Shifted domino tableaux

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Journées GT CombAlg GDR IM
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Outline

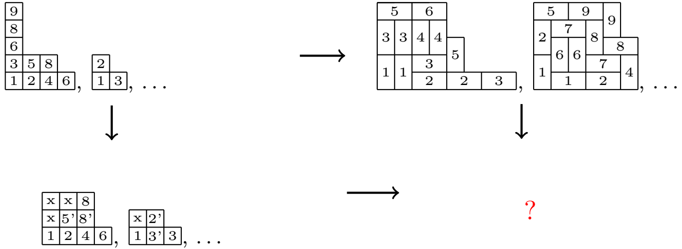
Introduction

Young tableaux: (Young)

- Schur functions
- plactic monoid (Lascoux/Schützenberger)

domino tableaux:

- product of two Schur functions
- super plactic monoid (Carré/Leclerc)



shifted Young tableaux: (Sagan/Worley)

- Q-Schur functions
- shifted plactic monoid (Serrano)

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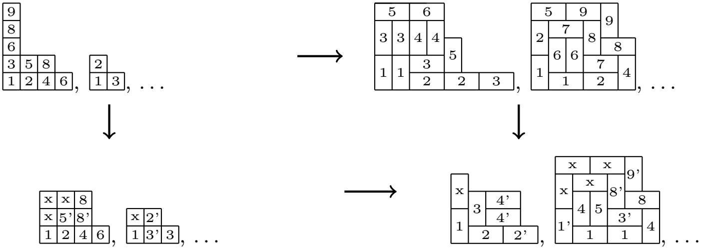

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- Q-Schur functions
- shifted plactic monoid (Serrano)

shifted domino tableaux:

- product of two Q-Schur functions
- super shifted plactic monoid

Partitions

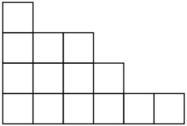
A partition λ of an integer n is:

- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$,
- $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$,
- $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

A Young diagram is:

- a set of square cells,
- the cells are adjusted down and left,
- the i^{th} row contains λ_i cells.

The Young diagram associated with the partition (6, 4, 3, 1) is:



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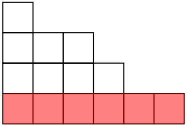

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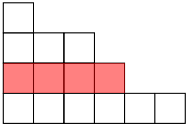

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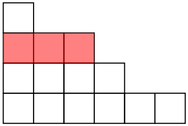

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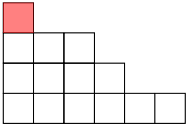
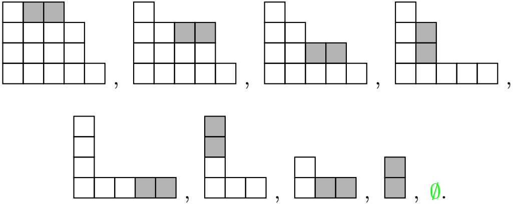


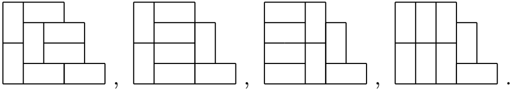
Paving a partition by dominoes

Two adjacent cells form a domino (*i.e.*,  or ).



, $\emptyset$.

Thus $(5, 4, 4, 3)$ is pavable. We give below some pavings of $(5, 4, 4, 3)$:



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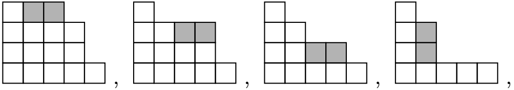

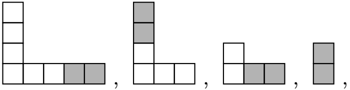 $\emptyset$.

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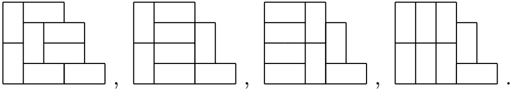
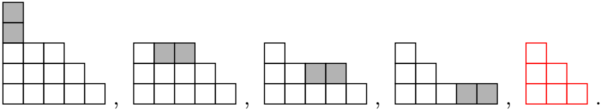


Note that the partition $(5,4,3,1,1)$ is not pavable.



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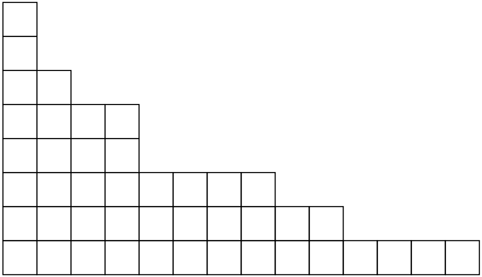

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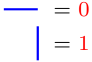

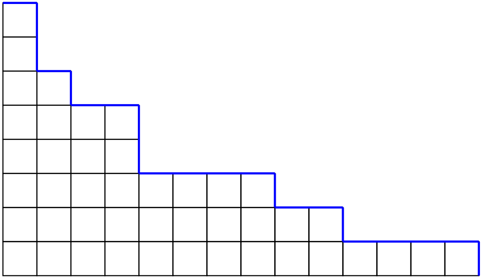

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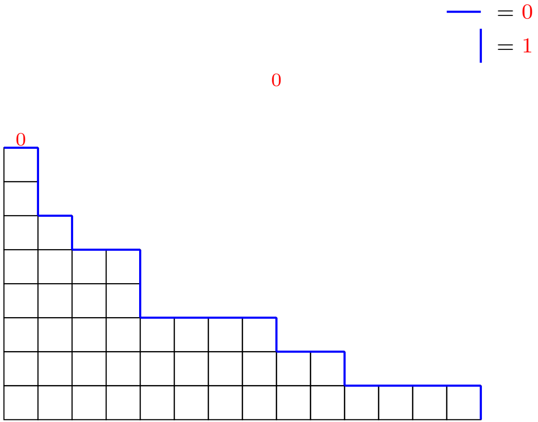

