Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

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Sperner's Theorem

- $[n] = \{1,2,\ldots,n\}$ for $n \in \mathbb{N}$
- **antichain**: set of pairwise incomparable subsets of $[n]$

Theorem (E. Sperner, 1928)

The maximal size of an antichain of $[n]$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Sperner's Theorem

- ***k*-family**: family of subsets of $[n]$ that can be written as a union of at most k antichains

Theorem (P. Erdős, 1945)

The maximal size of a k-family of $[n]$ is the sum of the k largest binomial coefficients.

A Generalization

- poset perspective:
  - antichain of $[n] \longleftrightarrow$ antichain in the Boolean lattice $\mathcal{B}_n$
  - binomial coefficients $\longleftrightarrow$ rank numbers of $\mathcal{B}_n$

- $\mathcal{P}$ .. graded poset of rank n
- ***k*-Sperner**: size of a k-family does not exceed sum of k largest rank numbers
- **strongly Sperner**: k-Sperner for all $k \leq n$

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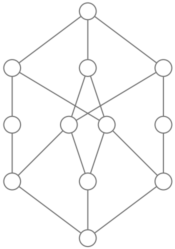
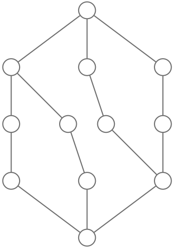
Examples

- a strongly Sperner poset



Examples

- a Sperner poset that is not 2-Sperner



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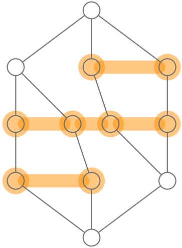

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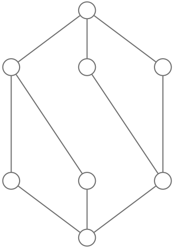

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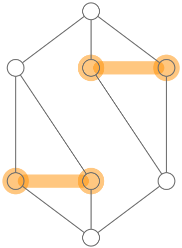

Examples

- strongly Sperner posets:
  - Boolean lattices
  - divisor lattices
  - lattices of noncrossing set partitions
  - Bruhat posets of finite Coxeter groups
  - weak order lattice of H_3
- non-Sperner posets:
  - lattices of set partitions
  - geometric lattices

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- non-Sperner posets:
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  - geometric lattices (certain bond lattices of graphs)

Outline

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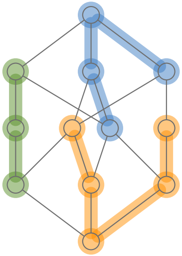
Poset Decompositions

- $\mathcal{P}$ .. graded poset of rank n
- **decomposition**: partition of $\mathcal{P}$ into connected subposets



Poset Decompositions

- $\mathcal{P}$ .. graded poset of rank n
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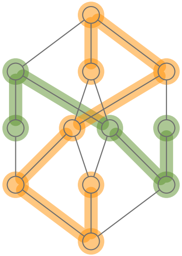

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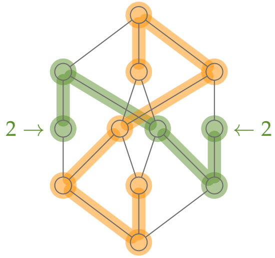

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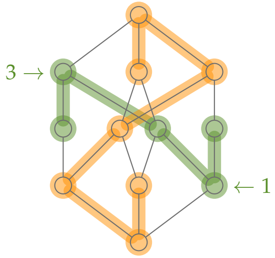

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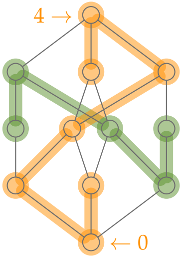

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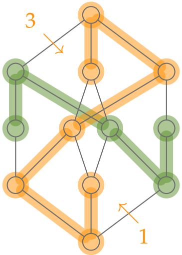
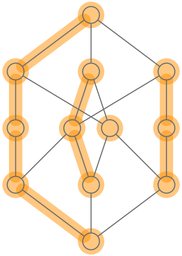


Poset Decompositions

- $\mathcal{P}$ .. graded poset of rank n
- **symmetric chain decomposition**: symmetric decomposition where parts are chains



Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank n

Theorem

If $\mathcal{P}$ admits a symmetric chain decomposition, then $\mathcal{P}$ is strongly Sperner.

Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank n

Theorem

If $\mathcal{P}$ and $\mathcal{Q}$ admit a symmetric chain decomposition, then so does $\mathcal{P} \times \mathcal{Q}$.

Symmetric Chain Decompositions

- $\mathcal{P}$ .. graded poset of rank n; N_i .. size of i^{th} rank
- **rank-symmetric**: $N_i = N_{n-i}$
- **rank-unimodal**: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$
- **Peck**: strongly Sperner, rank-symmetric, rank-unimodal

Theorem

If $\mathcal{P}$ admits a symmetric chain decomposition, then $\mathcal{P}$ is Peck.

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Outline

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Classification of Irreducible Complex Reflection Groups

- one infinite family $G(de,e,n)$:
  - monomial $(n\times n)$-matrices
  - non-zero entries are $(de)^{\text{th}}$ roots of unity
  - product of non-zero entries is d^{th} root of unity
- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$

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- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$
- well-generated complex reflection groups:
  - $G(1,1,n), n \geq 1$
  - $G(d,1,n), d \geq 2, n \geq 1$
  - $G(d,d,n), d,n \geq 2$
  - 26 exceptional groups

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- 34 exceptional groups $G_4, G_5, \ldots, G_{37}$

- finite Coxeter groups:
  - $G(1,1,n) \cong A_{n-1}$
  - $G(2,1,n) \cong B_n$
  - $G(2,2,n) \cong D_n$
  - $G(d,d,2) \cong I_2(d)$
  - $G_{24} = H_3, G_{28} = F_4, G_{30} = H_4, G_{35} = E_6, G_{36} = E_7, G_{37} = E_8$

Outline

Noncrossing Partitions

- W .. complex reflection group; T .. reflections of W; c .. Coxeter element
- **absolute length**: $\ell_T(w) = \min\{k \mid w = t_1 t_2 \cdots t_k, t_i \in T\}$
- **absolute order**: $u \leq_T v$ if and only if
$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$
- ***W*-noncrossing partitions**:
$$NC_W(c) = \{w \in W \mid w \leq_T c\}$$
- write $\mathcal{NC}_W(c) = \big(NC_W(c), \leq_T\big)$

Outline

Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

- $W = G(1,1,n) \cong \mathfrak{S}_n$; T .. transpositions; $c = (1\ 2\ \ldots\ n)$
- $\mathcal{NC}_{G(1,1,n)}(c)$ is isomorphic to the lattice of noncrossing set partitions of $[n]$
- $R_k = \{w \in NC_{G(1,1,n)}(c) \mid w(1) = k\}$, $\mathcal{R}_k = (R_k, \leq_T)$

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- $\uplus$ .. disjoint set union; $\mathbf{2}$ .. 2-chain

Lemma (R. Simion & D. Ullmann, 1991)

We have $\mathcal{R}_1 \uplus \mathcal{R}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-1)}$, *and* $\mathcal{R}_i \cong \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i+1)}$ *whenever* $3 \leq i \leq n$. *Moreover, this decomposition is symmetric.*

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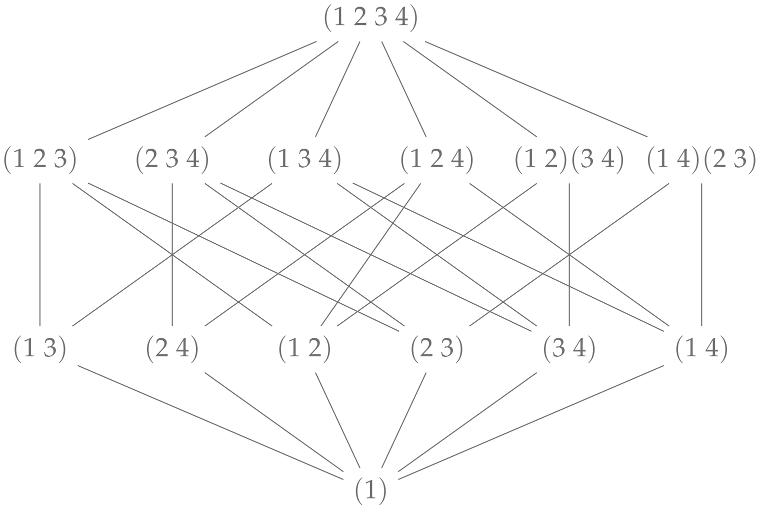
- $\uplus$ .. disjoint set union; **2** .. 2-chain

Theorem (R. Simion & D. Ullmann, 1991)

The lattice $\mathcal{NC}_{G(1,1,n)}$ admits a symmetric chain decomposition for each $n \geq 1$.

