SCD and SSI for NCP

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Chain Decor

Complex Reflect Groups Noncrossing

Partitions
SCD of

 ${}^{l\mathbb{C}}G(d_id_in)$ The Group  $G(d_id_in)$ A First

Decomposition A Second Decomposition

SSP of  $\mathcal{NC}_{\mathcal{V}}$ 

# Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

#### Henri Mühle

LIAFA (Université Paris Diderot)

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Journées du GT Combinatoire Algébrique du GDR IM

## Sperner's Theorem

•  $[n] = \{1, 2, ..., n\}$  for  $n \in \mathbb{N}$ 

• antichain: set of pairwise incomparable subsets of [n]

## Theorem (E. Sperner, 1928)

*The maximal size of an antichain of* [n] *is*  $\binom{n}{\lfloor \frac{n}{n} \rfloor}$ .

## Sperner's Theorem

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Complex Reflec Groups

Noncrossing Partitions

SCD of  $\mathcal{NC}_{G(d,d,n)}$ 

The Group G(d,d,
A First
Decomposition
A Second
Decomposition

SSP of  $\mathcal{NC}_{\mathsf{W}}$ 

• *k*-family: family of subsets of [*n*] that can be written as a union of at most *k* antichains

## Theorem (P. Erdős, 1945)

The maximal size of a k-family of [n] is the sum of the k largest binomial coefficients.

## A Generalization

SCD and SSP for NCP

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Groups
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Partitions

SCD OF  $NC_{G(d,d,n)}$ The Group G(d,d,n)A First Decomposition

Decomposition SSP of  $NC_W$ 

- poset perspective:
  - antichain of  $[n] \longleftrightarrow$  antichain in the Boolean lattice  $\mathcal{B}_n$
  - binomial coefficients  $\longleftrightarrow$  rank numbers of  $\mathcal{B}_n$

- $\mathcal{P}$  .. graded poset of rank n
- *k*-Sperner: size of a *k*-family does not exceed sum of *k* largest rank numbers
- **strongly Sperner**: k-Sperner for all  $k \le n$

## A Generalization

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A Second
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SSP of  $\mathcal{NC}_W$ 

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#### Motivation

Chain Decor positions

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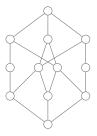
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The Group G(d,d,n)A First
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• a strongly Sperner poset



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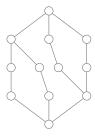
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SSP of  $NC_W$ 

• a Sperner poset that is not 2-Sperner



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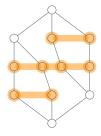
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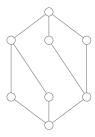
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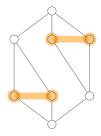
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Decomposition

SSP of  $\mathcal{NC}_{\mathcal{V}}$ 

#### • strongly Sperner posets:

- Boolean lattices
- divisor lattices
- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of  $H_3$
- non-Sperner posets:
  - lattices of set partitions
  - geometric lattices

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(symmetric chain decompositions)

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(of very large sets...)

geometric lattices

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- non-Sperner posets:
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(of very large sets...)

 geometric lattices (certain bond lattices of graphs)

#### Outline

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SSP of *NC* 

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2 Symmetric Chain Decompositions

Noncrossing Partition Lattices

- Complex Reflection Groups
- Noncrossing Partitions
- 4 Symmetric Chain Decompositions of  $\mathcal{NC}_{G(d,d,n)}$ 
  - The Group G(d, d, n)
  - A First Decomposition
  - A Second Decomposition
- Strong Sperner Property of  $\mathcal{NC}_W$

## Outline

SCD and SSP for NCP

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SSP of  $\mathcal{NC}_{W}$ 

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- Noncrossing Partition Lattices
  - Complex Reflection Groups
  - Noncrossing Partitions
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- lacksquare Strong Sperner Property of  $\mathcal{NC}_W$

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Noncrossing Partitions

SCD of NC G(d,d,n) The Group G(d,d,n). A First Decomposition A Second

SSP of  $NC_V$ 

- $\mathcal{P}$  .. graded poset of rank n
- **decomposition**: partition of  $\mathcal{P}$  into connected subposets



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Complex Reflecti Groups Noncrossing

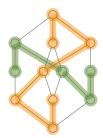
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SSP of NC

•  $\mathcal{P}$  .. graded poset of rank n

• symmetric decomposition: parts sit in  $\mathcal{P}$  symmetrically, i.e. match minimal and maximal elements so that ranks add up to n



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Chain Decompositions

Complex Reflect Groups Noncrossing

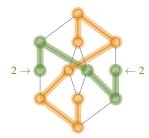
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•  $\mathcal{P}$  .. graded poset of rank n

• **symmetric decomposition**: parts sit in  $\mathcal{P}$  symmetrically, i.e. match minimal and maximal elements so that ranks add up to n



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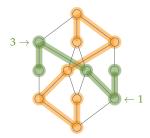
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Decomposition