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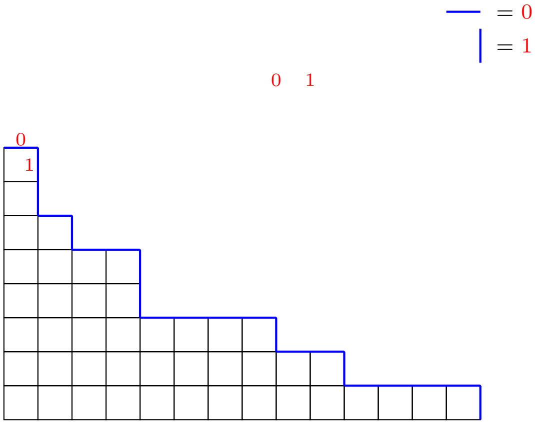

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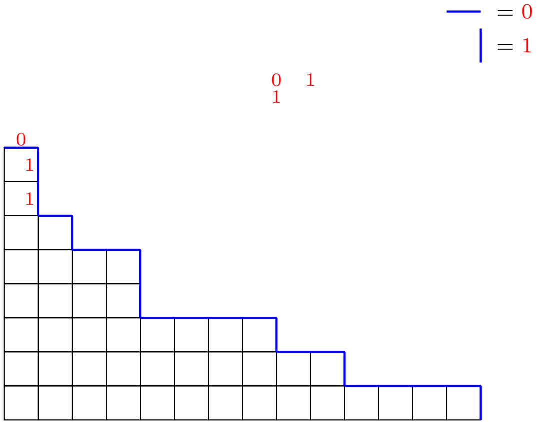

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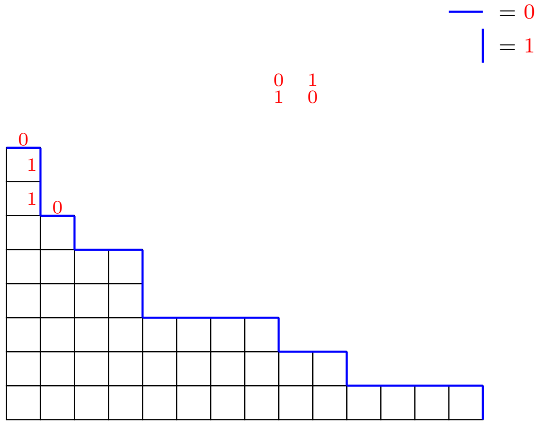

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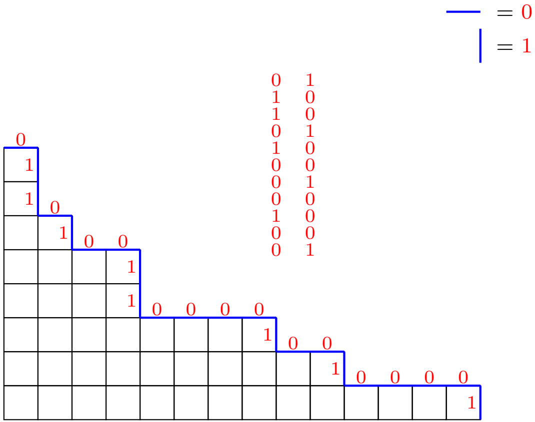

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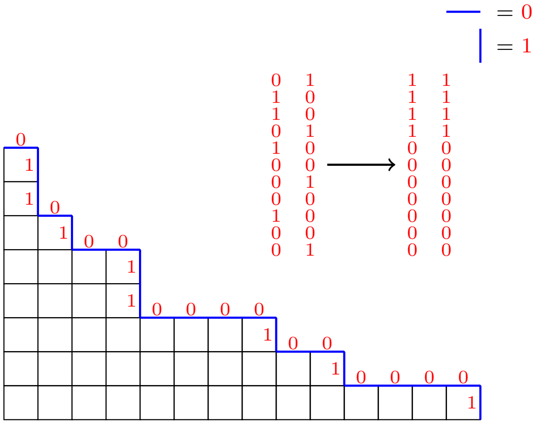

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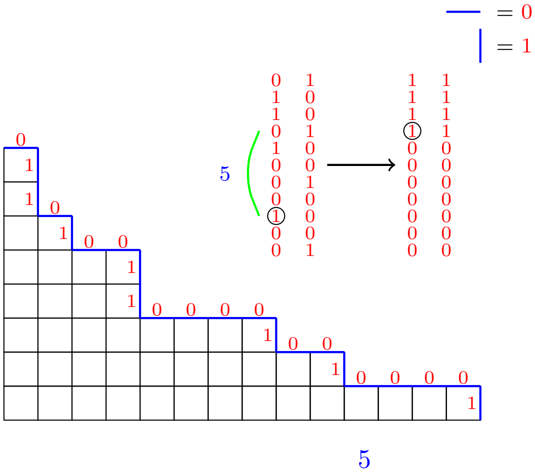

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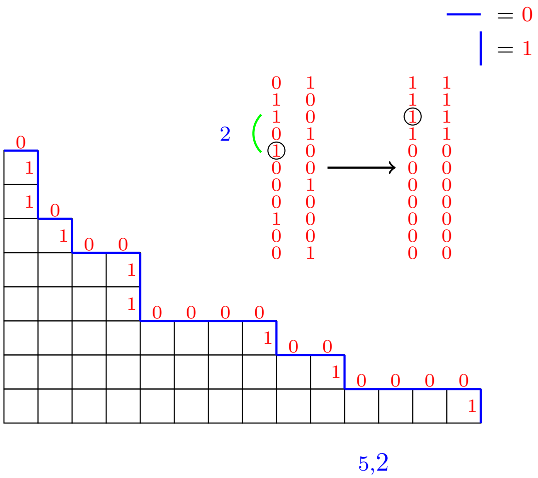

