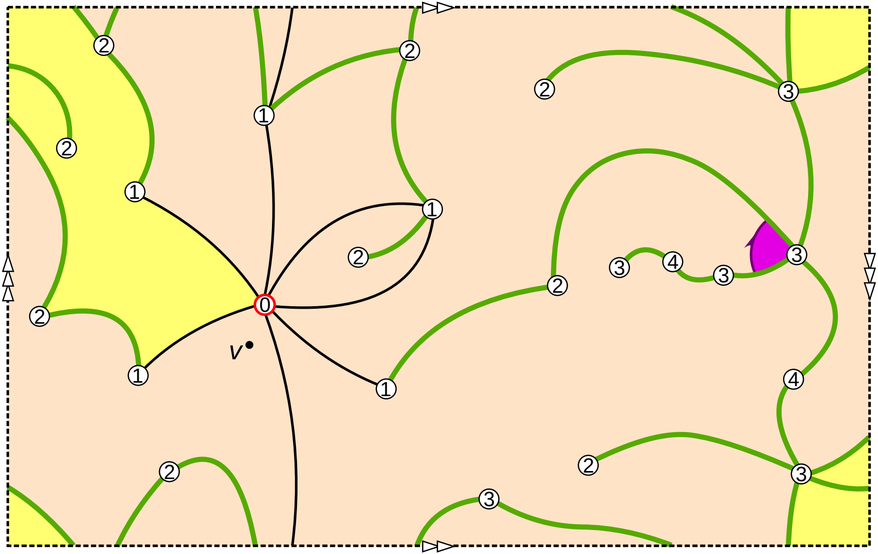

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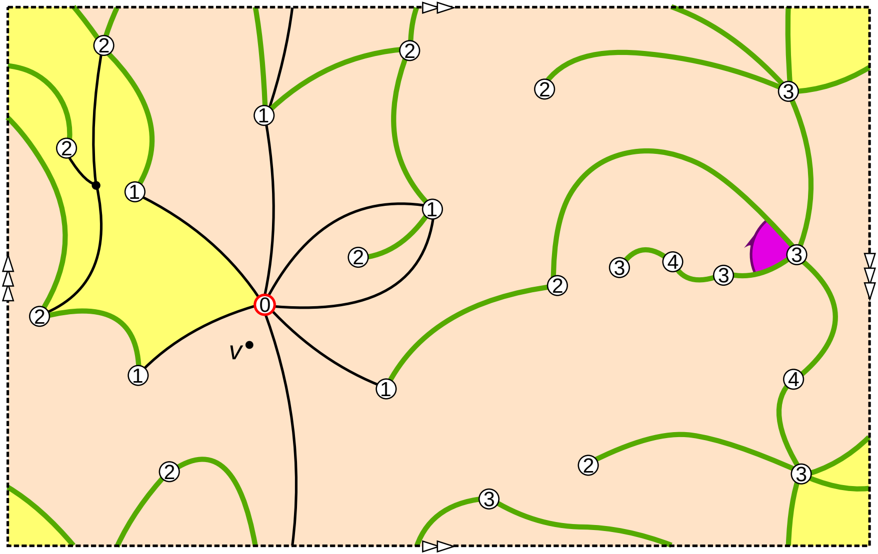

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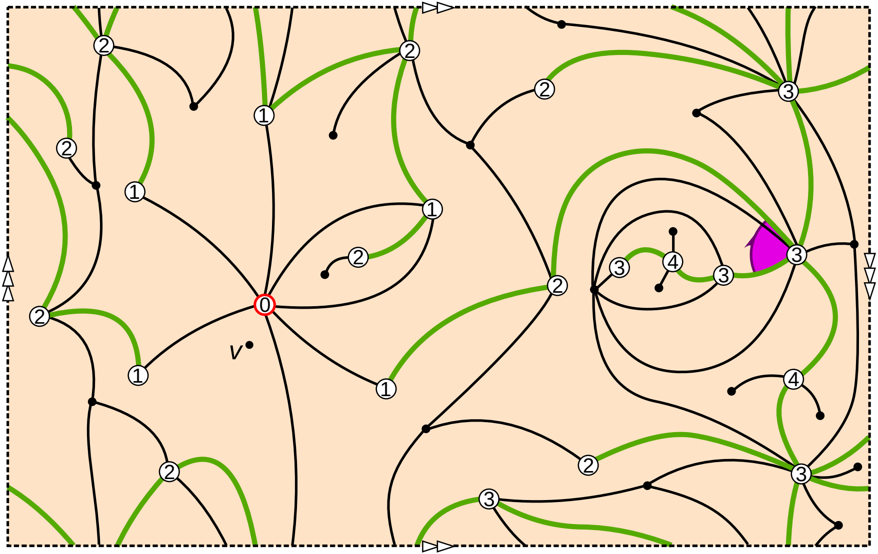

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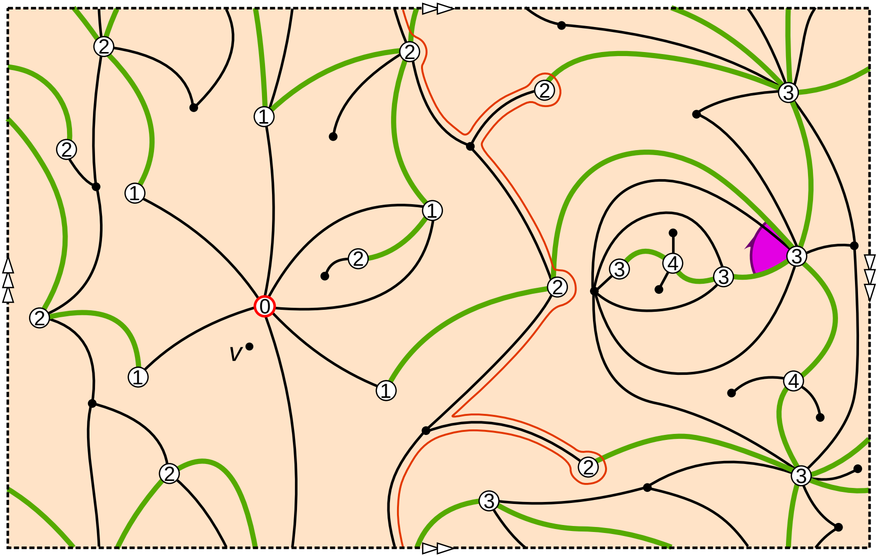

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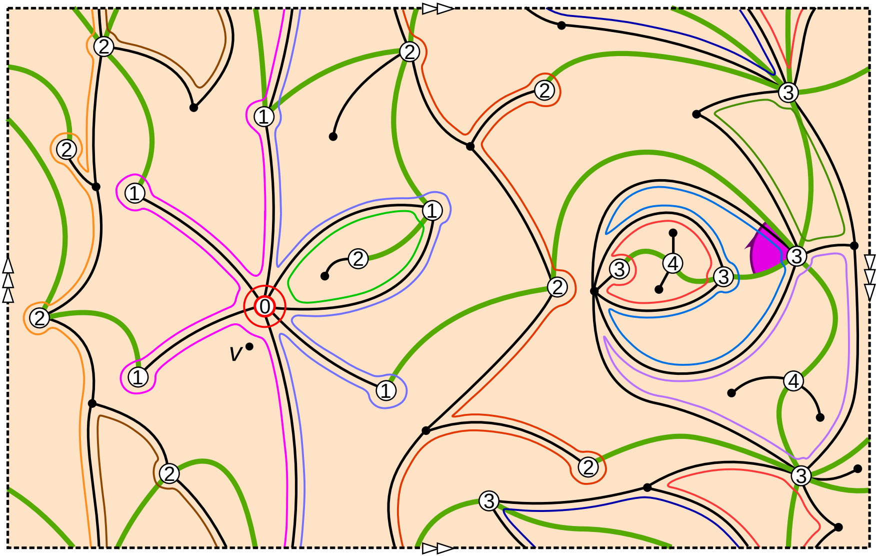

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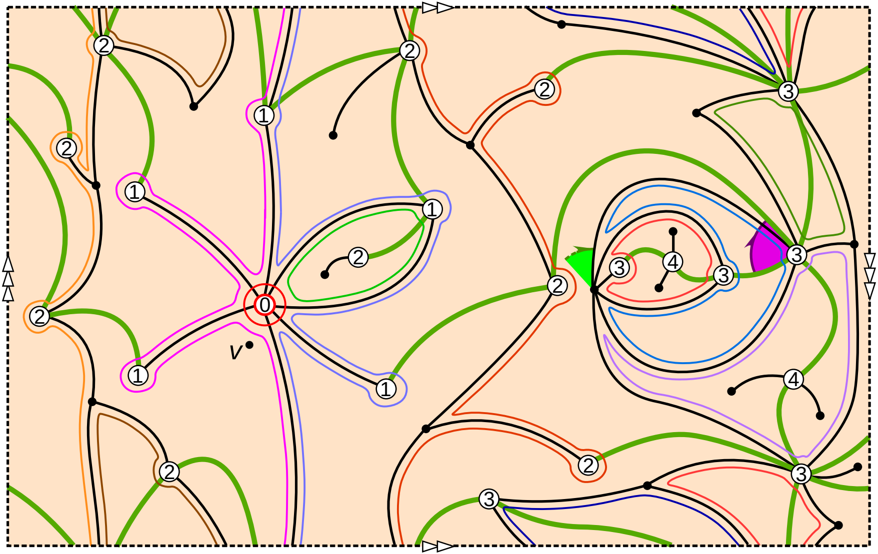

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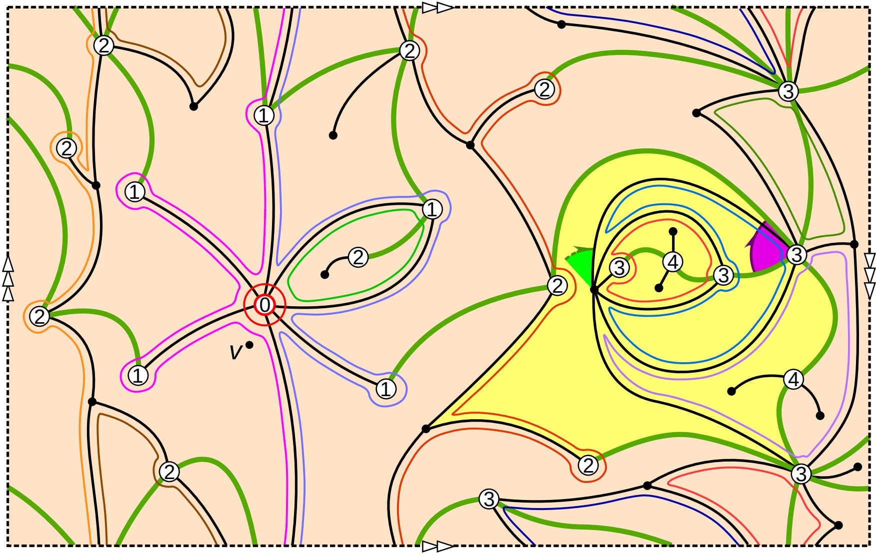

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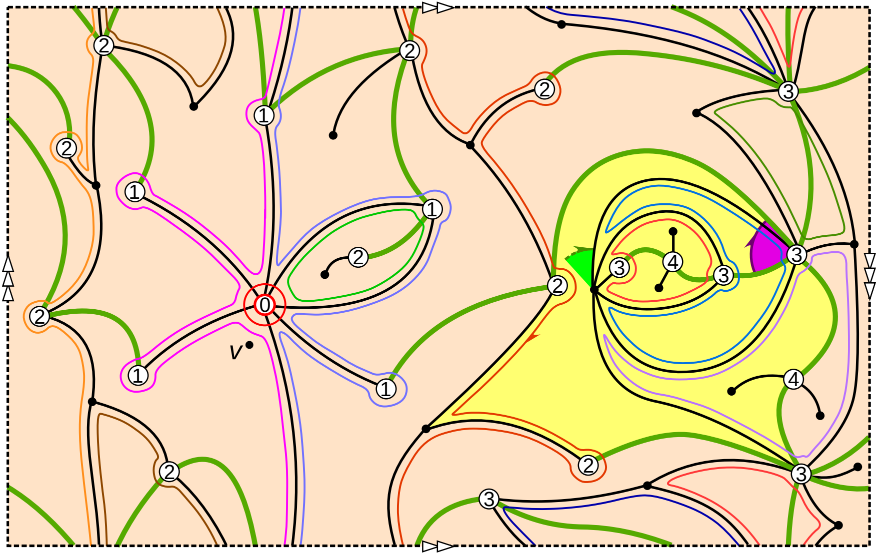

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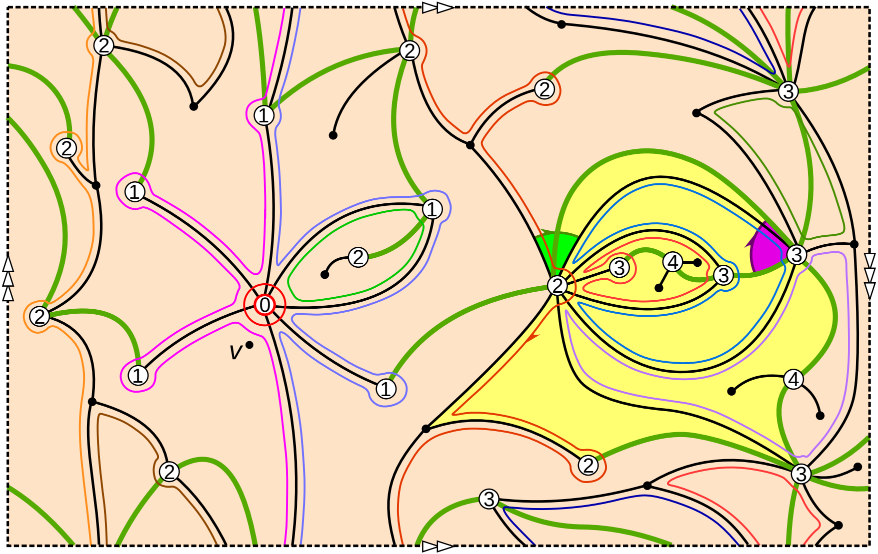

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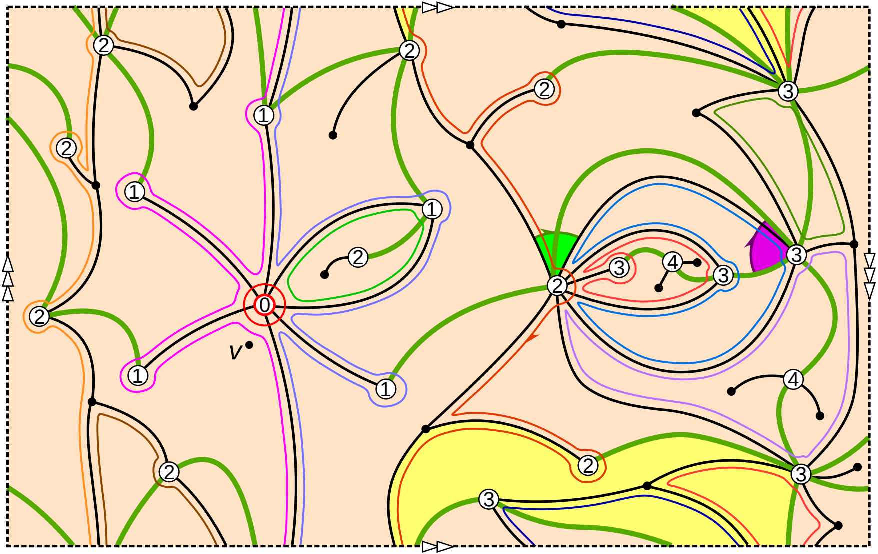

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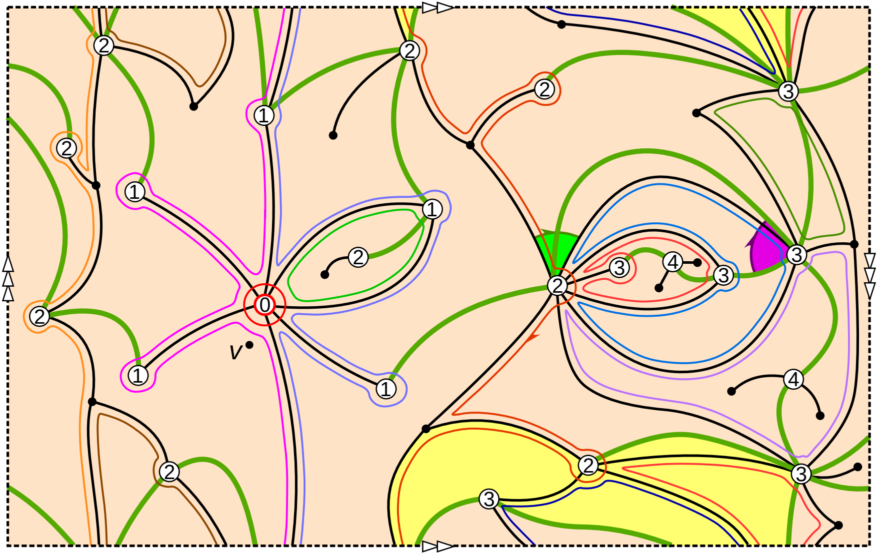