SSP of  $\mathcal{NC}_1$ 

- $\mathcal{P}$  .. graded poset of rank n
- symmetric decomposition: parts sit in  $\mathcal{P}$  symmetrically, i.e. match minimal and maximal elements so that ranks add up to n



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Chain Decompositions

Complex Reflect Groups Noncrossing

Noncrossing Partitions

NCG(d,d,n)The Group G(d,d,n)A First
Decomposition

SSP of  $\mathcal{NC}_{1}$ 

•  $\mathcal{P}$  .. graded poset of rank n

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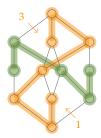
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SSP of NC

•  $\mathcal{P}$  .. graded poset of rank n

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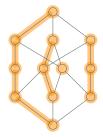
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 $\mathcal{NC}_{G(d,d,n)}$ The Group G(d,d,n)A First
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SSP of  $\mathcal{NC}_{1}$ 

- $\mathcal{P}$  .. graded poset of rank n
- symmetric chain decomposition: symmetric decomposition where parts are chains



•  $\mathcal{P}$  .. graded poset of rank n

#### Theorem

If P admits a symmetric chain decomposition, then P is strongly Sperner.

•  $\mathcal{P}$  .. graded poset of rank n

#### Theorem

If P and Q admit a symmetric chain decomposition, then so does  $\mathcal{P} \times \mathcal{Q}$ .

- $\mathcal{P}$  .. graded poset of rank n;  $N_i$  .. size of  $i^{th}$  rank
- rank-symmetric:  $N_i = N_{n-i}$
- rank-unimodal:  $N_0 \leq \cdots \leq N_i \geq \cdots \geq N_n$
- Peck: strongly Sperner, rank-symmetric, rank-unimodal

#### Theorem

If P admits a symmetric chain decomposition, then P is Peck.

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Symmetric Chain Decom positions

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SCD of  $^{NC}G(d_id_in)$ The Group  $G(d_id_in)$ A First Decomposition

Decomposition SSP of  $NC_W$ 

•  $\mathcal{P}$  .. graded poset of rank n;  $N_i$  .. size of  $i^{th}$  rank

• rank-symmetric:  $N_i = N_{n-i}$ 

• rank-unimodal:  $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$ 

 Peck: strongly Sperner, rank-symmetric, rank-unimodal

#### Theorem

*If*  $\mathcal{P}$  *and*  $\mathcal{Q}$  *are Peck, then so is*  $\mathcal{P} \times \mathcal{Q}$ .

## Outline

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Noncrossing Partition Lattices

- Complex Reflection Groups
- Noncrossing Partitions

Symmetric Chain Decompositions of  $\mathcal{NC}_{G(d,d,n)}$ 

- The Group G(d, d, n)
- A First Decomposition
- A Second Decomposition

lacktriangle Strong Sperner Property of  $\mathcal{NC}_W$ 

### Outline

SCD and SSP for NCP

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  - Complex Reflection Groups
  - Noncrossing Partitions
- 4 Symmetric Chain Decompositions of  $\mathcal{NC}_{G(d,d,n)}$ 
  - The Group G(d, d, n)
  - A First Decomposition
  - A Second Decomposition
- **(5)** Strong Sperner Property of  $\mathcal{NC}_W$

# Classification of Irreducible Complex Reflection Groups

SCD and SSP for NCP

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Symmetric Chain Decor

NCP Complex Reflection Groups

Noncrossing Partitions

 $NC_{G(d,d,n)}$ 

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SSP of NC 141

- one infinite family G(de, e, n):
  - monomial  $(n \times n)$ -matrices
  - non-zero entries are (de)<sup>th</sup> roots of unity
  - $\bullet$  product of non-zero entries is  $d^{th}$  root of unity
- 34 exceptional groups  $G_4, G_5, \ldots, G_{37}$

# Classification of Irreducible Complex Reflection Groups

SCD and SSF for NCP

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Motivation

Symmetric Chain Decom positions

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SCD of

The Group G(d,d,A First
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SSP of  $\mathcal{NC}_{W}$ 

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  - non-zero entries are  $(de)^{th}$  roots of unity
  - product of non-zero entries is *d*<sup>th</sup> root of unity
- 34 exceptional groups  $G_4, G_5, \ldots, G_{37}$
- well-generated complex reflection groups:
  - $G(1,1,n), n \ge 1$
  - $G(d,1,n), d \ge 2, n \ge 1$
  - $G(d,d,n),d,n \ge 2$
  - 26 exceptional groups

# Classification of Irreducible Complex Reflection Groups

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Motivation Symmetric

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Complex Reflection Groups
Noncrossing