Example: $\mathcal{NC}_{\mathfrak{S}_4}\big((1\,2\,3\,4)\big)$



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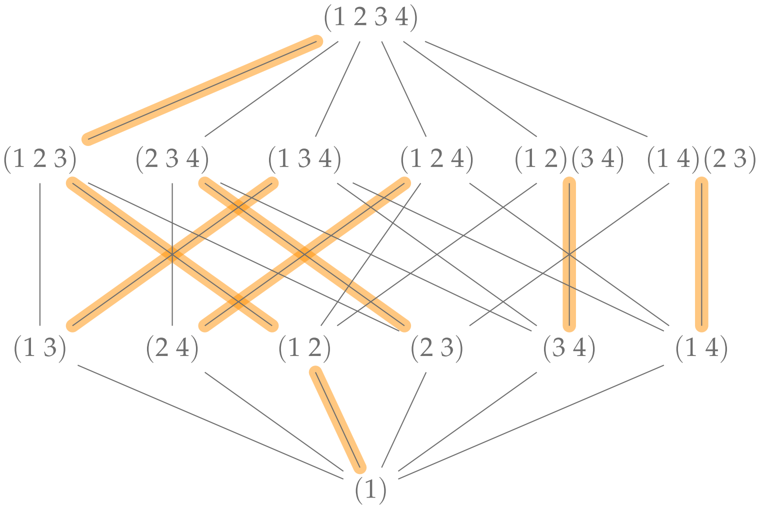

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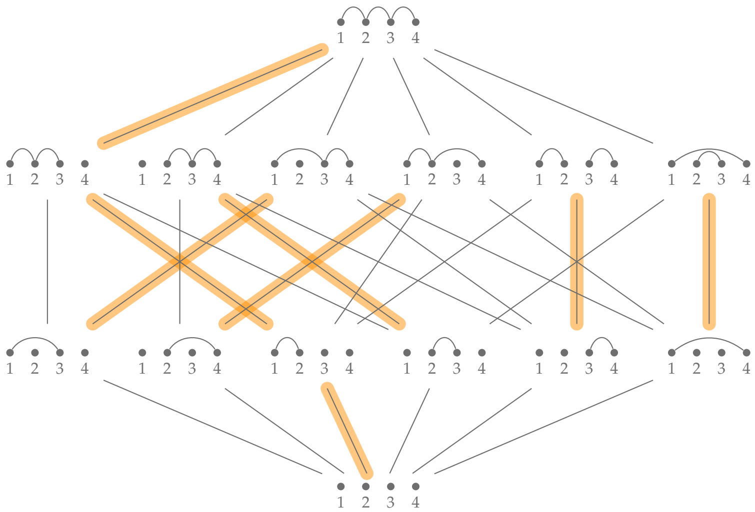


Outline

The Groups $G(d,d,n), d,n \geq 2$

- subgroups of $\mathfrak{S}_{dn}$, permuting elements of

$$\left\{1^{(0)},\ldots,n^{(0)},1^{(1)},\ldots,n^{(1)},\ldots,1^{(d-1)},\ldots,n^{(d-1)}\right\}$$

- $w \in G(d,d,n)$ satisfies $w\left(k^{(s)}\right) = \pi(k)^{(s+t_k)}$
  - $\sum_{k=1}^{n} t_k \equiv 0 \pmod{d}$
  - $\pi \in \mathfrak{S}_n$, and t_k depends on w and k