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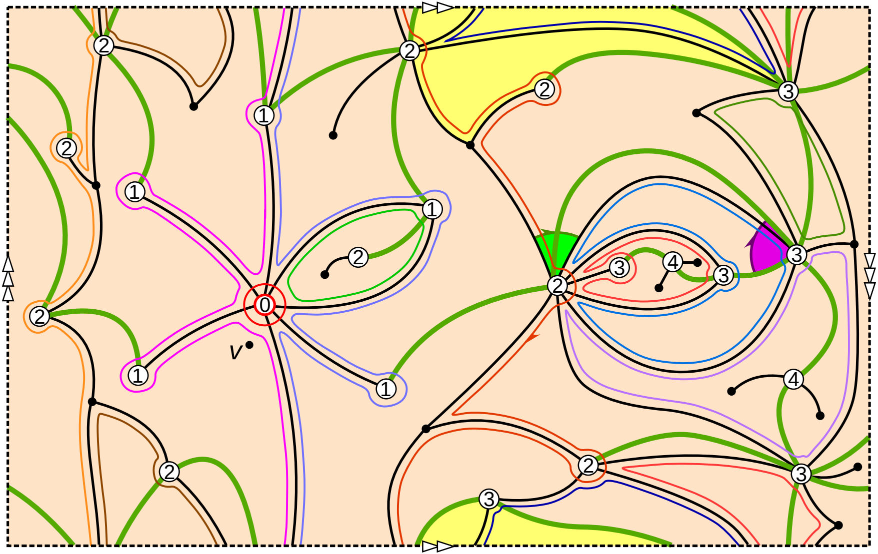

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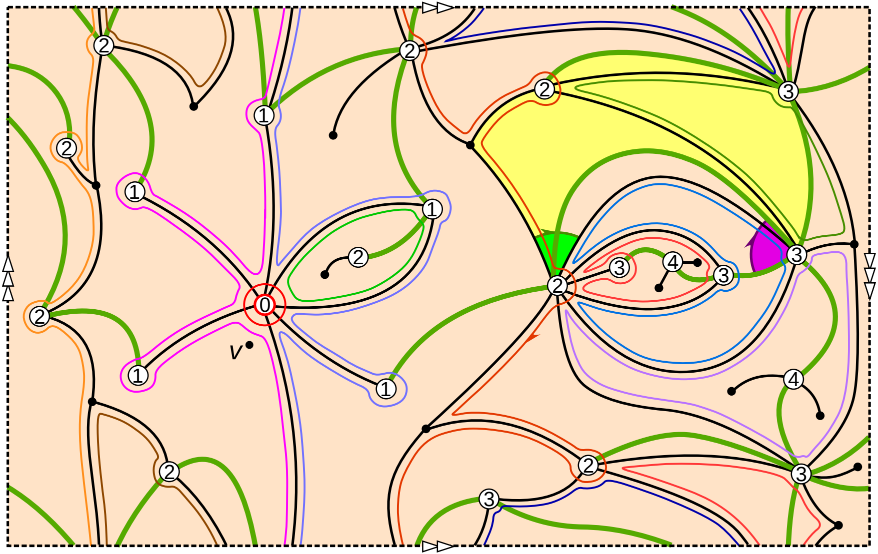

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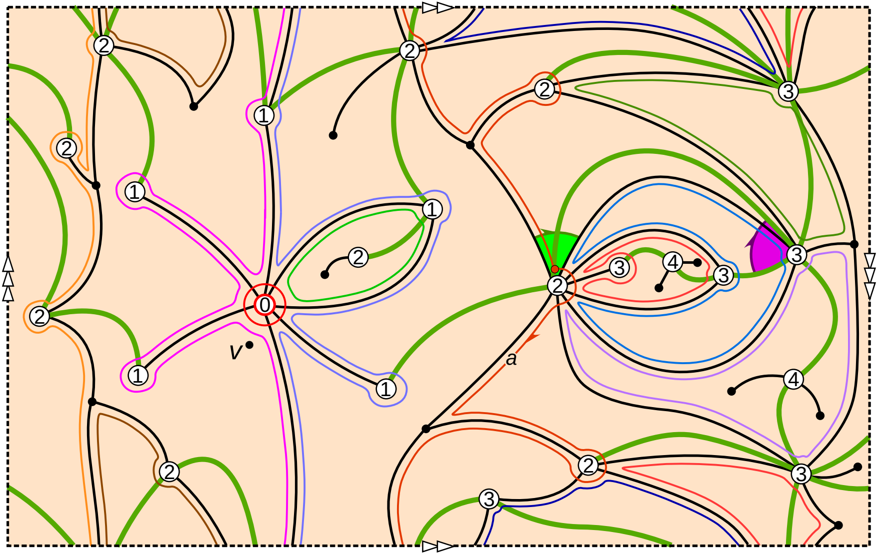

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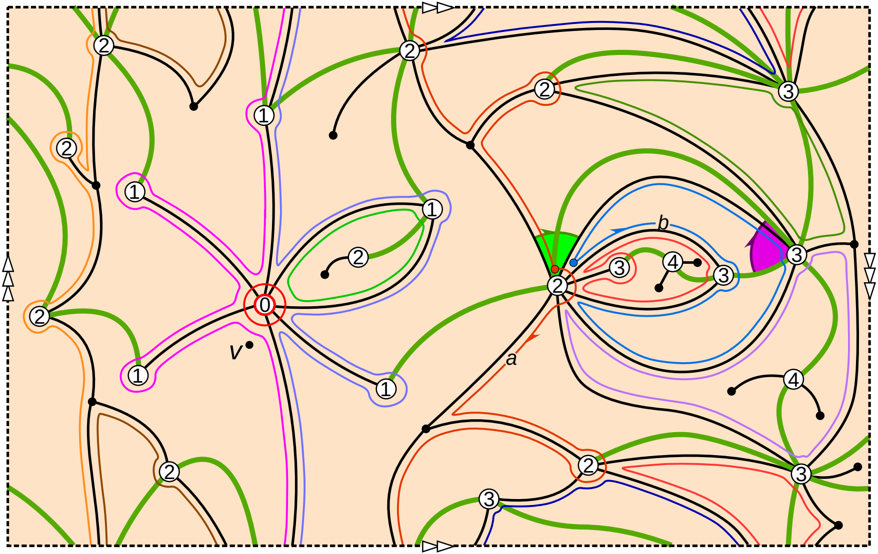

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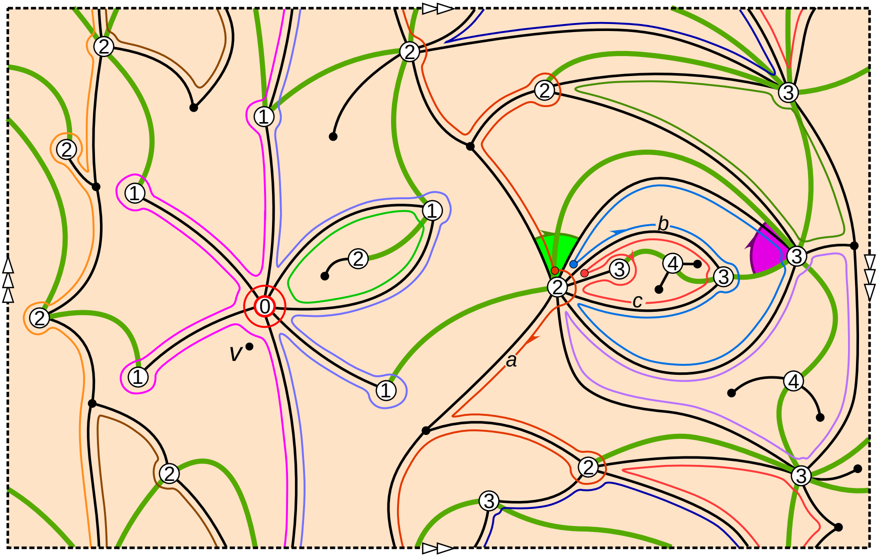

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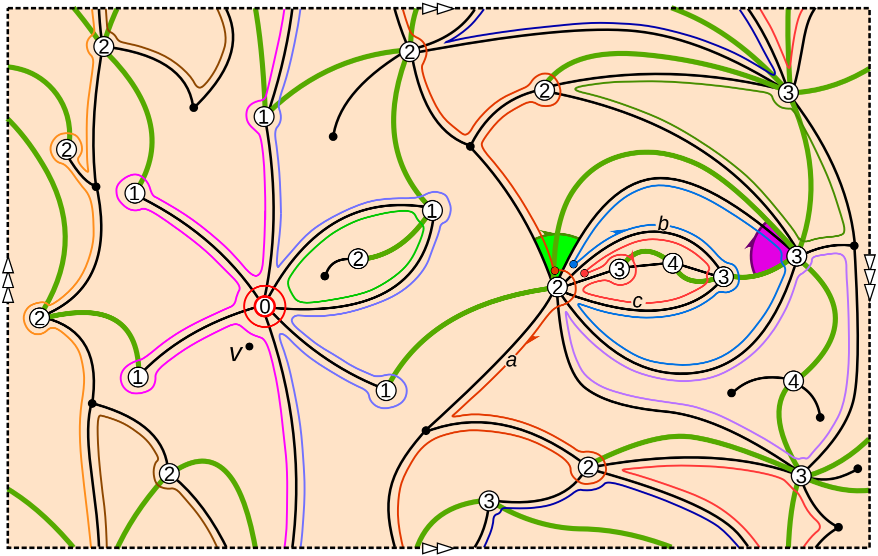

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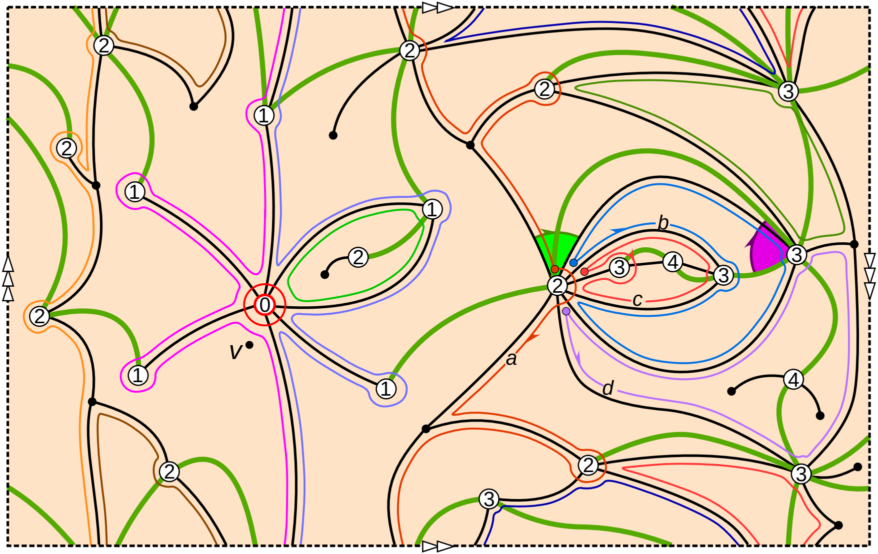

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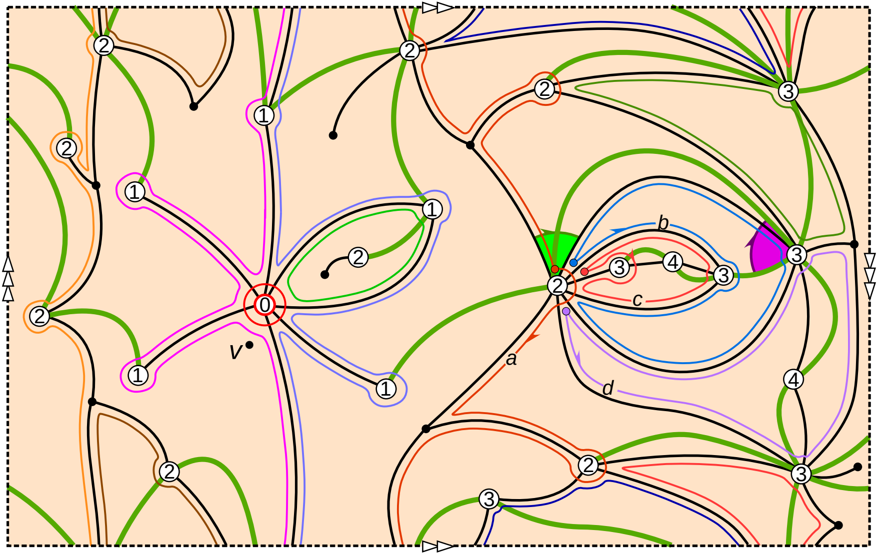

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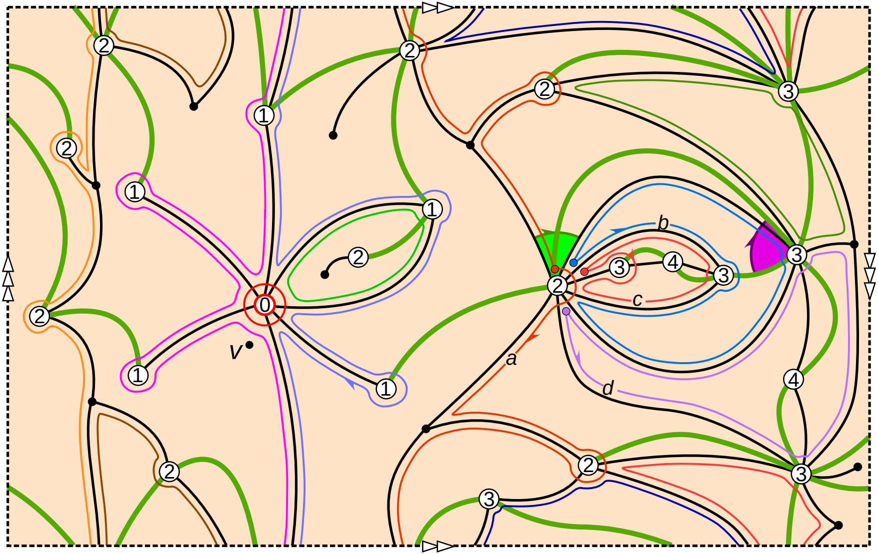

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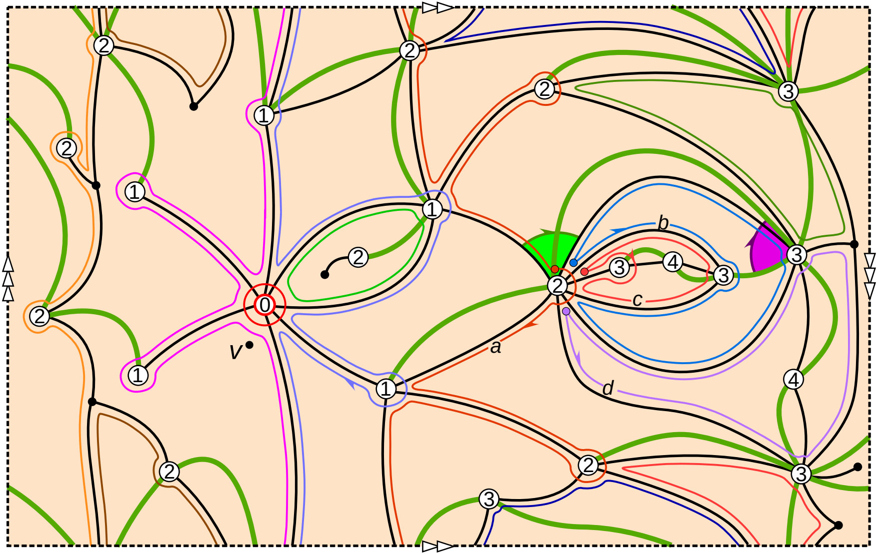

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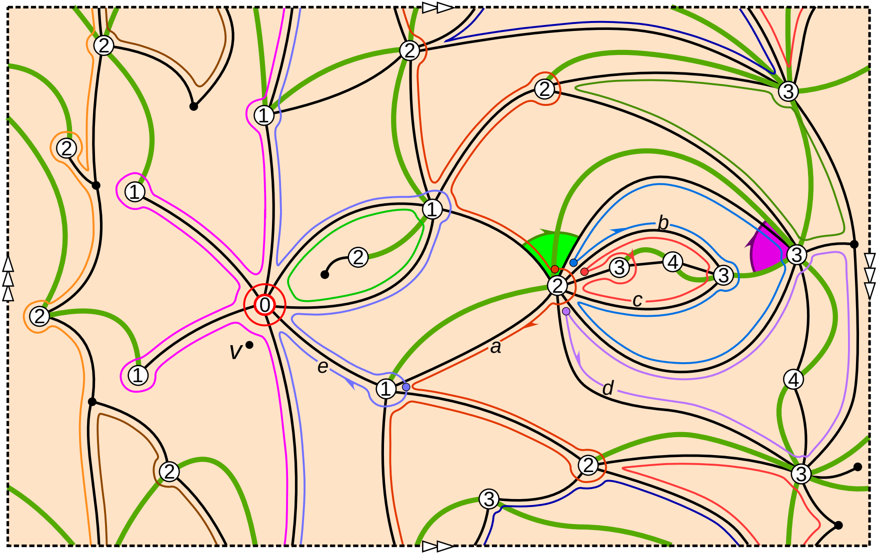

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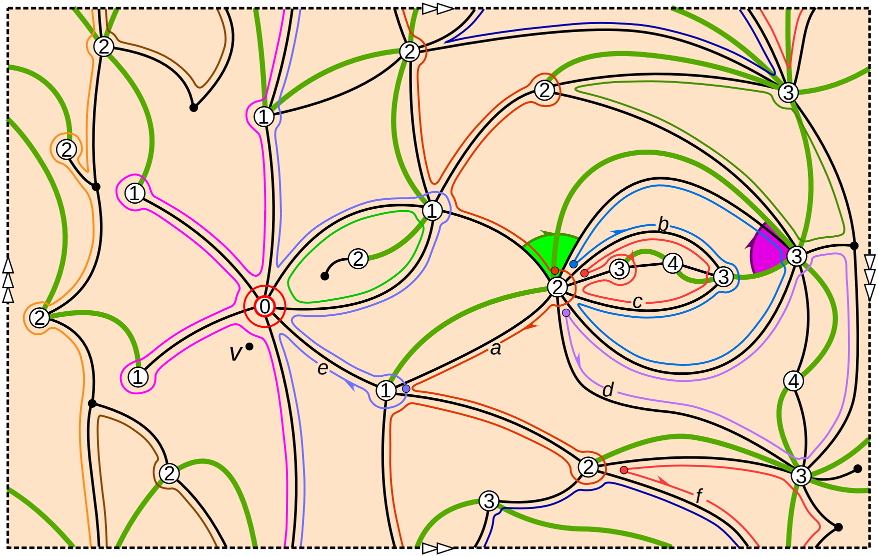