SCD of  $NC_{G(d)}$ 

A First
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Decomposition

SSP of  $\mathcal{NC}_W$ 

- one infinite family G(de, e, n):
  - monomial  $(n \times n)$ -matrices
  - non-zero entries are  $(de)^{th}$  roots of unity
  - product of non-zero entries is *d*<sup>th</sup> root of unity
- 34 exceptional groups  $G_4, G_5, \ldots, G_{37}$
- finite Coxeter groups:

• 
$$G(1,1,n) \cong A_{n-1}$$

• 
$$G(2,1,n) \cong B_n$$

• 
$$G(2,2,n) \cong D_n$$

$$G(d,d,2) \cong I_2(d)$$

• 
$$G_{24} = H_3$$
,  $G_{28} = F_4$ ,  $G_{30} = H_4$ ,  $G_{35} = E_6$ ,  $G_{36} = E_7$ ,  $G_{37} = E_8$ 

#### Outline

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**(5)** Strong Sperner Property of  $\mathcal{NC}_W$ 

# **Noncrossing Partitions**

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A First
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SSP of  $\mathcal{NC}_V$ 

- *W* .. complex reflection group; *T* .. reflections of *W*; *c* .. Coxeter element
- absolute length:  $\ell_T(w) = \min\{k \mid w = t_1t_2\cdots t_k, t_i \in T\}$
- **absolute order**:  $u \leq_T v$  if and only if

$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$

• W-noncrossing partitions:

$$NC_W(c) = \{ w \in W \mid w \leq_T c \}$$

• write  $\mathcal{NC}_W(c) = (\mathcal{NC}_W(c), \leq_T)$ 

## Outline

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- **4** Symmetric Chain Decompositions of  $\mathcal{NC}_{G(d,d,n)}$ 
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  - A First Decomposition
  - A Second Decomposition
- **Strong Sperner Property of**  $\mathcal{NC}_{\mathcal{W}}$

# Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

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#### SCD of

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SSP of NCu

- $W = G(1,1,n) \cong \mathfrak{S}_n$ ; T .. transpositions;  $c = (1 \ 2 \ \dots \ n)$
- $\mathcal{NC}_{G(1,1,n)}(c)$  is isomorphic to the lattice of noncrossing set partitions of [n]
- $R_k = \{ w \in NC_{G(1,1,n)}(c) \mid w(1) = k \}, R_k = (R_k, \leq_T)$

# Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

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SSP of NC

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•  $R_k = \{ w \in NC_{G(1,1,n)}(c) \mid w(1) = k \}, \mathcal{R}_k = (R_k, \leq_T)$ 

⊎ .. disjoint set union; 2 .. 2-chain

#### Lemma (R. Simion & D. Ullmann, 1991)

We have  $\mathcal{R}_1 \uplus \mathcal{R}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-1)}$ , and  $\mathcal{R}_i \cong \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i+1)}$  whenever  $3 \leq i \leq n$ . Moreover, this decomposition is symmetric.

# Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

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ullet .. disjoint set union;  ${f 2}$  .. 2-chain

#### Theorem (R. Simion & D. Ullmann, 1991)

The lattice  $NC_{G(1,1,n)}$  admits a symmetric chain decomposition for each  $n \geq 1$ .

# Example: $\mathcal{NC}_{\mathfrak{S}_4}((1\ 2\ 3\ 4))$

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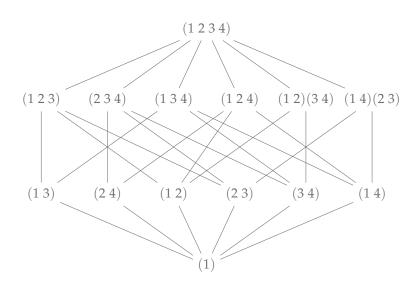
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# Example: $\mathcal{NC}_{\mathfrak{S}_4}((1\ 2\ 3\ 4))$

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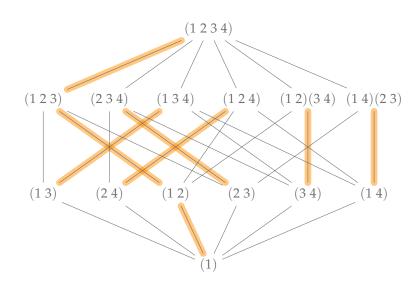
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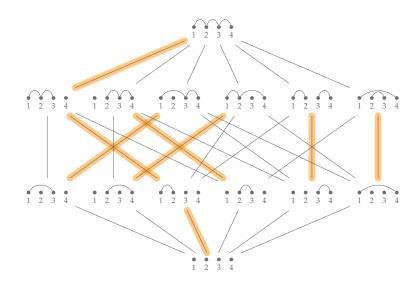
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**4** Symmetric Chain Decompositions of  $\mathcal{NC}_{G(d,d,n)}$ 

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**5** Strong Sperner Property of  $\mathcal{NC}_W$ 

# The Groups G(d, d, n), $d, n \ge 2$

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• subgroups of  $\mathfrak{S}_{dn}$ , permuting elements of

$$\left\{1^{(0)}, \dots, n^{(0)}, 1^{(1)}, \dots, n^{(1)}, \dots, 1^{(d-1)}, \dots, n^{(d-1)}\right\}$$

