A bijection for rooted maps on general surfaces

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Uniwersytet Wrocławski

joint work with

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I. Maps

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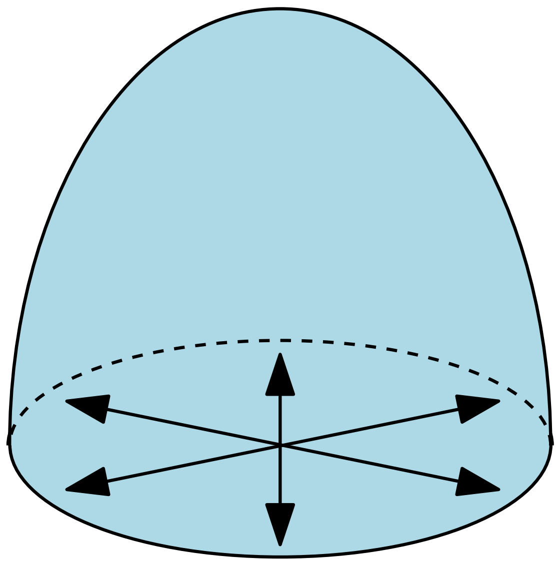
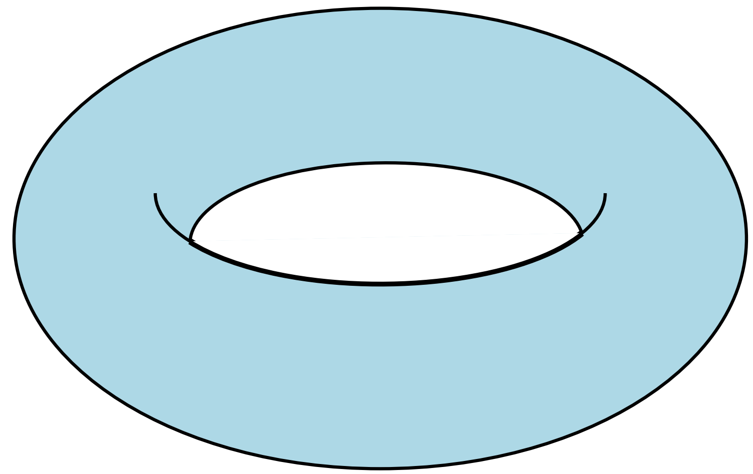


Projective plane



Torus

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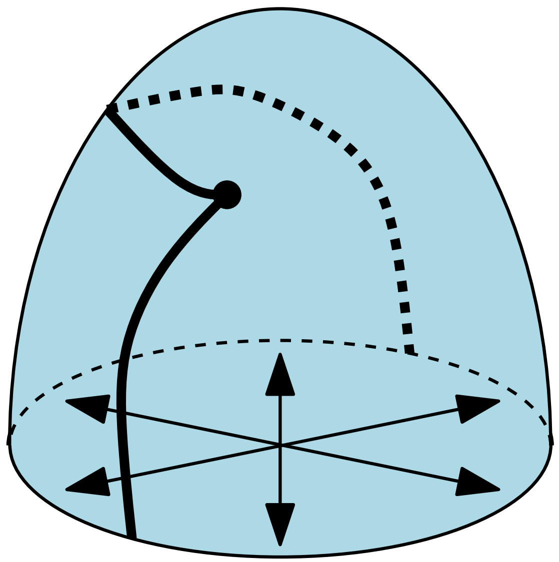
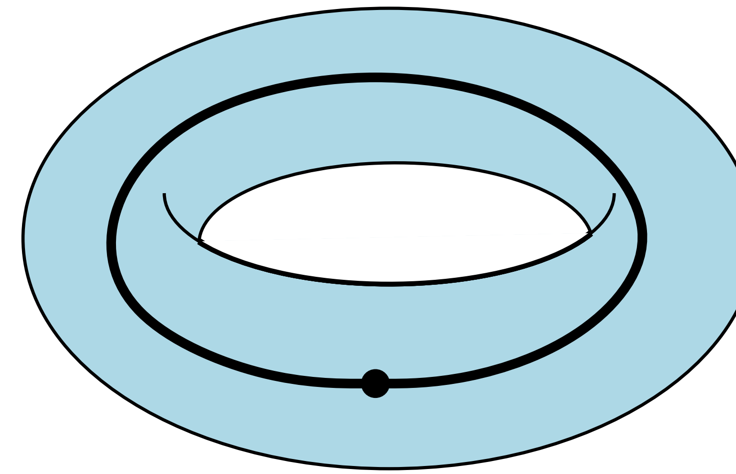


This is a map



This is not a map!

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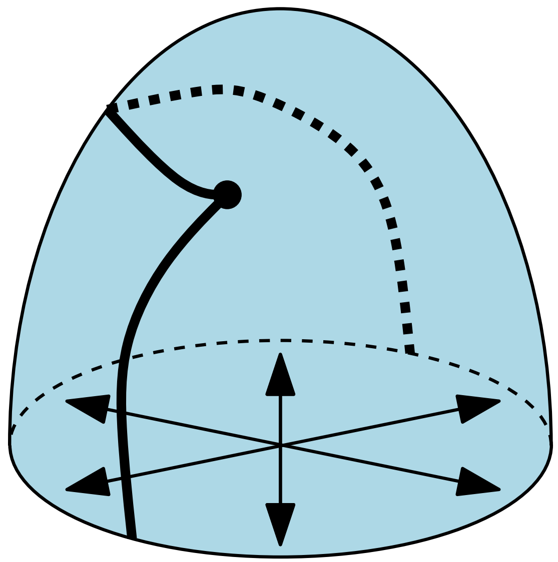

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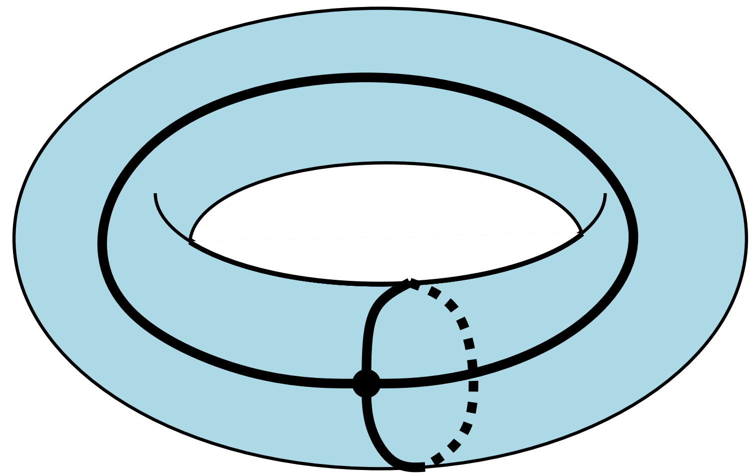

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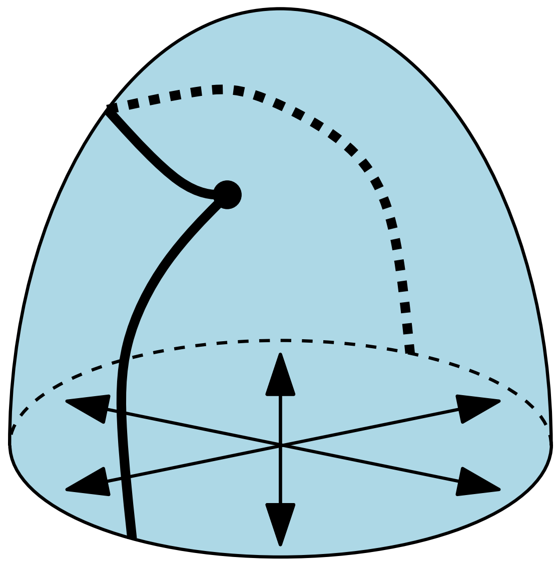

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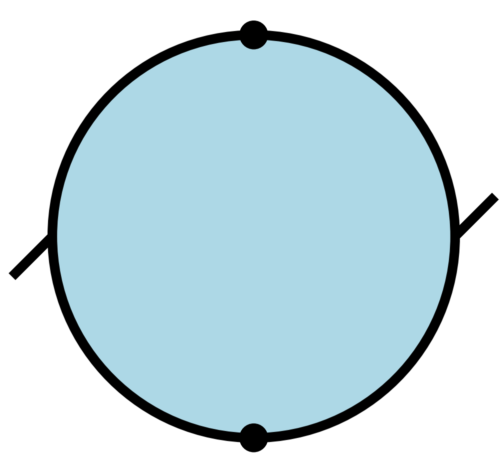

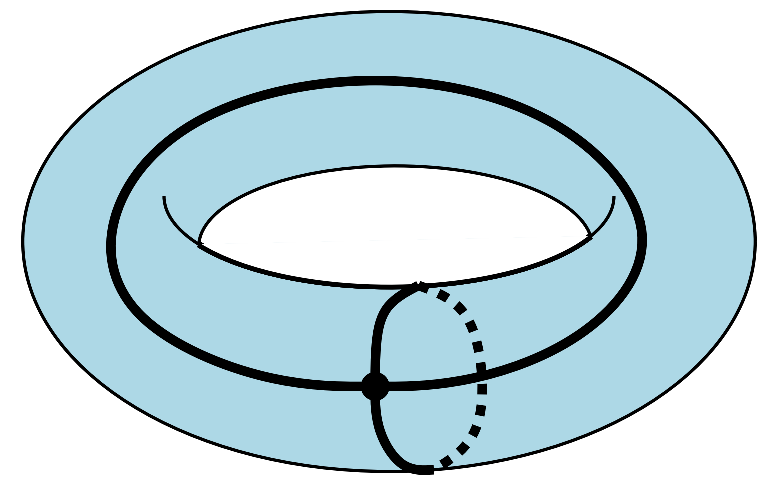

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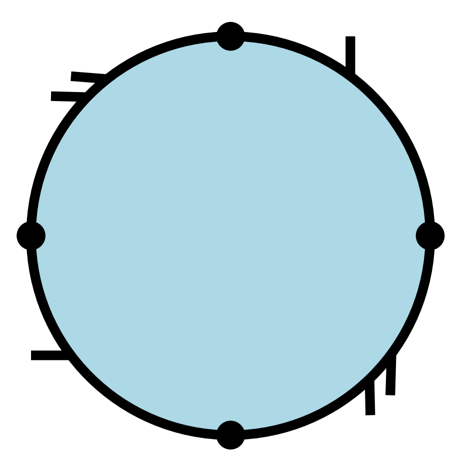

Orientable vs. non-orientable

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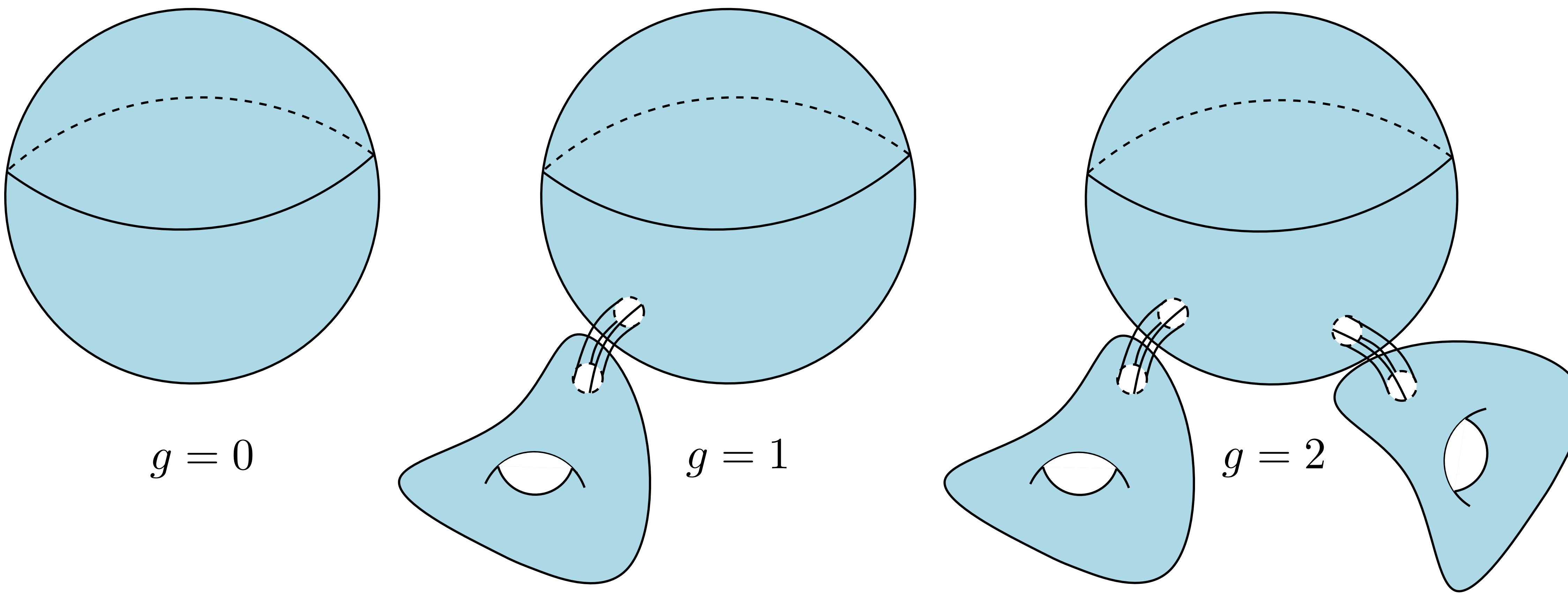

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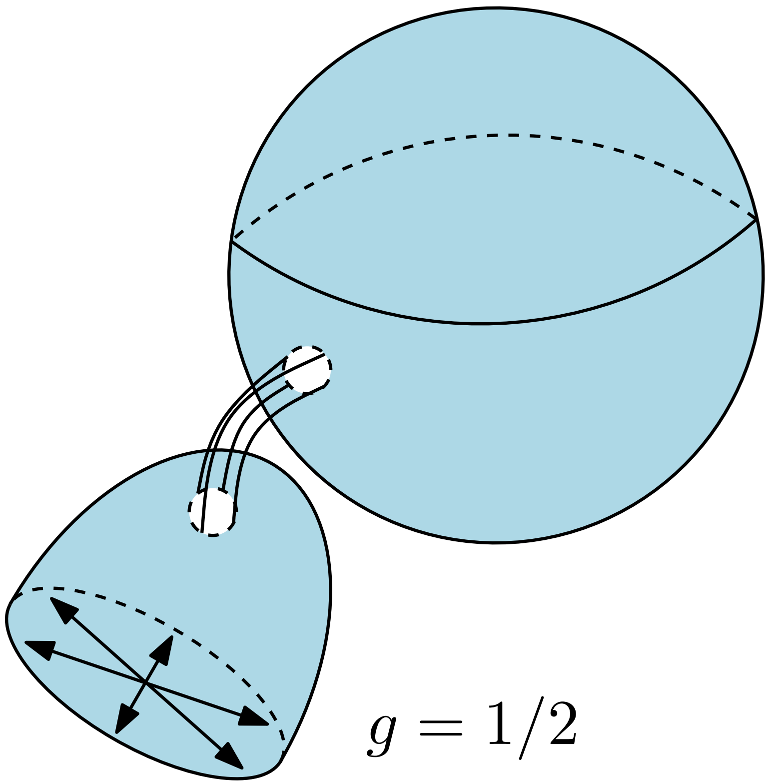

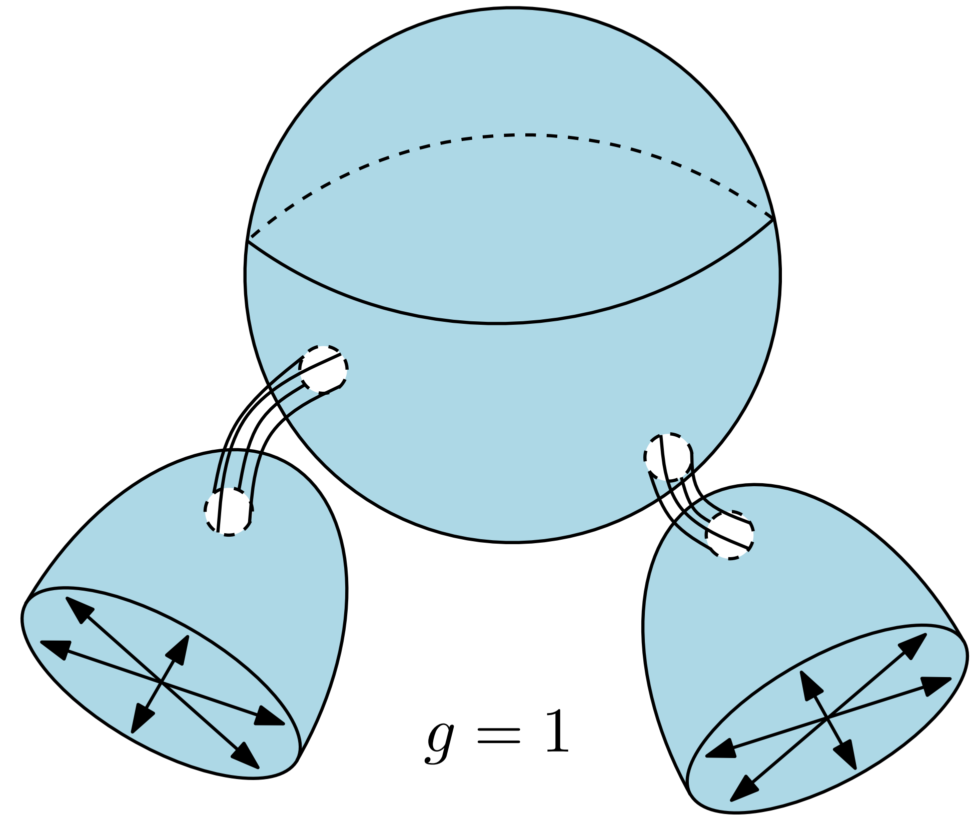

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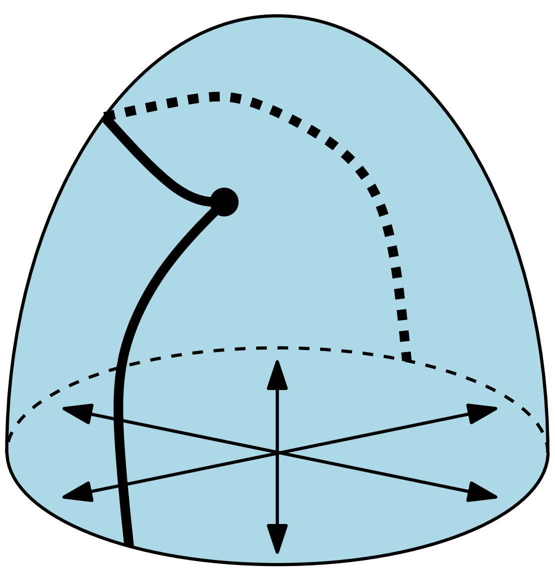
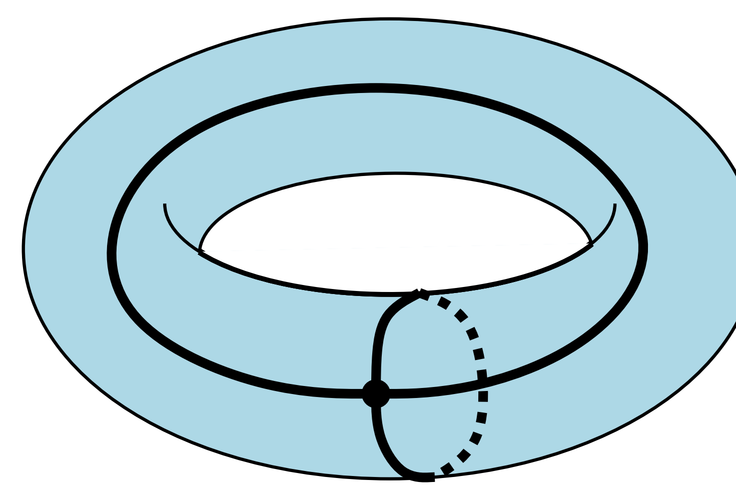


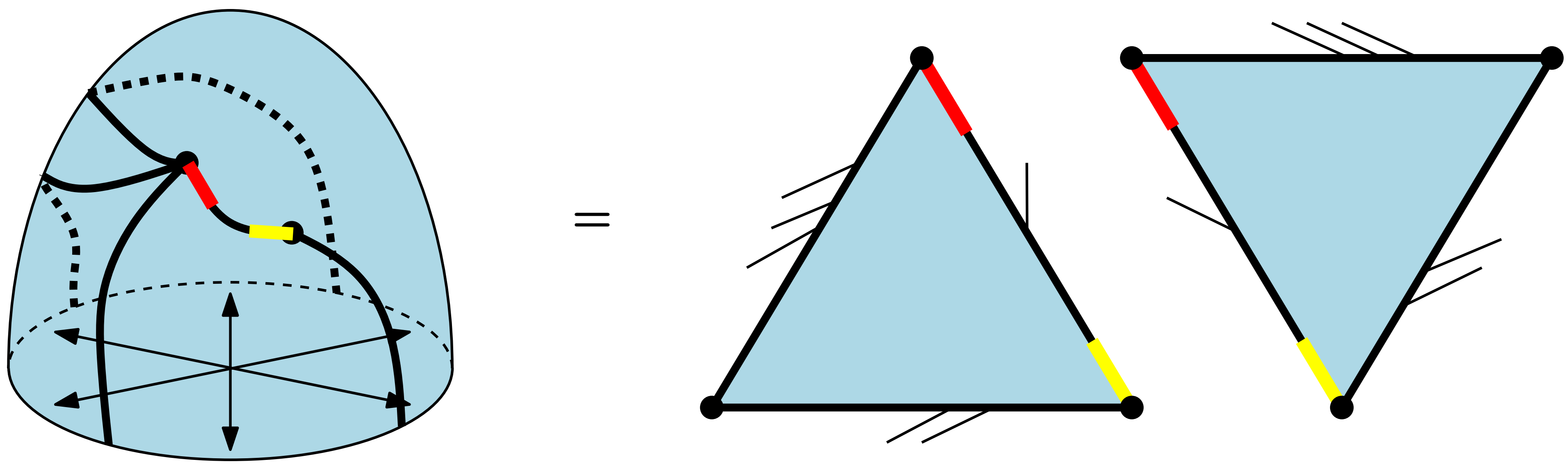
Non-orientable map of type $1/2$



Orientable map of type 1

Rooted maps

Each edge consists of two half-edges.