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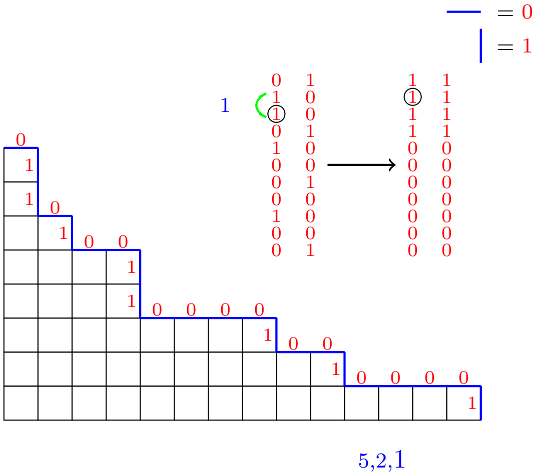

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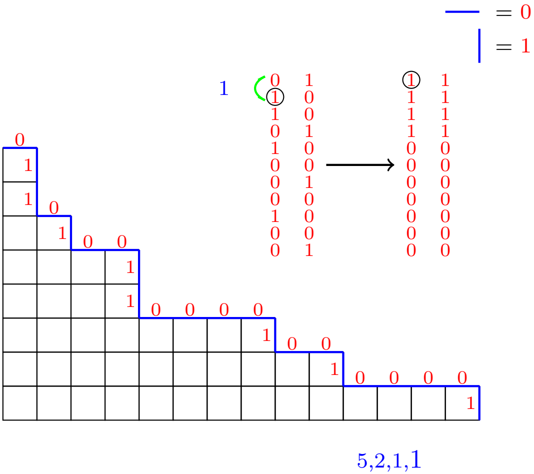

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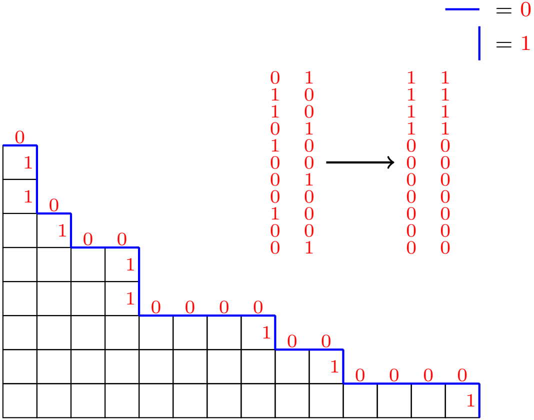

(5,2,1,1)

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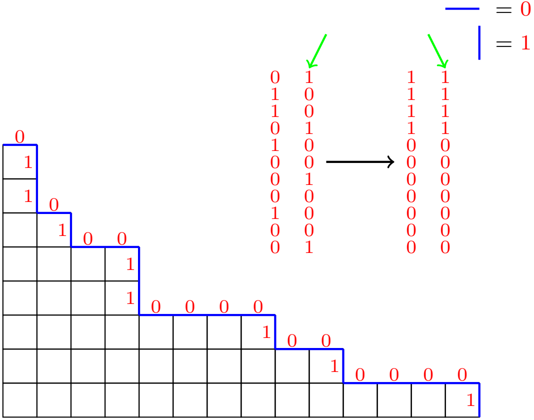

(5,2,1,1) (7,4,2)

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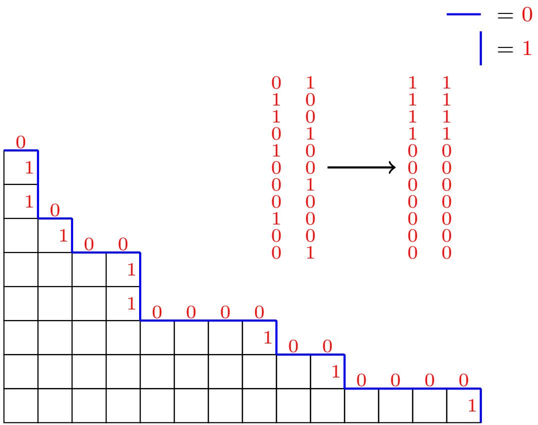

Hence, the 2-quotient of $(14, 10, 8, 4, 4, 2, 1, 1)$ is ((5,2,1,1),(7,4,2)).

Young tableaux

A Young tableau is:

- a filling of a Young diagram with positive integers,

5					
4	9	9			
2	4	7	8		
1	3	6	6	7	8

Young tableaux

A Young tableau is:

- a filling of a Young diagram with positive integers,
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$\vee$

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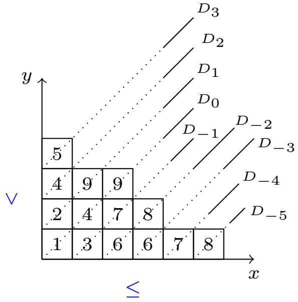

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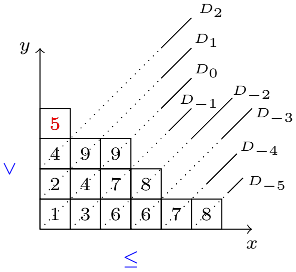

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The diagonal reading is: 5

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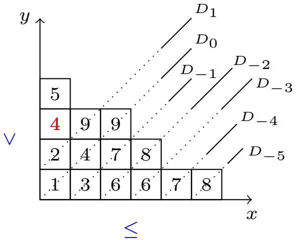

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The diagonal reading is: 5/4

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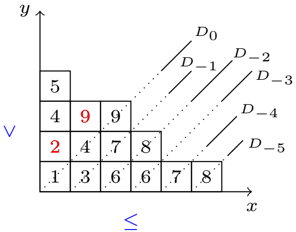

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The diagonal reading is: 5/4/2,9

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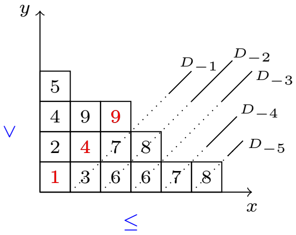

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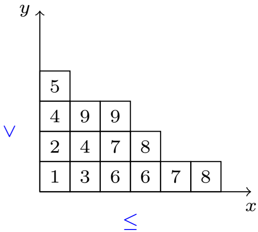

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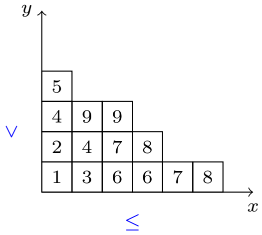


$D_k = x + k$

The diagonal reading is: 5/4/2,9/1,4,9/3,7/6,8/6/7/8.