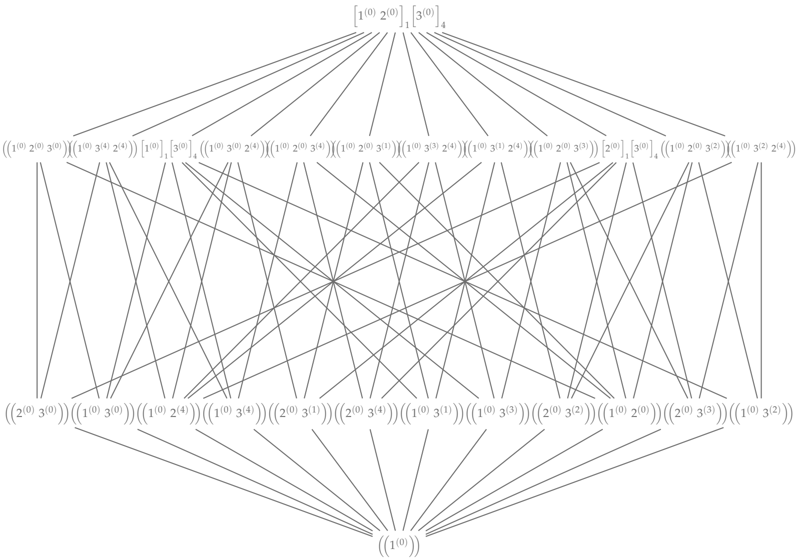
The Lattices $\mathcal{NC}_{G(d,d,n)}, d, n \geq 2$

- Coxeter element

$$c = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \zeta_d & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \zeta_d^{d-1} \end{pmatrix},$$

where $\zeta = e^{2\pi\sqrt{-1}/d}$

Example: $d = 5, n = 3$



Outline

A First Decomposition

- $R_k^{(s)} = \left\{ w \in NC_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$
- $\mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T \right)$

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Lemma (, 2015)

The sets $R_1^{(s)}$ and $R_k^{(s')}$ are empty for $2 \leq s < d$ as well as $2 \leq k < n$ and $1 \leq s' < d-1$.

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Lemma (, 2015)

The poset $\mathcal{R}_1^{(0)} \uplus \mathcal{R}_2^{(0)}$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(d,d,n-1)}$. Moreover, its least element has length 0, and its greatest element has length n.

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Lemma (, 2015)

The poset $\mathcal{R}_n^{(s)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-1)}$ for $0 \leq s < d$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A First Decomposition

- $R_k^{(s)} = \Bigl\{ w \in NC_{G(d,d,n)}(c) \mid w\bigl(1^{(0)}\bigr) = k^{(s)} \Bigr\}$
- $\mathcal{R}_k^{(s)} = \bigl(R_k^{(s)}, \leq_T \bigr)$

Lemma (, 2015)

The poset $\mathcal{R}_i^{(0)}$ is isomorphic to $\mathcal{NC}_{G(d,d,n-i+1)} \times \mathcal{NC}_{G(1,1,i-2)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

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Lemma (, 2015)

The poset $\mathcal{R}_i^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-i)} \times \mathcal{NC}_{G(d,d,i-1)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

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Lemma (, 2015)

The poset $\mathcal{R}_1^{(1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 2, and its greatest element has length $n-1$.

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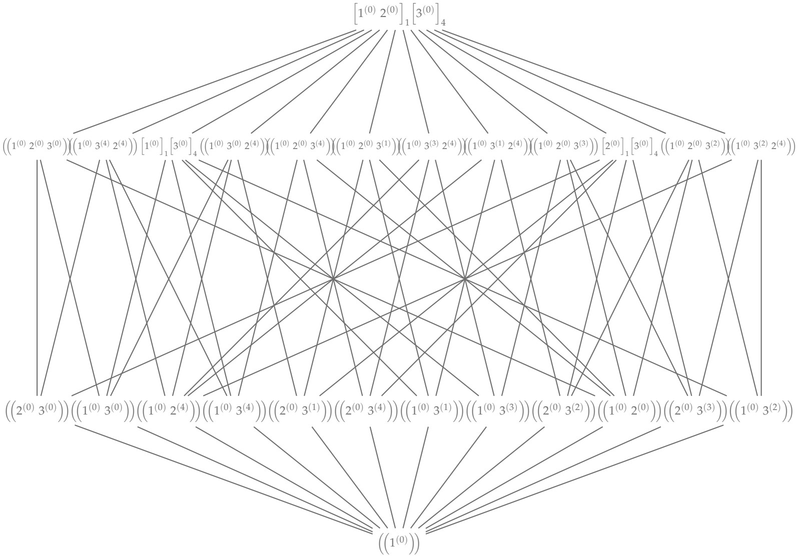
Lemma (, 2015)

The poset $\mathcal{R}_2^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-2$.