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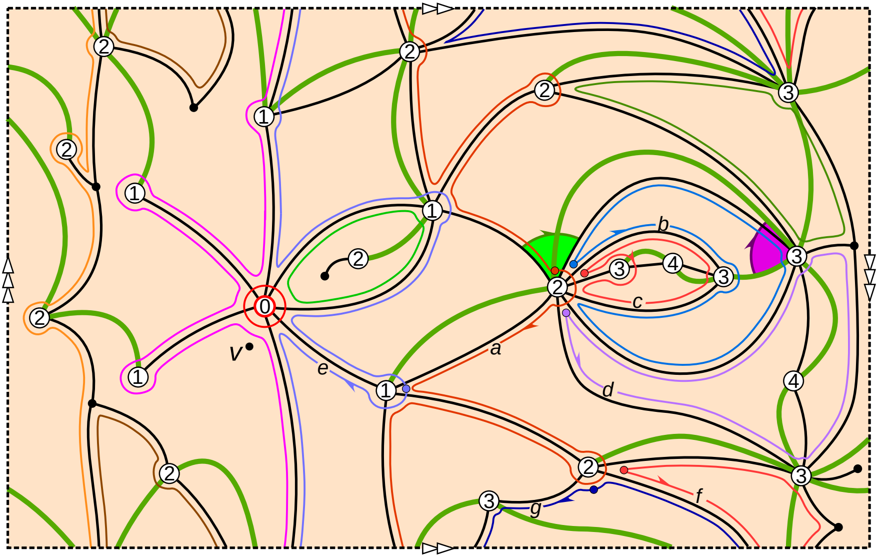

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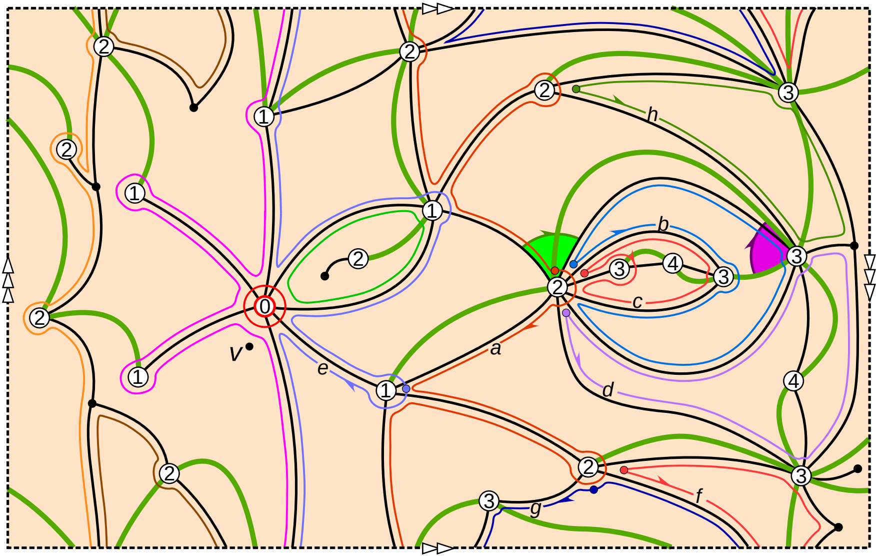

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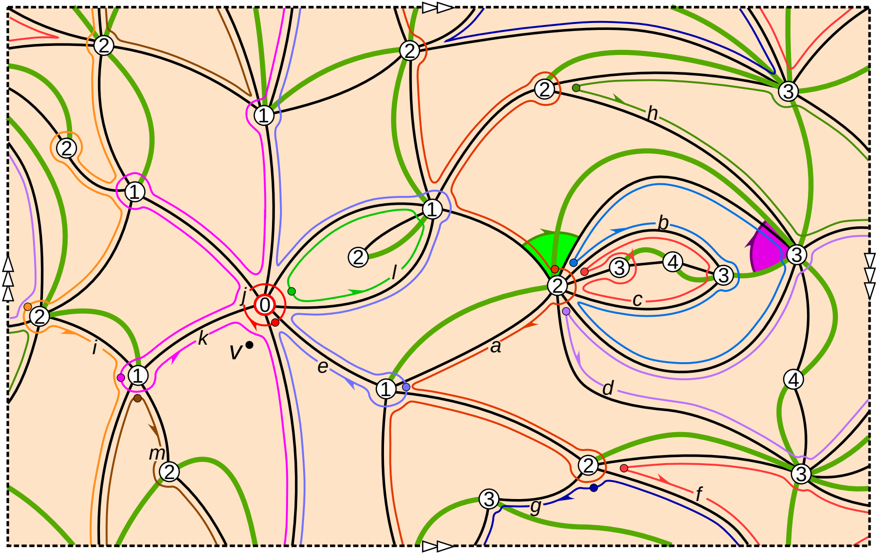

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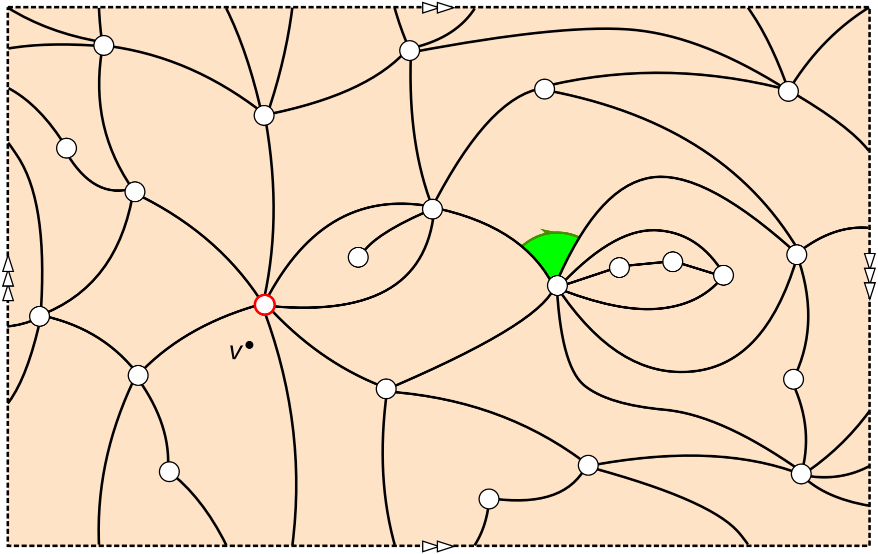
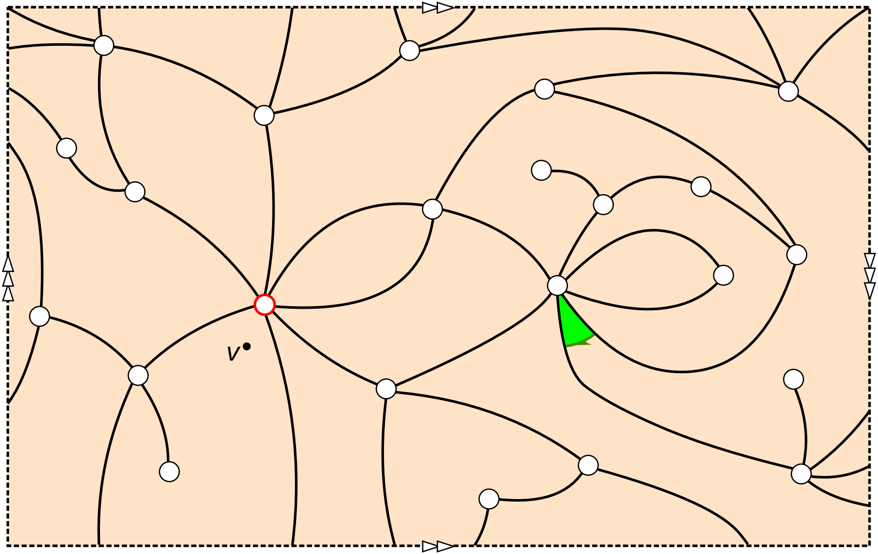


From pointed bipartite maps to unicellular mobiles



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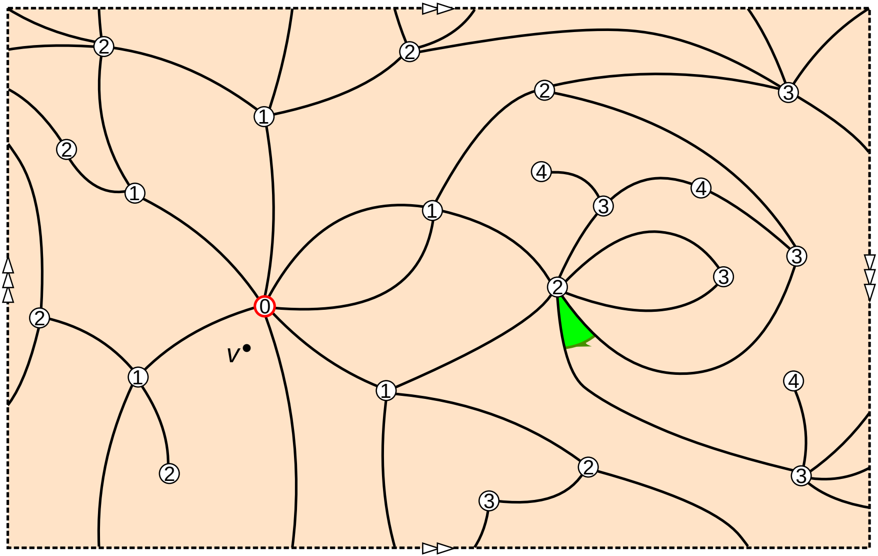

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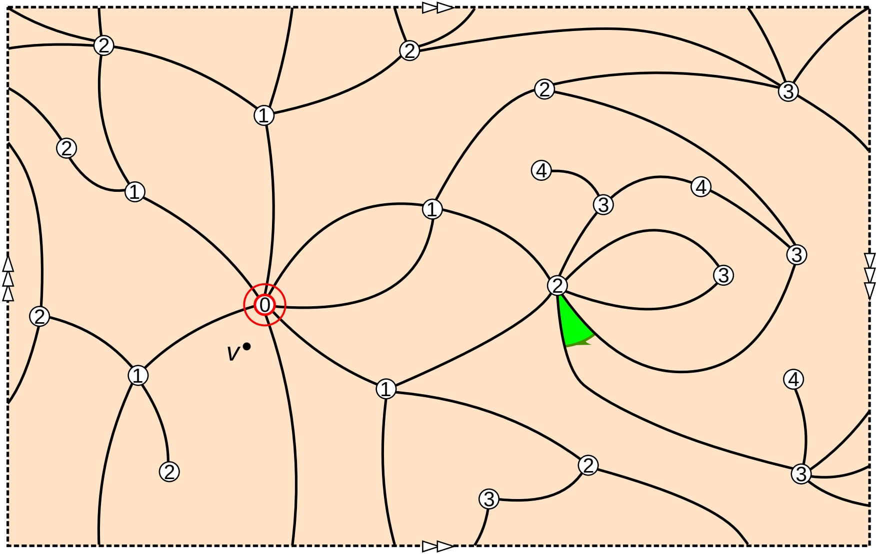

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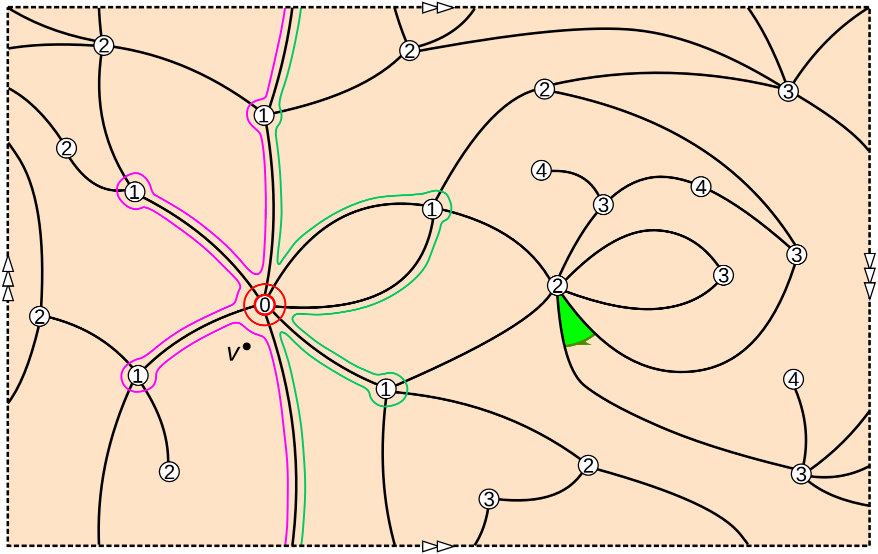

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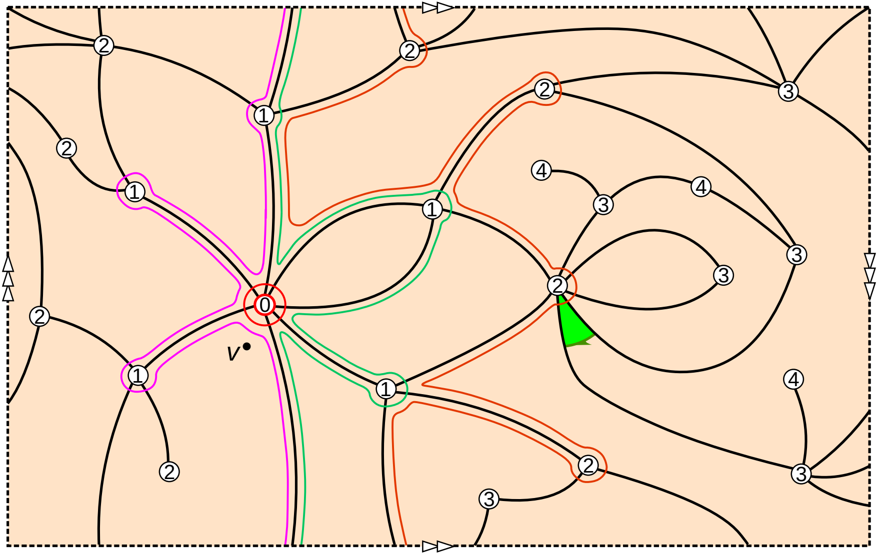

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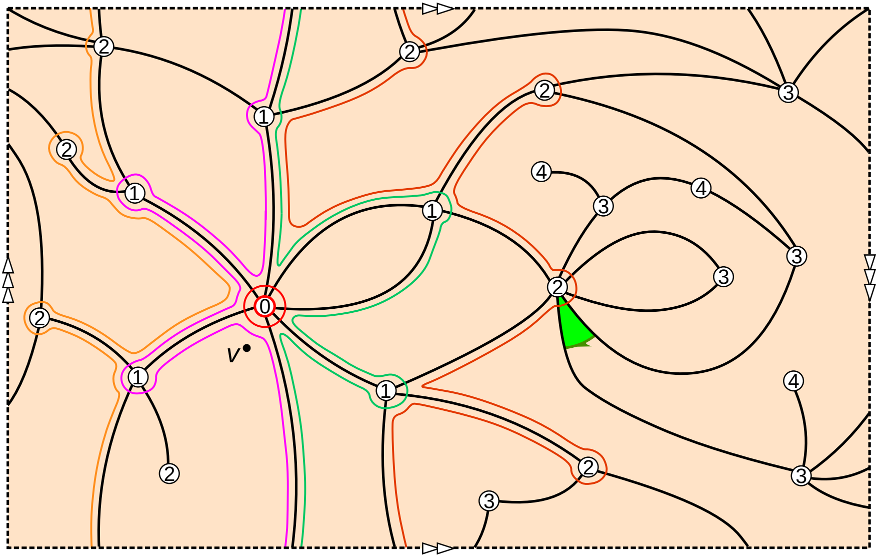

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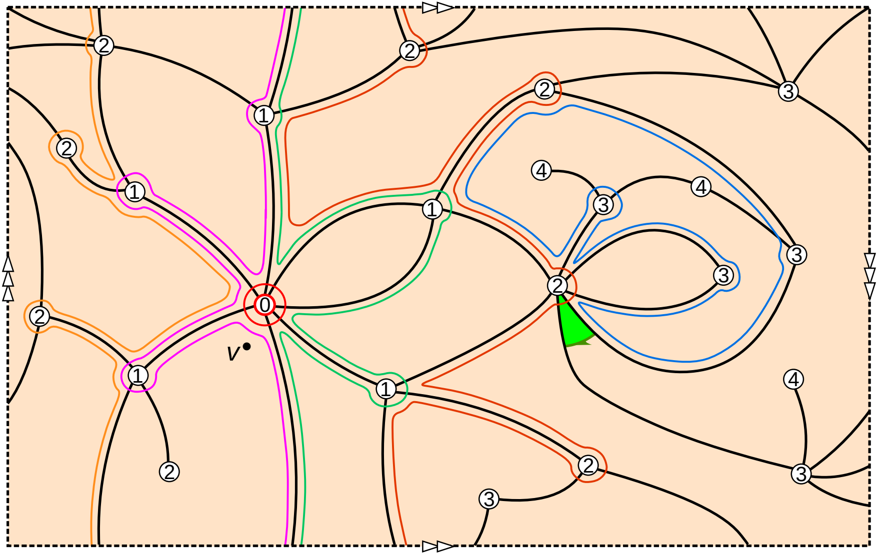