- $w \in G(d, d, n)$  satisfies  $w(k^{(s)}) = \pi(k)^{(s+t_k)}$ 
  - $\bullet \ \sum_{k=1}^n t_k \equiv 0 \ (\bmod \ d)$
  - $\pi \in \mathfrak{S}_n$ , and  $t_k$  depends on w and k

# The Lattices $\mathcal{NC}_{G(d,d,n)}$ , $d,n \geq 2$

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#### Coxeter element

$$c = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \zeta_d & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \zeta_d^{d-1} \end{pmatrix},$$

where 
$$\zeta = e^{2\pi\sqrt{-1}/d}$$

#### Example: *d*= 5,*n*= 3

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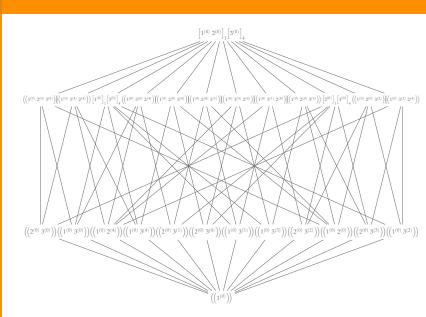
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- **4** Symmetric Chain Decompositions of  $\mathcal{NC}_{G(d,d,n)}$ 
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SSP of NCW

$$\bullet \ R_k^{(s)} = \left\{ w \in NC_{G(d,d,n)}(c) \mid w(1^{(0)}) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

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SSP of  $\mathcal{NC}_{W}$ 

$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

#### Lemma (\*\*, 2015)

The sets  $R_1^{(s)}$  and  $R_k^{(s')}$  are empty for  $2 \le s < d$  as well as  $2 \le k < n$  and  $1 \le s' < d - 1$ .

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SSP of  $\mathcal{NC}_{\mathsf{W}}$ 

$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

#### Lemma (\*\*, 2015)

The poset  $\mathcal{R}_1^{(0)} \uplus \mathcal{R}_2^{(0)}$  is isomorphic to  $\mathbf{2} \times \mathcal{NC}_{G(d,d,n-1)}$ . Moreover, its least element has length 0, and its greatest element has length n.

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SSP of  $\mathcal{NC}_{\mathcal{W}}$ 

 $\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$ 

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

## Lemma (\*\*, 2015)

The poset  $\mathcal{R}_n^{(s)}$  is isomorphic to  $\mathcal{NC}_{G(1,1,n-1)}$  for  $0 \leq s < d$ . Moreover, its least element has length 1, and its greatest element has length n-1.

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SSP of  $\mathcal{NC}_W$ 

 $\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$ 

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

## Lemma (%, 2015)

The poset  $\mathcal{R}_i^{(0)}$  is isomorphic to  $\mathcal{NC}_{G(d,d,n-i+1)} \times \mathcal{NC}_{G(1,1,i-2)}$  whenever  $3 \leq i < n$ . Moreover, its least element has length 1, and its greatest element has length n-1.

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 $\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$ 

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

## Lemma (\*\*, 2015)

The poset  $\mathcal{R}_i^{(d-1)}$  is isomorphic to  $\mathcal{NC}_{G(1,1,n-i)} \times \mathcal{NC}_{G(d,d,i-1)}$  whenever  $3 \leq i < n$ . Moreover, its least element has length 1, and its greatest element has length n-1.

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$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

### Lemma (\*\*, 2015)

The poset  $\mathcal{R}_1^{(1)}$  is isomorphic to  $\mathcal{NC}_{G(1,1,n-2)}$ . Moreover, its least element has length 2, and its greatest element has length n-1.

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SSP of  $\mathcal{NC}_{\mathcal{W}}$ 

$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

## Lemma (\*\*, 2015)

The poset  $\mathcal{R}_2^{(d-1)}$  is isomorphic to  $\mathcal{NC}_{G(1,1,n-2)}$ . Moreover, its least element has length 1, and its greatest element has length n-2.

#### Example: *d*= 5,*n*= 3

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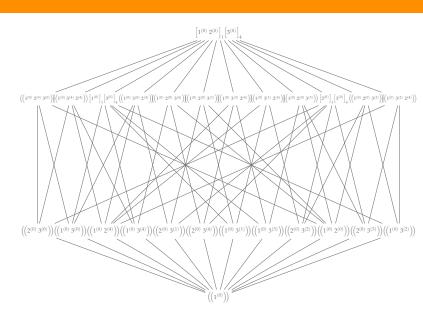
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### Example: *d*= 5,*n*= 3