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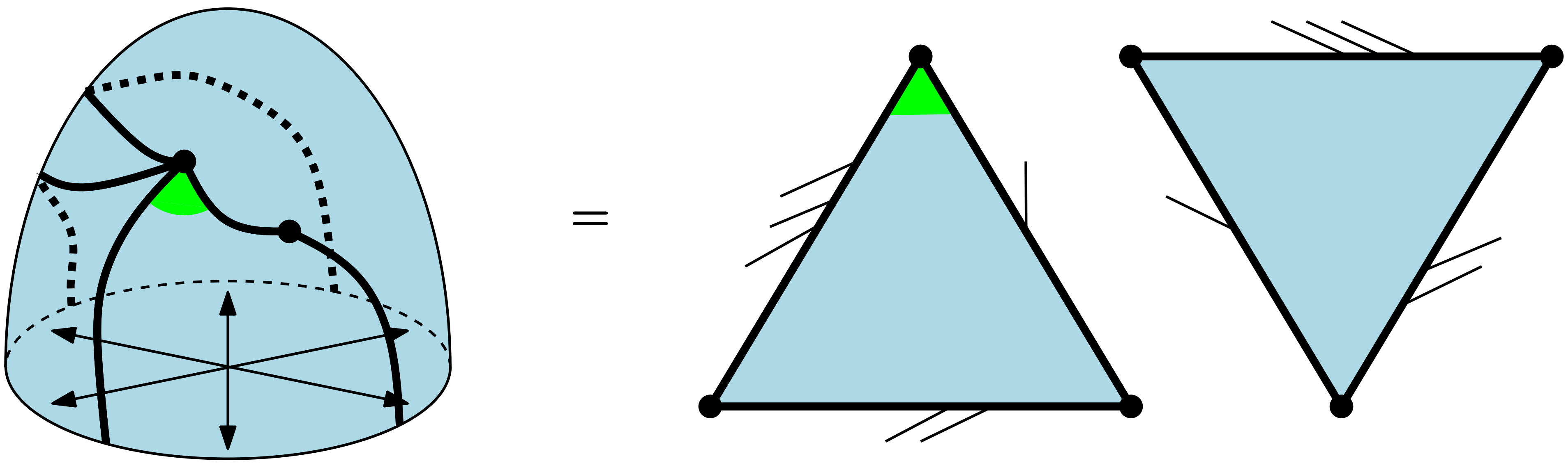

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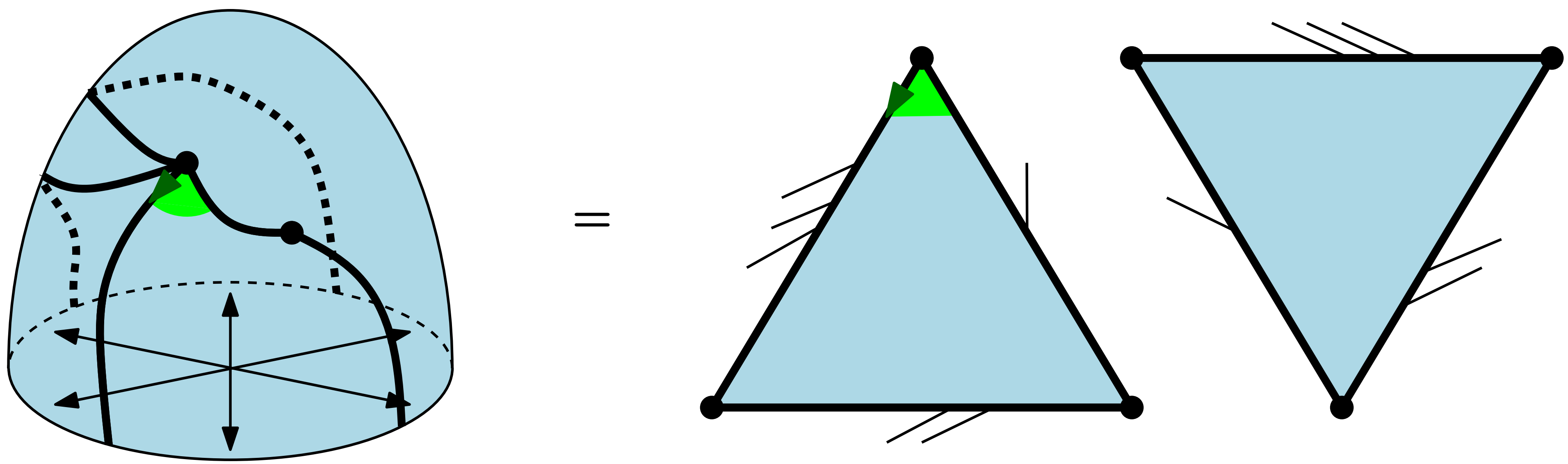

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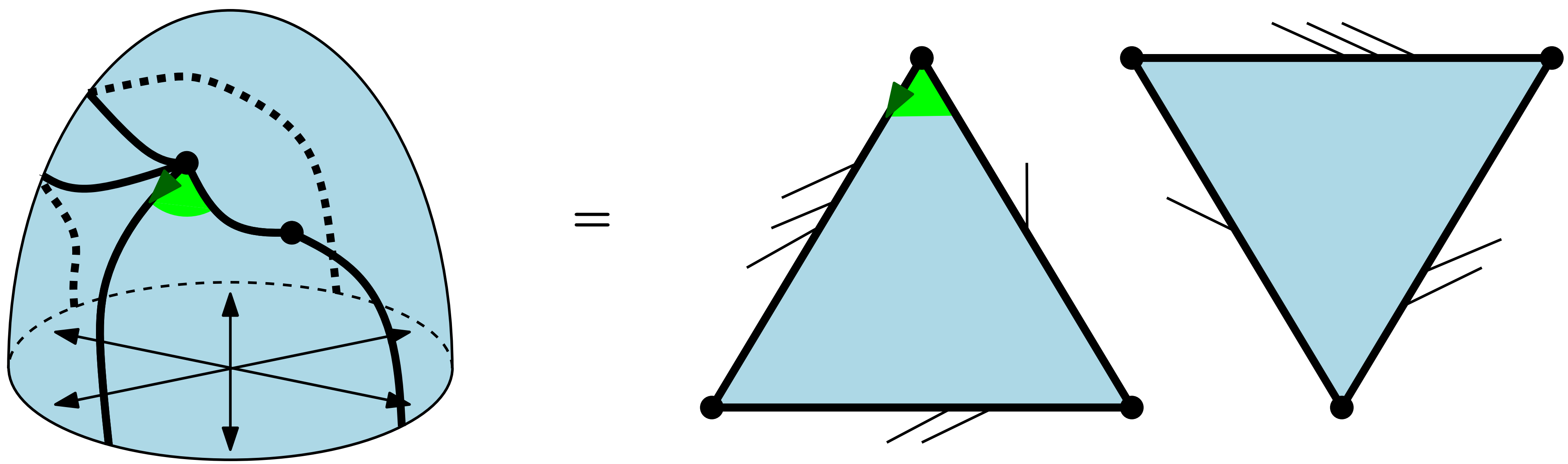

Remark:
Tutte noticed that maps are much simpler to enumerate, when rooted, because of the lack of symmetry. From now on, all maps will be rooted!

Maps with n edges vs. bipartite quadrangulations with n faces

Map M is bipartite if vertices can be colored by two different colors ($V_\bullet(M)$ - set of black vertices, $V_\circ(M)$ - set of white vertices, root vertex is black by convention) such that each edge connects two vertices of different colors.

Quadrangulation is a map with all faces of degree 4.

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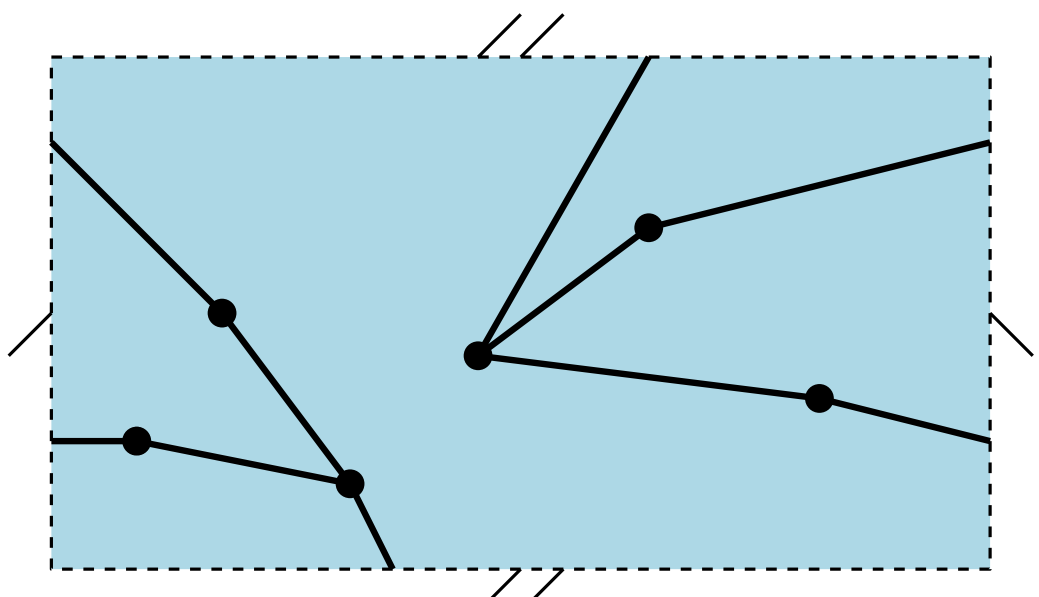

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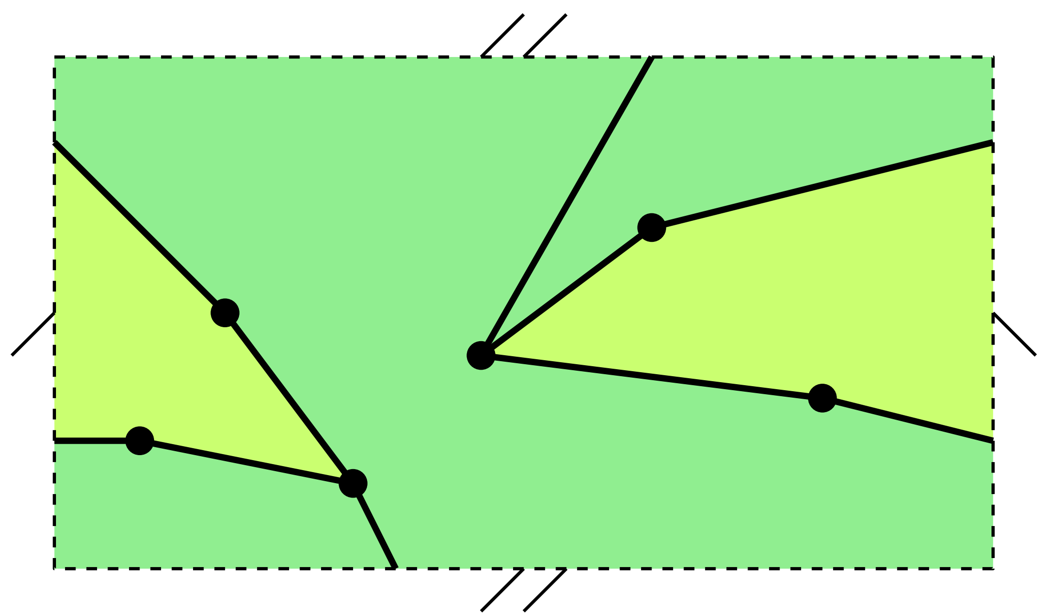

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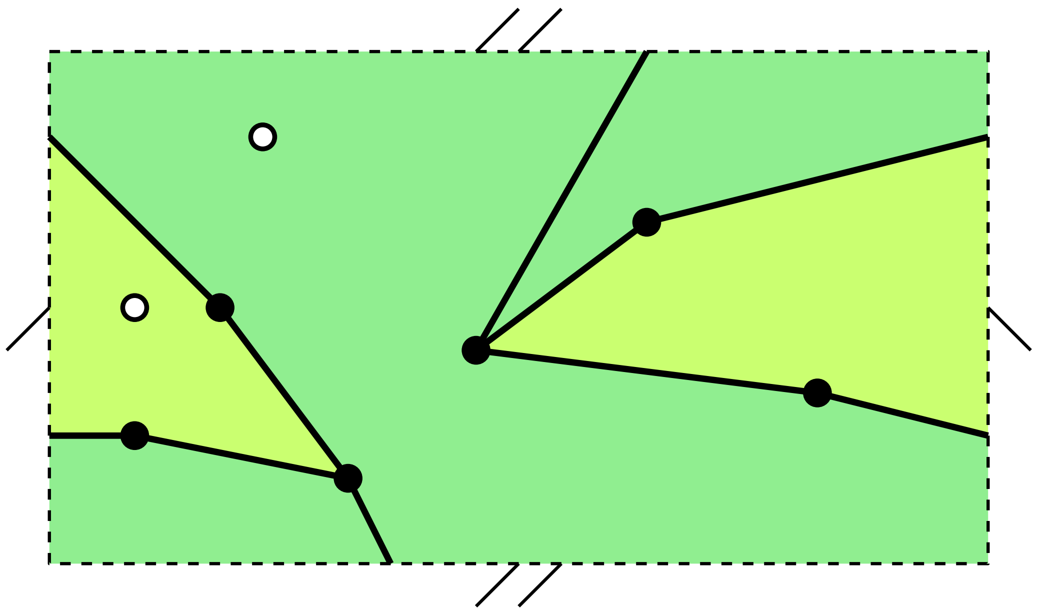

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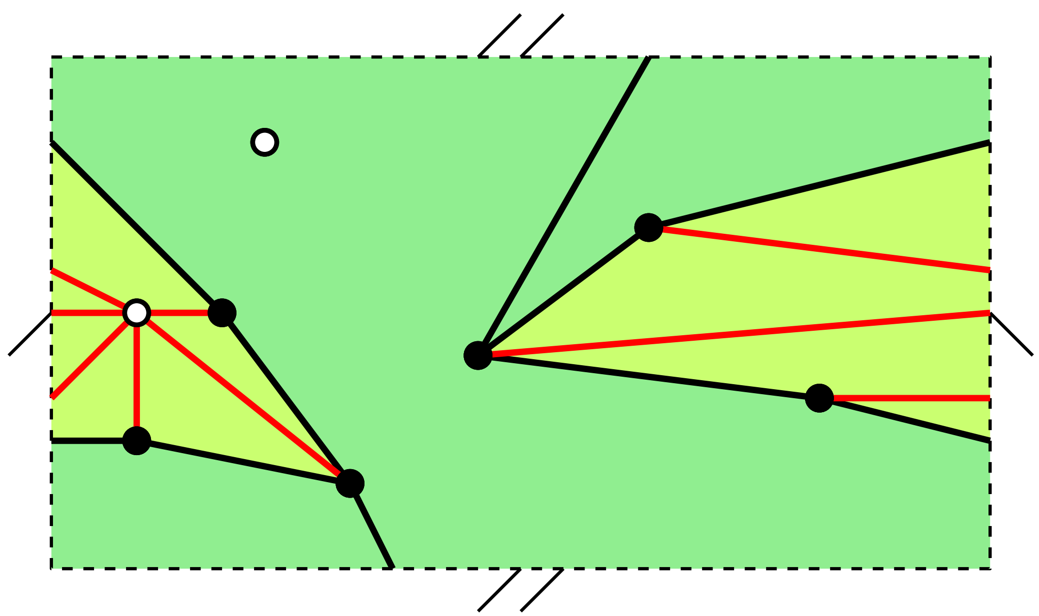

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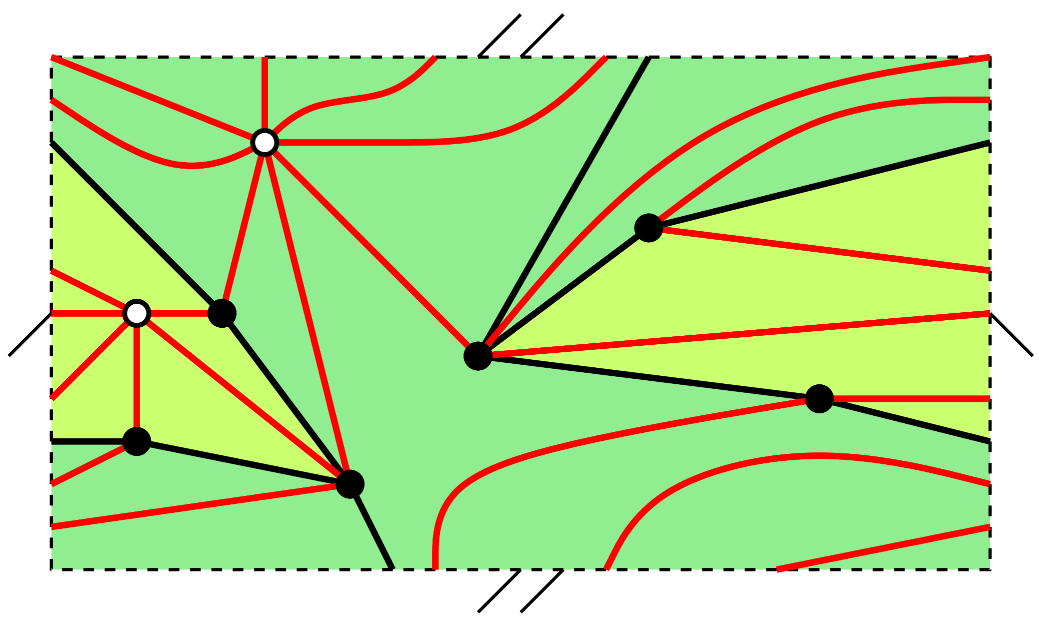

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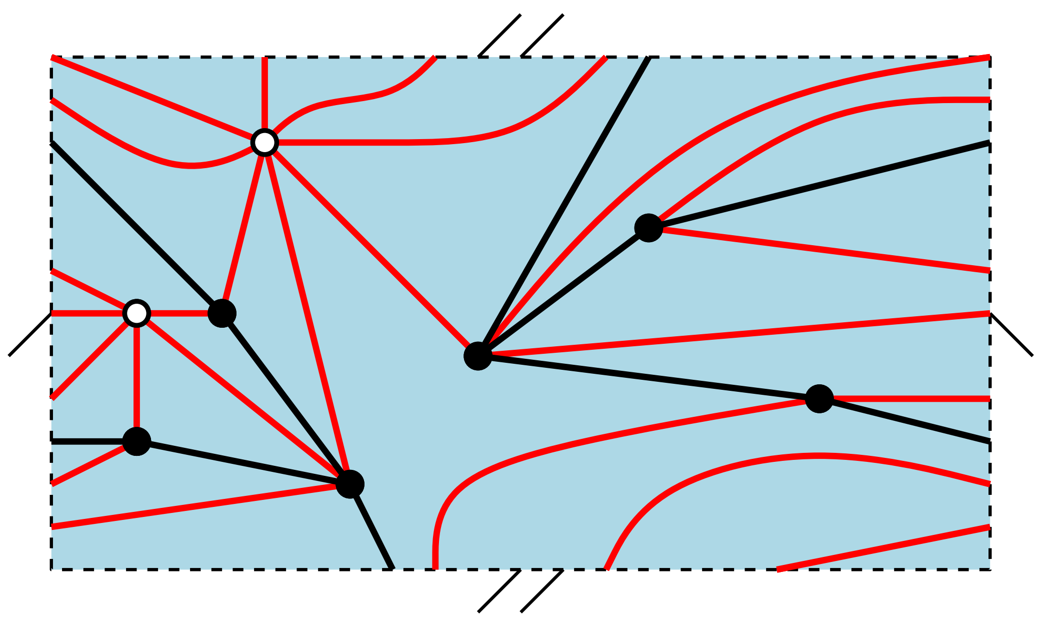

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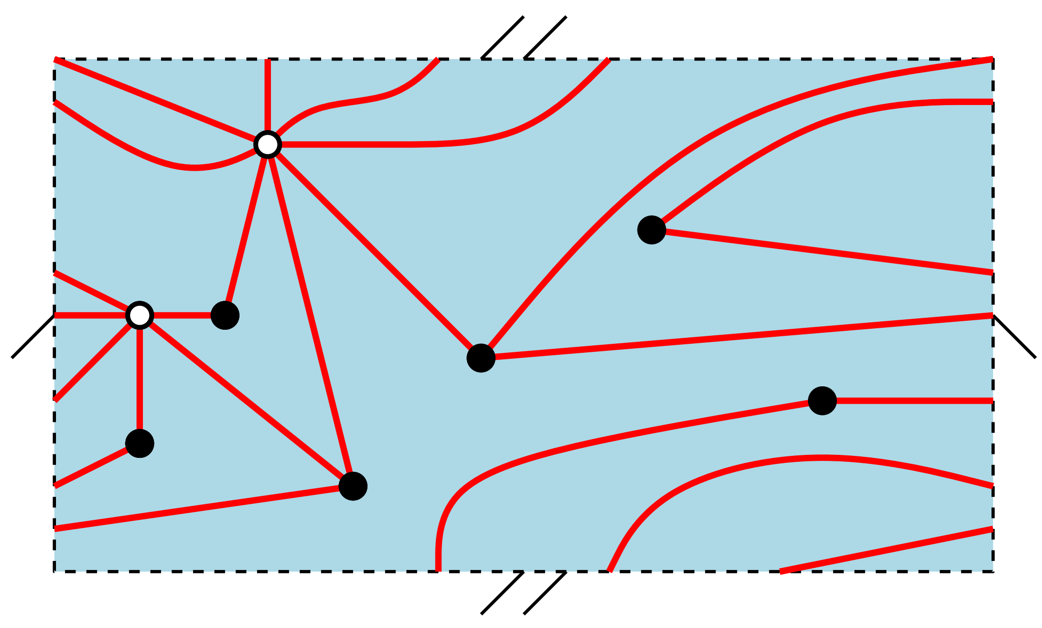

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Find a bijection between maps and some objects with a WELL-UNDERSTOOD structure!