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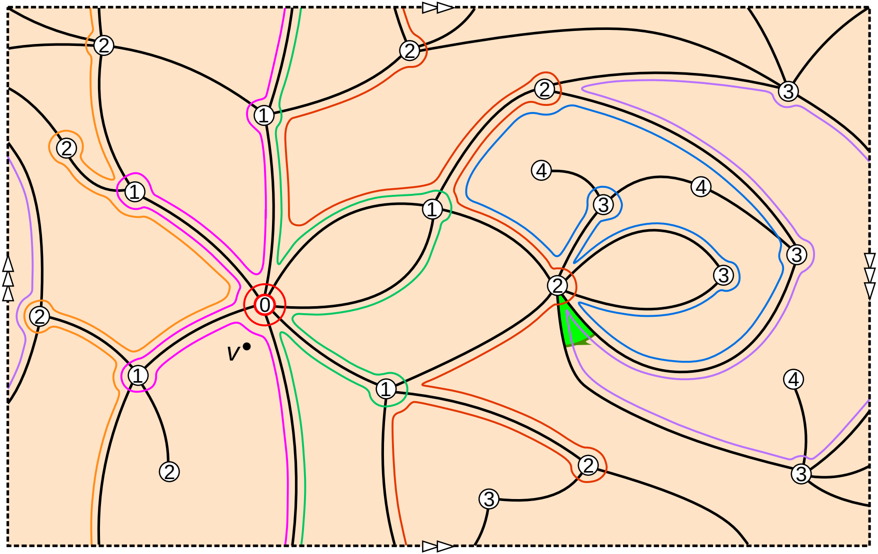

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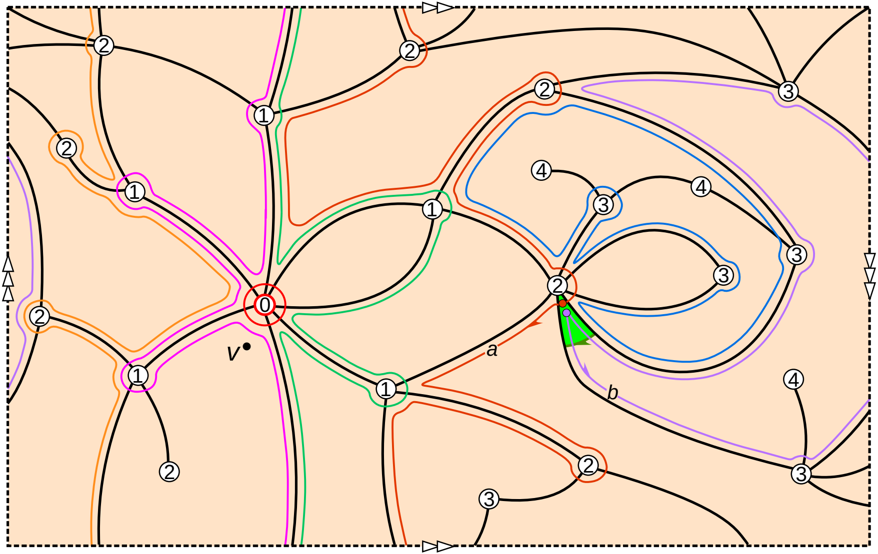
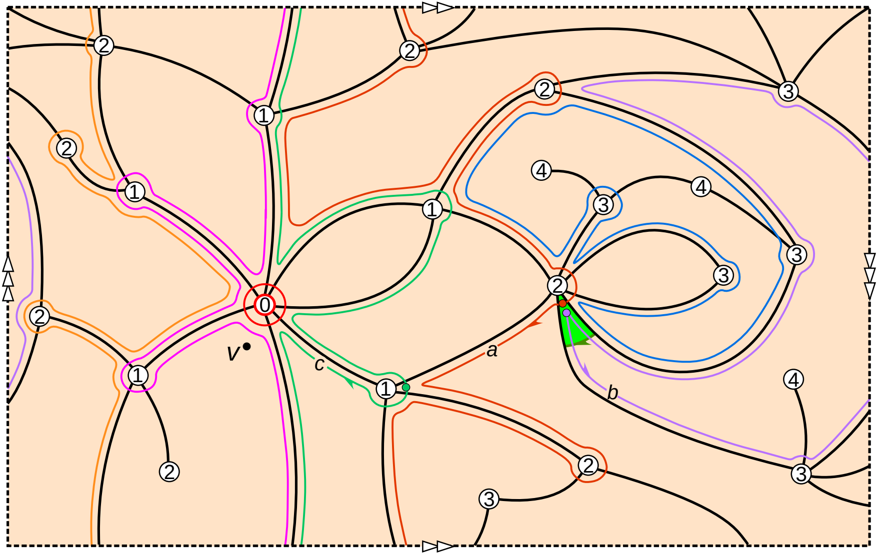
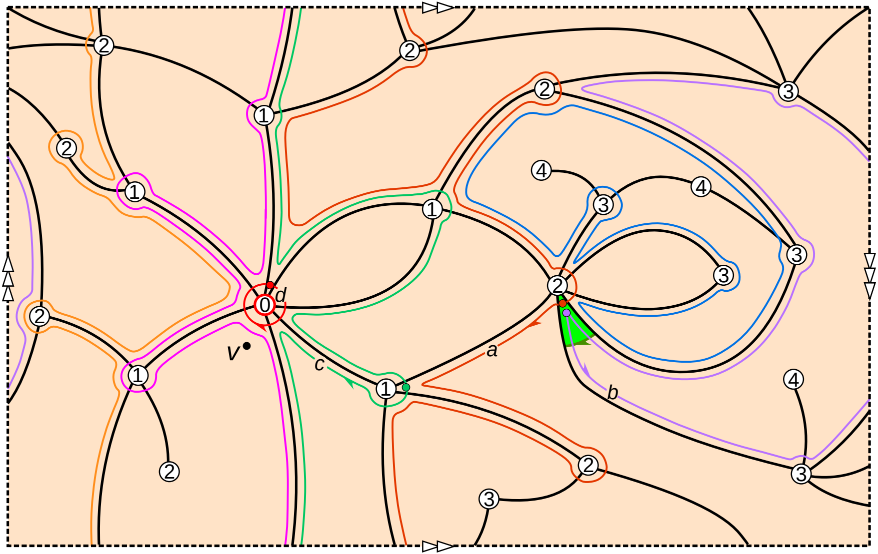


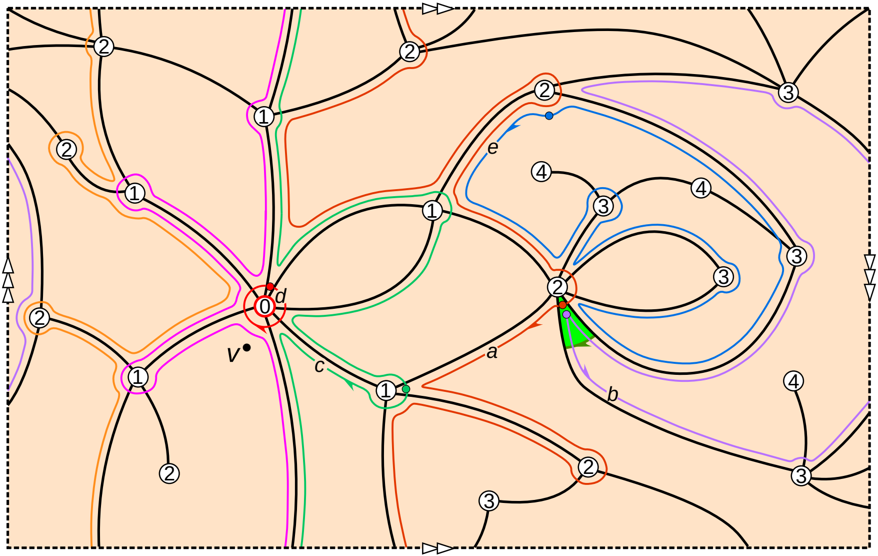
From pointed bipartite maps to unicellular mobiles



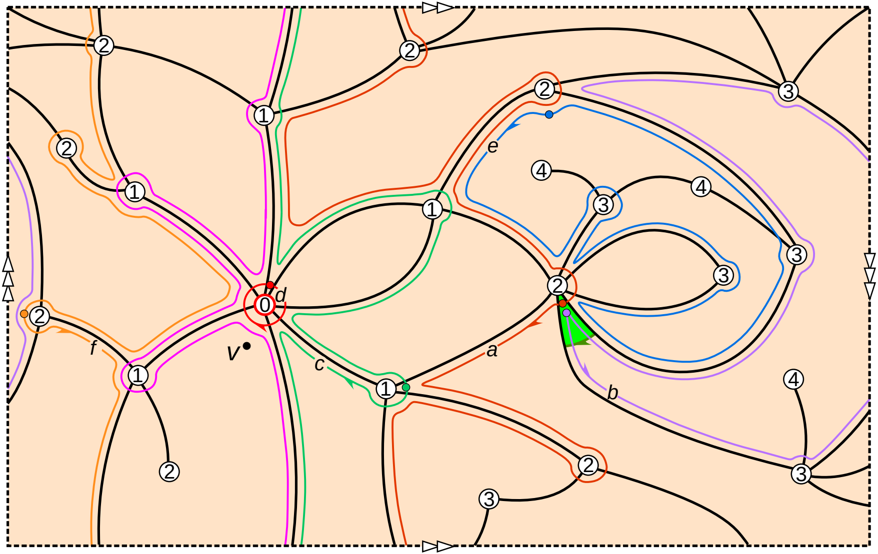
From pointed bipartite maps to unicellular mobiles



From pointed bipartite maps to unicellular mobiles



From pointed bipartite maps to unicellular mobiles



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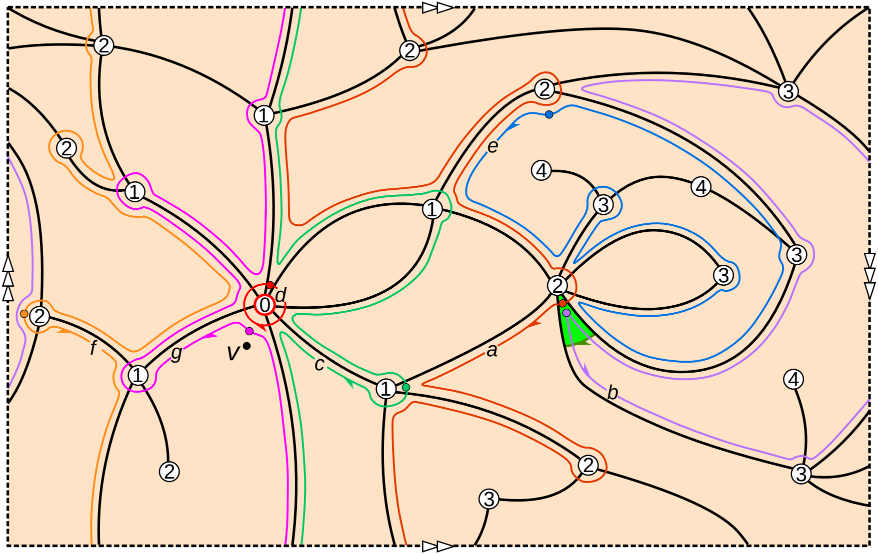

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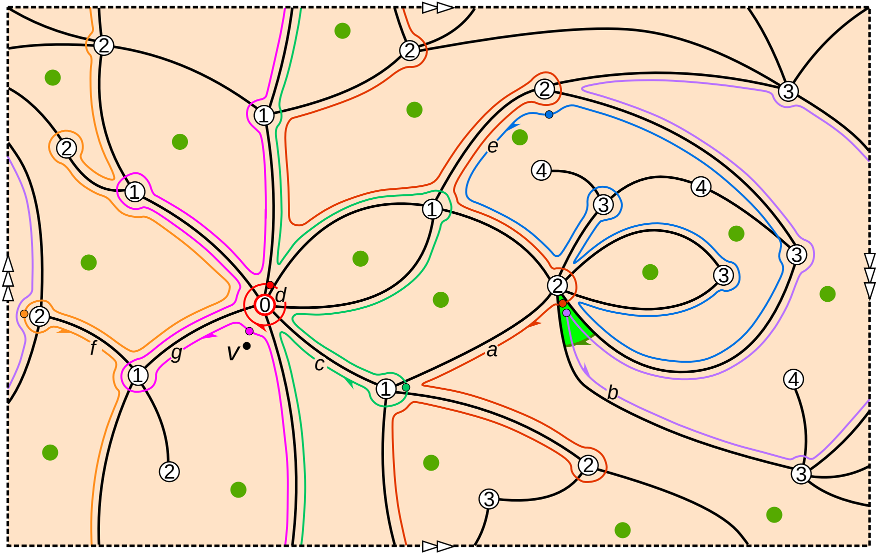

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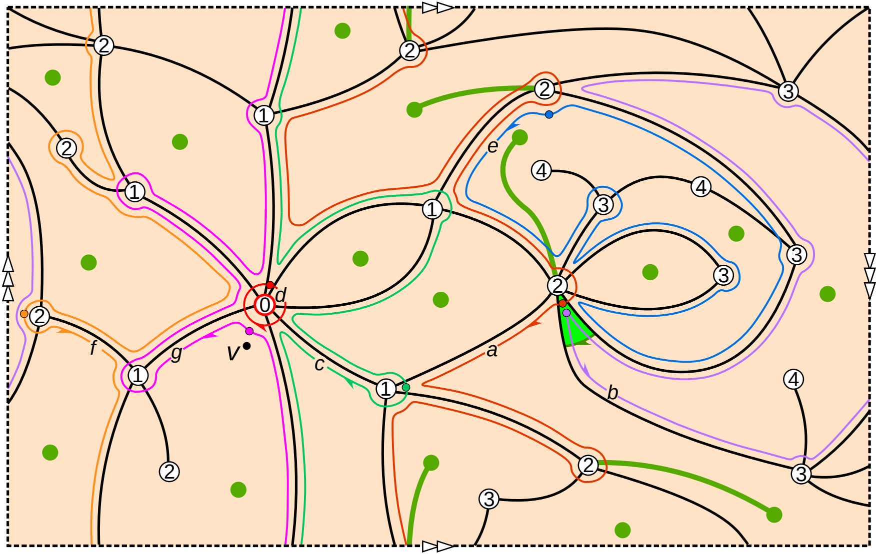

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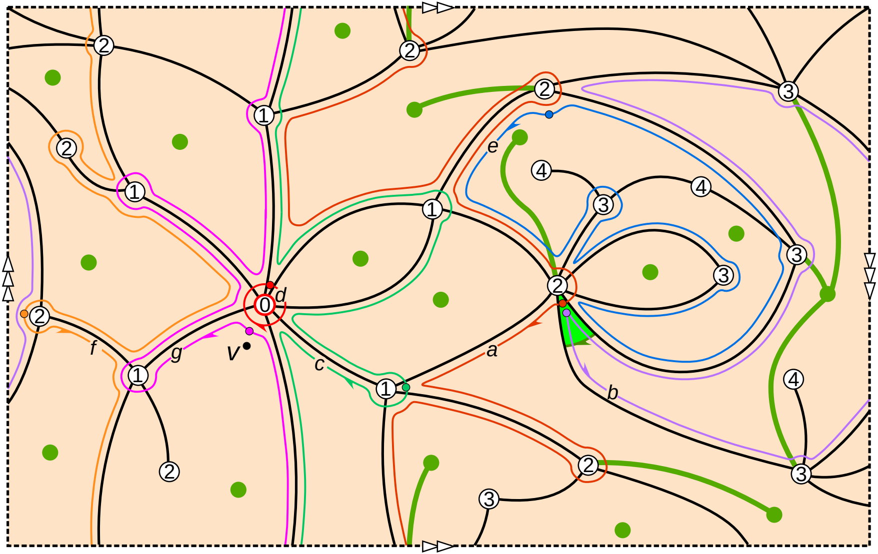

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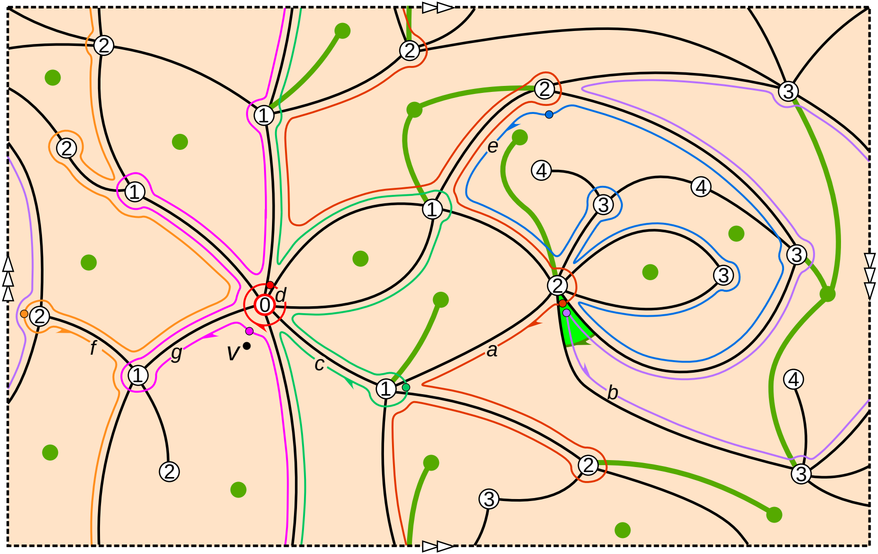

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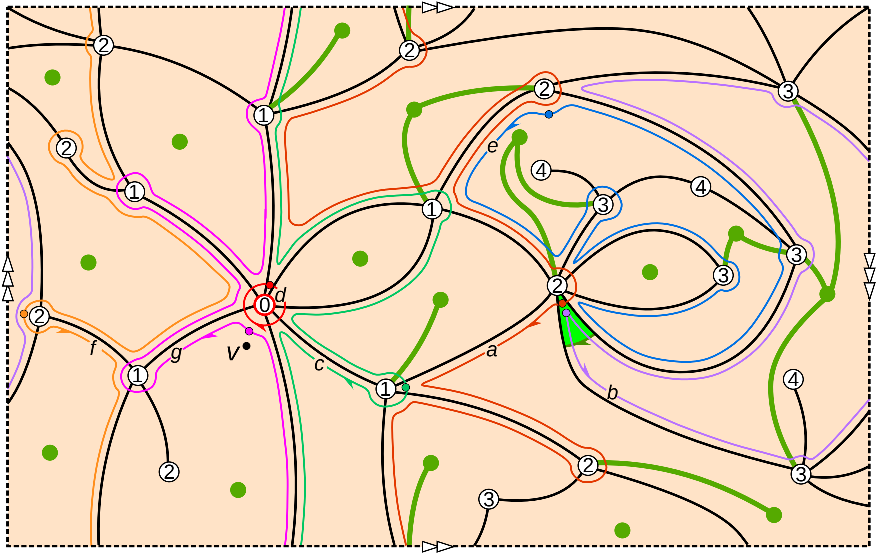