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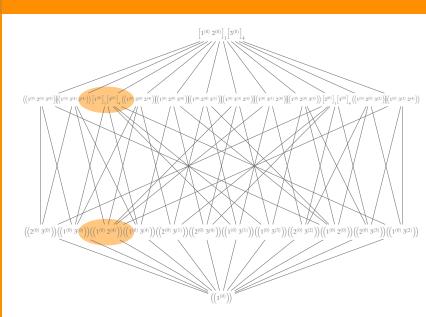
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  - Strong Sperner Property of  $\mathcal{NC}_W$

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ullet bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to NC_{G(d,d,n)}(c), \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)}\right)\right) x$$

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- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 1^{(0)} \ n^{(d-2)} \right) \right) x$$

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SSP of  $NC_V$ 

• bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

• consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 1^{(0)} \ n^{(d-2)} \right) \right) x$$

- this map is an injective involution
- its image consists of permutations  $w \in R_n^{(d-1)}$  with  $w\left(n^{(d-1)}\right) = 1^{(0)}$

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SSP of  $NC_V$ 

- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 1^{(0)} \ n^{(d-2)} \right) \right) x$$

- this map is an injective involution
- its image is the interval

$$\left[\left(\left(1^{(0)} n^{(d-1)}\right)\right), \left(\left(1^{(0)} n^{(d-1)}\right)\right) \left(\left(2^{(0)} \dots (n-1)^{(0)}\right)\right)\right]_{T}$$

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SSP of  $\mathcal{NC}_{W}$ 

• bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

• consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 1^{(0)} \ n^{(d-2)} \right) \right) x$$

#### Lemma (¥, 2015)

The interval 
$$(f_1(R_1^{(1)}), \leq_T)$$
 is isomorphic to  $\mathcal{NC}_{G(1,1,n-2)}$ .

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SSP of  $NC_{W}$ 

- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 1^{(0)} \ n^{(d-2)} \right) \right) x$$

• define  $D_1 = R_1^{(1)} \uplus f_1(R_1^{(1)})$ , and  $\mathcal{D}_1 = (D_1, \leq_T)$ 

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SSP of  $\mathcal{NC}_{V}$ 

• bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

• consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 1^{(0)} \ n^{(d-2)} \right) \right) x$$

• define  $D_1 = R_1^{(1)} \uplus f_1(R_1^{(1)})$ , and  $\mathcal{D}_1 = (D_1, \leq_T)$ 

#### Lemma (¥, 2015)

The poset  $\mathcal{D}_1$  is isomorphic to  $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$ . Moreover, its least element has length 1, and its greatest element has length n-1.

- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to NC_{G(d,d,n)}(c), \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)}\right)\right) x$$

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- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 2^{(0)} \ n^{(0)} \right) \right) x$$

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- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(\hat{d}-1)}, \quad x \mapsto \left( \left( 2^{(0)} \ n^{(0)} \right) \right) x$$

- this map is an injective involution
- its image consists of permutations  $w \in R_n^{(d-1)}$  with  $w\left(n^{(d-1)}\right) = 2^{(d-1)}$

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SSP of  $\mathcal{NC}_{\mathsf{W}}$ 

• bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 2^{(0)} \ n^{(0)} \right) \right) x$$

- this map is an injective involution
- its image is the interval

$$\left[ \left( \left( 1^{(0)} \ n^{(d-1)} \ 2^{(d-1)} \right) \right), \left( \left( 1^{(0)} \ n^{(d-1)} \ 2^{(d-1)} \ \dots \ (n-1)^{(d-1)} \right) \right) \right]_T$$

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SSP of  $\mathcal{NC}_W$ 

• bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 2^{(0)} \ n^{(0)} \right) \right) x$$

#### Lemma (¥, 2015)

The interval 
$$(f_2(R_2^{(d-1)}), \leq_T)$$
 is isomorphic to  $\mathcal{NC}_{G(1,1,n-2)}$ .

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- bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 2^{(0)} \ n^{(0)} \right) \right) x$$

• define 
$$D_2 = R_2^{(d-1)} \uplus f_2(R_2^{(d-1)})$$
, and  $\mathcal{D}_2 = (D_2, \leq_T)$ 

# A Second Decomposition

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SSP of  $\mathcal{NC}_1$ 

• bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left( \left( 2^{(0)} \ n^{(0)} \right) \right) x$$

• define 
$$D_2 = R_2^{(d-1)} \uplus f_2(R_2^{(d-1)})$$
, and  $\mathcal{D}_2 = (D_2, \leq_T)$ 

#### Lemma (¥, 2015)

The poset  $\mathcal{D}_2$  is isomorphic to  $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$ . Moreover, its least element has length 1, and its greatest element has length n-1.

# A Second Decomposition

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SSP of  $\mathcal{NC}_W$ 

• bad parts:  $R_1^{(1)}$  and  $R_2^{(d-1)}$ 

• define 
$$D = R_n^{(d-1)} \setminus \left( f_1\left(R_1^{(1)}\right) \uplus f_2\left(R_2^{(d-1)}\right) \right)$$
, and  $\mathcal{D} = (D, \leq_T)$ 

### Lemma (¥, 2015)

The poset  $\mathcal{D}$  is isomorphic to  $\biguplus_{i=3}^{n-1} \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i)}$ . Morever, its minimal elements have length 2, and its maximal elements have length n-2.