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Understanding a geometry of a random surface:
- growing maps as a discrete model of a continuous manifold,
- geometry of a random surface = geometry of a random map, when its size tends to infinity,
- bijection helps to understand a discrete surface as a metric space!

II. Bijections for bipartite quadrangulations and tree–like labeled structures

Labeled and well-labeled maps

A map is called labeled if its vertices are labeled by integers such that:

- the root vertex has label 1;
- if two vertices are linked by an edge, their labels differ by at most 1.

If in addition we have:

- all the vertex labels are positive,

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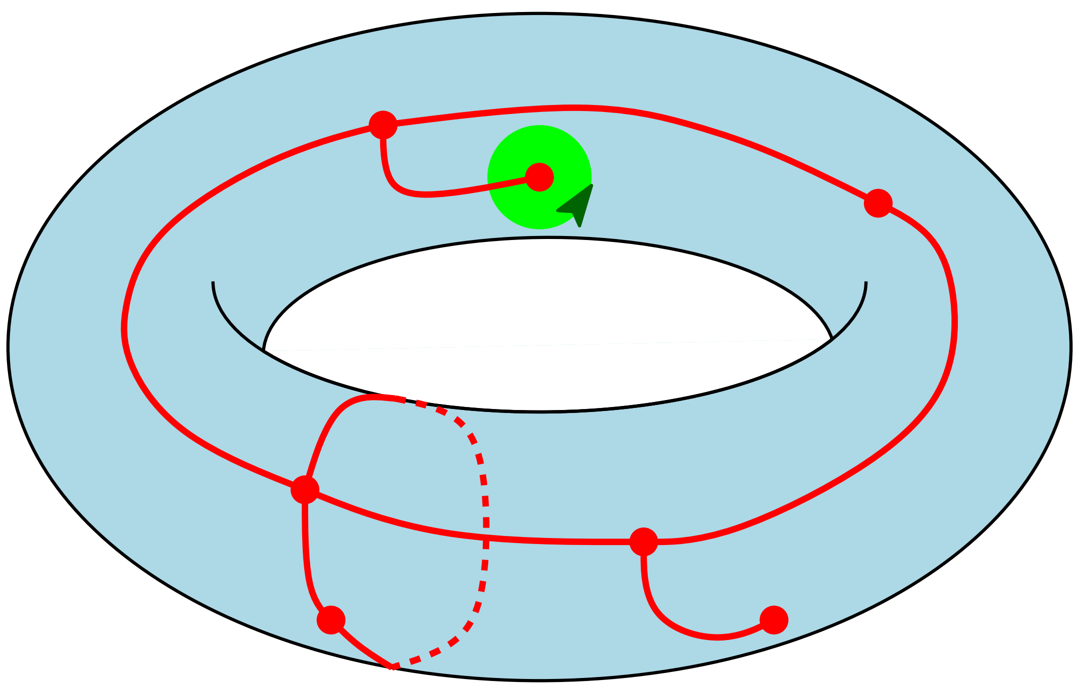

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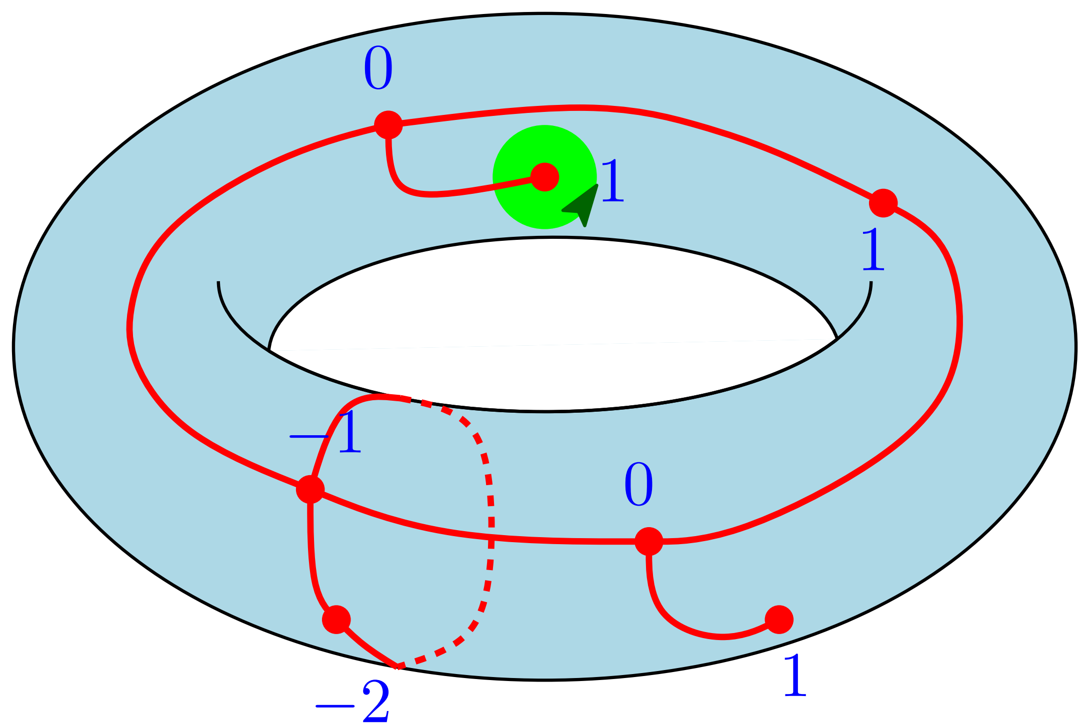

labeled map on a torus

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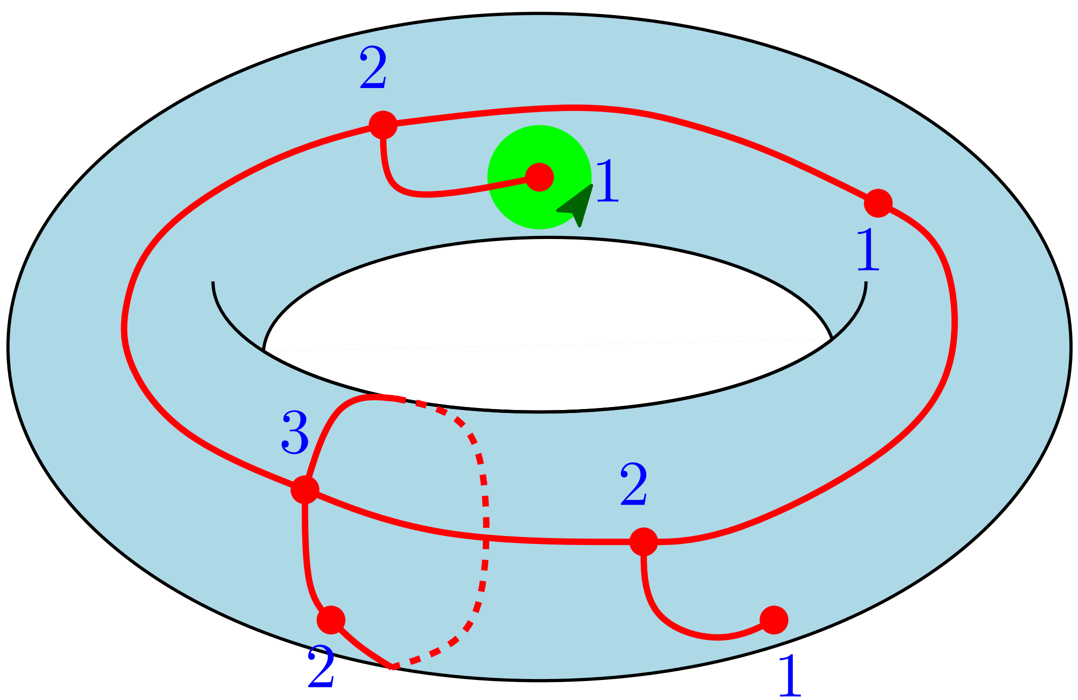

well-labeled map on a torus

Orientable case

Theorem [Marcus, Schaeffer 1996]
There exists a bijection between:

- rooted, bipartite quadrangulations on ORIENTABLE surface $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)$;
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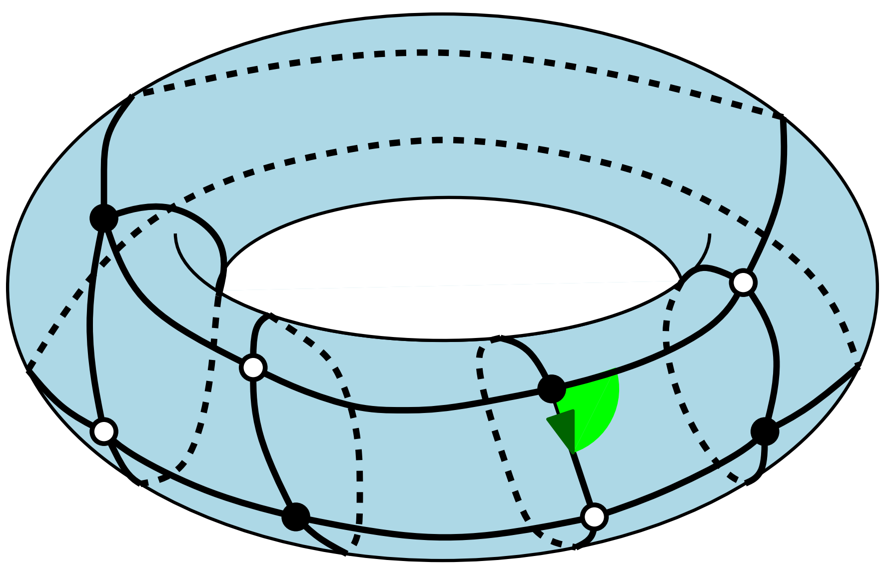

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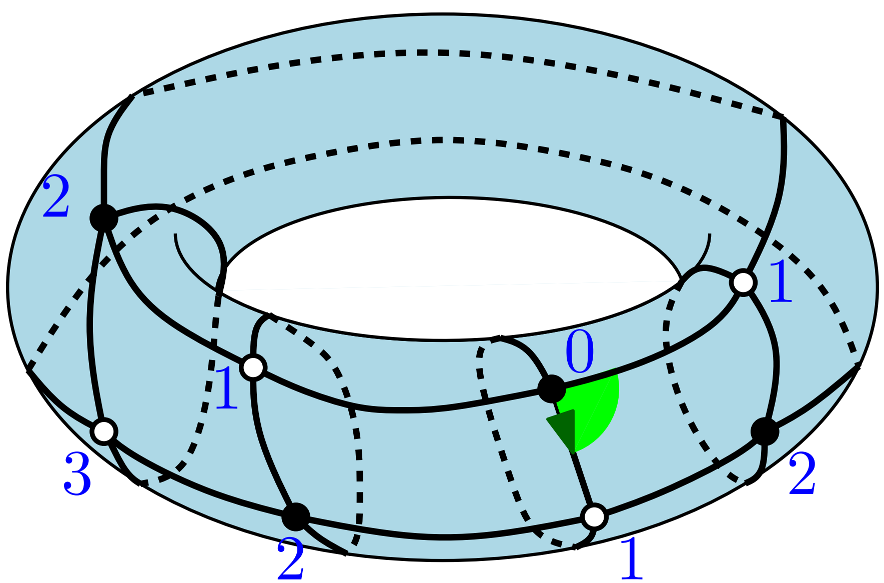

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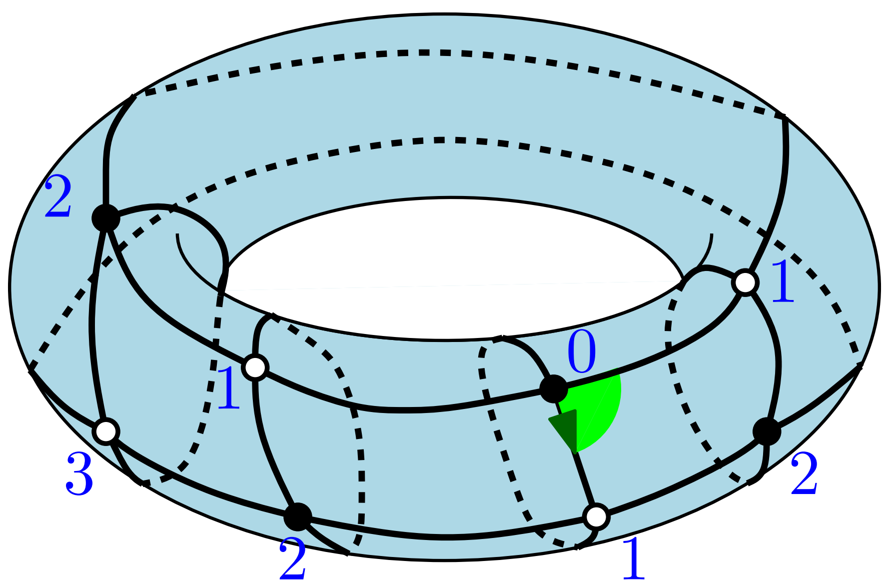

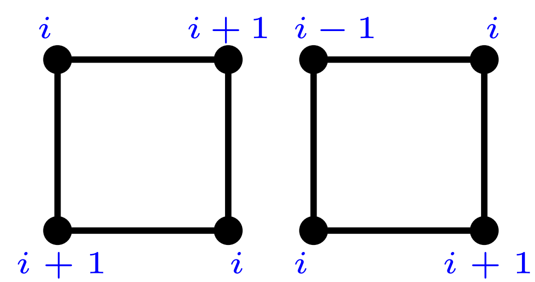

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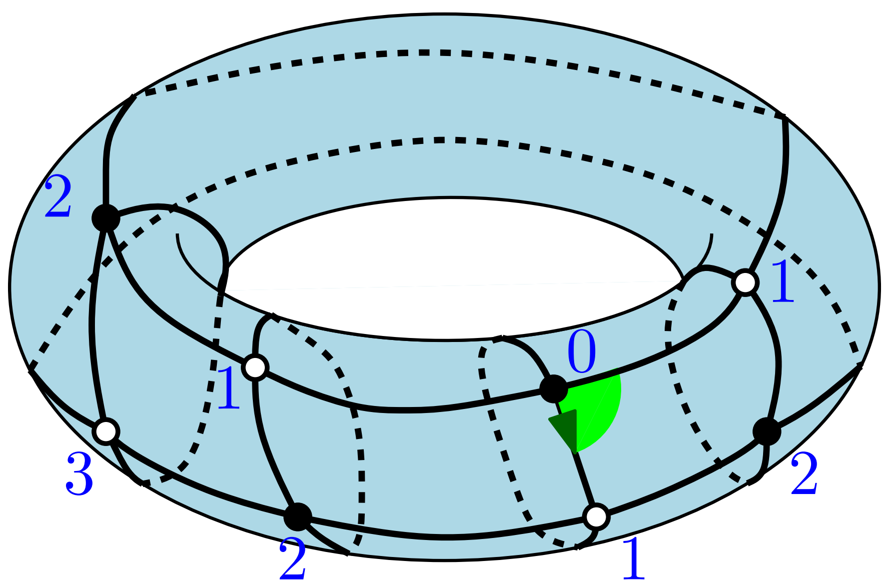

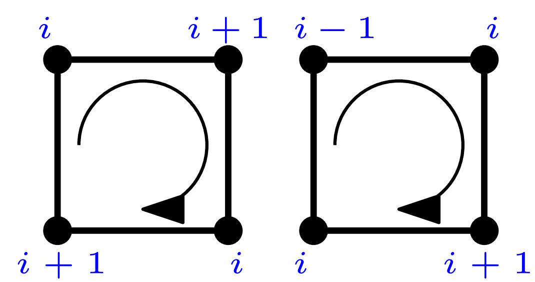

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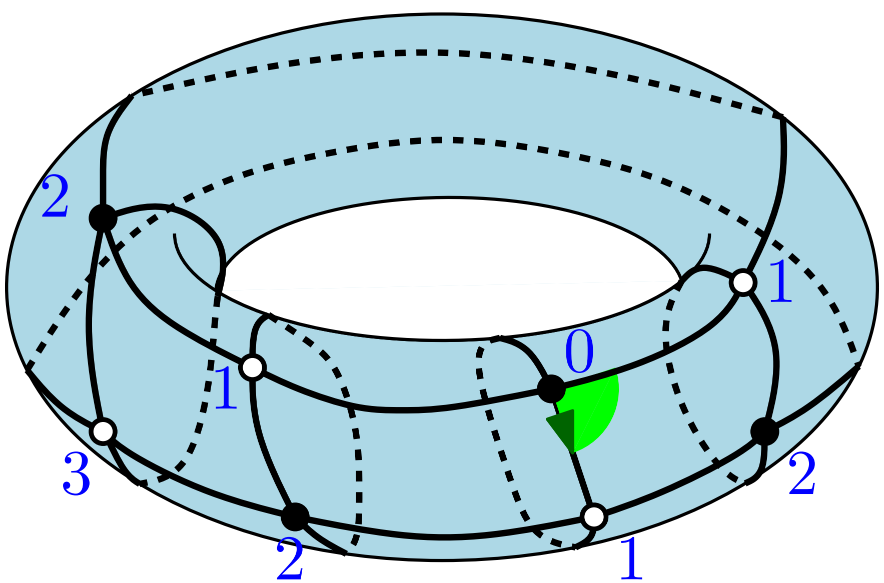

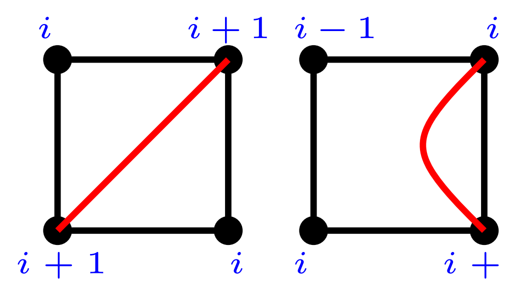

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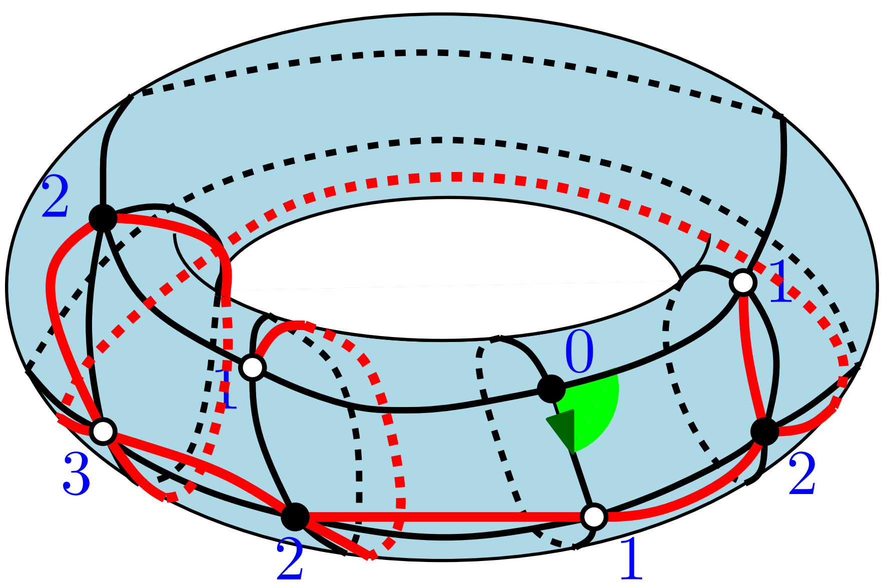

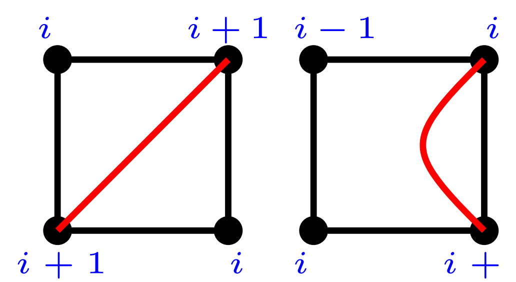

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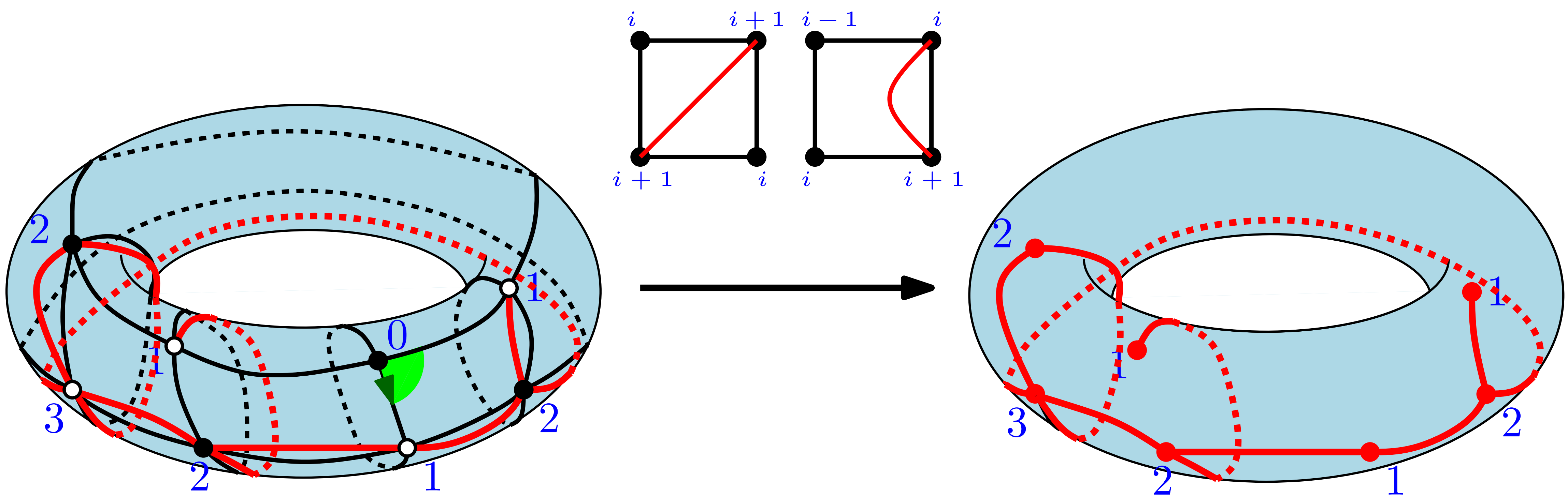