Conversely, given a diagonal reading, **we can construct the associated Young tableau**.

Schur functions

Given a Young tableau t, its corresponding monomial is:

$$x^t = \prod_{i \in t} x_i.$$

For each partition λ, the Schur function s_λ is:

$$s_\lambda = \sum_t x^t$$

where the sum runs over all Young tableaux t of shape λ.

For example:

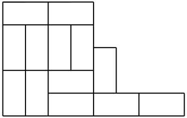
$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3,$$

where the monomial $x_1 x_2 x_3$ is obtained from the two following Young tableaux:

$$\begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\ , \quad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\ .$$

Domino tableaux

Given a paved partition λ, a domino tableau is:



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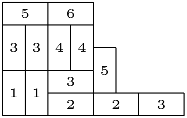

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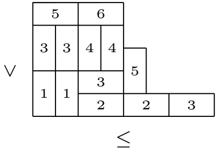

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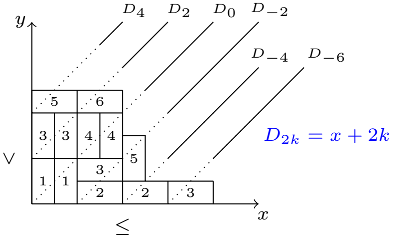
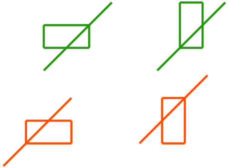


two types of dominoes:



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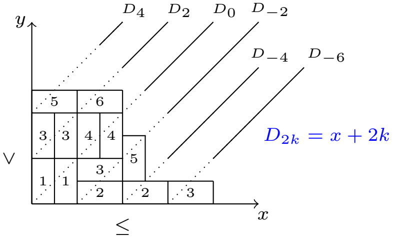

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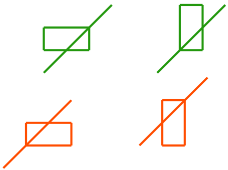

Theorem (Stanton, White 1985)

Given a pavable partition λ of 2-quotient (μ, ν), the set of domino tableaux of shape λ and the set of pairs of Young tableaux (t_1, t_2) of shape (μ, ν) are in bijection.

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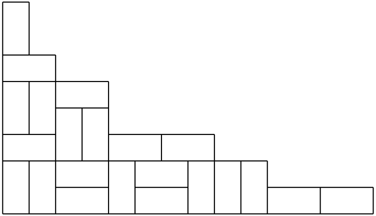

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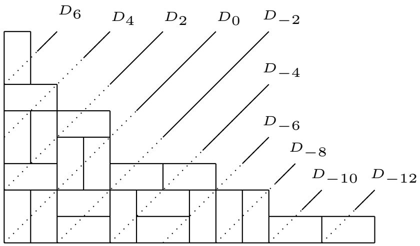

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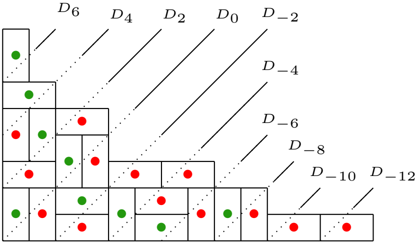

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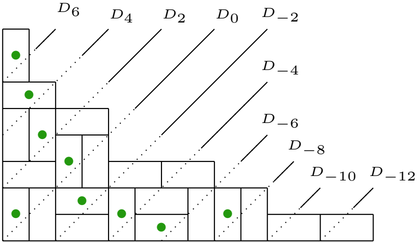
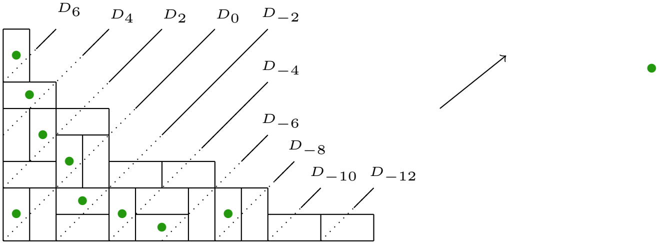


2-quotient of a pavable partition

The 2-quotient of a pavable partition λ is a pair of partitions (μ, ν) obtained by:



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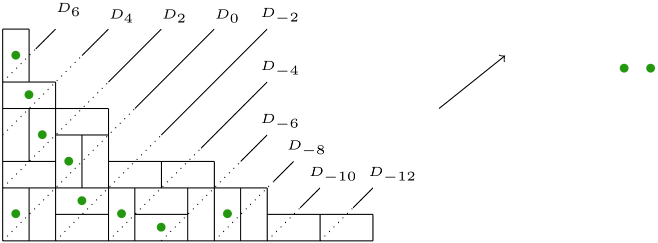

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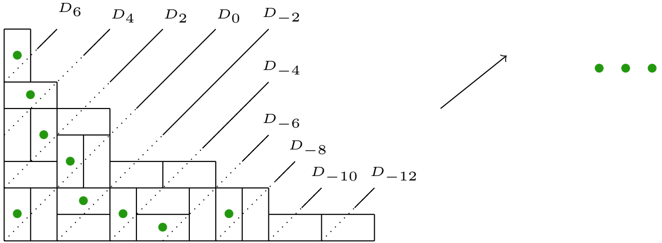