#### The Main Result

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SSP of NC

#### Theorem (¥, 2015)

For  $d, n \geq 2$  the lattice  $NC_{G(d,d,n)}$  admits a symmetric chain decomposition. Consequently, it is Peck.

### Example: *d*= 5,*n*= 3

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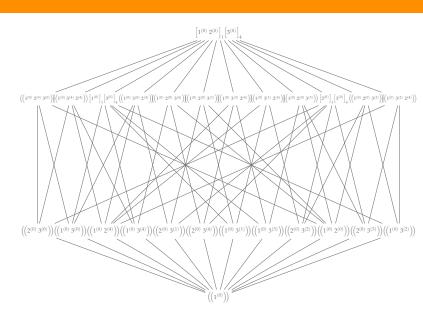
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#### Example: *d*= 5,*n*= 3

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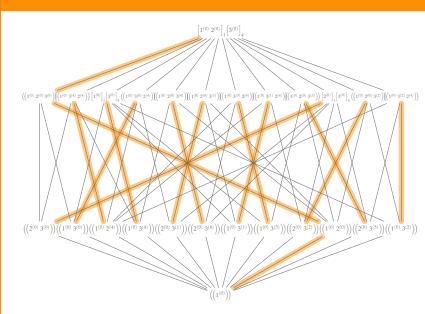
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- so far:  $\mathcal{NC}_{G(1,1,n)}$  and  $\mathcal{NC}_{G(d,d,n)}$  admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?

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• so far:  $\mathcal{NC}_{G(1,1,n)}$  and  $\mathcal{NC}_{G(d,d,n)}$  admit symmetric chain decompositions

what about the other well-generated complex reflection groups?

#### Theorem (V. Reiner, 1997)

The lattice  $NC_{G(2,1,n)}$  admits a symmetric chain decomposition for any  $n \ge 1$ .

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- so far:  $\mathcal{NC}_{G(1,1,n)}$  and  $\mathcal{NC}_{G(d,d,n)}$  admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?
- we have  $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$  for  $d \geq 2$  and  $n \geq 1$

#### Theorem (V. Reiner, 1997)

The lattice  $NC_{G(2,1,n)}$  admits a symmetric chain decomposition for any  $n \ge 1$ .

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- SSP of  $\mathcal{NC}_W$

- ullet so far:  $\mathcal{NC}_{G(1,1,n)}$  and  $\mathcal{NC}_{G(d,d,n)}$  admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?
- we have  $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$  for  $d \geq 2$  and  $n \geq 1$
- only the 26 exceptional groups remain

#### Theorem (V. Reiner, 1997)

The lattice  $NC_{G(2,1,n)}$  admits a symmetric chain decomposition for any  $n \ge 1$ .

SCD and SSP for NCP

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Motivation

Chain Decom

Complex Refle

Noncrossing Partitions

SCD of

NCG(d,d,n)The Group G(d,d,n)A First
Decomposition
A Second

SSP of MCw

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed

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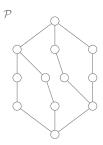
Groups
Noncrossing
Partitions

SCD of

 $\mathcal{NC}_G(d,d,n)$ The Group G(d,d,n)A First Decomposition

SSP of  $NC_W$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed



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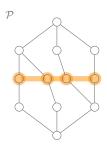
SCD of

The Group G(d,d,n)A First
Decomposition

A Second

SSP of  $\mathcal{NC}_W$ 

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- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed



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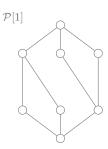
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The Group G(d, d, n)A First
Decomposition

Decomposition

SSP of  $\mathcal{NC}_W$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed



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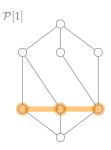
SCD of

 $\mathcal{NC}_{G(d,d,n)}$ The Group G(d,d,n)A First

A Second Decompositio

SSP of  $NC_W$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed



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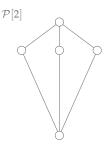
 $\mathcal{NC}_{G(d,d,n)}$ The Group G(d,d,n)

A Second
Decomposition

SSP of  $NC_W$ 

•  $\mathcal{P}$  .. graded poset of rank n

ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed



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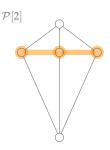
Partitions
CCD of

 $\mathcal{NC}_{G(d,d,n)}$ The Group G(d,d,n)A First

A Second Decomposition

SSP of  $NC_W$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed



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Symmetric
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NCF

Complex Reflections

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SCD of

NCG(d,d,n)The Group G(d,d,n)A First

A Second
Decomposition

SSP of  $NC_{IA}$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed

$$\mathcal{P}[3]$$

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Chain Decompositions

NCF

Complex Reflect Groups Noncrossing

Partitions

SCD of

The Group G(d,d,z)A First

A Second Decomposition

SSP of  $NC_N$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed

 $\mathcal{P}[3]$ 



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 $NC_{G(d,d,n)}$ 

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SSP of MCw

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed

 $\mathcal{P}[4]$ 

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The Group G(d, d)
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A Second
Decomposition

SSP of  $\mathcal{NC}_W$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed

#### Proposition (%, 2015)

A graded poset P of rank n is strongly Sperner if and only if P[i] is Sperner for all  $i \in \{0, 1, ..., n\}$ .