Orientable case

Theorem [Marcus, Schaeffer 1996]

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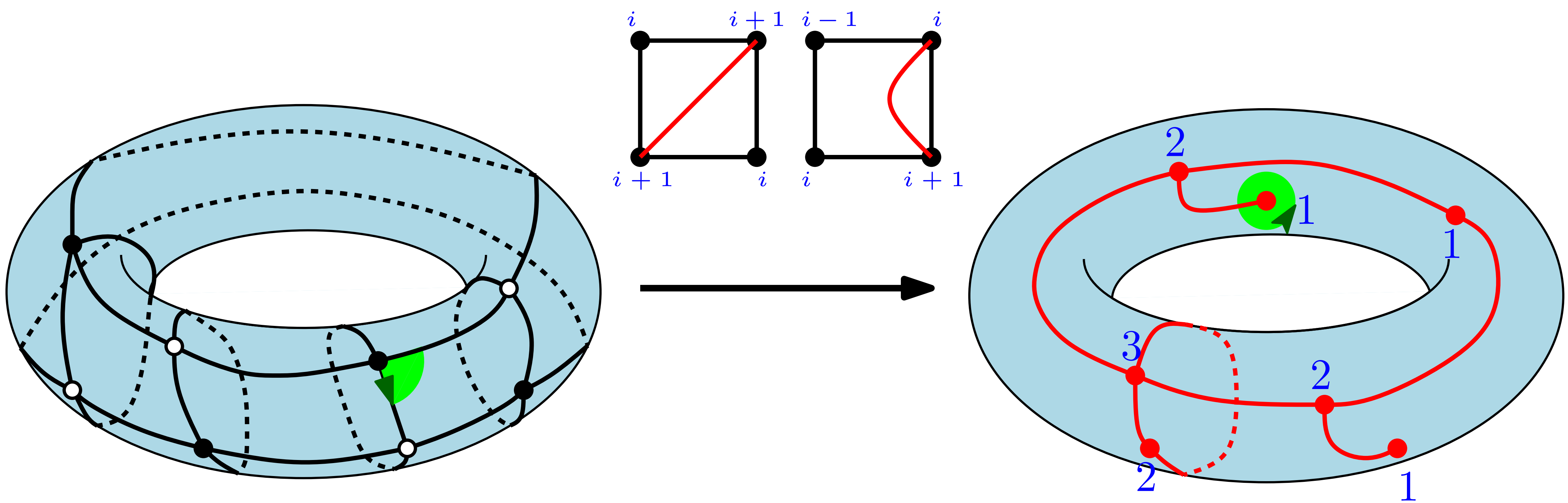

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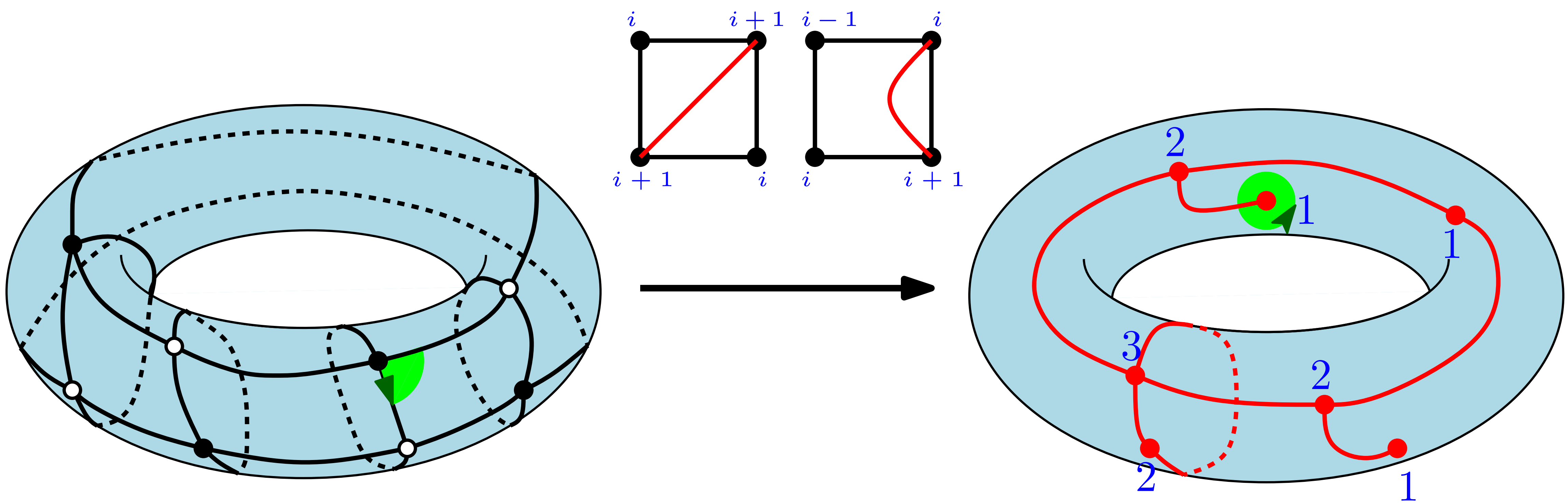

Are **non-orientable** maps different?

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Theorem [Chapuy, D. 2015]

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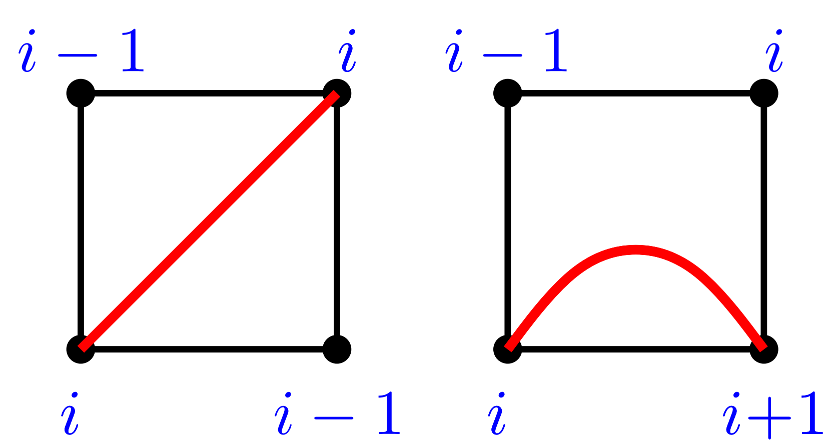

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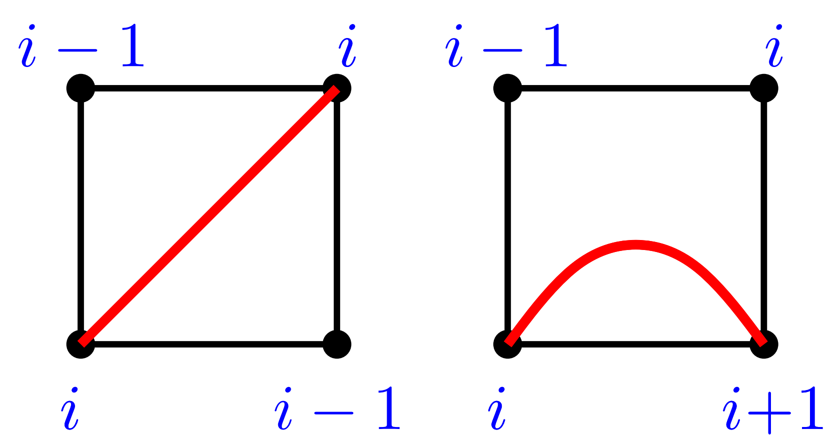

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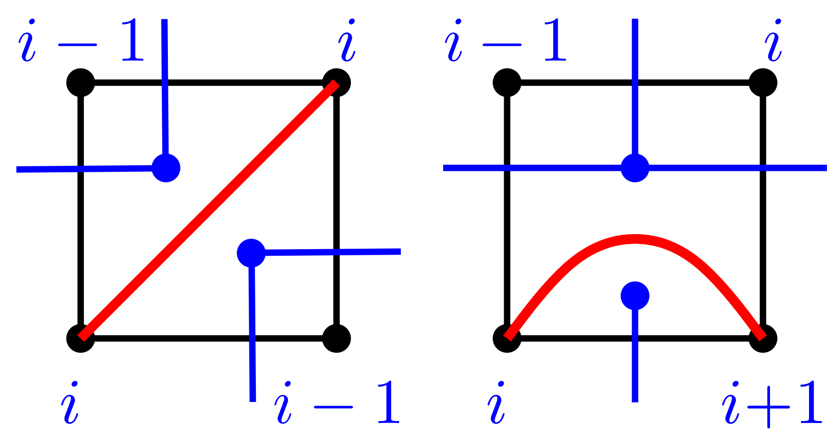

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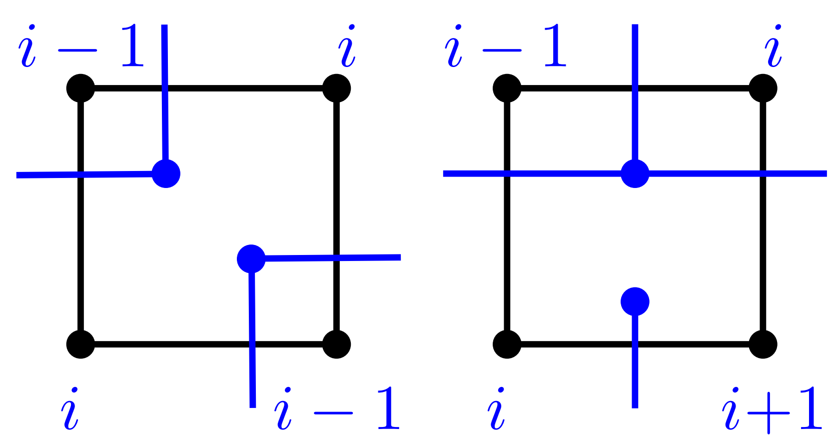

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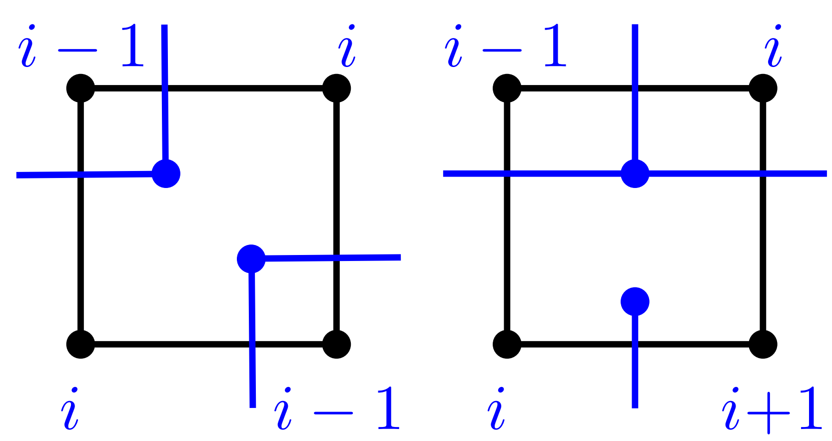

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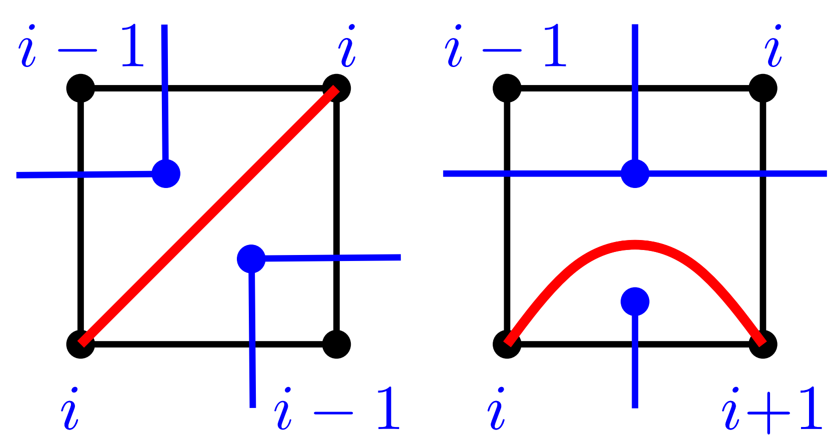

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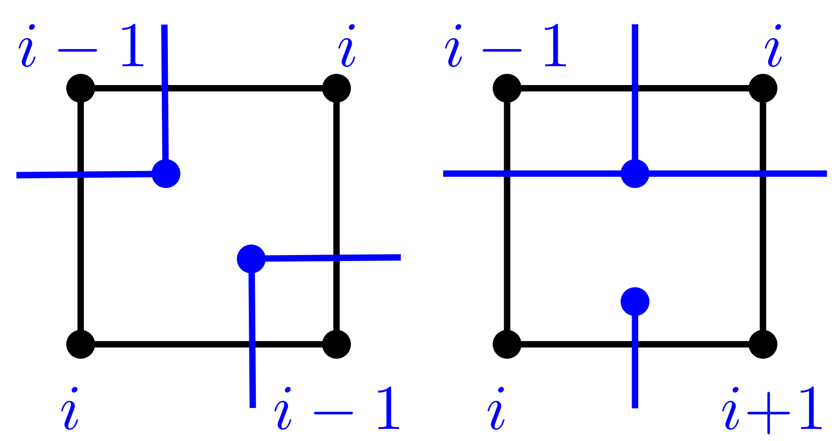
• If the construction of blue graph is local then it is invertible and it leads to a **BIJECTION!**



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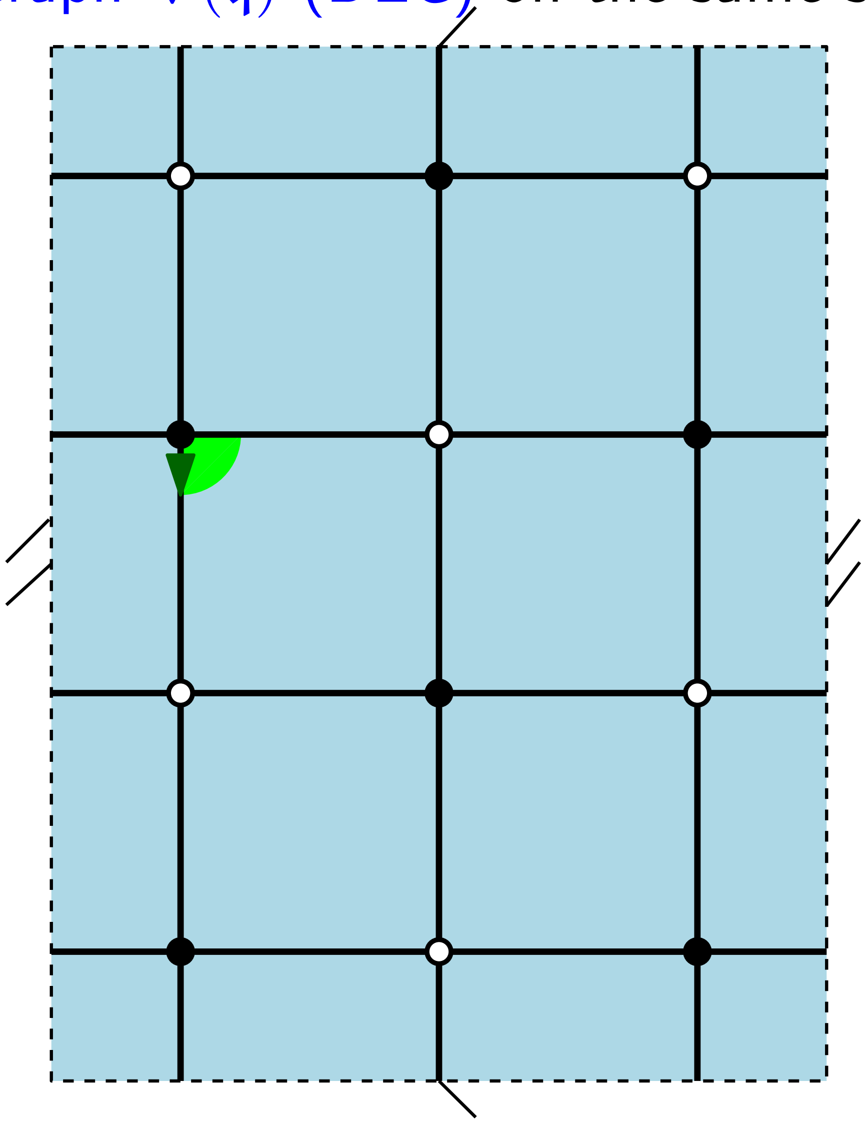

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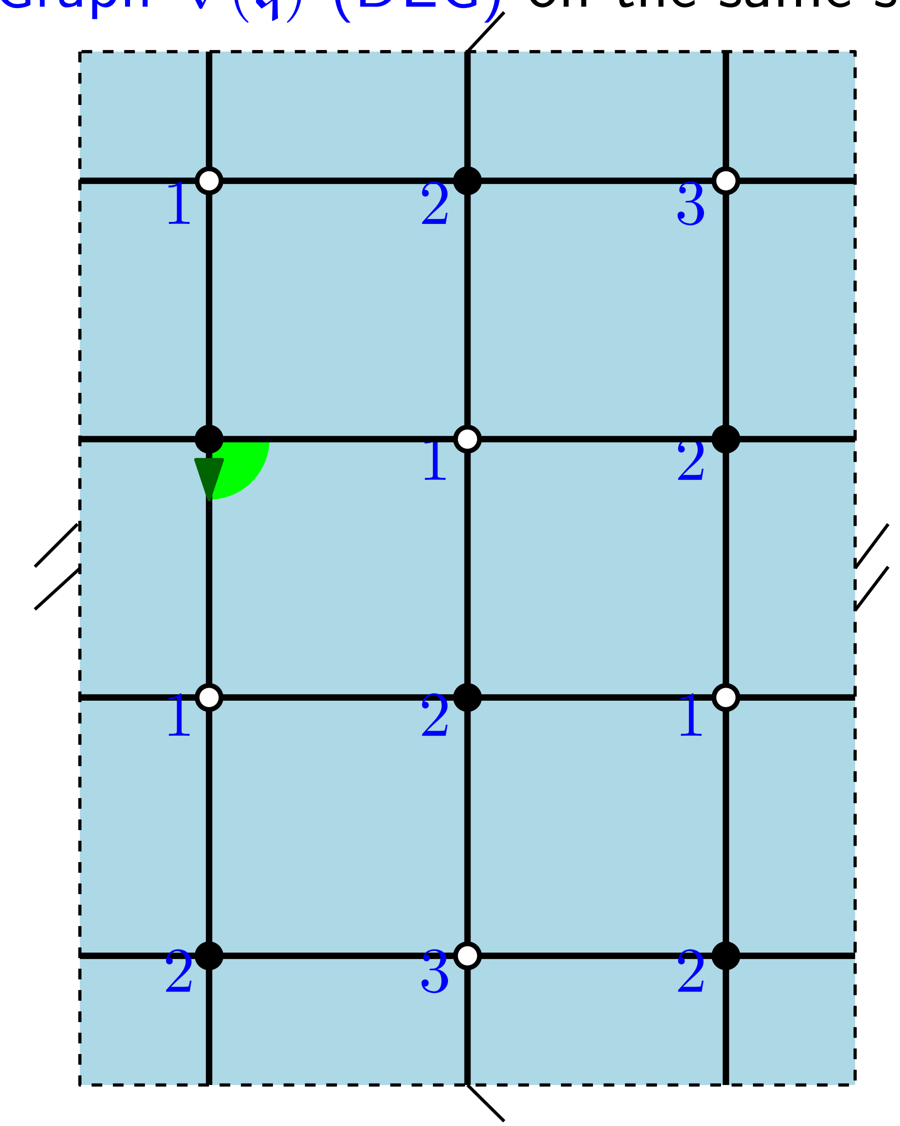

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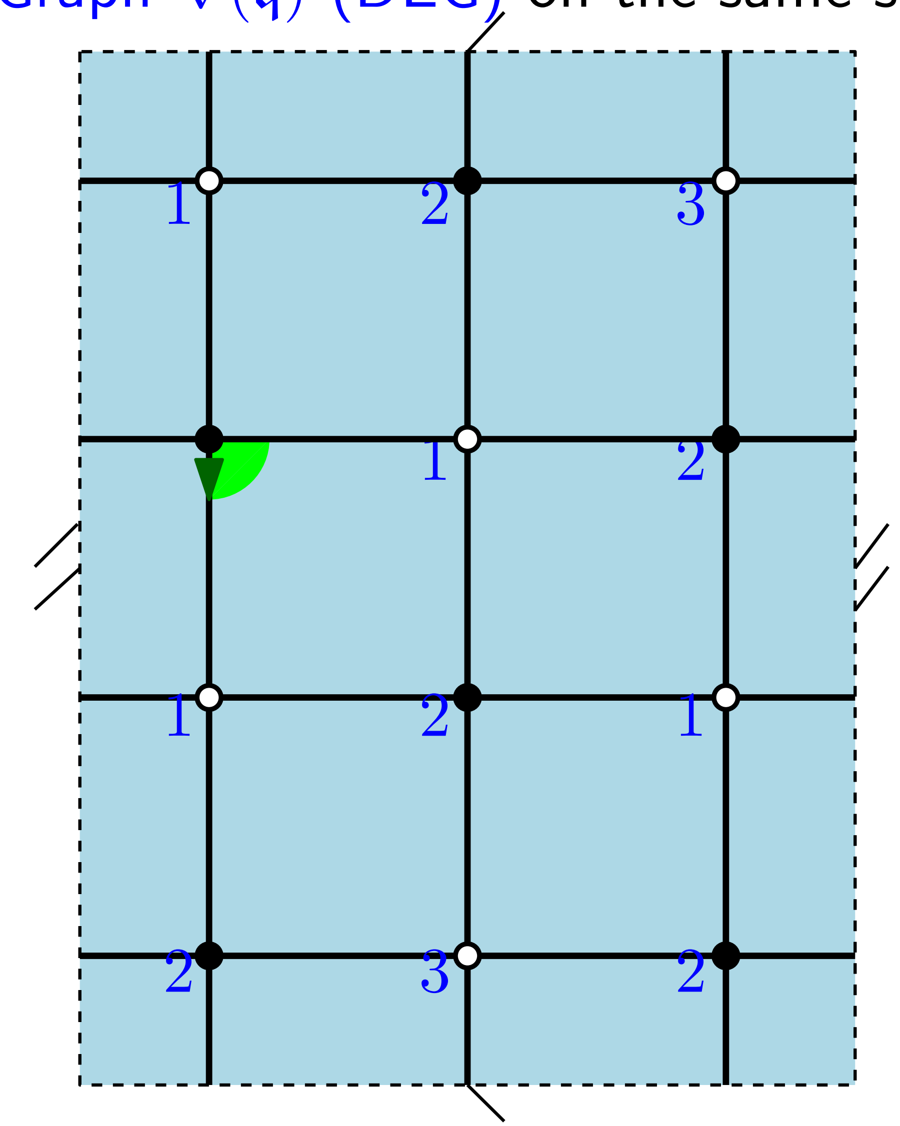