Example: $d = 5, n = 3$



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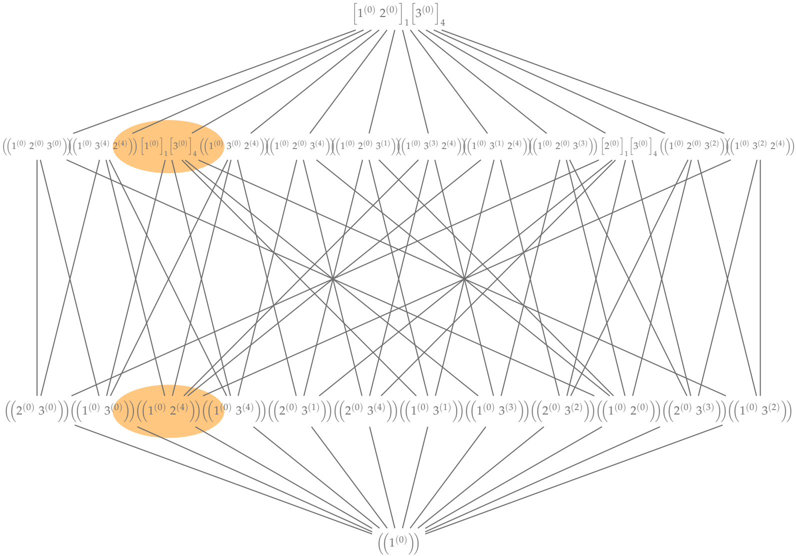

Outline

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

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- consider the map
  $f_1 : R_1^{(1)} \to \mathcal{NC}_{G(d,d,n)}(c), \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$

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- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with $w\Big(n^{(d-1)}\Big) = 1^{(0)}$

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- this map is an injective involution
- its image is the interval
$$\Big[\Big(\big(1^{(0)}\ n^{(d-1)}\big)\Big), \Big(\big(1^{(0)}\ n^{(d-1)}\big)\Big)\Big(\big(2^{(0)}\ \ldots\ (n-1)^{(0)}\big)\Big)\Big]_T$$

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- consider the map
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Lemma (🌬, 2015)

The interval $\Bigl(f_1\bigl(R_1^{(1)}\bigr), \leq_T\Bigr)$ *is isomorphic to* $\mathcal{NC}_{G(1,1,n-2)}$.

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- consider the map
  $f_1 : R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \Big(\big(1^{(0)}\ n^{(d-2)}\big)\Big)x$
- define $D_1 = R_1^{(1)} \uplus f_1\Big(R_1^{(1)}\Big)$, and $\mathcal{D}_1 = (D_1, \leq_T)$

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Lemma (, 2015)

The poset $\mathcal{D}_1$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2 : R_2^{(d-1)} \to NC_{G(d,d,n)}(c), \quad x \mapsto \Big(\big(2^{(0)}\ n^{(0)}\big)\Big)x$$

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- its image consists of permutations $w \in R_n^{(d-1)}$ with $w\Big(n^{(d-1)}\Big) = 2^{(d-1)}$

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- this map is an injective involution
- its image is the interval

$$\left[\left(\left(1^{(0)}\; n^{(d-1)}\; 2^{(d-1)}\right)\right), \left(\left(1^{(0)}\; n^{(d-1)}\; 2^{(d-1)}\; \ldots\; (n-1)^{(d-1)}\right)\right)\right]_T$$

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map
$$f_2 : R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \Big(\big(2^{(0)}\ n^{(0)}\big)\Big)x$$

Lemma (🏠, 2015)

The interval $\Big(f_2\big(R_2^{(d-1)}\big), \leq_T\Big)$ *is isomorphic to* $\mathcal{NC}_{G(1,1,n-2)}$.