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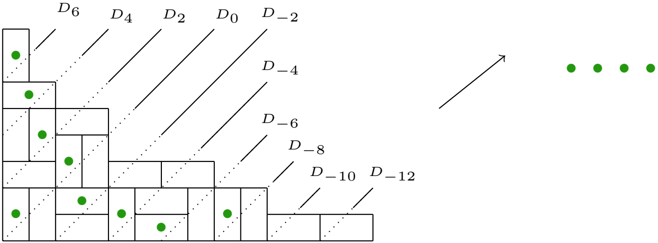

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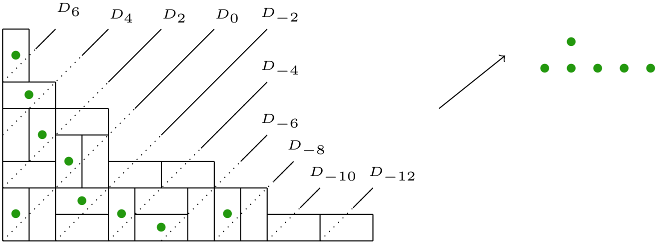

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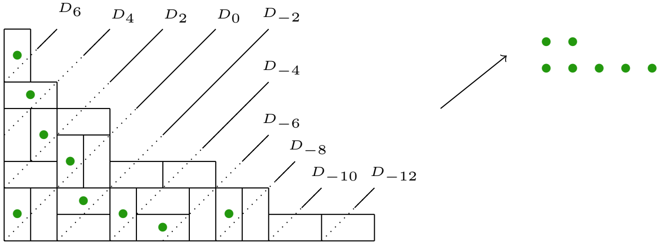

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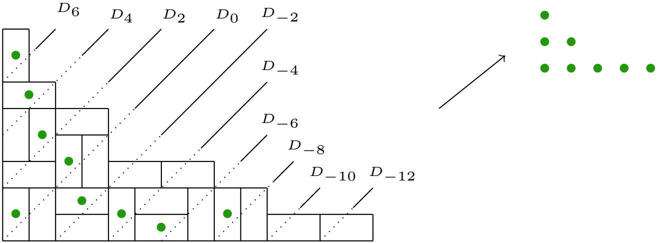

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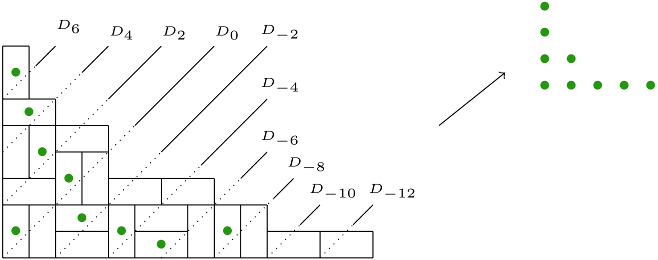

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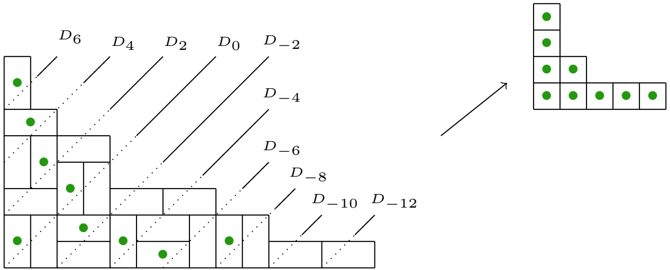

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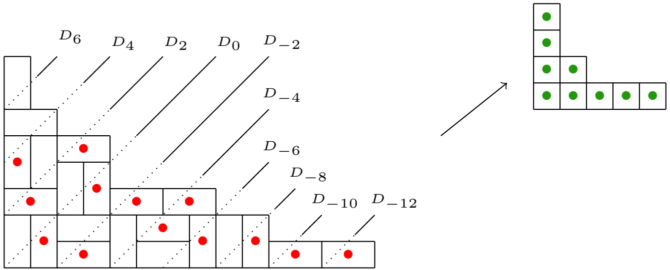

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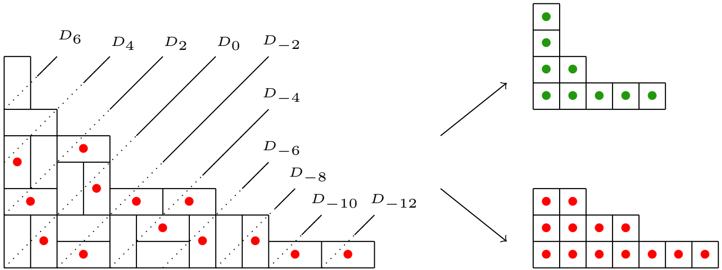
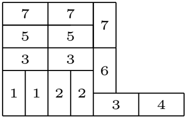


Hence, the 2-quotient of $(14, 10, 8, 4, 4, 2, 1, 1)$ is $((5, 2, 1, 1), (7, 4, 2))$.

Sketch of proof [Carré, Leclerc 1993] (1/2)



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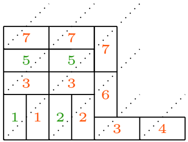
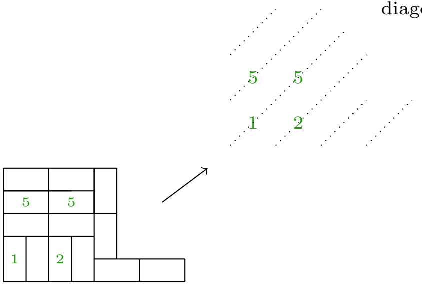


Sketch of proof [Carré, Leclerc 1993] (1/2)

diagonal reading: $5/1, 5/2$



$$\longrightarrow \quad t_1 = \begin{array}{|c|c|}\hline 5 & 5 \\ \hline 1 & 2 \\ \hline \end{array}$$

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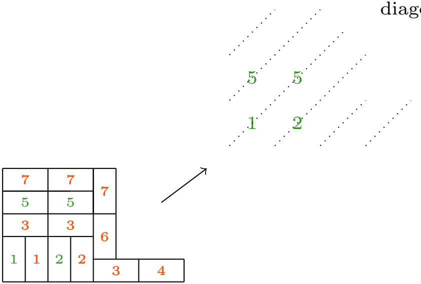