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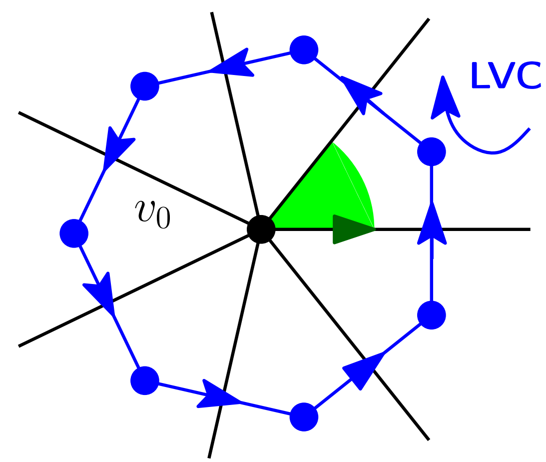

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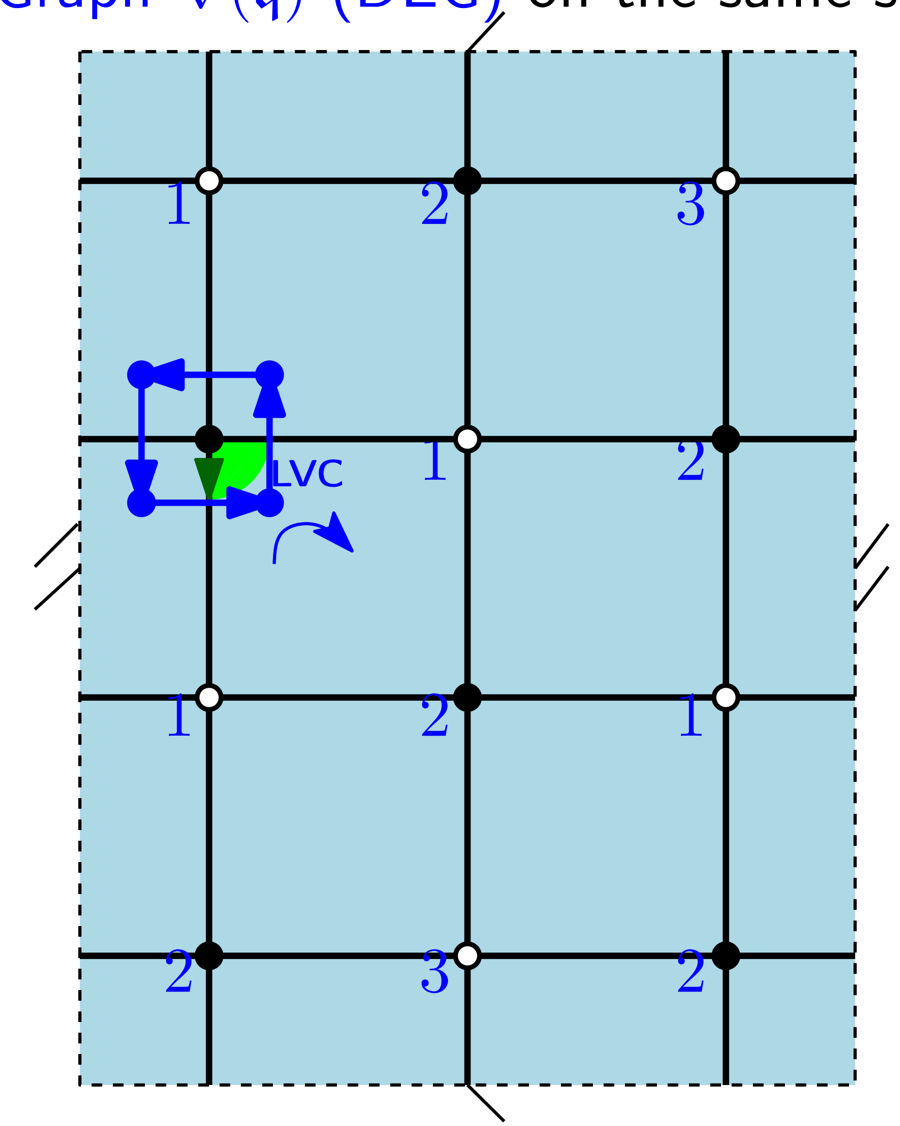

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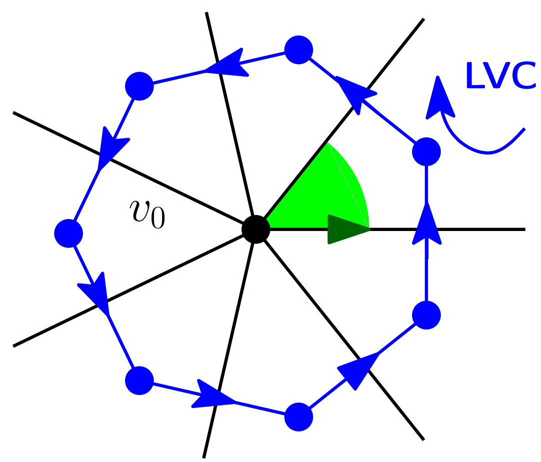

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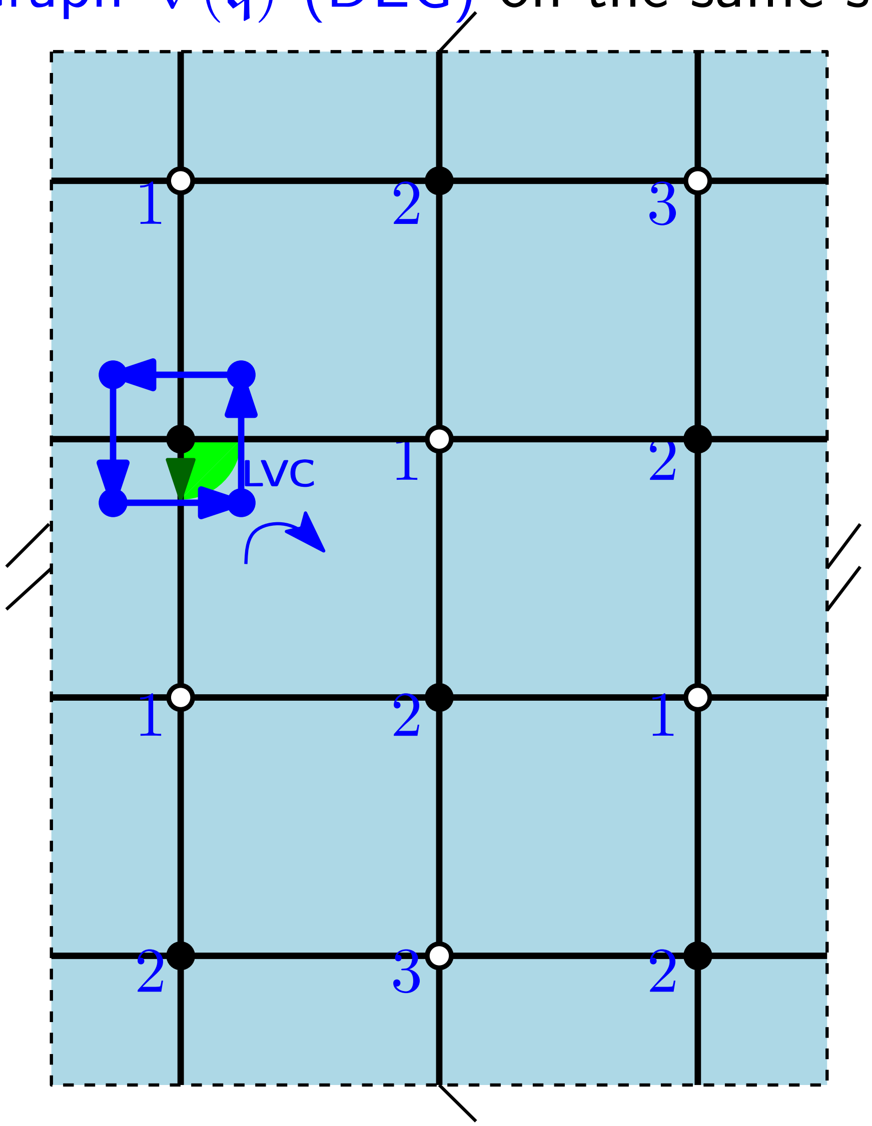

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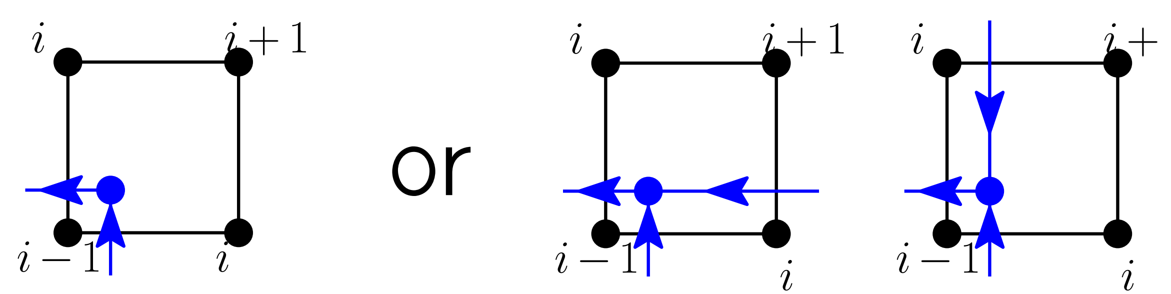

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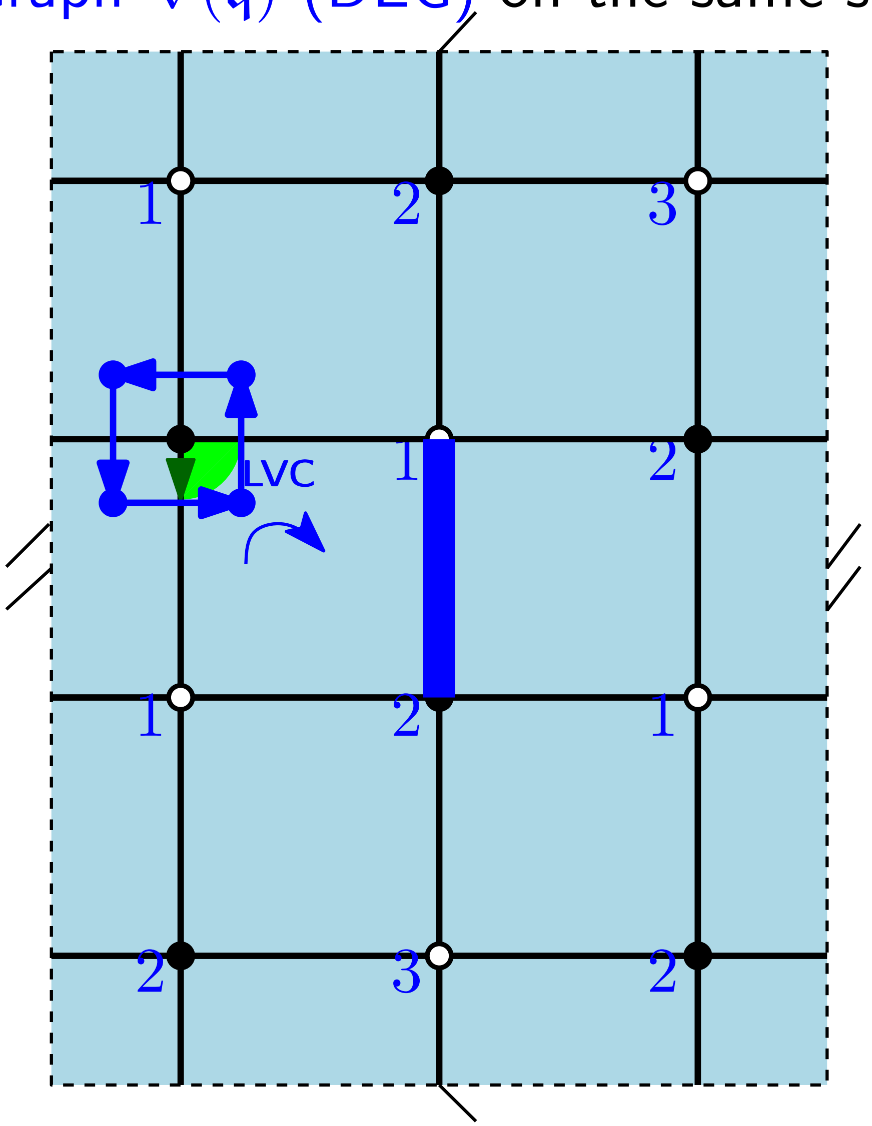

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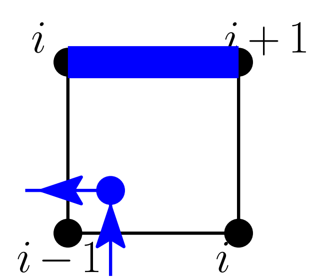

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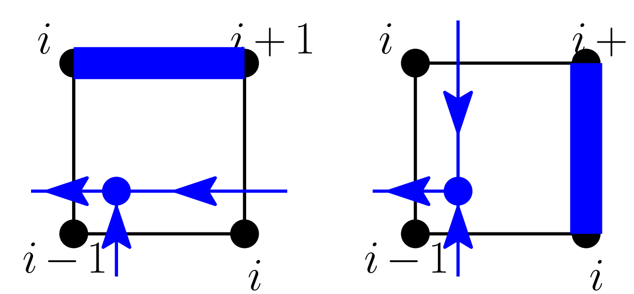

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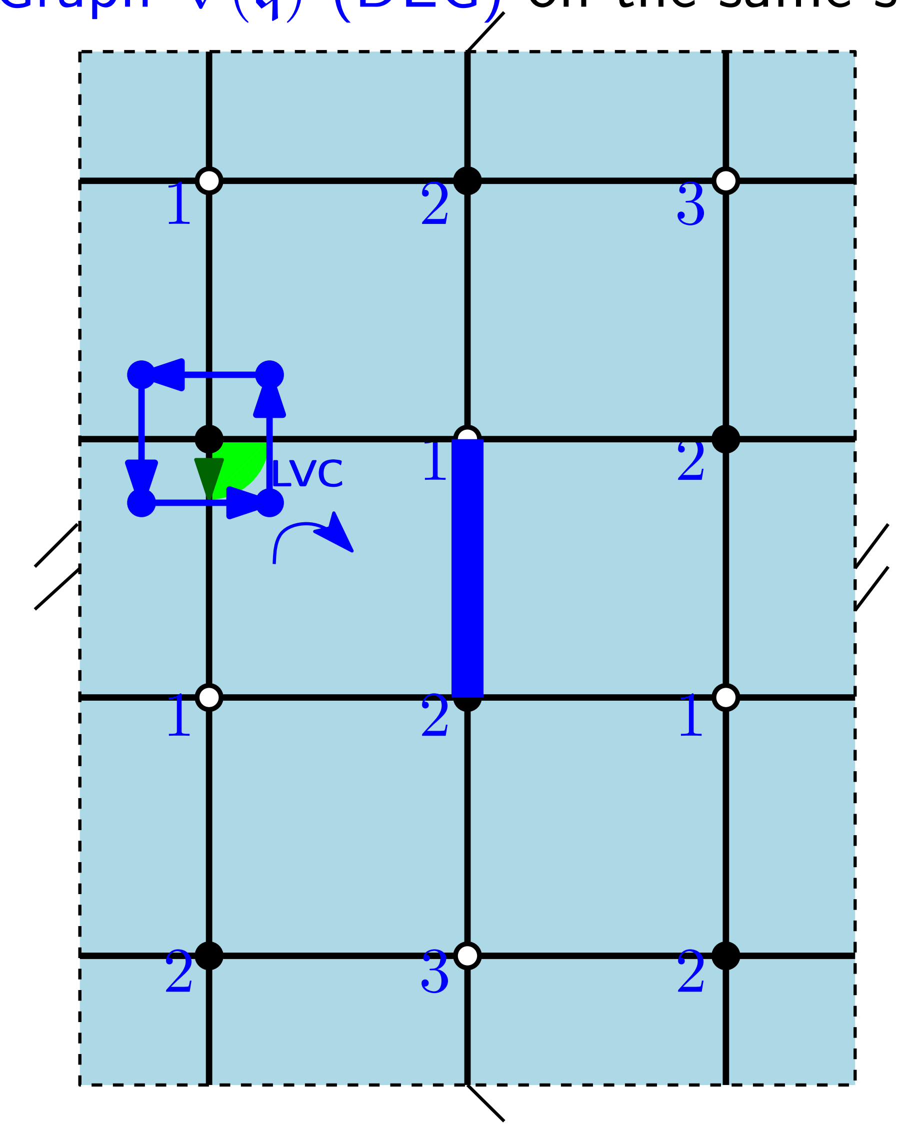

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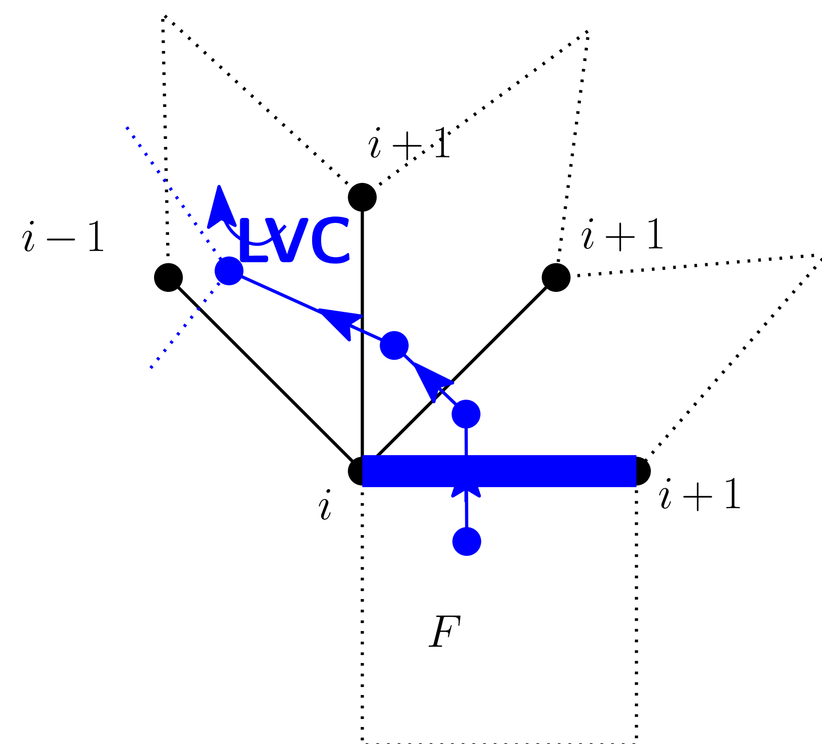

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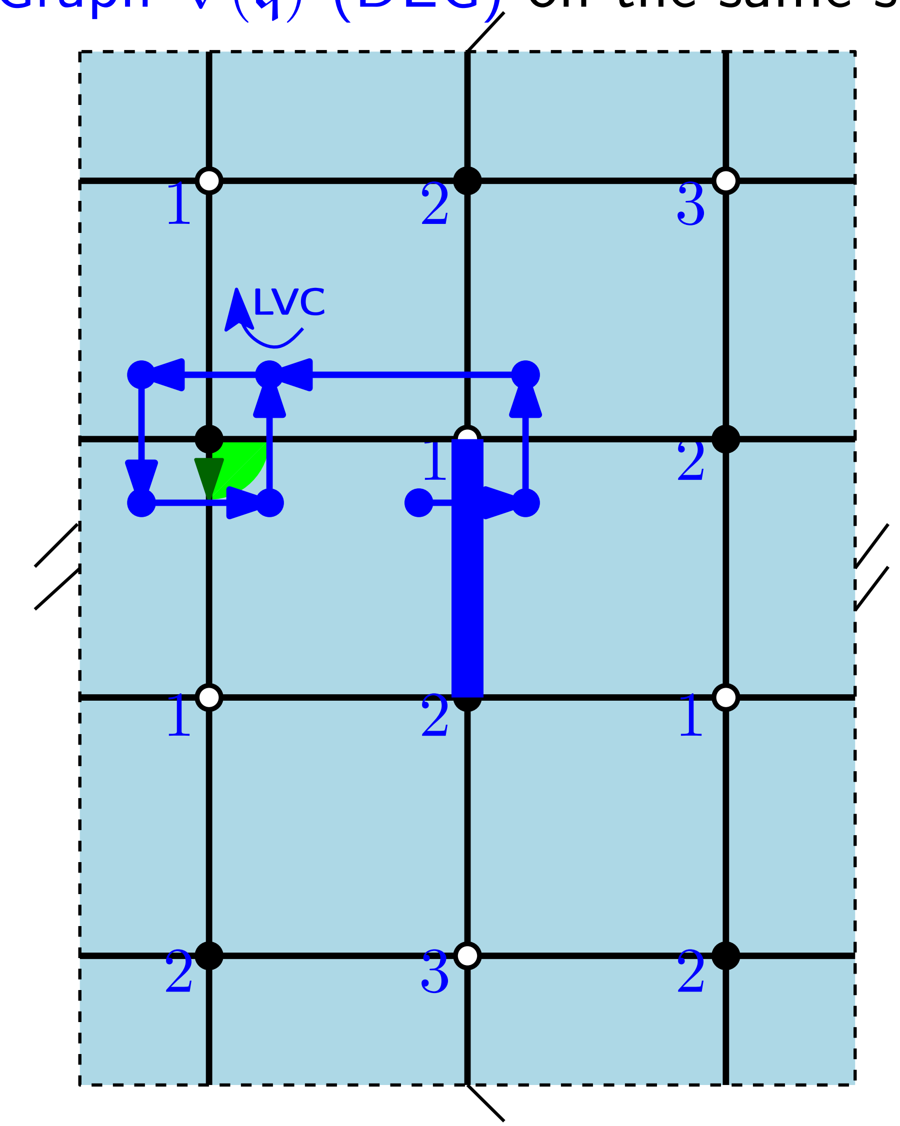