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- define $D_2 = R_2^{(d-1)} \uplus f_2\Big(R_2^{(d-1)}\Big)$, and $\mathcal{D}_2 = (D_2, \leq_T)$

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Lemma (, 2015)

The poset $\mathcal{D}_2$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length $n-1$.

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- define $D = R_n^{(d-1)} \setminus \left(f_1\left(R_1^{(1)}\right) \uplus f_2\left(R_2^{(d-1)}\right)\right)$, and $\mathcal{D} = (D, \leq_T)$

Lemma (, 2015)

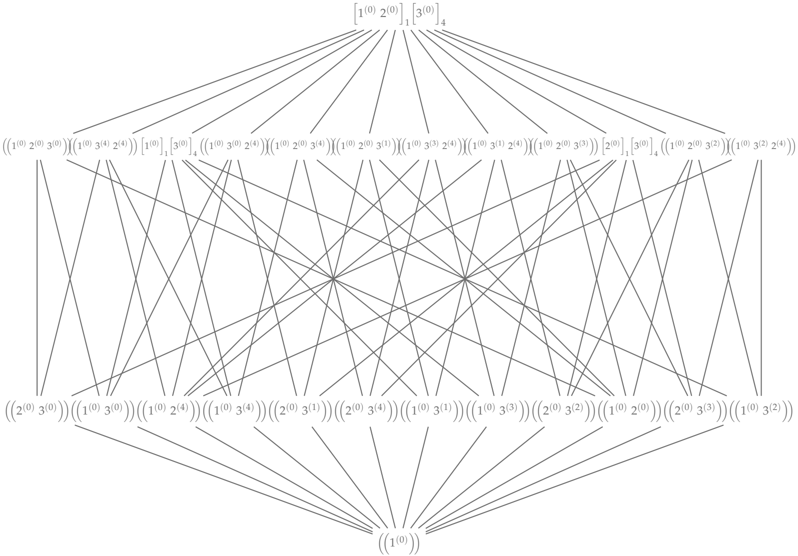
The poset $\mathcal{D}$ is isomorphic to $\biguplus_{i=3}^{n-1} \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i)}$. Morever, its minimal elements have length 2, and its maximal elements have length $n-2$.

The Main Result

Theorem (, 2015)

For $d, n \geq 2$ the lattice $\mathcal{NC}_{G(d,d,n)}$ admits a symmetric chain decomposition. Consequently, it is Peck.

Example: $d = 5, n = 3$



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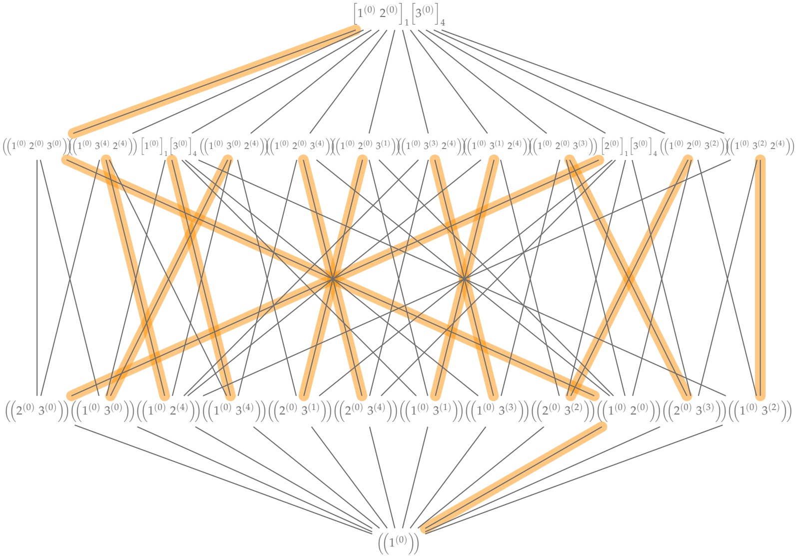

Outline

The Remaining Cases

- so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?

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The lattice $\mathcal{NC}_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \geq 1$.