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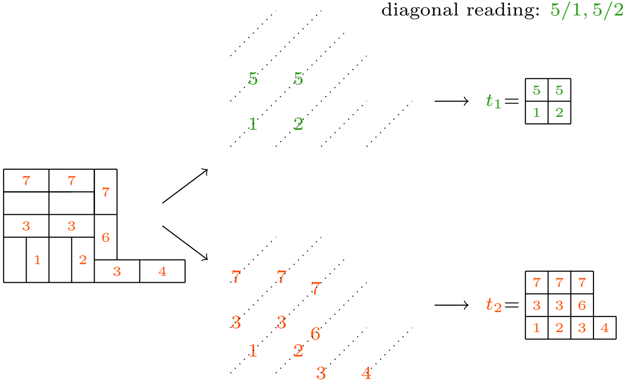

diagonal reading: $7/3, 7/1, 3, 7/2, 6/3/4$

We obtain two Young tableaux (t_1, t_2) of shape $((2,2),(4,3,3))$.

Sketch of proof [Carré, Leclerc 1993] (2/2)

Given a pair of Young tableaux:

$$\left(\begin{array}{|c|c|} \hline 5 & 5 \\ \hline 1 & 2 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c} \cline{1-3} 7 & 7 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 3 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \right)$$

Sketch of proof [Carré, Leclerc 1993] (2/2)

Given a pair of Young tableaux:

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we construct the associated domino tableau:

Sketch of proof [Carré, Leclerc 1993] (2/2)

Given a pair of Young tableaux:

$$\left(\begin{array}{|c|c|}\hline 5&5\\\hline 1&2\\\hline\end{array} \;,\; \begin{array}{|c|c|c|c|}\cline{1-3} 7&7&7&\multicolumn{1}{c}{}\\\cline{1-3} 3&3&6&\multicolumn{1}{c}{}\\\hline 1&2&3&4\\\hline\end{array} \right)$$

we construct the associated domino tableau:

$$\left(\boxed{1} \;,\; \boxed{1} \right) \rightarrow \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}$$

Sketch of proof [Carré, Leclerc 1993] (2/2)

Given a pair of Young tableaux:

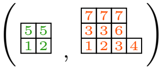
$$\left(\begin{array}{|c|c|} \hline 5 & 5 \\ \hline 1 & 2 \\ \hline \end{array} \ , \ \begin{array}{|c|c|c|c} \hline 7 & 7 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 3 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \right)$$

we construct the associated domino tableau:

$$\left(\boxed{1} \ , \ \boxed{1} \right) \rightarrow \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \ , \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \ , \ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right) \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \end{array} \ ,$$

Sketch of proof [Carré, Leclerc 1993] (2/2)

Given a pair of Young tableaux:



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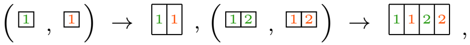

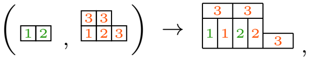

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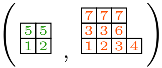

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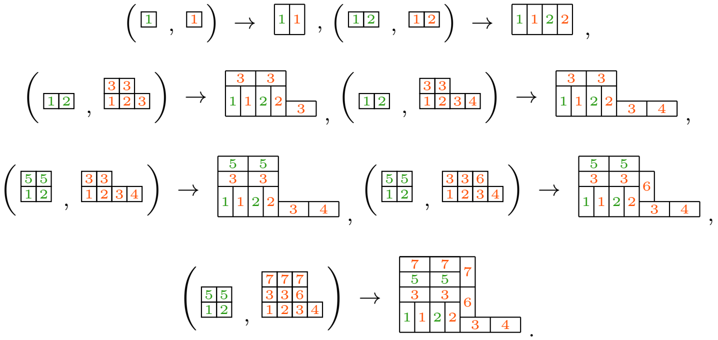


In terms of symmetric functions

Theorem (Carré, Leclerc 1993)

Let λ be a partition of 2-quotient (μ, ν). One has

$$\sum_T x^T = s_\mu s_\nu$$

where the sum runs over all domino tableaux of shape λ.

Let λ and θ be two partitions and $K^{(1)}_{\lambda\theta}$ be the number of domino tableaux of shape λ and evaluation θ.

Corollary (Carré, Leclerc 1993)

Let λ be a partition. Then

$$\sum_T x^T = \sum_\theta K^{(1)}_{\lambda\theta} m_\theta$$

where the first sum runs over all domino tableaux of shape λ and the second sum runs over all partitions θ.

The numbers $K^{(1)}_{\lambda\theta}$ are the domino analogues of the Kostka numbers.

Super plactic monoid

Given the totally ordered infinite alphabets $A_1 := \{a_1^1 < a_2^1 < a_3^1 < \cdots\}$ and $A_2 := \{a_1^2 < a_2^2 < a_3^2 < \cdots\}$. The Super Plactic monoid is the quotient of the free monoid $(A_1 \cup A_2)^*$ by the relations:

$$a_j^\epsilon a_i^\epsilon a_k^\epsilon \equiv a_j^\epsilon a_k^\epsilon a_i^\epsilon \text{ for } i < j \leq k \text{ and } \epsilon \in \{1, 2\},$$

$$a_i^\epsilon a_k^\epsilon a_j^\epsilon \equiv a_k^\epsilon a_i^\epsilon a_j^\epsilon \text{ for } i \leq j < k \text{ and } \epsilon \in \{1, 2\},$$

$$a_i^1 a_j^2 \equiv a_j^2 a_i^1 \quad \text{for any positive integers } i \text{ and } j.$$

Carré and Leclerc proved that **each super plactic class is represented by a unique domino tableau**.

Summarizing

Schur functions	Young tableaux	plactic monoid
	$\downarrow$ Carré, Leclerc	
product of two Schur functions	domino tableaux	super plactic monoid

. . . .