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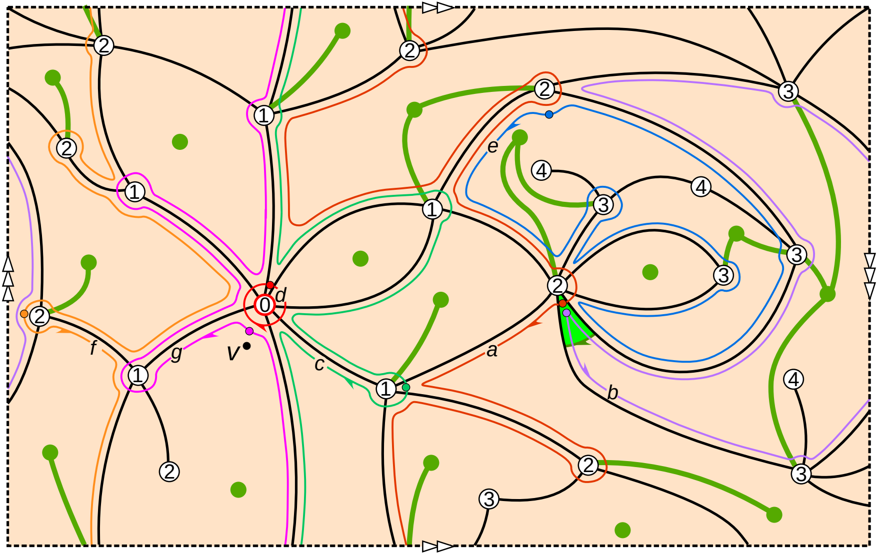

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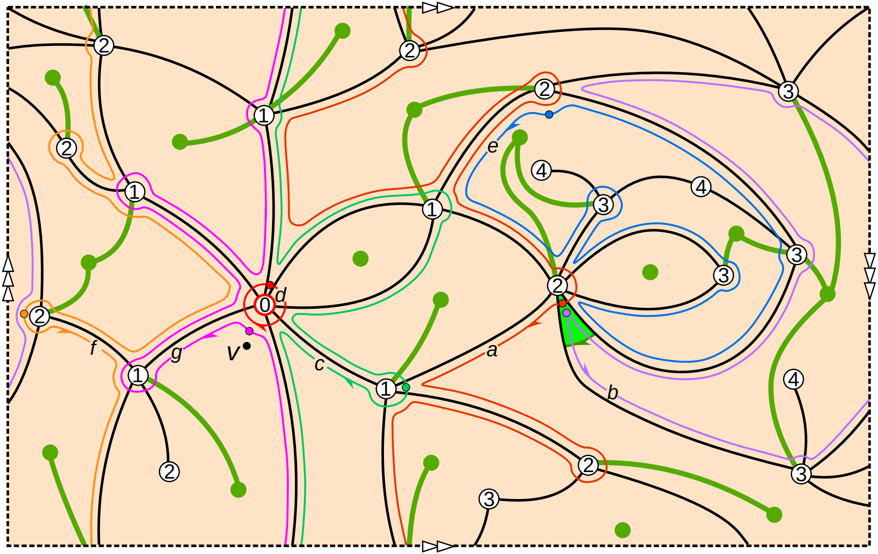

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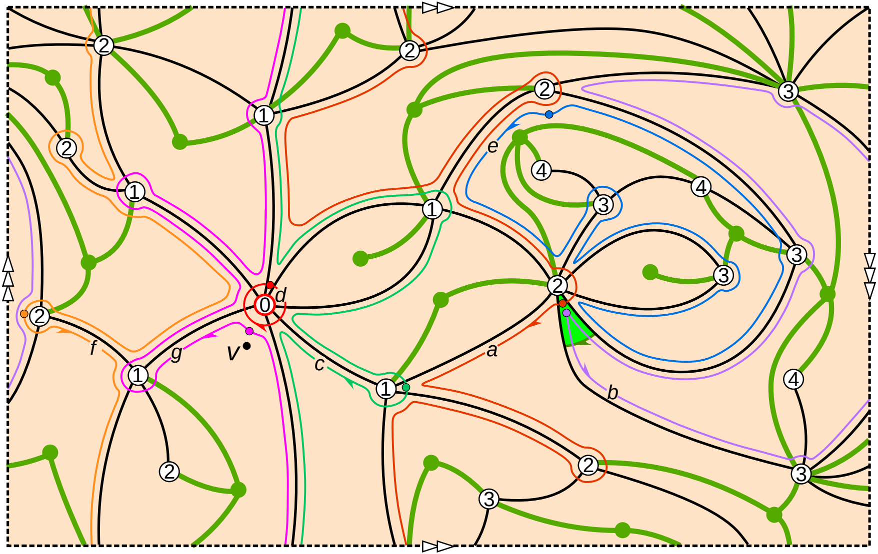

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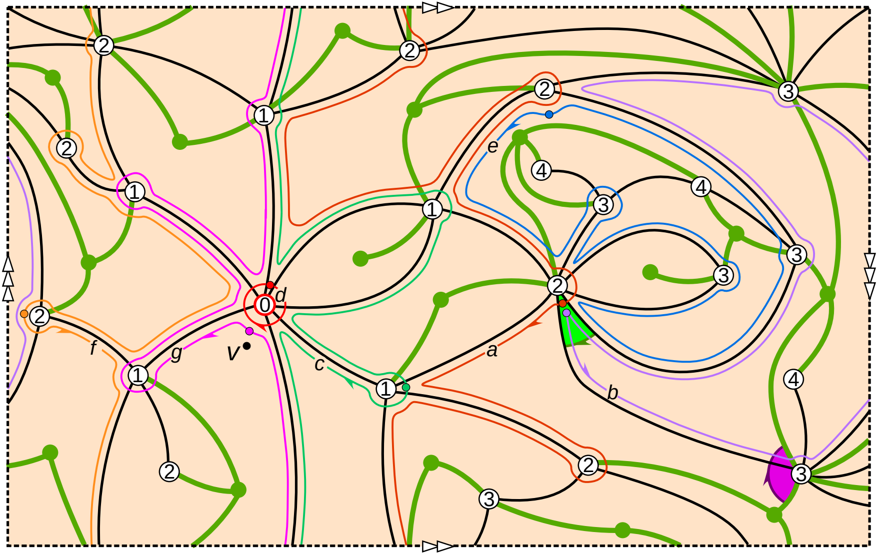
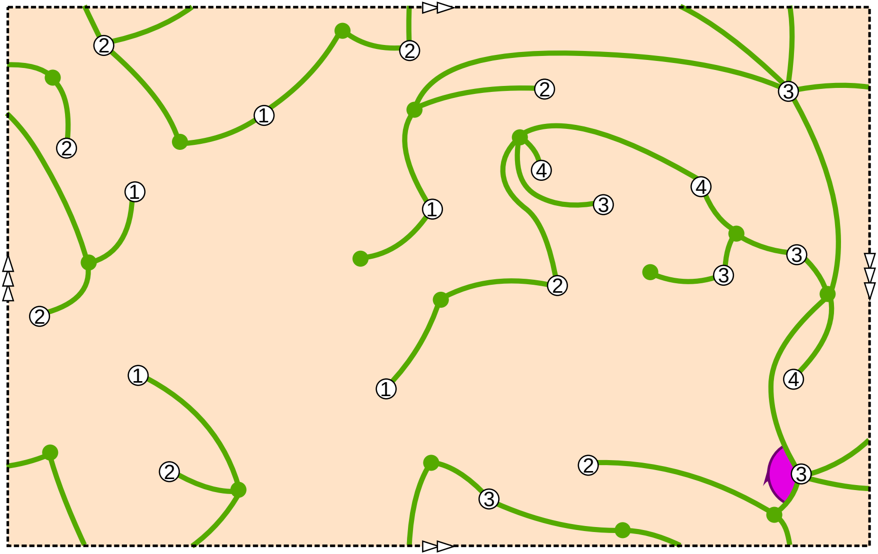


From pointed bipartite maps to unicellular mobiles



First properties of the encoding object

Definition

A labeled unicellular mobile is a pair $(\mathfrak{u}, \mathfrak{l})$ such that

- $\mathfrak{u}$ is a one-face map with vertex set $V_\bullet(\mathfrak{u}) \sqcup V_\circ(\mathfrak{u})$ and only edges linking vertices from $V_\bullet(\mathfrak{u})$ to vertices from $V_\circ(\mathfrak{u})$;
- $\mathfrak{l} : V_\circ(\mathfrak{u}) \to \mathbb{N}$ is a function with minimum 1.

We will only consider corners of $\mathfrak{u}$ that are incident to labeled vertices.

Definition

An $\{i, j\}$-arc is a contiguous interval of two or more subsequent corners $c_1, \dots, c_k$ of $\mathfrak{u}$ such that

- $\{\mathfrak{l}(c_1), \mathfrak{l}(c_k)\} = \{i, j\}$;
- for $1 < r < k$, $\mathfrak{l}(c_r) > i \vee j$.

The level is the number $i \vee j$ and the arc is called trivial if $k = 2$.

First properties of the encoding object

Lemma

The following properties are equivalent:

(a) *For all $i \geq 1$, every nontrivial arc at level i contains a corner with label $i+1$.*

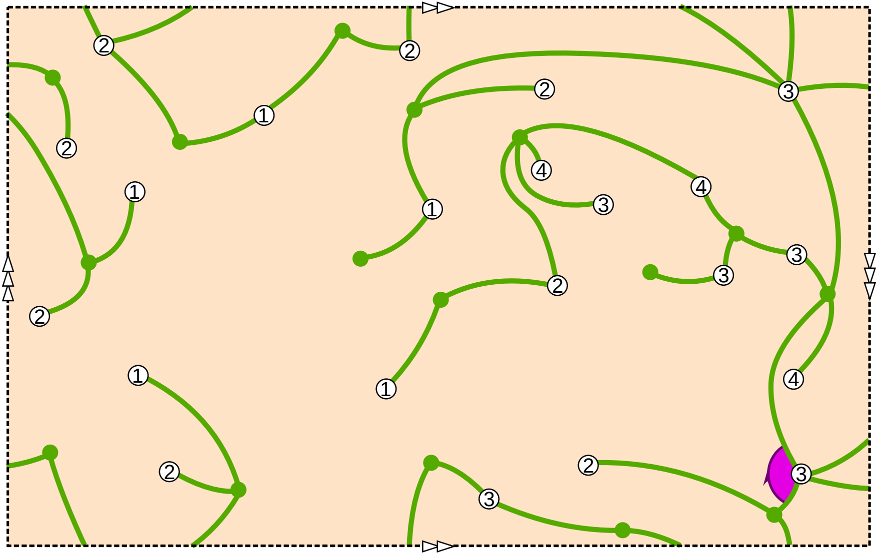
(b) *For all $i \geq 1$, every nontrivial arc at level i has a range of internal corner labels of the form $\{i+1, i+2, \ldots, m\}$ for some m.*

(c) *For all $i \geq 2$ and every corner with label i, either the first subsequent corner with label strictly smaller than i has label $i-1$ or the last preceding corner with label strictly smaller than i has label $i-1$.*

Moreover, these properties imply that

- *for all $i \geq 2$, every arc at level $i \geq 2$ is included in a unique arc at level $i-1$;*
- *every corner labeled $i \geq 2$ is included in a unique arc at level $i-1$.*

From unicellular mobiles to pointed bipartite maps



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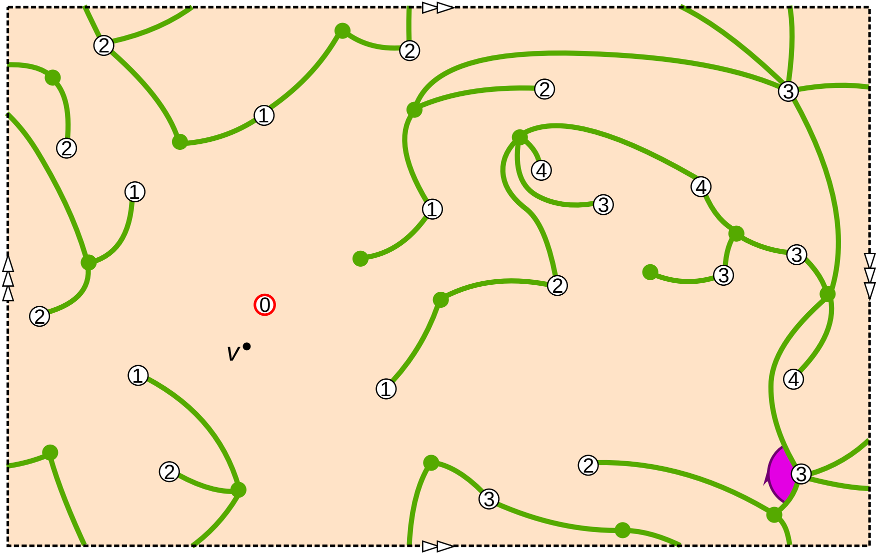

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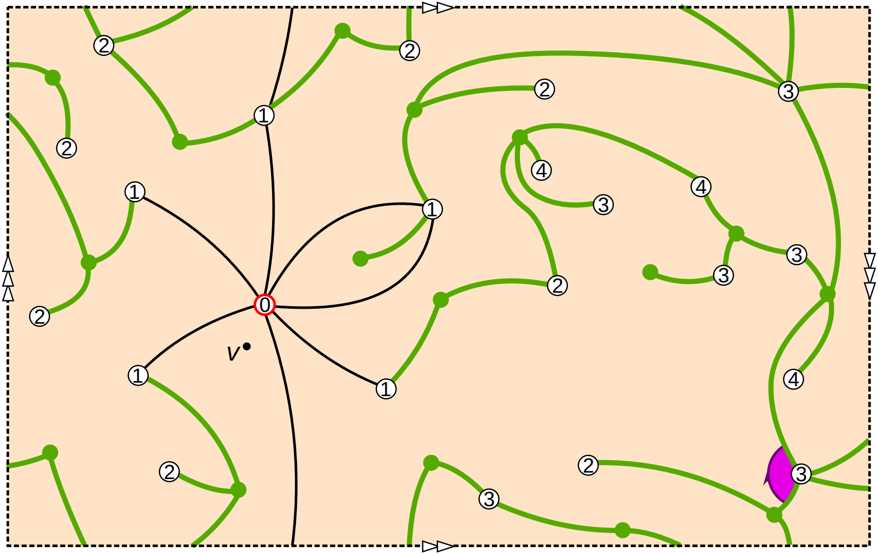

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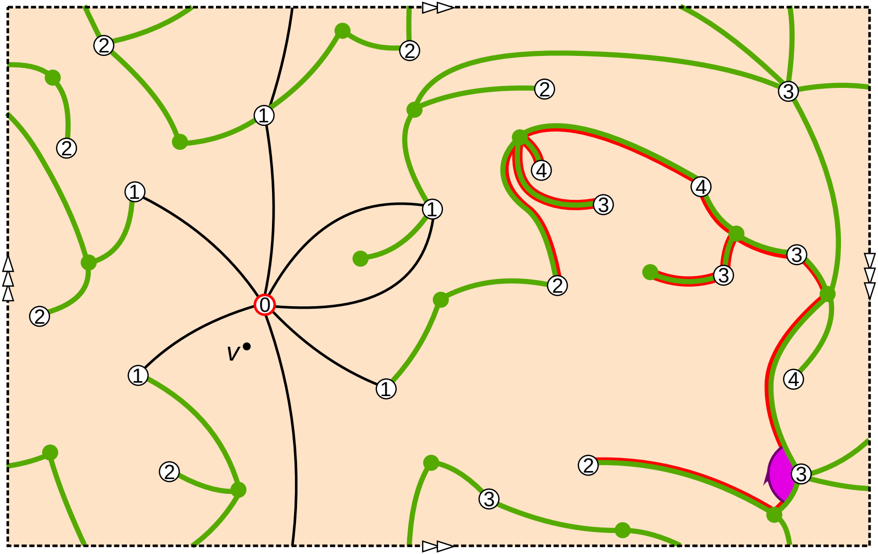
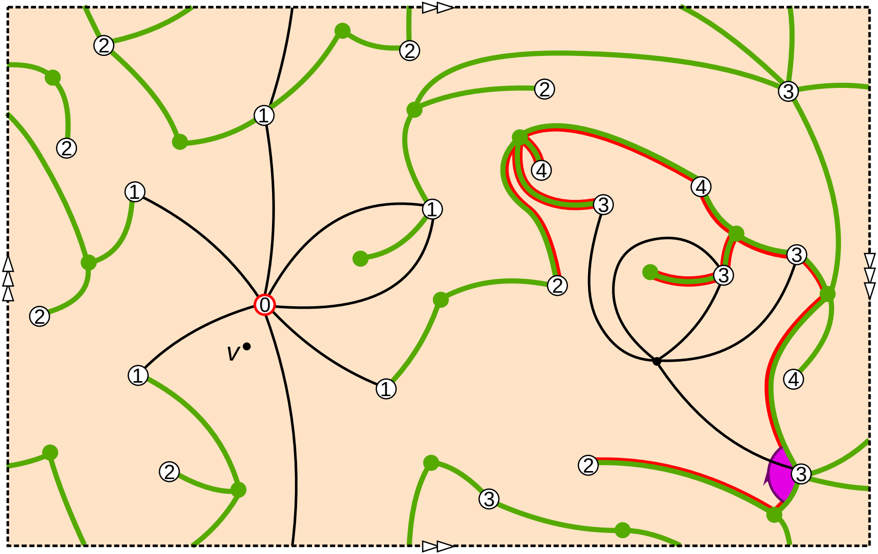
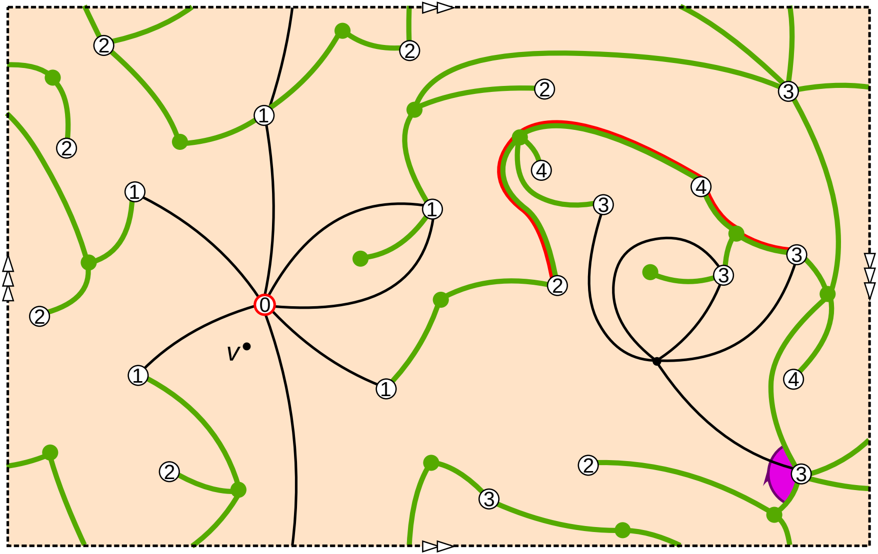