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- only the 26 exceptional groups remain

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- $\mathcal{P}$ .. graded poset of rank n
- $\mathcal{P}[i]$ .. subposet of $\mathcal{P}$ with i largest ranks removed

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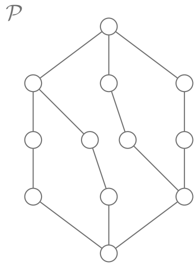

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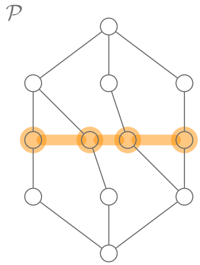

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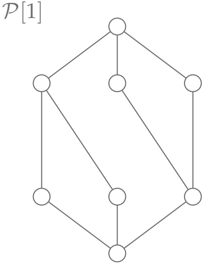

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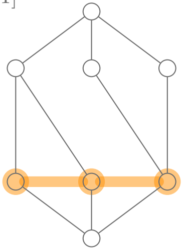
$\mathcal{P}[1]$



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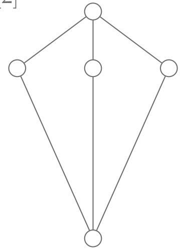

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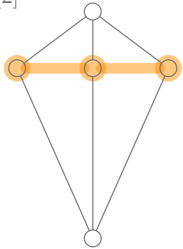
$\mathcal{P}[2]$



A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank n
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$\mathcal{P}[3]$

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A Decomposition Argument

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$\mathcal{P}[4]$

A Decomposition Argument

- $\mathcal{P}$ .. graded poset of rank n
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Proposition (, 2015)

A graded poset $\mathcal{P}$ of rank n is strongly Sperner if and only if $\mathcal{P}[i]$ is Sperner for all $i \in \{0, 1, \ldots, n\}$.

- antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for $s < i$

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- SAGE has a fast implementation to compute the size of the largest antichain of a poset

A Decomposition Argument

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The lattice $\mathcal{NC}_W$ is Peck for any well-generated exceptional complex reflection group W.

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The lattice $\mathcal{NC}_W$ is Peck for any well-generated complex reflection group W.

m-Divisible Noncrossing Partition Posets

- W .. well-generated complex reflection group; c .. Coxeter element of W
- ***m*-divisible noncrossing partition**: m-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$

$$(w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c$$

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$$(w)_m = (w_1, w_2, \ldots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \cdots \leq_T w_m \leq_T c$$

$$\partial(w)_m = \left[w_1; w_1^{-1}w_2, w_2^{-1}w_3, \ldots, w_{m-1}^{-1}w_m, w_m^{-1}c\right]$$

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- ***m*-divisible noncrossing partition**: m-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$
- ***m*-delta sequence**: sequence of "differences" of elements in a multichain
- partial order: $(u)_m \leq (v)_m$ if and only if $\partial(u)_m \leq_T \partial(v)_m$ $\rightsquigarrow \mathcal{NC}_W^{(m)}(c)$

Question (D. Armstrong, 2009)

Are the posets $\mathcal{NC}_W^{(m)}$ strongly Sperner for any W and any $m \geq 1$?

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- affirmative answer for $m = 1$

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m-Divisible Noncrossing Partition Posets

- affirmative answer for $m = 1$
- what about $m > 1$?
  - $\mathcal{NC}_W^{(m)}$ is antiisomorphic to an order ideal in $(\mathcal{NC}_W)^m$
  - $(\mathcal{NC}_W)^m$ is Peck
  - $\mathcal{NC}_W^{(m)}$ is not rank-symmetric $\rightsquigarrow$ no symmetric chain decomposition