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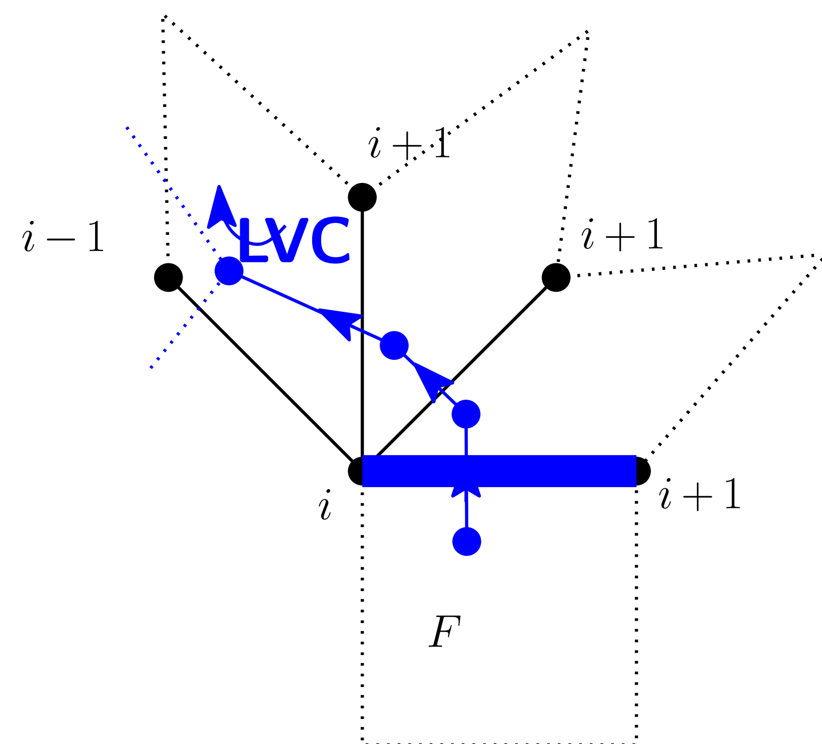

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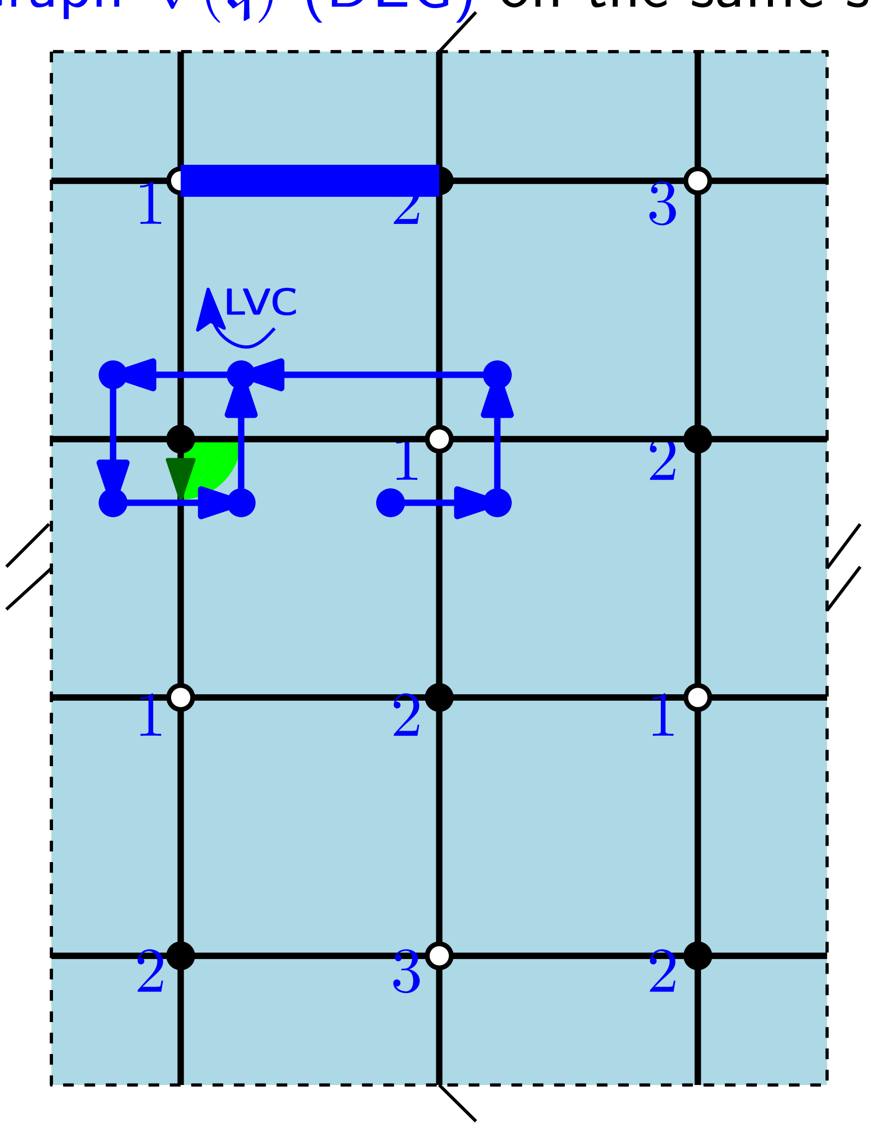

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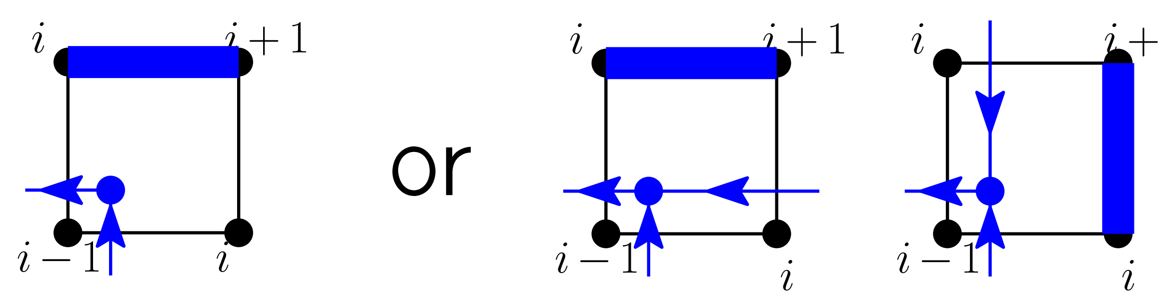

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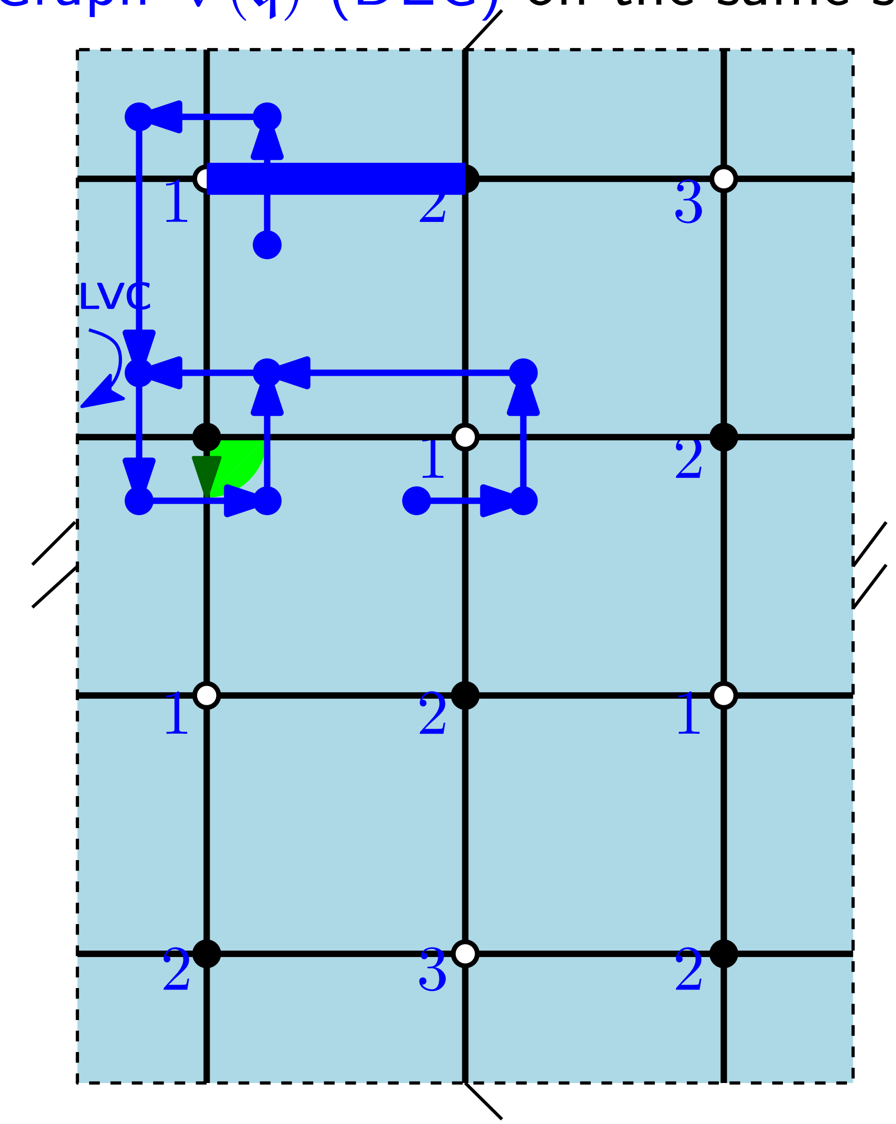

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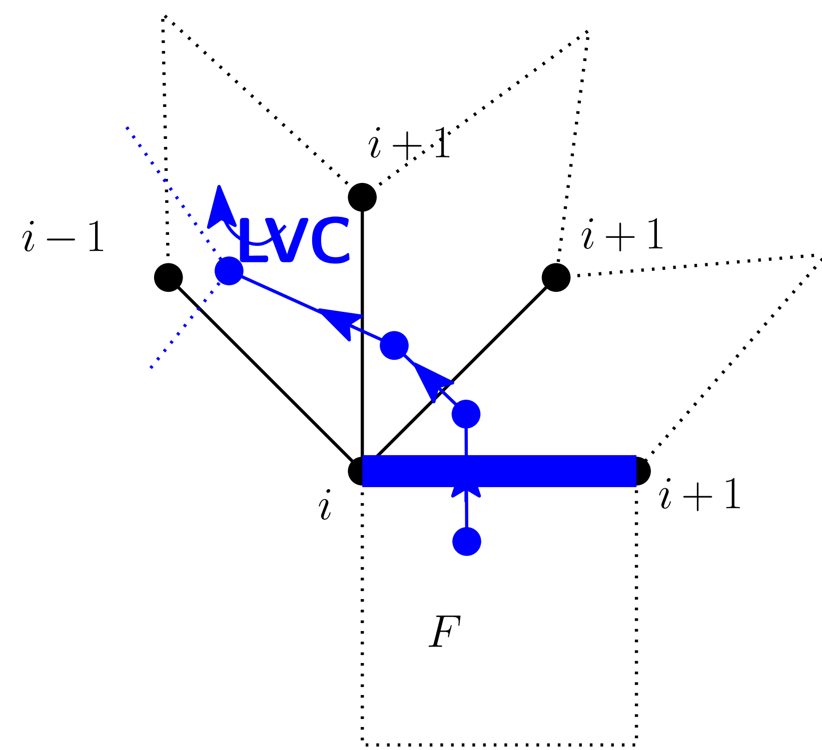

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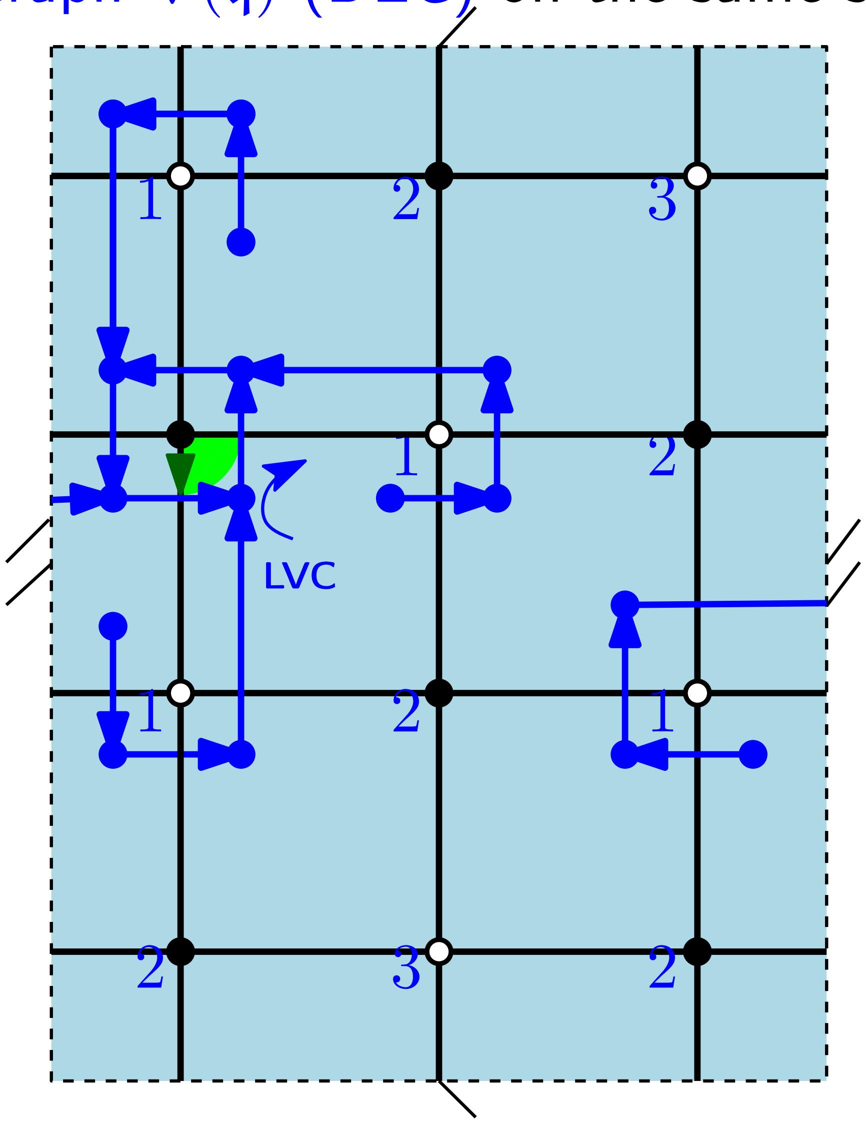

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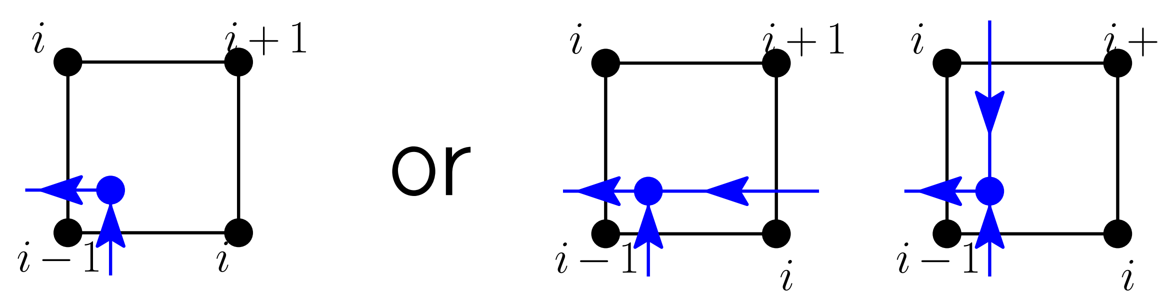

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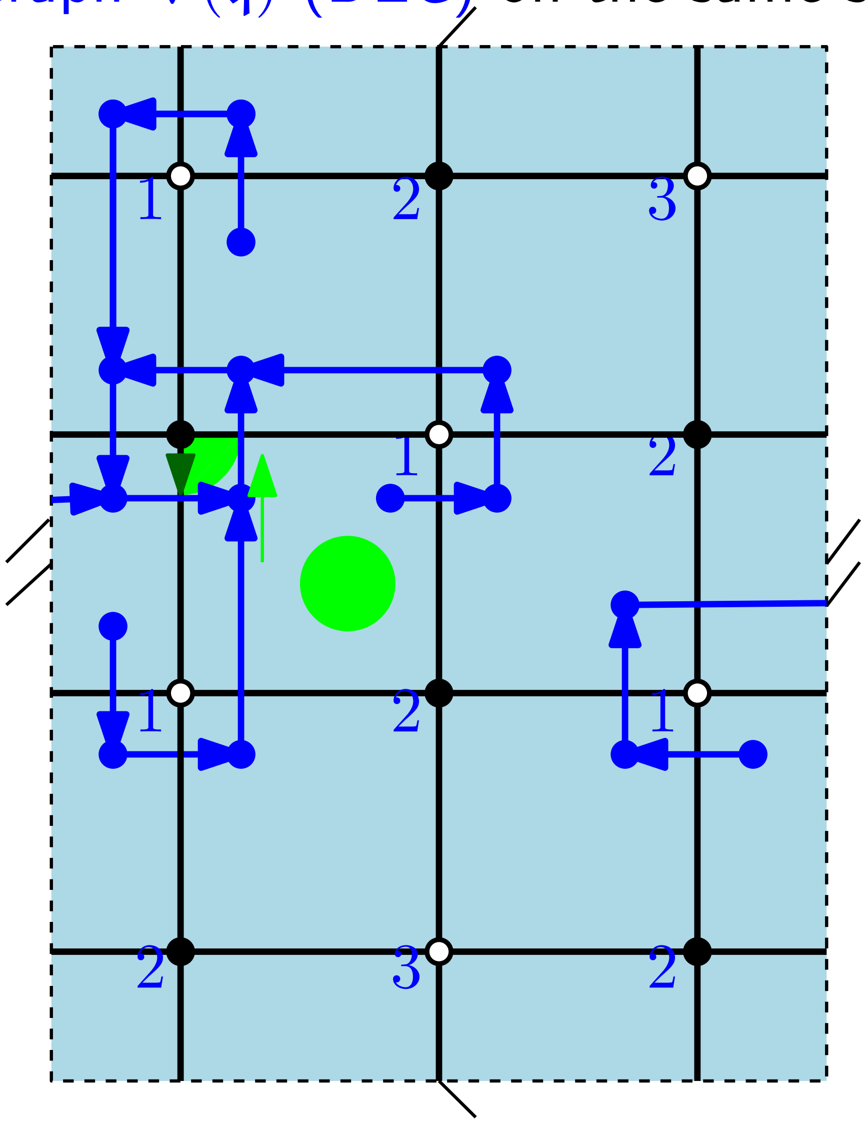

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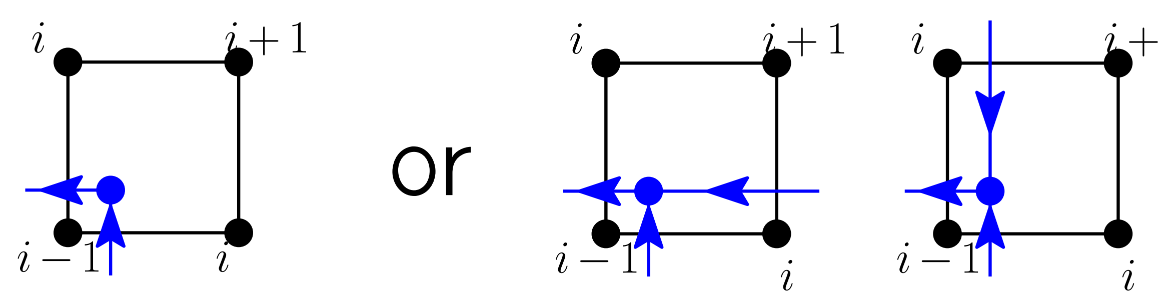