From unicellular mobiles to pointed bipartite maps



From unicellular mobiles to pointed bipartite maps



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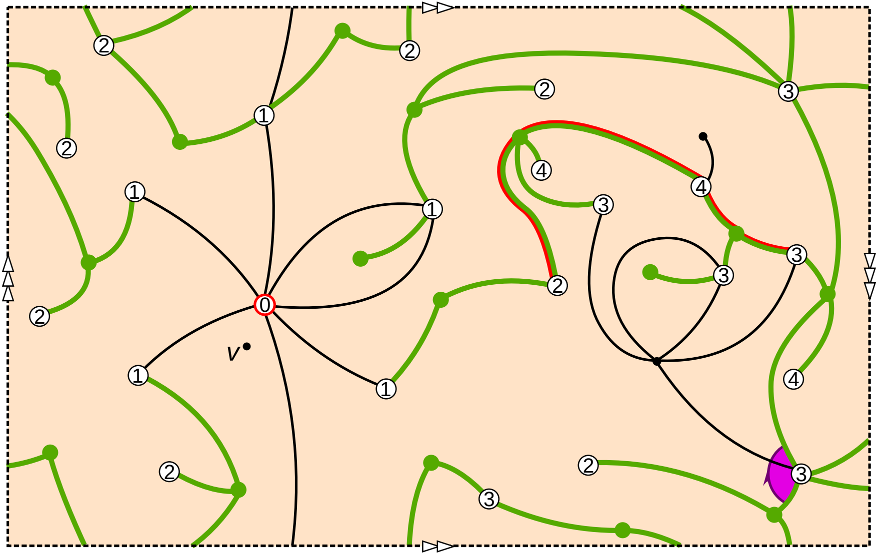

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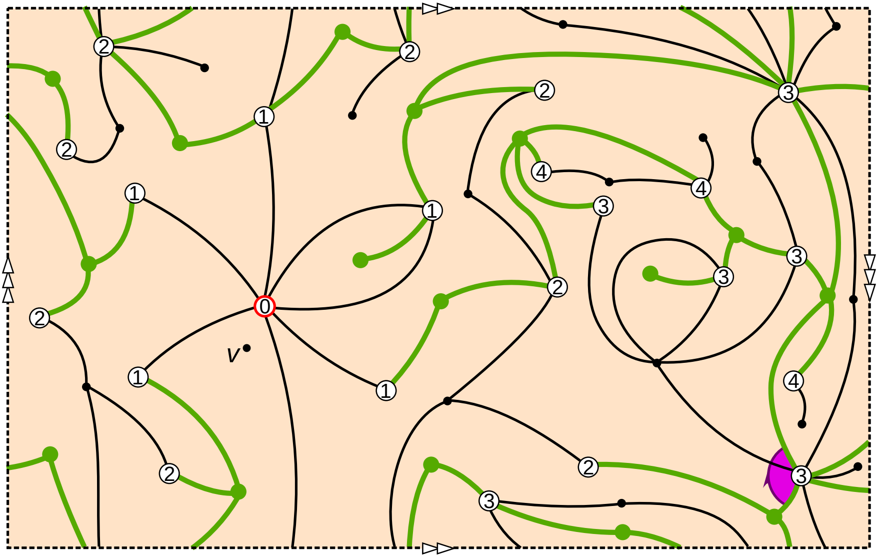

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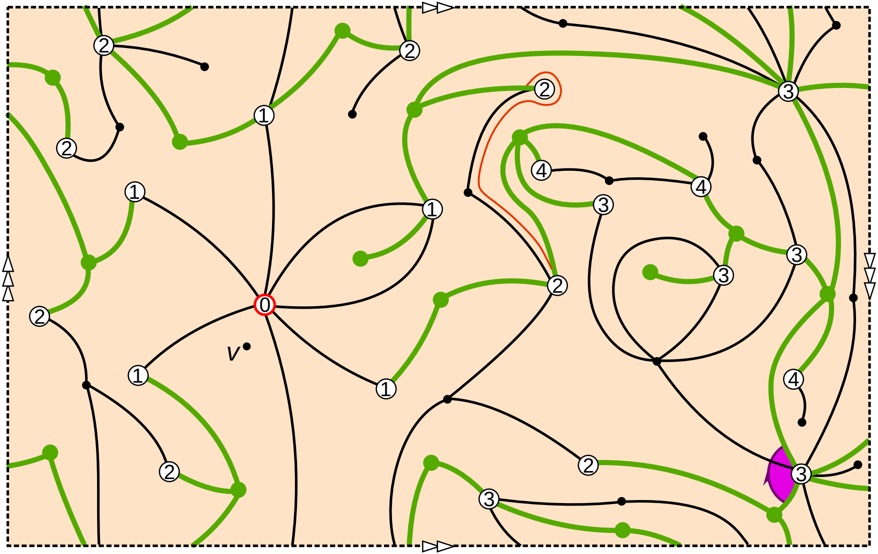

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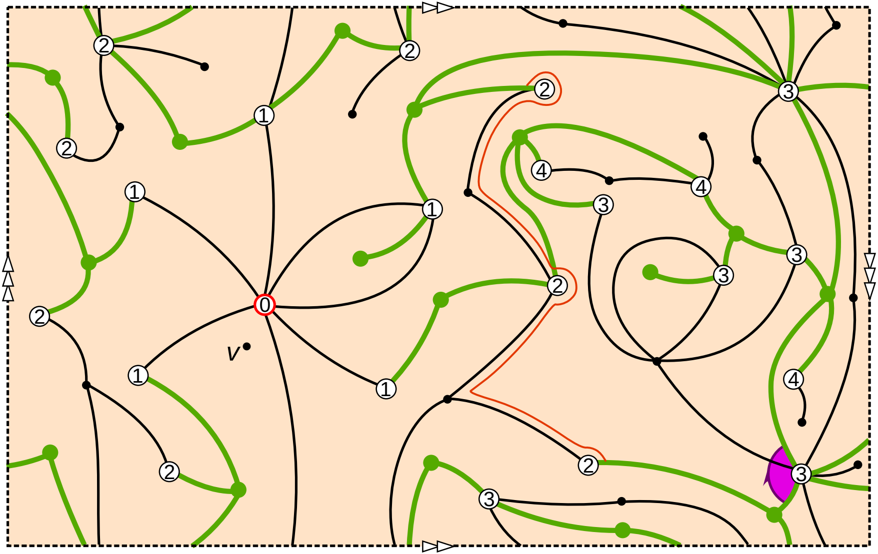

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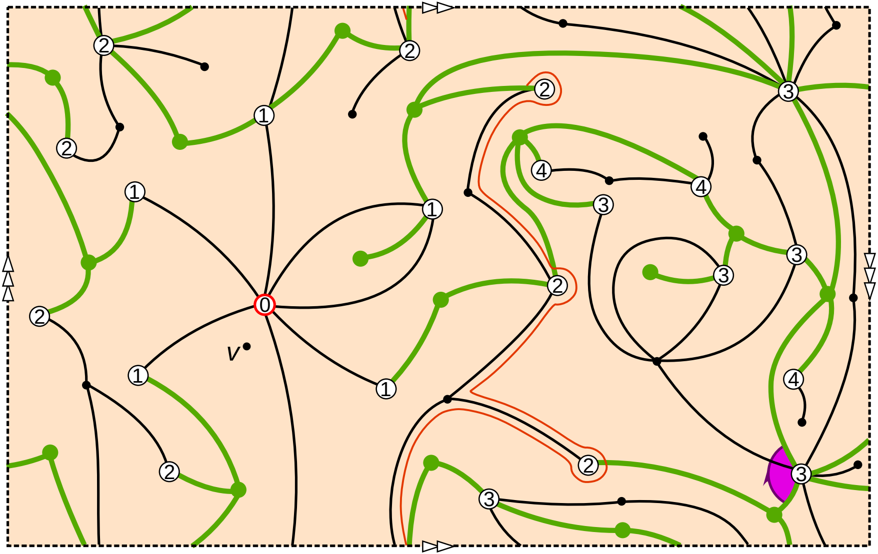

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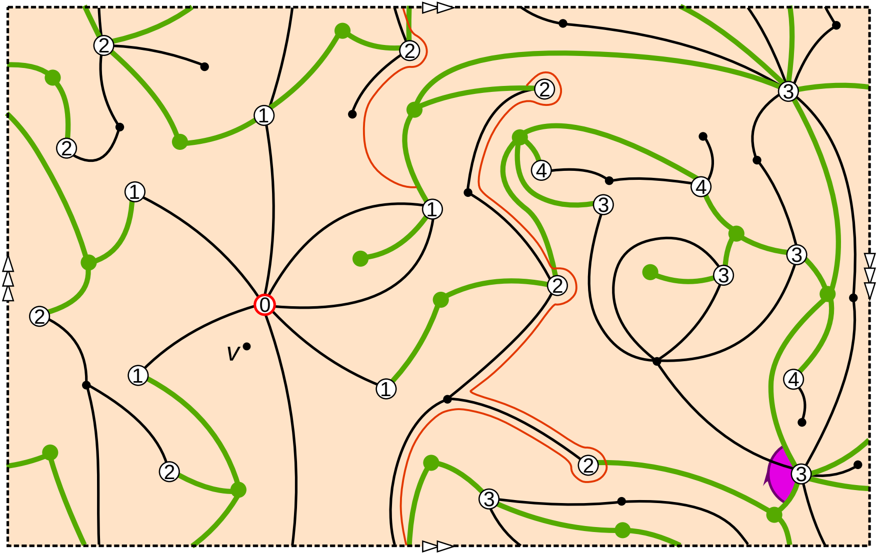

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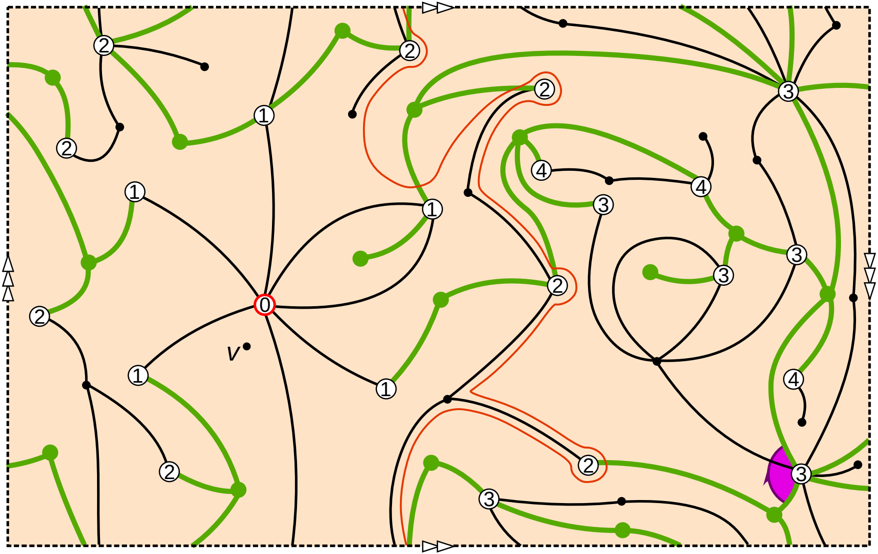

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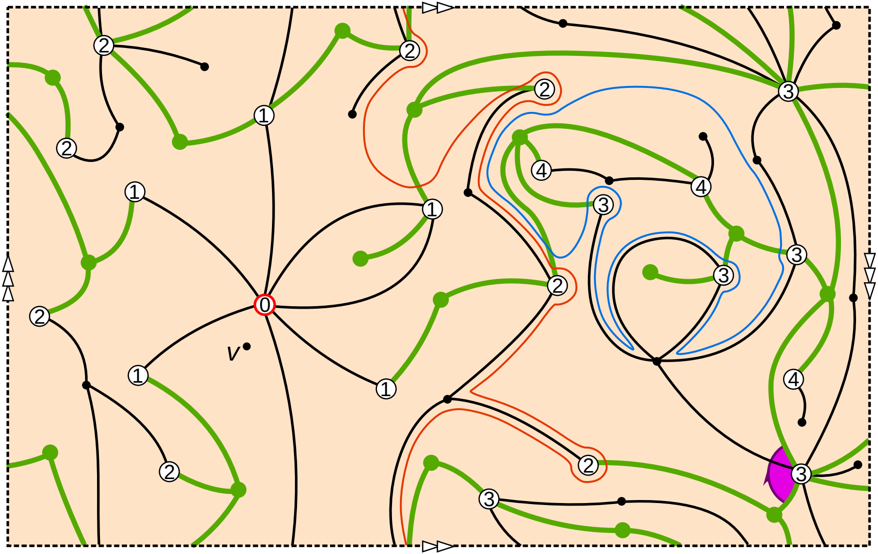
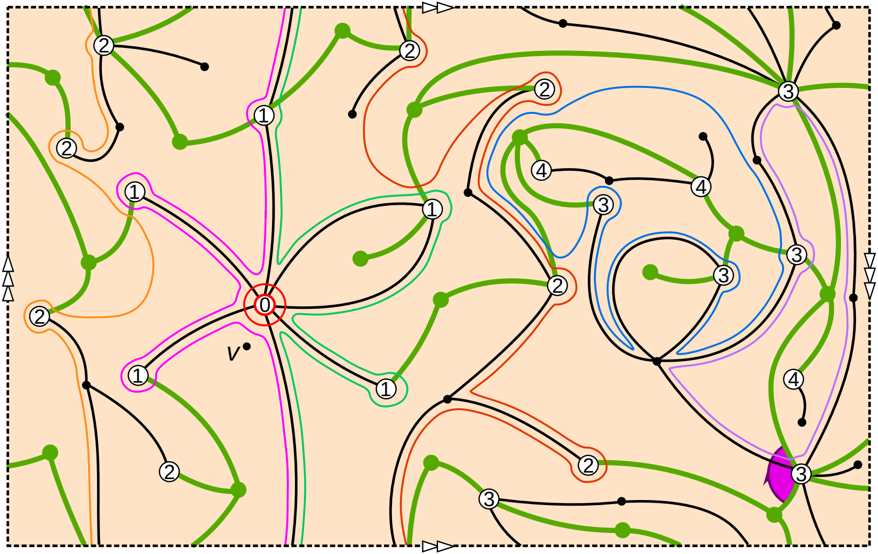


From unicellular mobiles to pointed bipartite maps

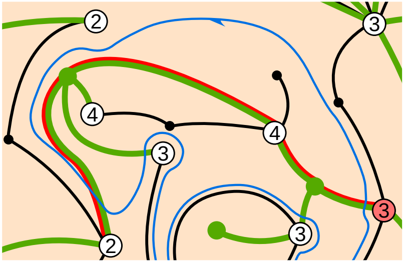


From unicellular mobiles to pointed bipartite maps

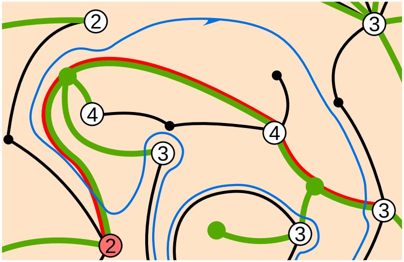
By construction, a loop at level i delimits arcs, which are either trivial at level $\leq i$ or nontrivial at level i.

Definition

A level loop at level i is well oriented if every nontrivial arc it delimits is visited starting from an extremity with label i.