• antichains in  $\mathcal{P}[i]$  are antichains in  $\mathcal{P}[s]$  for s < i

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Symmetric Chain Decom positions

NCP Complex Reflection Groups Noncrossing Partitions

 $VC_{G(d,d,n)}$ The Group G(d,d)A First
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Decomposition

SSP of  $\mathcal{NC}_W$ 

- $\mathcal{P}$  .. graded poset of rank n
- ullet  $\mathcal{P}[i]$  .. subposet of  $\mathcal{P}$  with i largest ranks removed

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SCD of

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Decomposition

SSP of NCw

 SAGE has a fast implementation to compute the size of the largest antichain of a poset

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Partitions

Partitions

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SSP of NCw

 SAGE has a fast implementation to compute the width of a poset

• SAGE has a fast implementation to compute the width of a poset

#### Theorem (\*\*, 2015)

The lattice  $NC_W$  is Peck for any well-generated exceptional complex reflection group W.

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CCD of  $NC_{G(d,d,n)}$ The Group G(d,d,n)A First Decomposition
A Second Decomposition

SSP of  $\mathcal{NC}_{\mathcal{N}}$ 

 SAGE has a fast implementation to compute the width of a poset

### Theorem (\*\*, 2015)

The lattice  $NC_W$  is Peck for any well-generated complex reflection group W.

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 $NC_{G(d,d,n)}$ The Group G(d,d,n)

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SSP of  $NC_{W}$ 

• *W* .. well-generated complex reflection group; *c* .. Coxeter element of *W* 

• *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions  $\rightsquigarrow NC_W^{(m)}(c)$ 

$$(w)_m = (w_1, w_2, \dots, w_m)$$
 with  $w_1 \leq_T w_2 \leq_T \dots \leq_T w_m \leq_T c$ 

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SCD of  $NC_{G(d,d,n)}$ The Group G(d,d,n)A First

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SSP of  $\mathcal{NC}_{W}$ 

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- *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions  $\rightsquigarrow NC_W^{(m)}(c)$
- *m*-delta sequence: sequence of "differences" of elements in a multichain

$$(w)_m = (w_1, w_2, \dots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \dots \leq_T w_m \leq_T c$$
  
$$\partial(w)_m = [w_1; w_1^{-1} w_2, w_2^{-1} w_3, \dots, w_{m-1}^{-1} w_m, w_m^{-1} c]$$

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- W .. well-generated complex reflection group; c ..
   Coxeter element of W
- *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions  $\rightsquigarrow NC_W^{(m)}(c)$
- *m*-delta sequence: sequence of "differences" of elements in a multichain
- partial order:  $(u)_m \le (v)_m$  if and only if  $\partial(u)_m \le_T \partial(v)_m \qquad \rightsquigarrow \mathcal{NC}_W^{(m)}(c)$

#### Question (D. Armstrong, 2009)

Are the posets  $\mathcal{NC}_{W}^{(m)}$  strongly Sperner for any W and any m > 1?

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SSP of  $\mathcal{NC}_{W}$ 

• affirmative answer for m = 1

#### Question (D. Armstrong, 2009)

Are the posets  $NC_W^{(m)}$  strongly Sperner for any W and any m > 1?

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 $NC_{G(d,d,n)}$ The Group G(d,d,n)A First

SSP of  $\mathcal{NC}_{W}$ 

- affirmative answer for m = 1
- what about m > 1?
  - $\mathcal{NC}_W^{(m)}$  is antiisomorphic to an order ideal in  $(\mathcal{NC}_W)^m$
  - $(\mathcal{NC}_W)^m$  is Peck
  - $\mathcal{NC}_W^{(m)}$  is not rank-symmetric  $\leadsto$  no symmetric chain decomposition

#### Question (D. Armstrong, 2009)

Are the posets  $NC_W^{(m)}$  strongly Sperner for any W and any m > 1?

# Example: $\mathcal{NC}_{\mathfrak{S}_4}^{(2)}$

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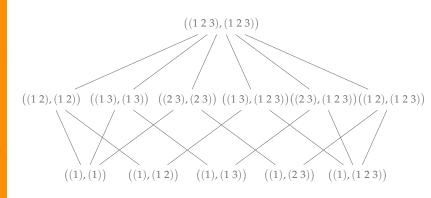
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Noncrossing Partitions

SCD of  $NC_{G(d,d,n)}$ The Group G(d,d,n)

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SSP of MC.