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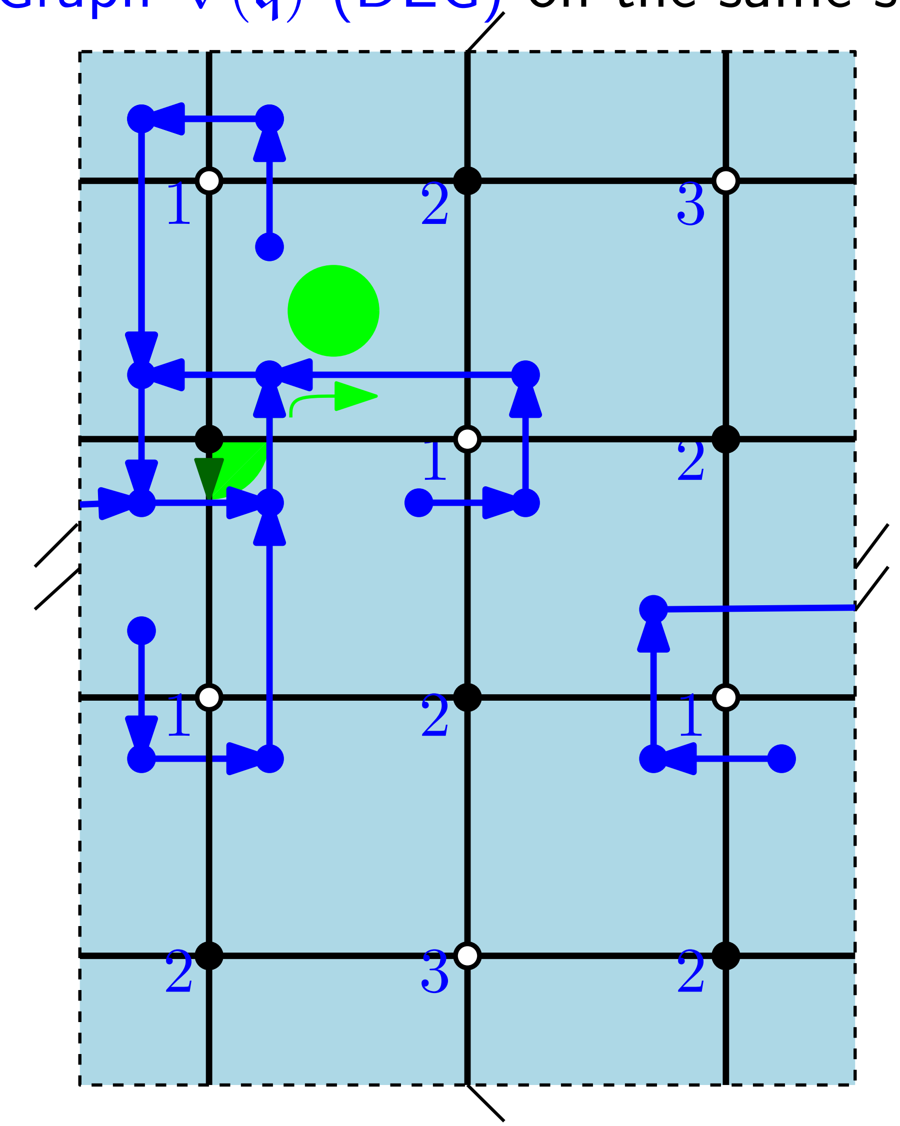

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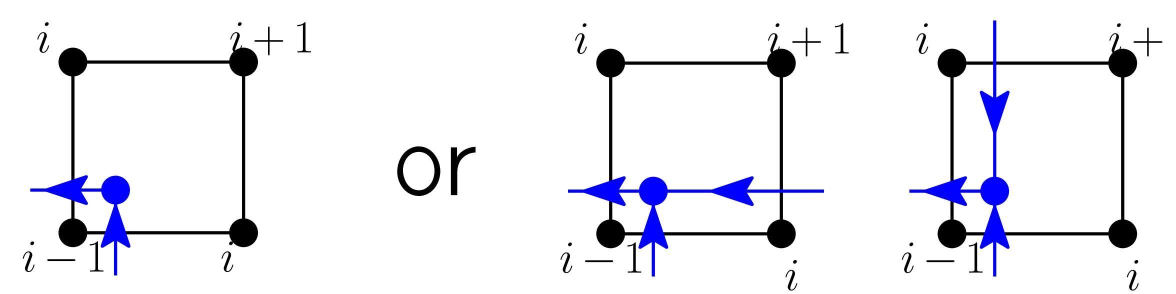

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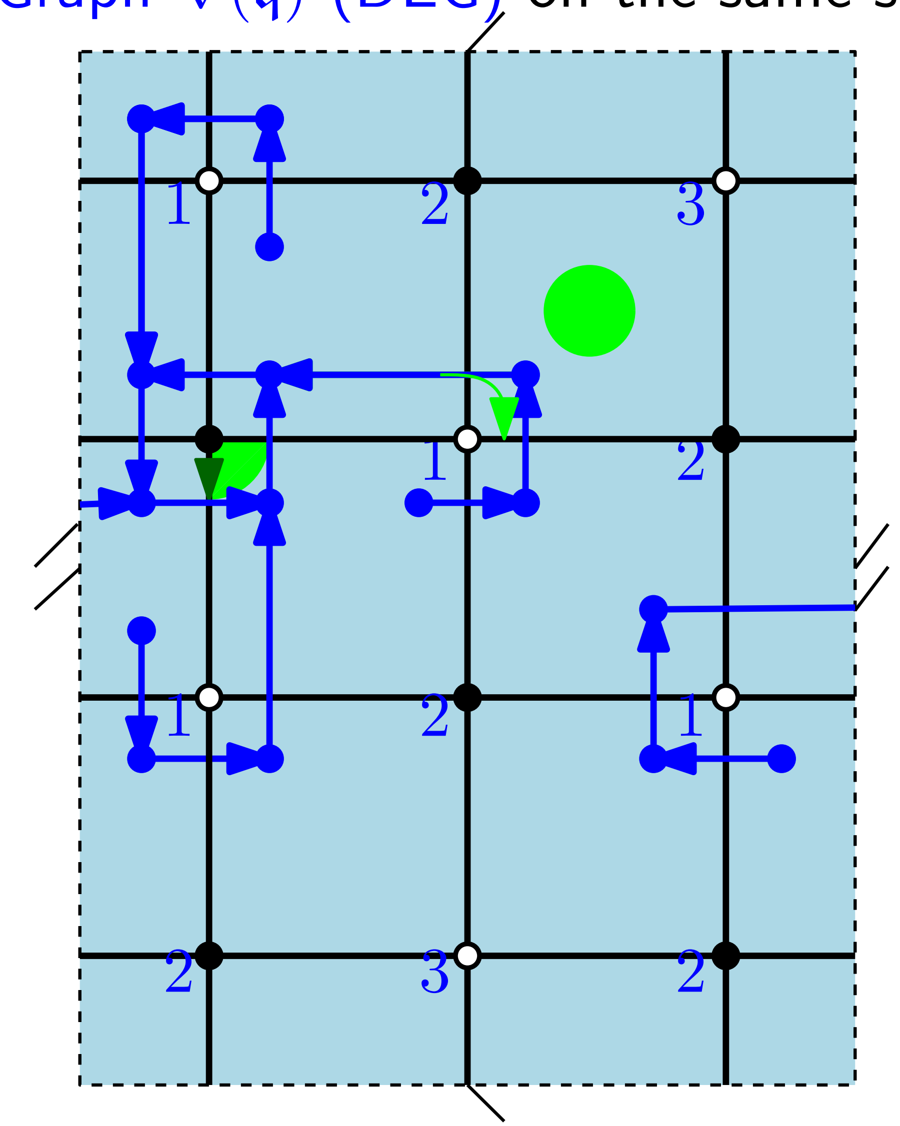

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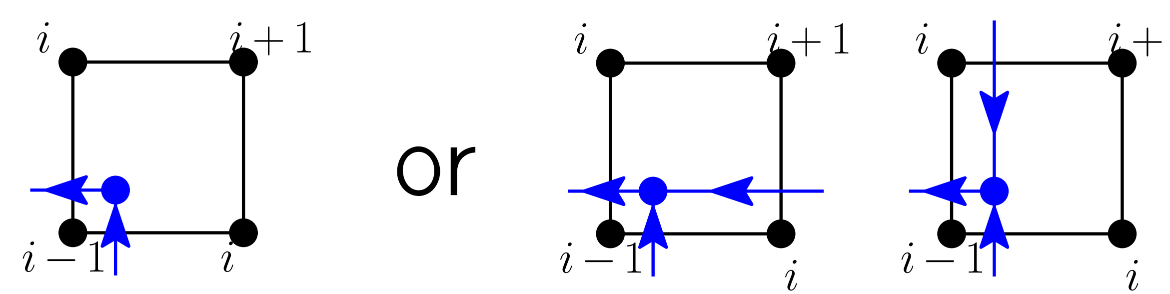

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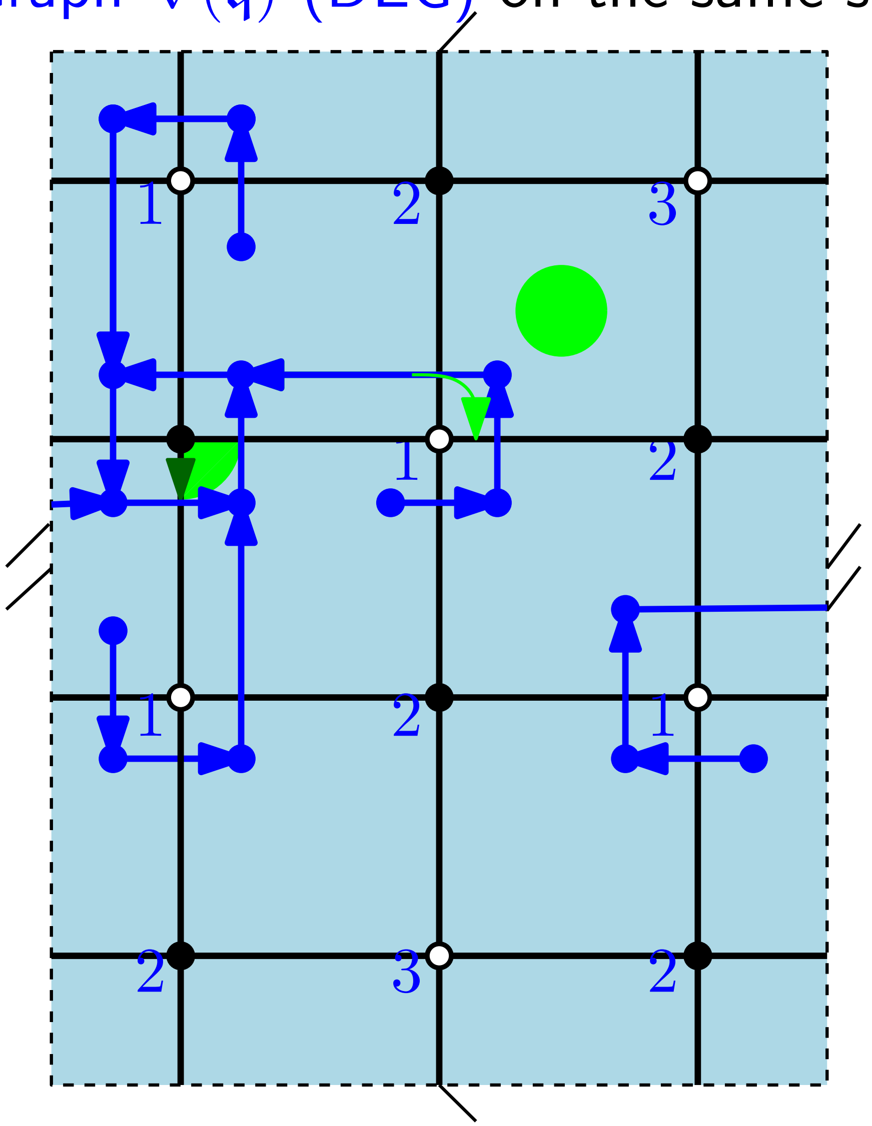

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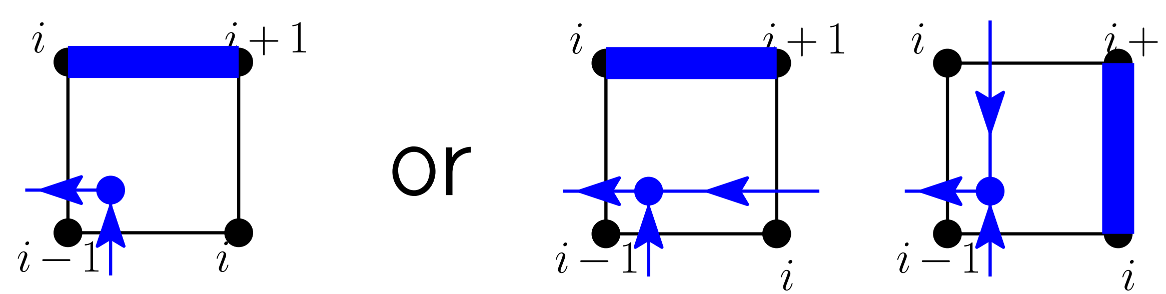

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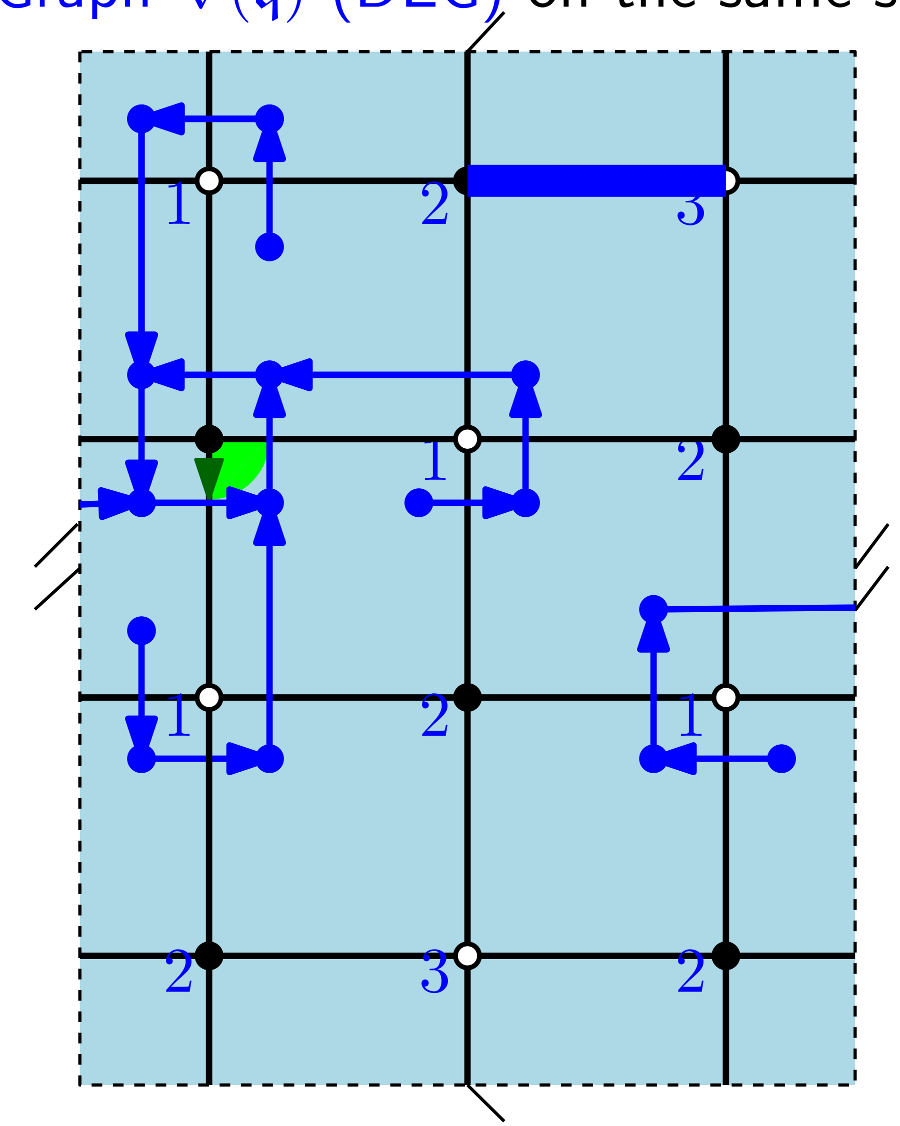

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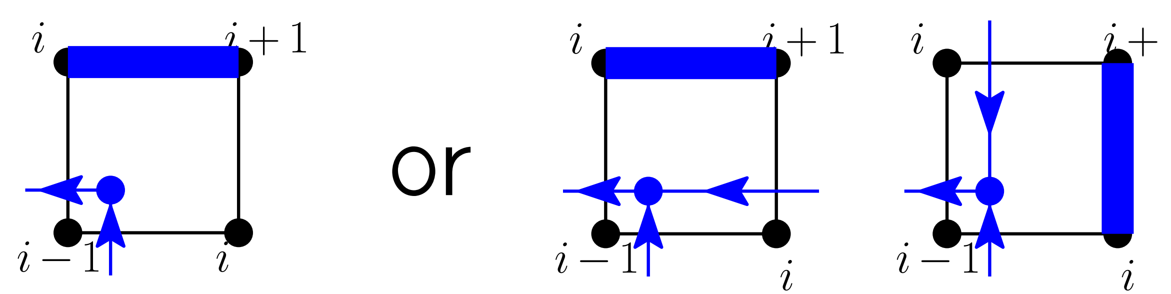

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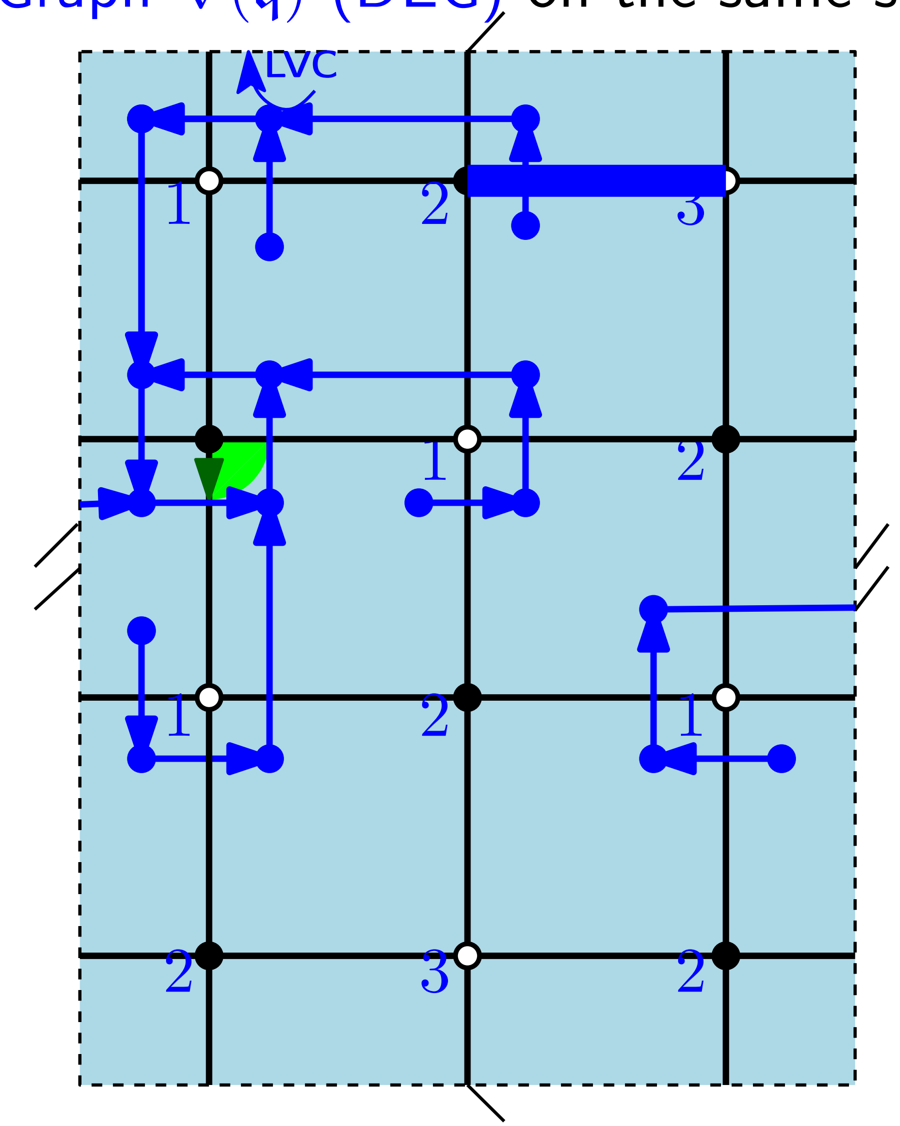

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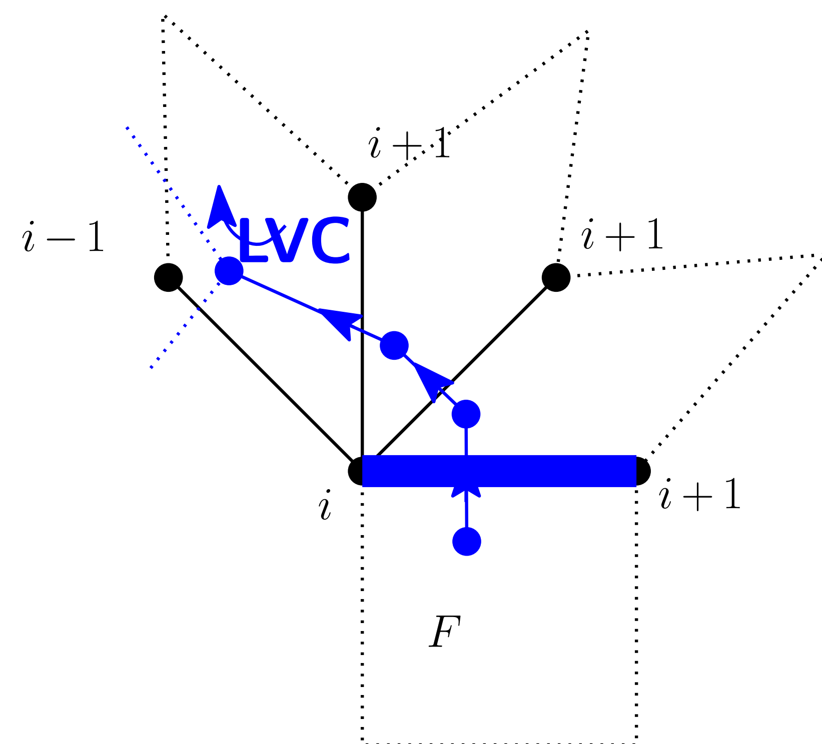