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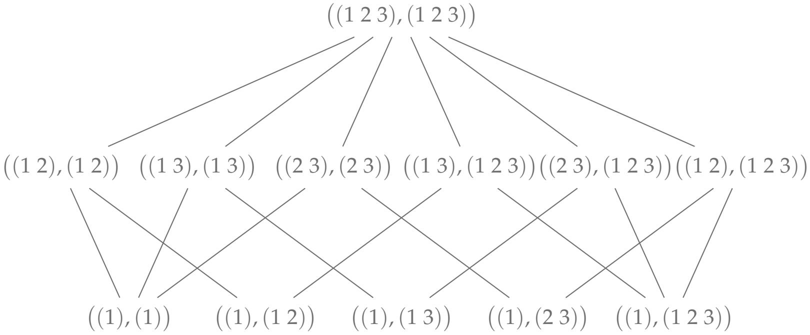
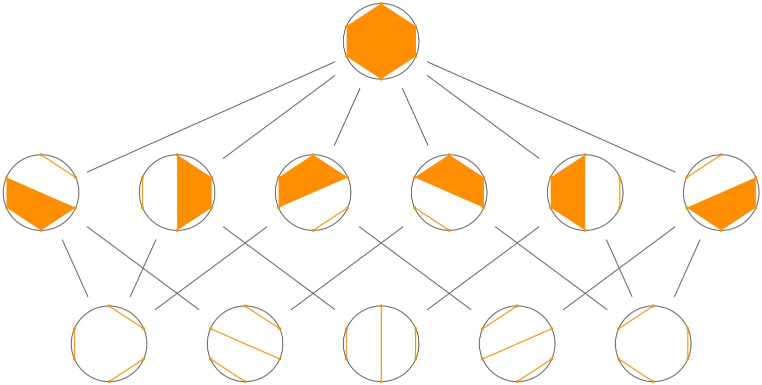
Example: $\mathcal{NC}_{\mathfrak{S}_4}^{(2)}$



Example: $\mathcal{NC}^{(2)}_{\mathfrak{S}_4}$



Thank You.

Interlude: A Convolution Formula

$$NC_{G(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \biguplus_{i=2}^{n-1} \left(R_i^{(0)} \uplus R_i^{(d-1)}\right) \uplus \biguplus_{s=0}^{d-1} R_n^{(s)}$$

Interlude: A Convolution Formula

Proposition (🏠, 2015)

For $n \geq 0$ we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

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$$\mathrm{Cat}_{G(d,d,n+2)} = 2 \cdot \mathrm{Cat}_{G(d,d,n+1)} + 2 \cdot \mathrm{Cat}_{G(1,1,n)} + d \cdot \mathrm{Cat}_{G(1,1,n+1)} + 2 \sum_{i=3}^{n+1} \mathrm{Cat}_{G(d,d,n-i+3)})\mathrm{Cat}_{G(1,1,i-2)}$$

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$$\mathrm{Cat}_{G(d,d,n+2)} = \left(\prod_{i=1}^{n+1} \frac{di + (n-1)d}{di}\right) \frac{n + (n-1)d}{n}$$

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$$\mathrm{Cat}_{G(d,d,n+2)} = \Big((n+1)d + n + 2\Big) \cdot \mathrm{Cat}_{G(1,1,n+1)}$$

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Interlude: A Convolution Formula

Proposition (, 2015)

For $n \geq 0$ we have $\displaystyle\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$nd \cdot \mathrm{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = 2 \cdot \sum_{i=0}^{n} (id + i + 1) \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

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$$nd \cdot \mathrm{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} =$$
$$2 \cdot \sum_{i=0}^{n} \Big(id \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)} \Big) + 2 \cdot \sum_{i=0}^{n} \binom{2i}{i} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

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$$nd \cdot \mathrm{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = 2d \cdot \sum_{i=0}^{n} \left(i \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)} \right) + \binom{2(n+1)}{n+1}$$

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$$\frac{n}{2} \cdot \mathrm{Cat}_{G(1,1,n+1)} = \sum_{i=0}^{n} i \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

Interlude: A Convolution Formula

Proposition (🌬, 2015)

For $n \geq 0$ we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\binom{2n+1}{n-1} = \sum_{i=0}^{n} i \cdot \mathrm{Cat}_{G(1,1,i)} \cdot \mathrm{Cat}_{G(1,1,n-i)}$$

Interlude: A Convolution Formula

Proposition (Y. Kong, 2000)

For $n \geq 0$ we have $$\sum_{i=0}^{n-1} Cat_{G(1,1,i)} \cdot \binom{2(n-i)}{n-i-1} = \binom{2n+1}{n-1}.$$