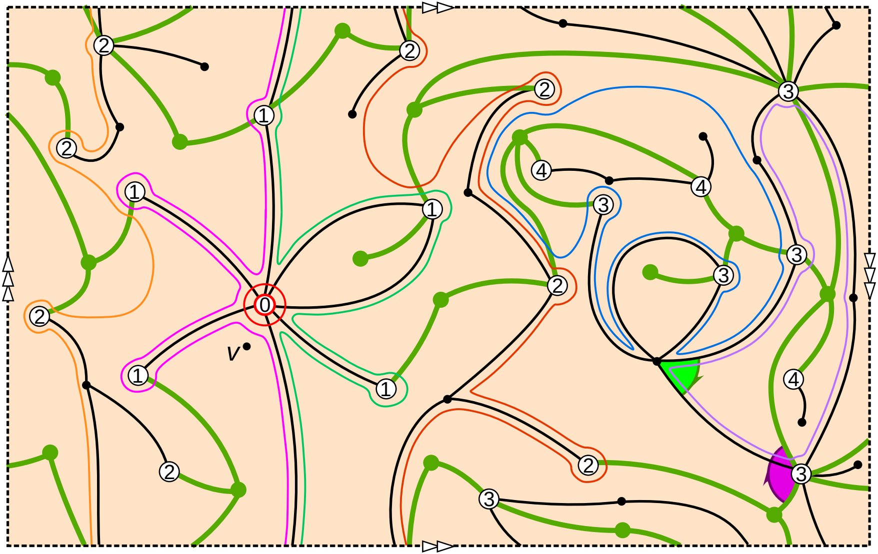


well oriented



not well oriented

From unicellular mobiles to pointed bipartite maps



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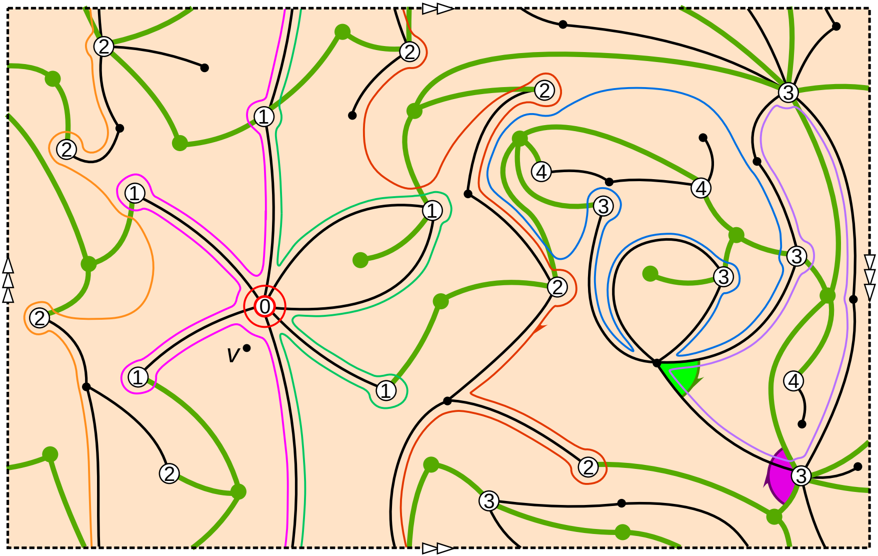

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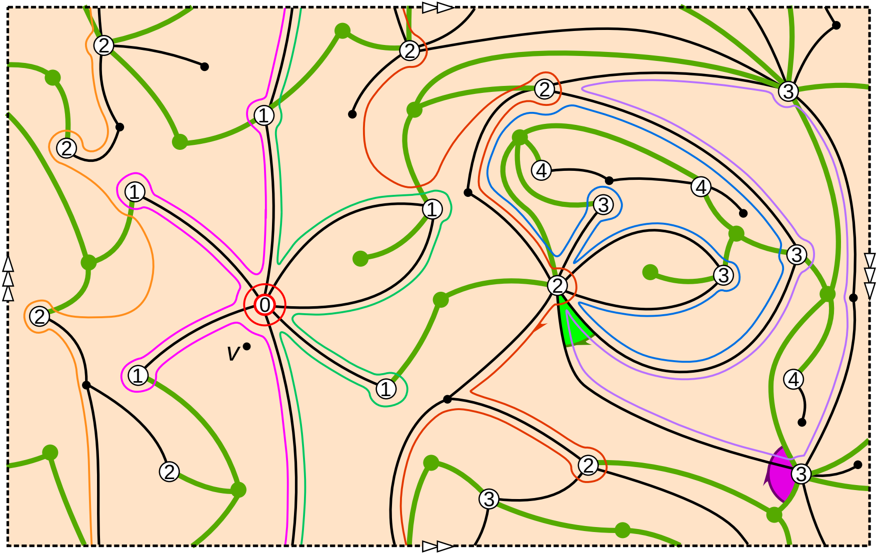

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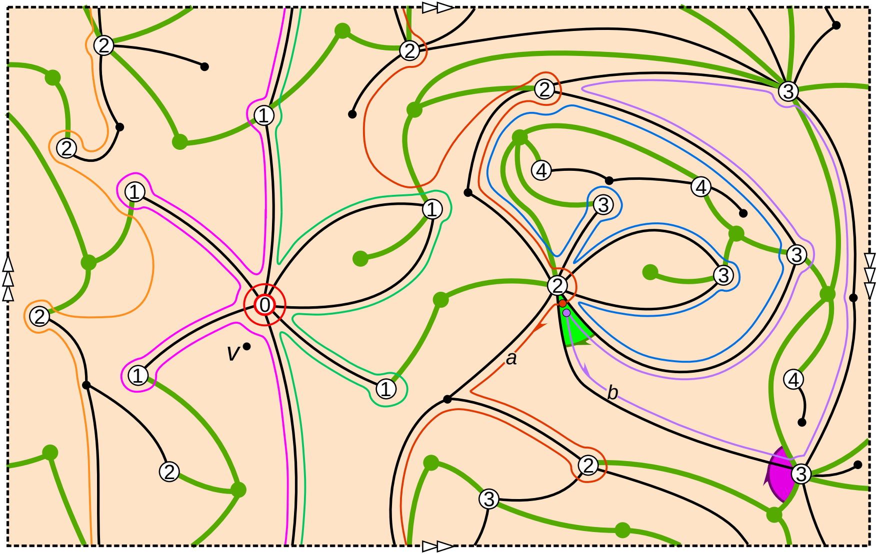

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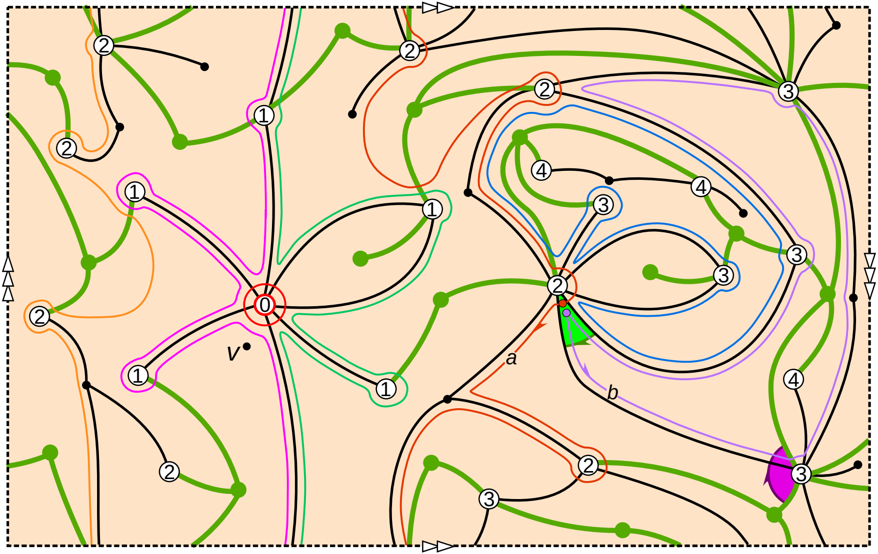

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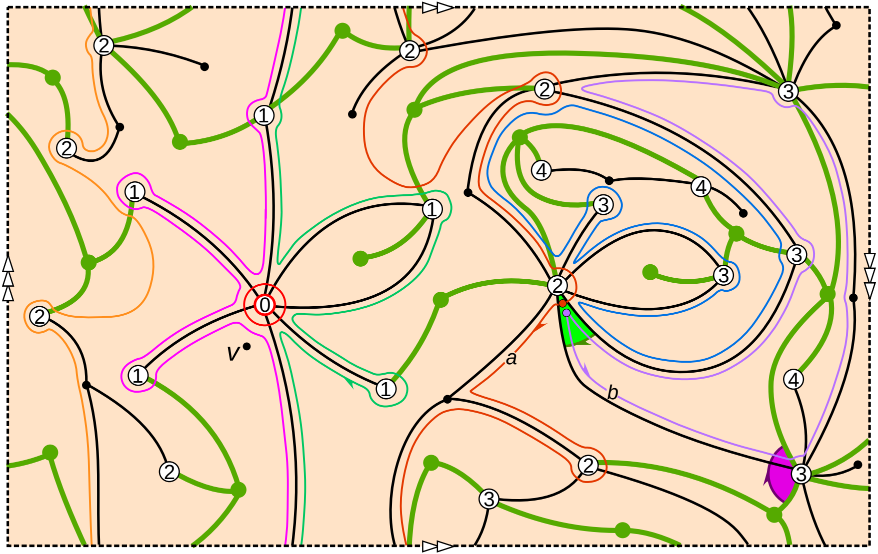

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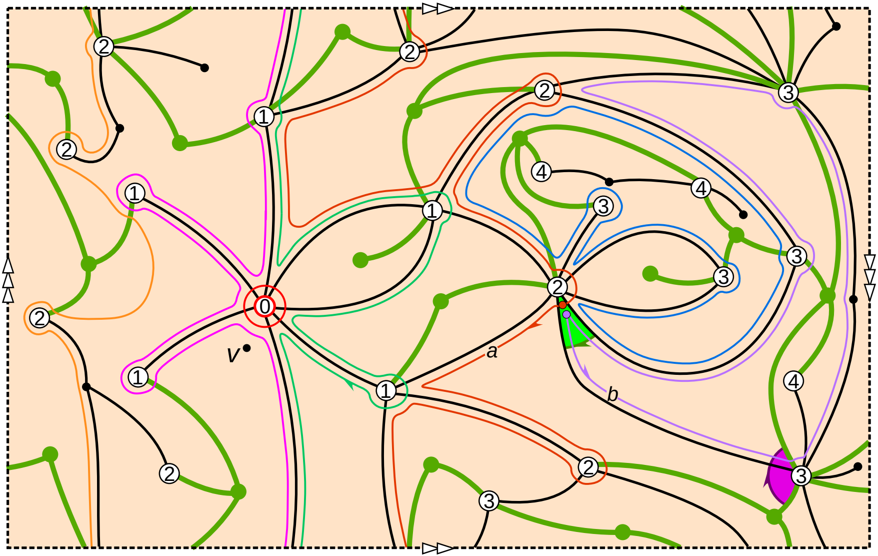

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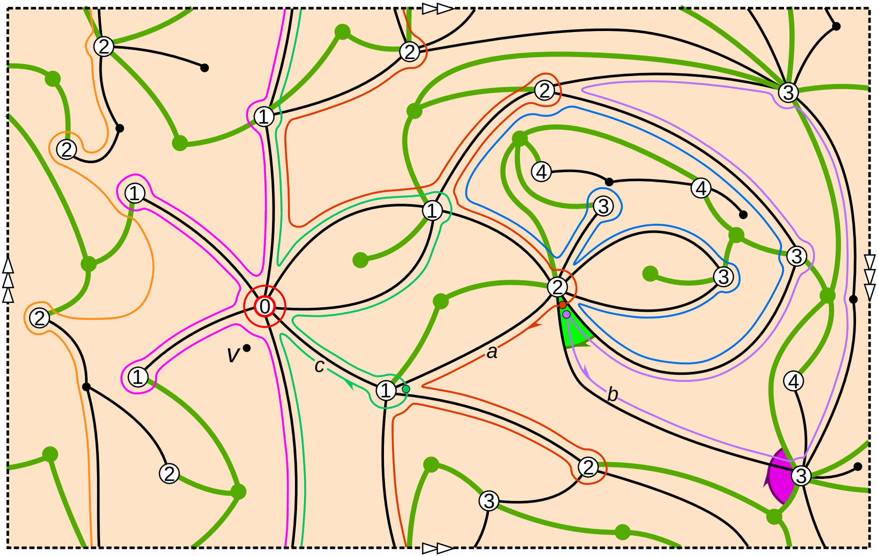

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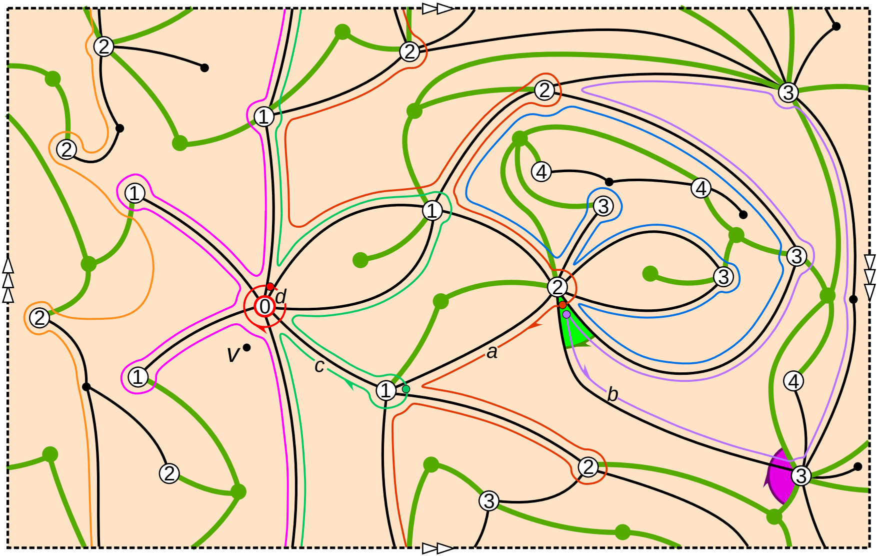

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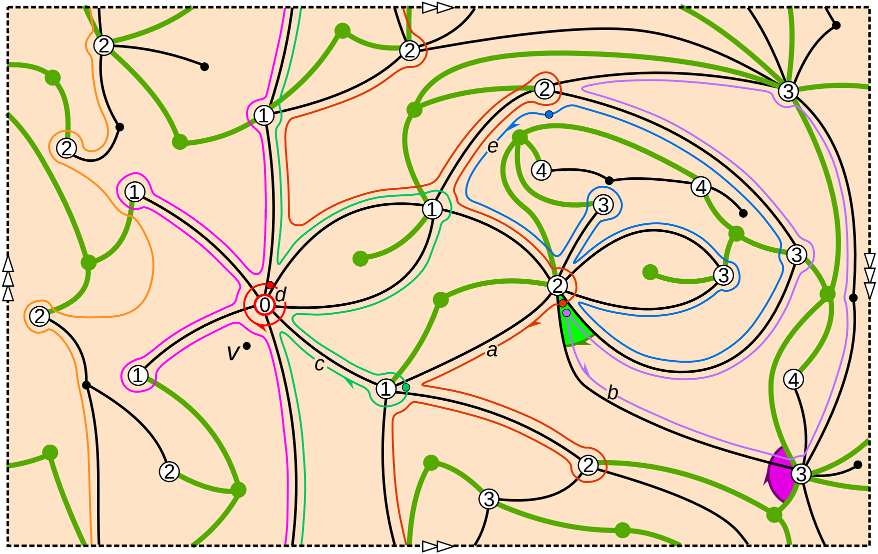

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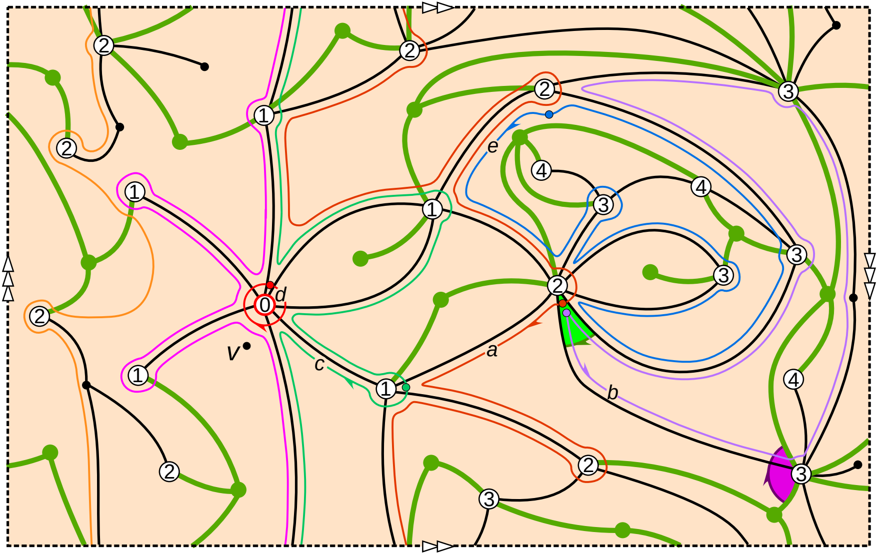

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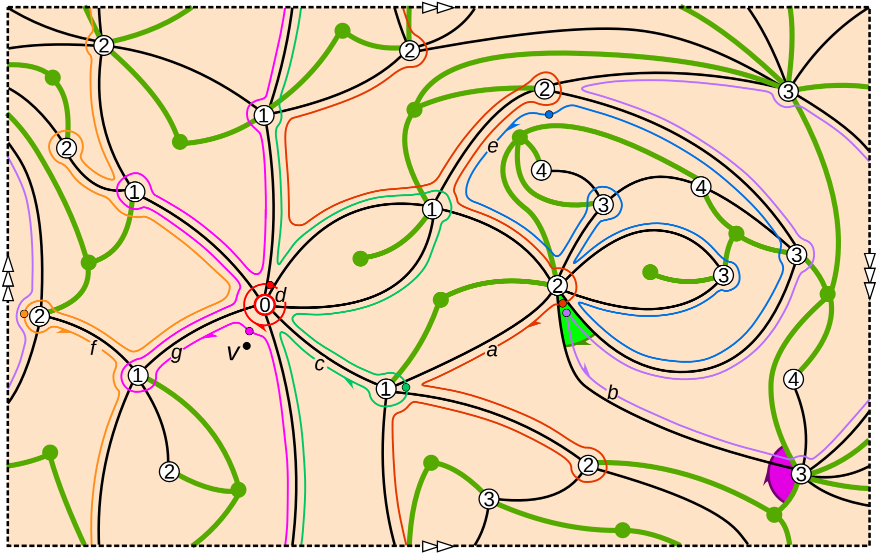