Q-Schur functions	shifted Young tableaux	shifted plactic monoid
	$\downarrow$ Carré, Leclerc?	
product of two Q-Schur functions	?	?

Summarizing

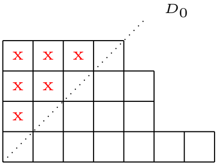
$$\begin{array}{ccc} \text{Schur functions} & \text{Young tableaux} & \text{plactic monoid} \\ & \downarrow{\scriptstyle \text{Carré, Leclerc}} & \\ \text{product of two Schur functions} & \text{domino tableaux} & \text{super plactic monoid} \end{array}$$

$$\begin{array}{ccc} \text{Q-Schur functions} & \text{shifted Young tableaux} & \text{shifted plactic monoid} \\ & \downarrow{\scriptstyle \text{Carré, Leclerc?}} & \\ \text{product of two Q-Schur functions} & \text{shifted domino tableaux} & \text{super shifted plactic monoid} \end{array}$$

Shifted Young tableaux

Given a partition λ of length ℓ satisfying $\lambda_\ell \geq \ell$, a shifted Young tableau is:

- a filling of the cells above D_0 by x,



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Given a partition λ of length ℓ satisfying $\lambda_\ell \geq \ell$, a shifted Young tableau is:

- a filling of the cells above D_0 by x,
- the remaining cells are filled with letters from $A' = \{1' < 1 < 2' < 2 < \cdots\}$,
- such that:

x	x	x	8			
x	x	7	$8'$	9		
x	$2'$	3	5	6		
1	$2'$	$3'$	3	4	5	5

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  - the rows and columns are non-decreasing,

$\leq$

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  - the rows and columns are non-decreasing,
  - a letter $\ell' \in \{1', 2', 3', \dots\}$ appears at most once in each row,

$$\cdots \cancel{\boxed{\ell'}\boxed{\ell'}} \cdots$$

$\leq$

x	x	x	8			
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$\leq$

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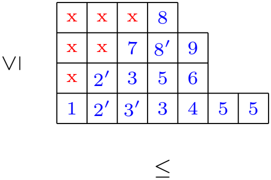
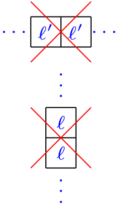
  - a letter $\ell \in \{1, 2, 3, \ldots\}$ appears at most once in each column.





Q-Schur functions

Given a shifted Young tableau t, its corresponding monomial is:

$$x^t = \prod_{\ell \in t} x_{|\ell|}, \text{ where } |\ell| = \ell \text{ and } |\ell'| = \ell.$$

For each partition λ, the Q-Schur function is:

$$Q_\lambda = \sum_t x^t$$

where the sum runs over all shifted Young tableaux t of shape λ.

For example:

$$Q_{(2,1)} = 4x_1^2x_2 + 4x_1x_2^2 + 4x_1^2x_3 + 4x_1x_3^2 + 4x_2^2x_3 + 4x_2x_3^2 + 8x_1x_2x_3,$$

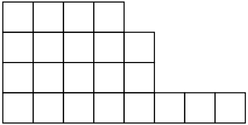
where the monomial $x_1^2x_2$ is obtained from the following four shifted Young tableaux:

$$\begin{array}{|c|c|}\hline \mathrm{X} & 2 \\ \hline 1 & 1 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|}\hline \mathrm{X} & 2 \\ \hline 1' & 1 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|}\hline \mathrm{X} & 2' \\ \hline 1 & 1 \\ \hline \end{array}\ , \quad \begin{array}{|c|c|}\hline \mathrm{X} & 2' \\ \hline 1' & 1 \\ \hline \end{array}\ .$$

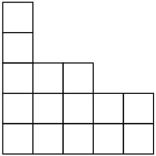
Shifted pavable partition

A partition λ of 2-quotient (μ, ν) is a shifted pavable partition if it satisfies the following two conditions:

- the last parts of μ and ν are greater than or equal to their lengths,



((2,2),(4,3))



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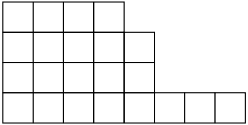

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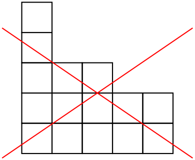

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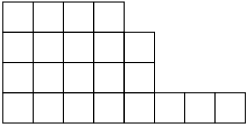

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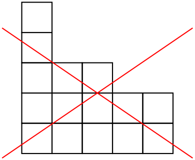
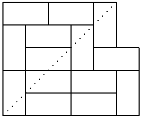
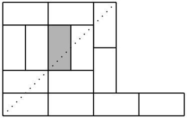


((3),(3,1,1))

- there is no vertical domino d on D_0, such that d has at its left only adjacent dominoes strictly above D_0,





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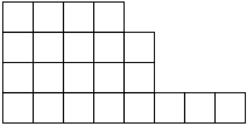

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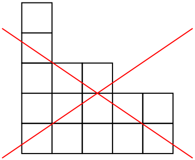

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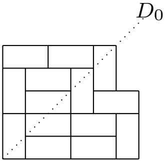

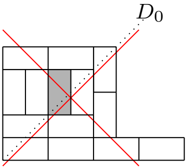

Shifted domino tableaux

Given a shifted pavable partition λ, a shifted domino tableaux is:

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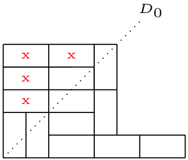

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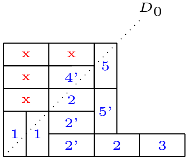

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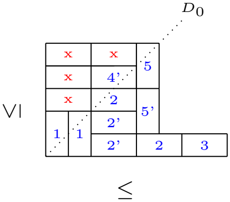
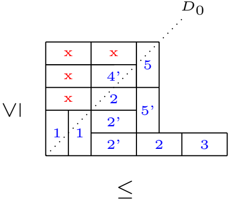
  - the rows and columns are non-decreasing,



Shifted domino tableaux

Given a shifted pavable partition λ, a shifted domino tableaux is:

- a filling of the dominoes above D_0 by x,
- the remaining ones are filled with letters from $A' = \{1' < 1 < 2' < 2 < \cdots\}$,
- such that:
  - the rows and columns are non-decreasing,
  - a letter $\ell \in \{1, 2, 3, \dots\}$ appears at most once in each column,
  - a letter $\ell' \in \{1', 2', 3', \dots\}$ appears at most once in each row.