# Example: $\mathcal{NC}_{\mathfrak{S}_4}^{(2)}$

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positions

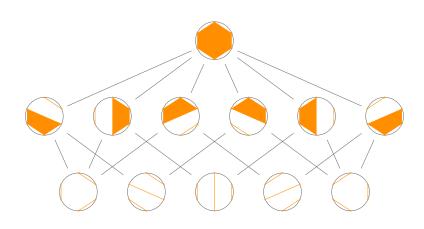
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Decomposition

SSP of NC IN



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Complex Reflection Groups

Noncrossing Partitions

SCD of

 $NC_{G(d,d,n)}$ The Group G(d,d,n)

A First
Decomposition

SSP of MC.

Thank You.

#### Interlude: A Convolution Formula

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$$NC_{G(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \biguplus_{i=2}^{n-1} \left( R_i^{(0)} \uplus R_i^{(d-1)} \right) \uplus \biguplus_{s=0}^{d-1} R_n^{(s)}$$

#### Interlude: A Convolution Formula

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#### Proposition (%, 2015)

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$NC_{G(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \biguplus_{i=2}^{n-1} \left( R_i^{(0)} \uplus R_i^{(d-1)} \right) \uplus \biguplus_{s=0}^{d-1} R_n^{(s)}$$

SCD and SSF for NCP

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= 2 \cdot \mathsf{Cat}_{G(d,d,n+1)} + 2 \cdot \mathsf{Cat}_{G(1,1,n)} + d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \sum_{i=3}^{n+1} \mathsf{Cat}_{G(d,d,n-i+3)} \mathsf{Cat}_{G(1,1,i-2)} \end{aligned}$$

SCD and SSF for NCP

For 
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 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=2}^{n+2} \mathsf{Cat}_{G(d,d,n-i+3)} \cdot \mathsf{Cat}_{G(1,1,i-2)} \end{aligned}$$

SCD and SSP for NCP

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$$\begin{split} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^n \mathsf{Cat}_{G(d,d,n-i+1)} \cdot \mathsf{Cat}_{G(1,1,i)} \end{split}$$

SCD and SSP for NCP

For 
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 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

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SCD and SSF for NCP

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$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^{n} \mathsf{Cat}_{G(d,d,i+1)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \\ \mathsf{Cat}_{G(d,d,n+2)} &= \left( \prod_{i=1}^{n+1} \frac{di + (n-1)d}{di} \right) \frac{n + (n-1)d}{n} \end{aligned}$$

SCD and SSP for NCP

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$\begin{split} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^{n} \mathsf{Cat}_{G(d,d,i+1)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \\ \mathsf{Cat}_{G(d,d,n+2)} &= \Big( (n+1)d + n + 2 \Big) \cdot \mathsf{Cat}_{G(1,1,n+1)} \end{split}$$

SCD and SSF for NCP

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^{n} \mathsf{Cat}_{G(d,d,i+1)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \\ \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ \left( nd + n + 2 \right) \cdot \mathsf{Cat}_{G(1,1,n+1)} \end{aligned}$$

SCD and SSP for NCP

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$nd \cdot \operatorname{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = 2 \cdot \sum_{i=0}^{n} (id+i+1) \cdot \operatorname{Cat}_{G(1,1,i)} \cdot \operatorname{Cat}_{G(1,1,n-i)}$$

SCD and SSF for NCP

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$\begin{split} nd \cdot \mathsf{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} &= \\ 2 \cdot \sum_{i=0}^n \left( id \cdot \mathsf{Cat}_{G(1,1,i)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \right) + 2 \cdot \sum_{i=0}^n \binom{2i}{i} \cdot \mathsf{Cat}_{G(1,1,n-i)} \end{split}$$

SCD and SSF for NCP

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$nd \cdot \operatorname{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = \\ 2d \cdot \sum_{i=0}^{n} \left( i \cdot \operatorname{Cat}_{G(1,1,i)} \cdot \operatorname{Cat}_{G(1,1,n-i)} \right) + \binom{2(n+1)}{n+1}$$

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For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$\frac{n}{2} \cdot \text{Cat}_{G(1,1,n+1)} = \sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)}$$

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For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$ .

$$\binom{2n+1}{n-1} = \sum_{i=0}^{n} i \cdot \operatorname{Cat}_{G(1,1,i)} \cdot \operatorname{Cat}_{G(1,1,n-i)}$$

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#### Proposition (Y. Kong, 2000)

For 
$$n \ge 0$$
 we have  $\sum_{i=0}^{n-1} Cat_{G(1,1,i)} \cdot {2(n-i) \choose n-i-1} = {2n+1 \choose n-1}$ .