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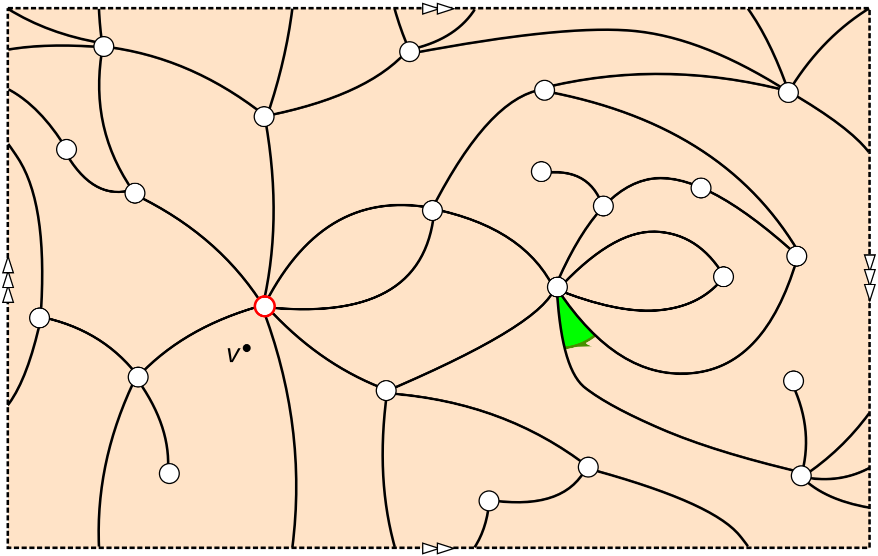


Corresponding quantities

Proposition

Let $(\mathfrak{m}, v^\bullet)$ be a pointed bipartite map and $(\mathfrak{u}, \mathfrak{l})$ the corresponding well-labeled unicellular mobile. Then

(i) $V(\mathfrak{m}) = V_\circ(\mathfrak{u}) \sqcup \{v^\bullet\}$ *and, for* $v \in V_\circ(\mathfrak{u})$, $\mathfrak{l}(v) = d_{\mathfrak{m}}(v, v^\bullet)$*;*

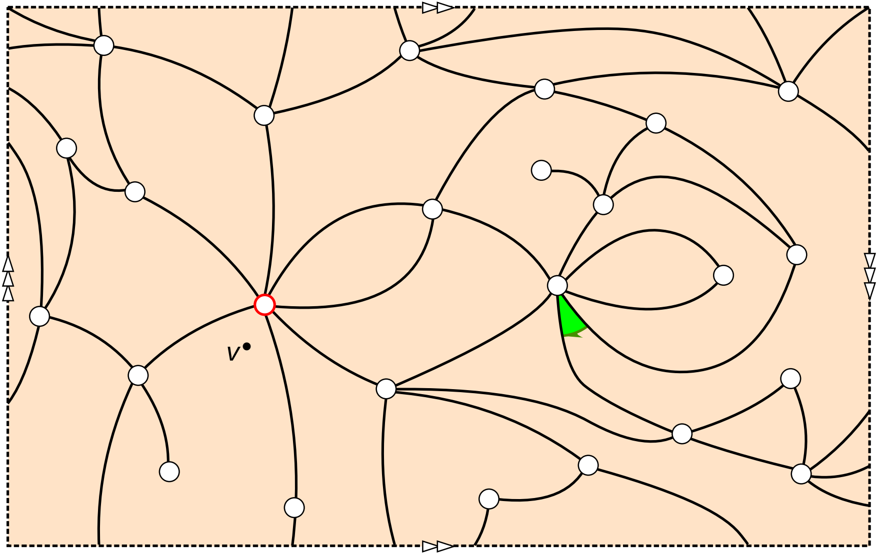
(ii) *the faces of* $\mathfrak{m}$ *correspond to* $V_\bullet(\mathfrak{u})$*: moreover, the degree of a face of* $\mathfrak{m}$ *is twice the degree of the corresponding vertex in* $V_\bullet(\mathfrak{u})$*;*

(iii) *the maps* $\mathfrak{m}$ *and* $\mathfrak{u}$ *have the same number of edges.*

The previous construction specializes into a bijection between

- bipartite maps with n faces marked 1, 2, ..., n and such that, for $1 \leq i \leq n$, the face marked i has degree $2\alpha_i$;
- pairs of a sign $+$ or $-$ and a well-labeled unicellular mobiles with n green vertices marked 1, 2, ..., n and such that, for $1 \leq i \leq n$, the green vertex marked i has degree α_i.

The construction for nonbipartite maps



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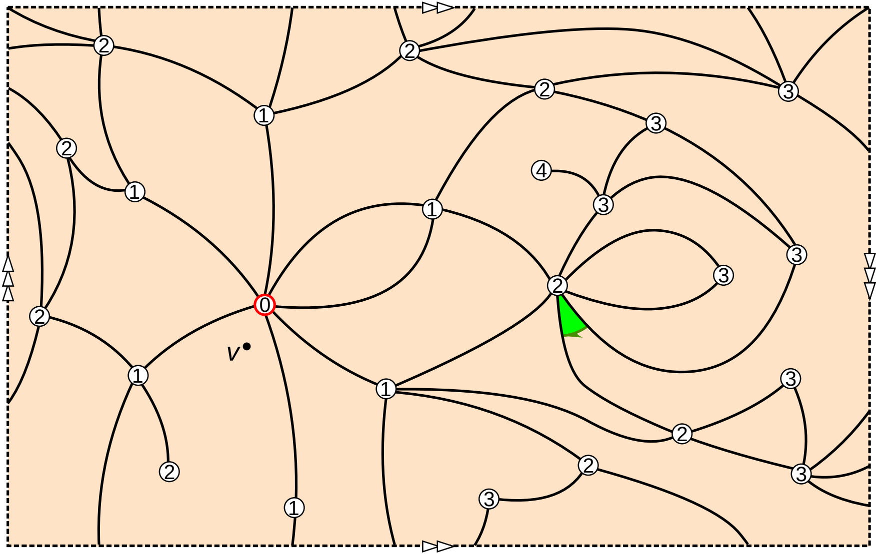

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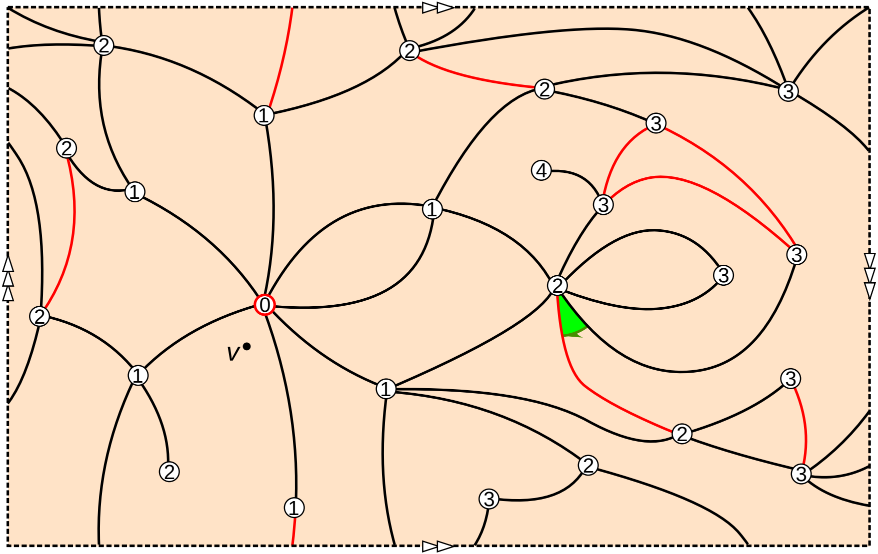

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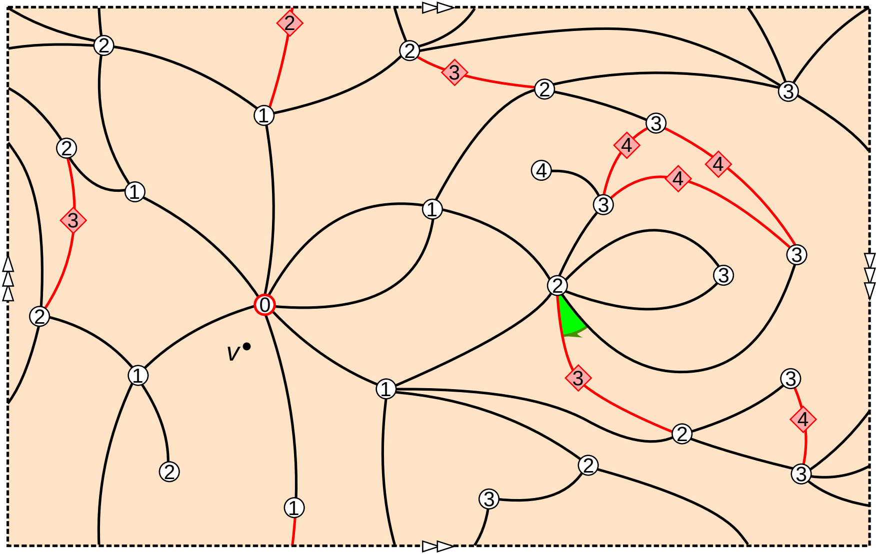

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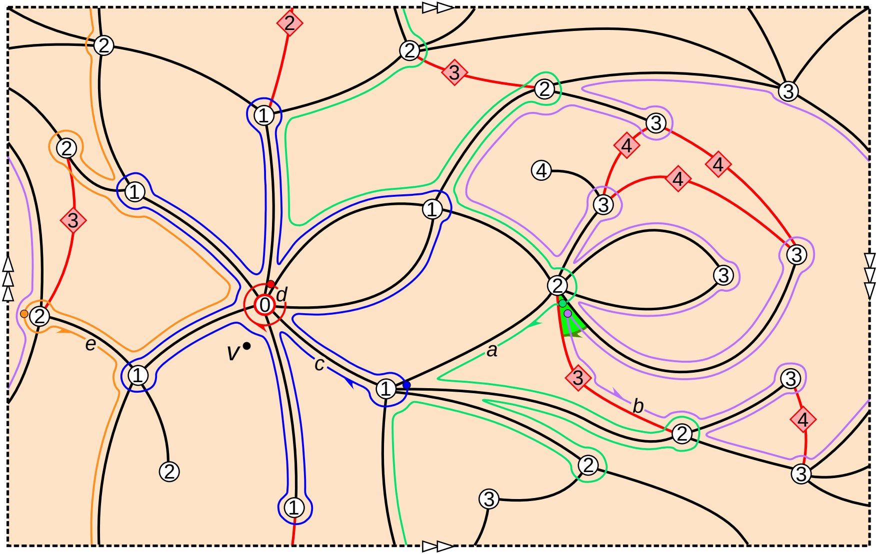

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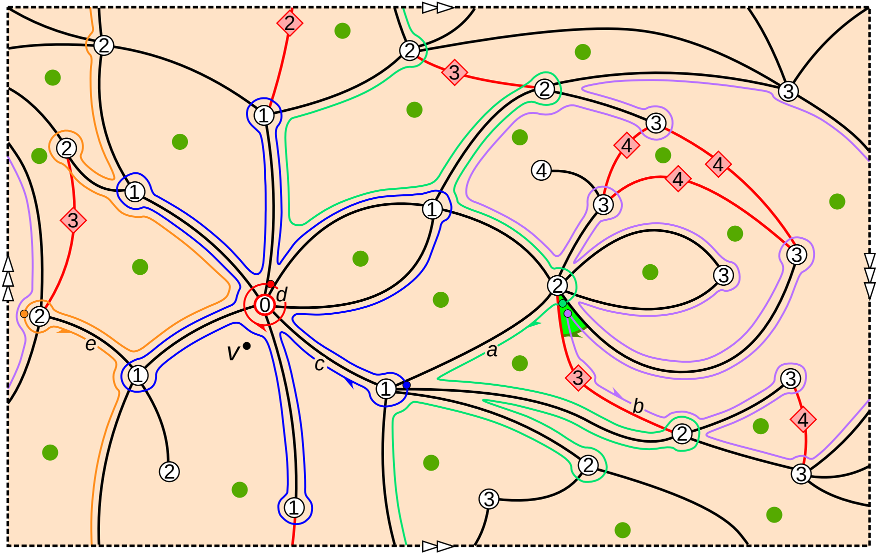

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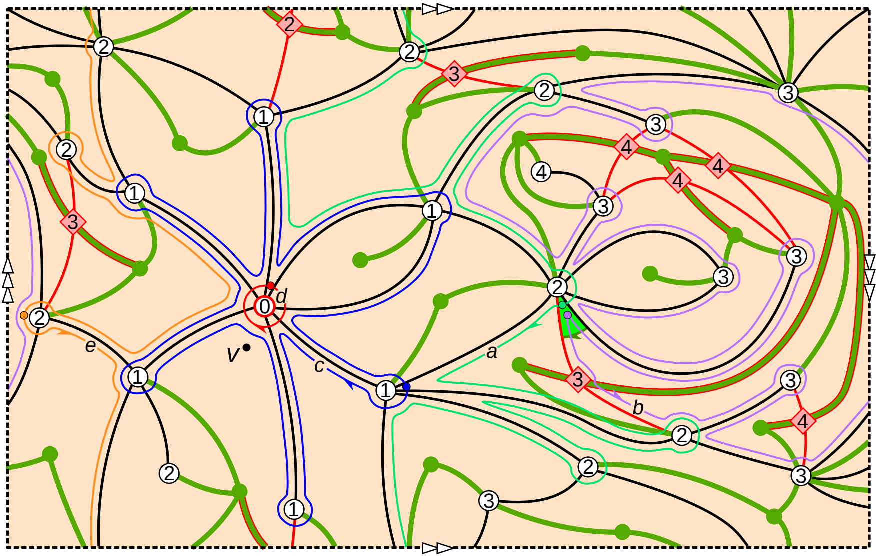

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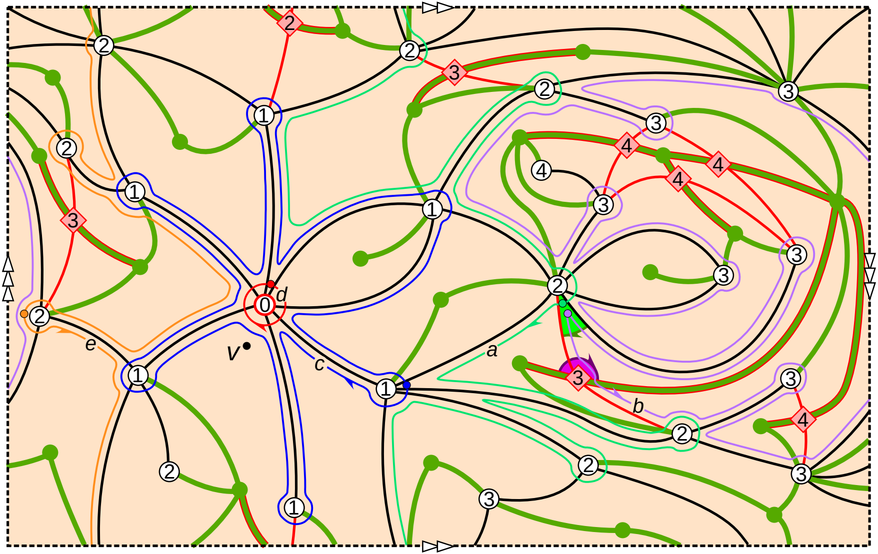

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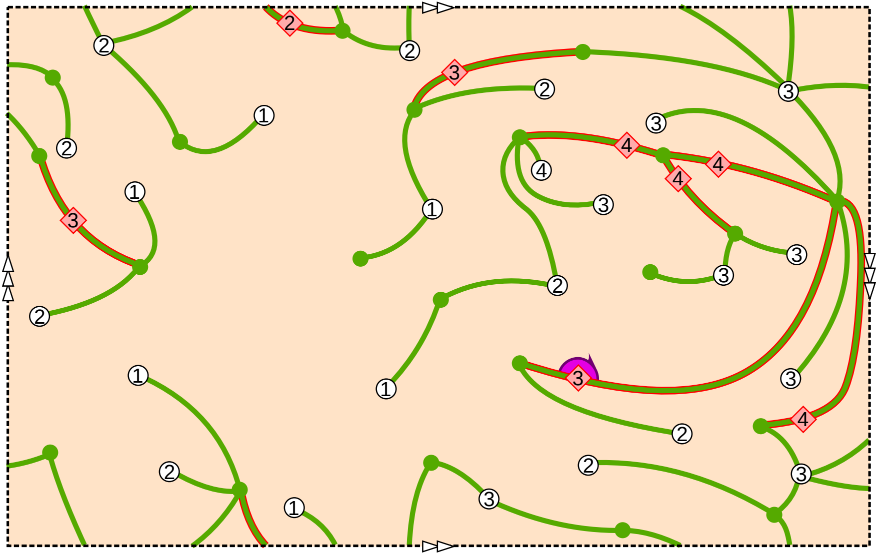

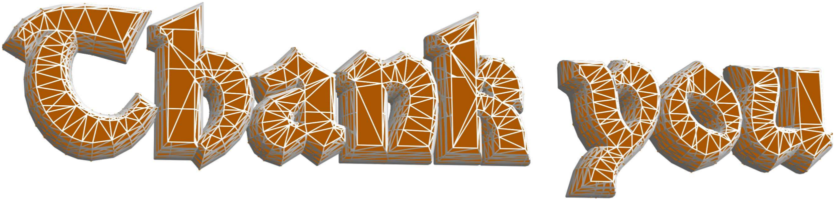
Thank you