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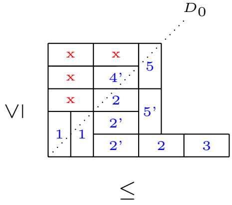
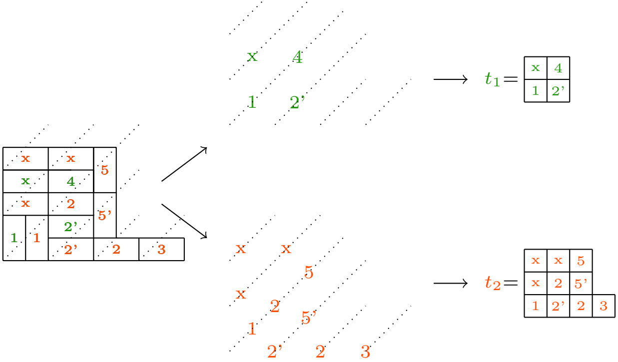


Theorem (C. 2015)

Given a valid paved partition λ of 2-quotient (μ, ν), the set of shifted domino tableaux of shape λ and the set of pairs of shifted Young tableaux (t_1, t_2) of shape (μ, ν) are in bijection.

Sketch of proof (1/2)

diagonal reading : $x/1, 4/2'$

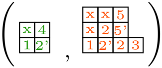


diagonal reading : $x/x, x/1, 2, 5/2', 5'/2/3$

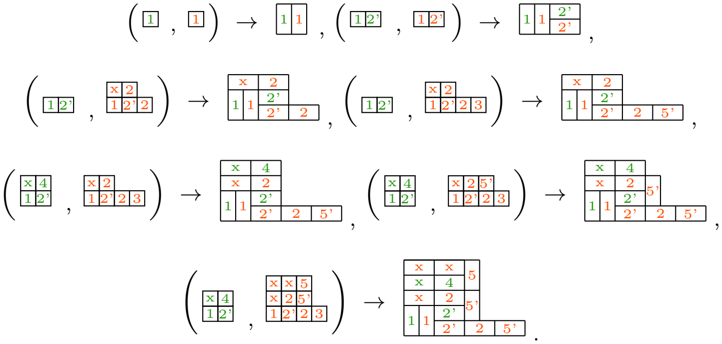
We obtain two shifted Young tableaux (t_1, t_2) of shape $((2,2),(4,3,3))$.

Sketch of proof (2/2)

Given a pair of shifted Young tableaux:



We construct the associated shifted domino tableau:



In terms of symmetric functions

Theorem (C. 2015)

Let λ be a valid paved partition of 2-quotient (μ, ν). One has

$$\sum_T x^T = Q_\mu Q_\nu$$

where the sum runs over all shifted domino tableaux of shape λ.

Let λ and θ be two partitions and $K^{(2)}_{\lambda\theta}$ be the number of shifted domino tableaux of shape λ and evaluation θ.

Corollary (C. 2015)

Let λ be a partition. Then

$$\sum_T x^T = \sum_\theta K^{(2)}_{\lambda\theta} m_\theta.$$

where the first sum runs over all shifted domino tableaux of shape λ and the second sum runs over all partitions θ.

The numbers $K^{(2)}_{\lambda\theta}$ can be seen as analogues of the Kostka numbers.

Super shifted plactic monoid

Let $A_1 := \{a_1^1 < a_2^1 < a_3^1 < \cdots\}$ and $A_2 := \{a_1^2 < a_2^2 < a_3^2 < \cdots\}$ be two totally ordered infinite alphabets. The super shifted plactic monoid is the quotient of the free monoid $(A_1 \cup A_2)^*$ by the relations:

$$
\begin{aligned}
&a_i^\epsilon a_j^\epsilon a_l^\epsilon a_k^\epsilon \equiv a_i^\epsilon a_l^\epsilon a_j^\epsilon a_k^\epsilon \text{ for } i \leq j \leq k < l \text{ and } \epsilon \in \{1,2\},\\
&a_i^\epsilon a_l^\epsilon a_k^\epsilon a_j^\epsilon \equiv a_i^\epsilon a_k^\epsilon a_j^\epsilon a_l^\epsilon \text{ for } i \leq j < k \leq l \text{ and } \epsilon \in \{1,2\},\\
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&a_k^\epsilon a_j^\epsilon a_l^\epsilon a_i^\epsilon \equiv a_k^\epsilon a_l^\epsilon a_j^\epsilon a_i^\epsilon \text{ for } i < j < k \leq l \text{ and } \epsilon \in \{1,2\},\\
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&a_j^\epsilon a_k^\epsilon a_l^\epsilon a_i^\epsilon \equiv a_j^\epsilon a_k^\epsilon a_i^\epsilon a_l^\epsilon \text{ for } i < j \leq k \leq l \text{ and } \epsilon \in \{1,2\},\\
&a_k^\epsilon a_i^\epsilon a_l^\epsilon a_j^\epsilon \equiv a_k^\epsilon a_l^\epsilon a_i^\epsilon a_j^\epsilon \text{ for } i \leq j < k \leq l \text{ and } \epsilon \in \{1,2\},\\
&a_i^1 a_j^2 \equiv a_j^2 a_i^1 \quad \text{for any positive integers } i \text{ and } j.
\end{aligned}
$$

Theorem (C. 2015)

Each super shifted plactic class is represented by a unique shifted domino tableau.

Perspectives

- express the coefficients of the shifted Littlewood-Richardson rule in terms of shifted domino tableaux,
- find an insertion algorithm for shifted domino tableaux.

Thank you