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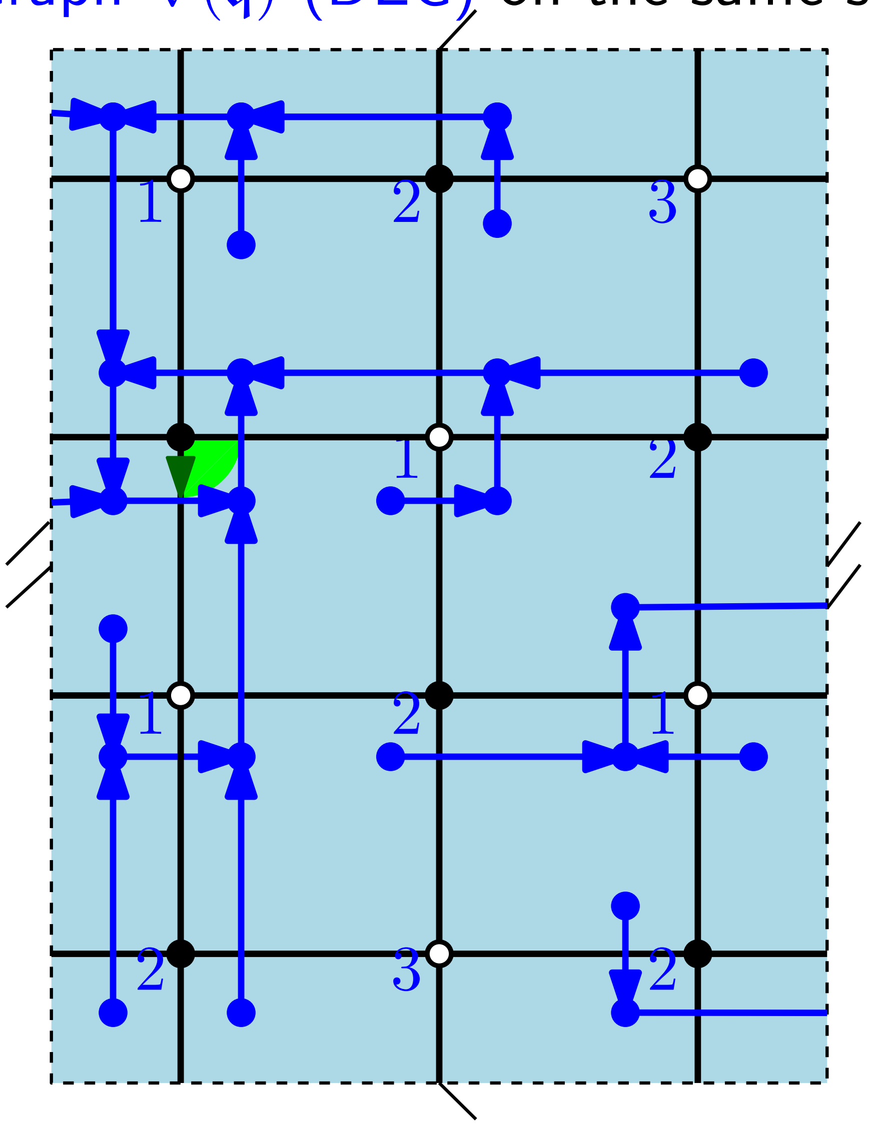

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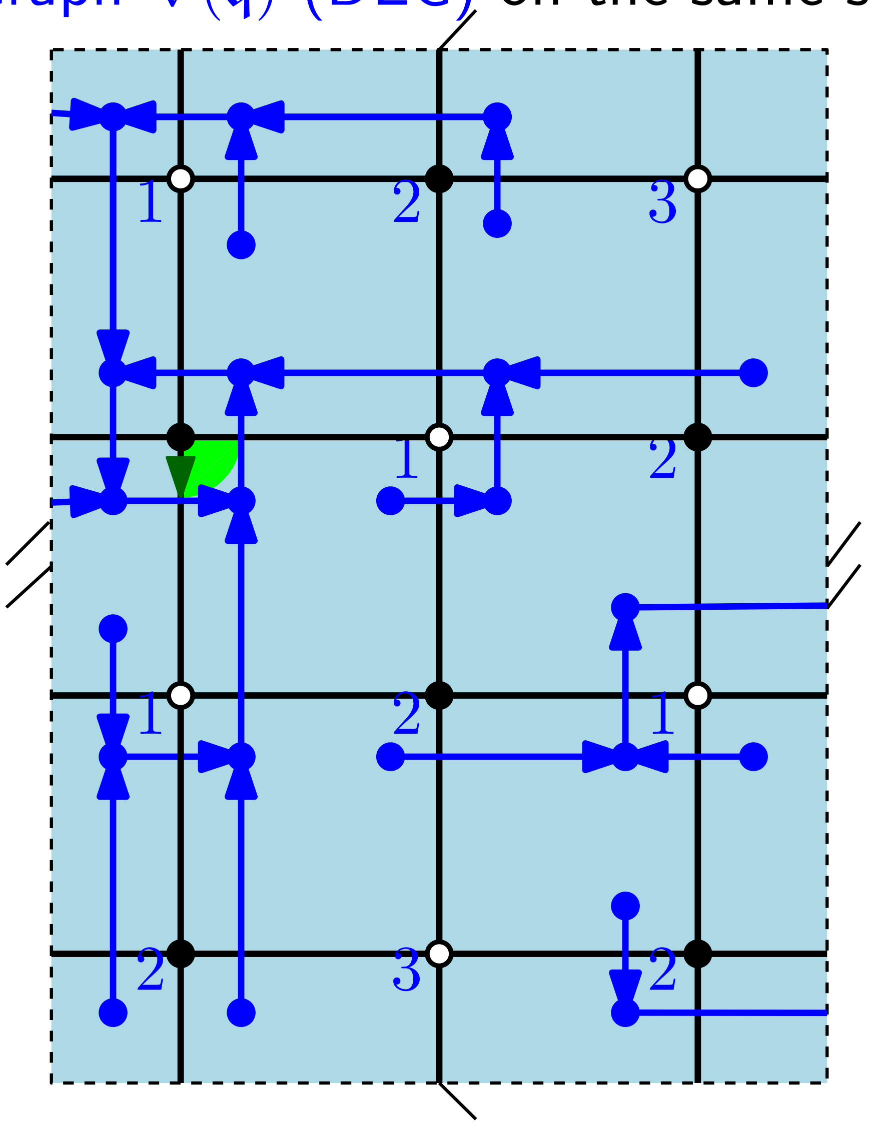
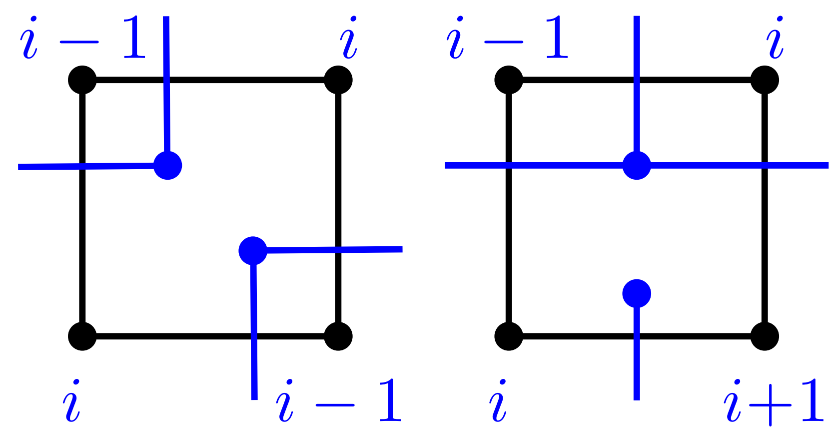


Proposition:
DEG $\nabla(\mathfrak{q})$ is formed by a unique oriented cycle encircling root vertex v_0, to which oriented trees are attached. After the construction of $\nabla(\mathfrak{q})$ is complete, each face of $\mathfrak{q}$ is of one of the two types:



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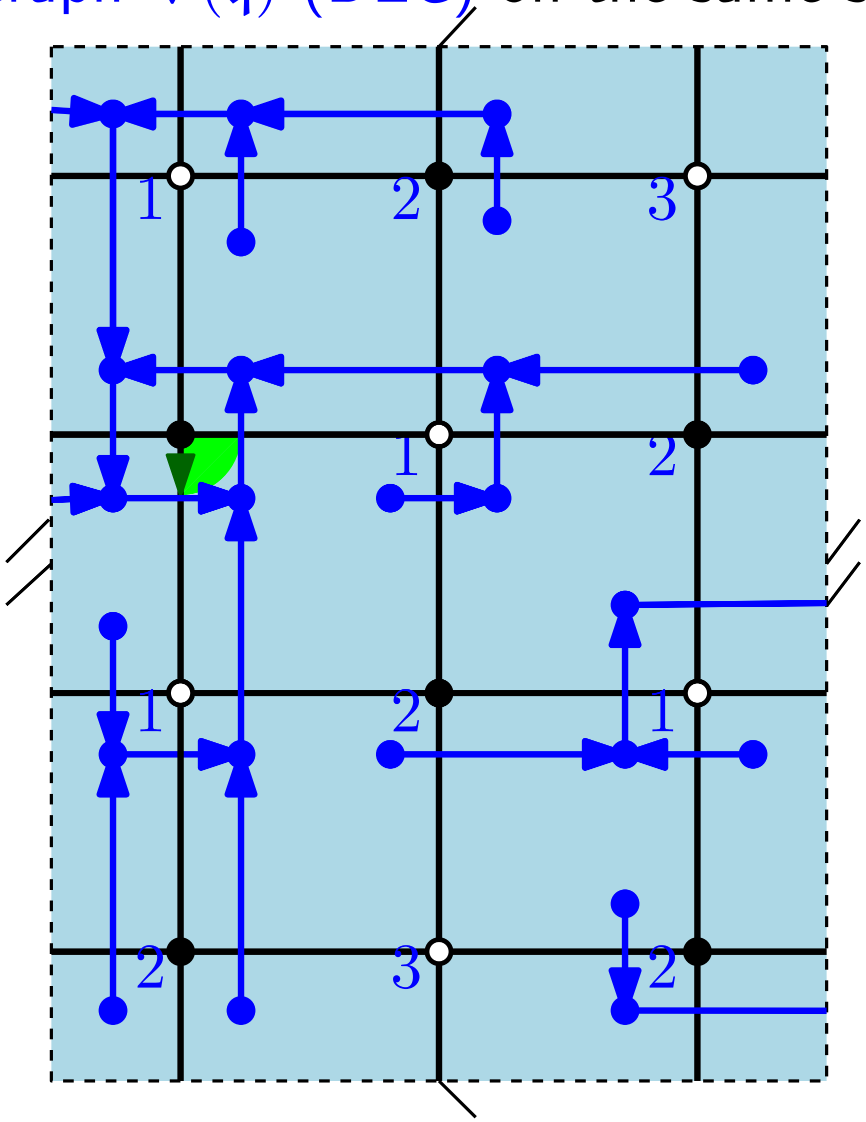
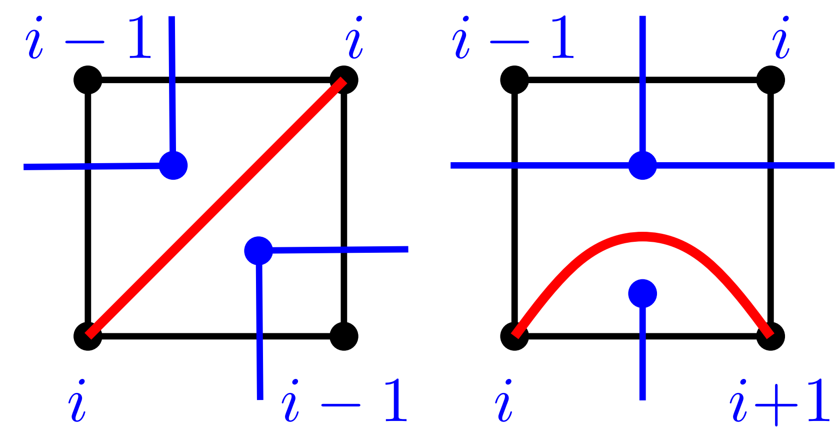
For a given quadrangulation $\mathfrak{q}$ we construct recursively a Dual Exploration Graph $\nabla(\mathfrak{q})$ (DEG) on the same surface:



Proposition:

DEG $\nabla(\mathfrak{q})$ is formed by a unique oriented cycle encircling root vertex v_0, to which oriented trees are attached. After the construction of $\nabla(\mathfrak{q})$ is complete, each face of $\mathfrak{q}$ is of one of the two types:



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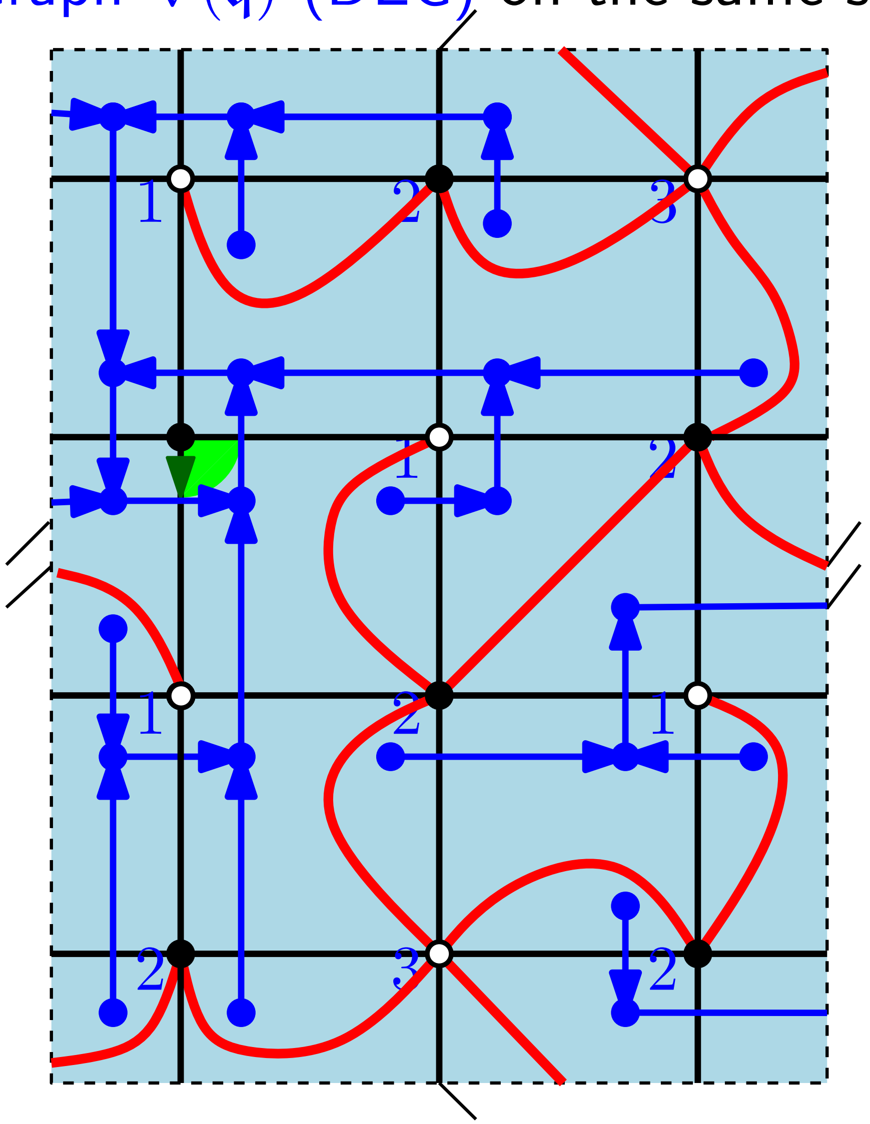

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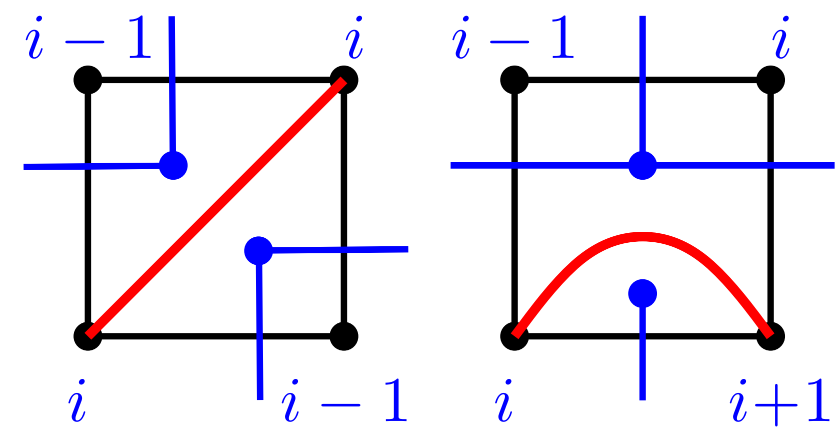

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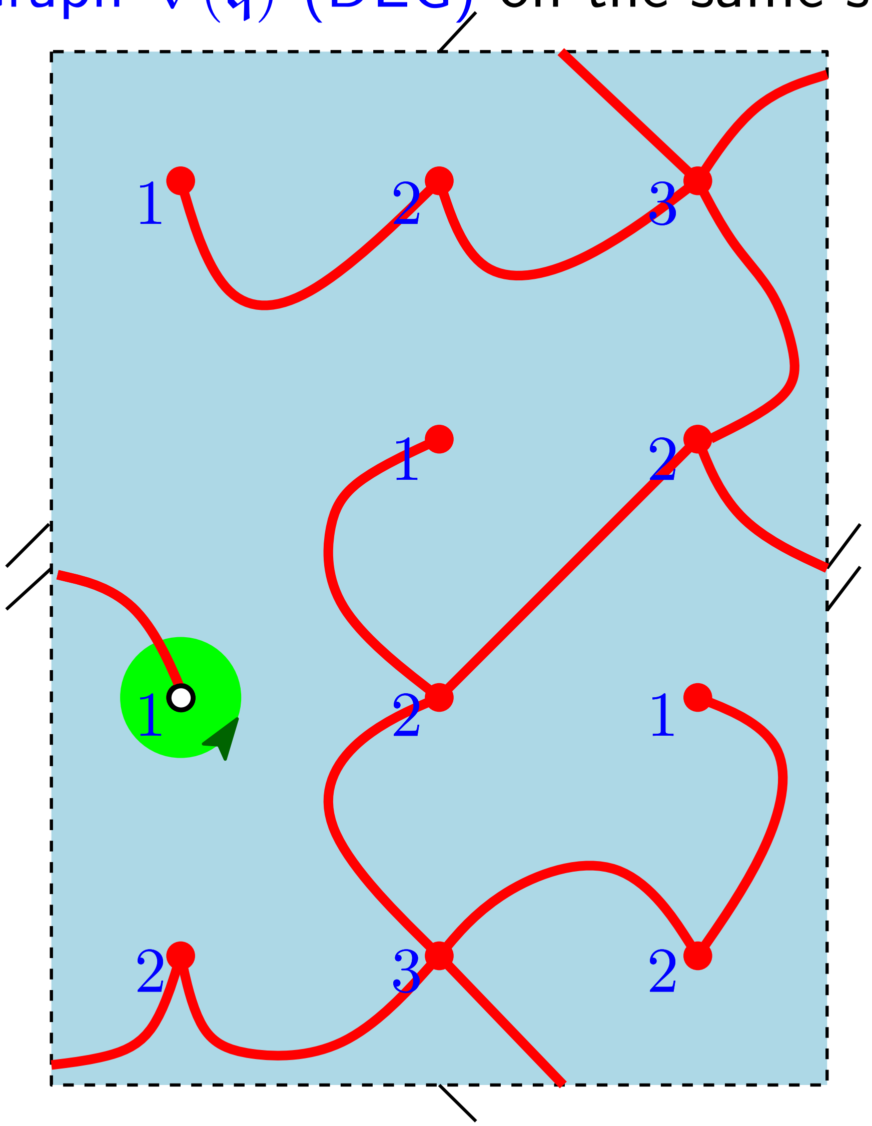

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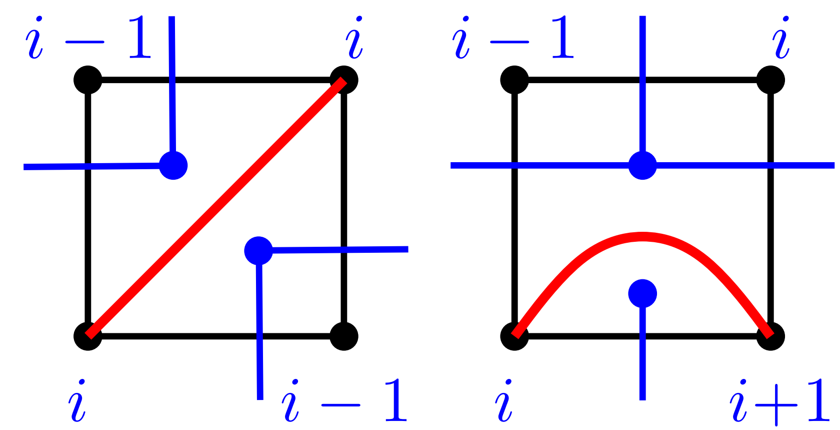

Corollary:
Red map $\phi(\mathfrak{q})$ is a one-face well-labeled rooted map with n edges, where n is the number of faces of $\mathfrak{q}$.

General case (III)

$\{$rooted, bipartite quadrangulations on $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)\}$

$$\leftrightarrow$$

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Double rooting trick and Hall's marriage theorem - see next slide!

General case (IV)

$(\mathfrak{q}, v_0)$ - pointed, rooted quadrangulation

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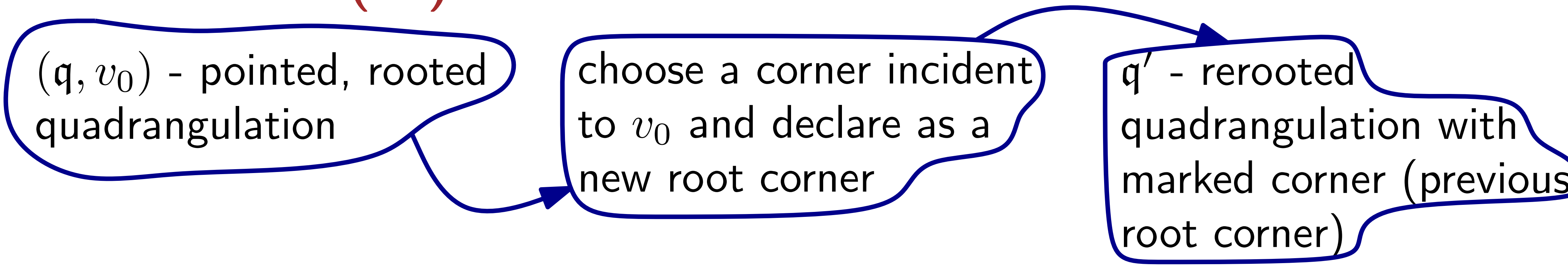

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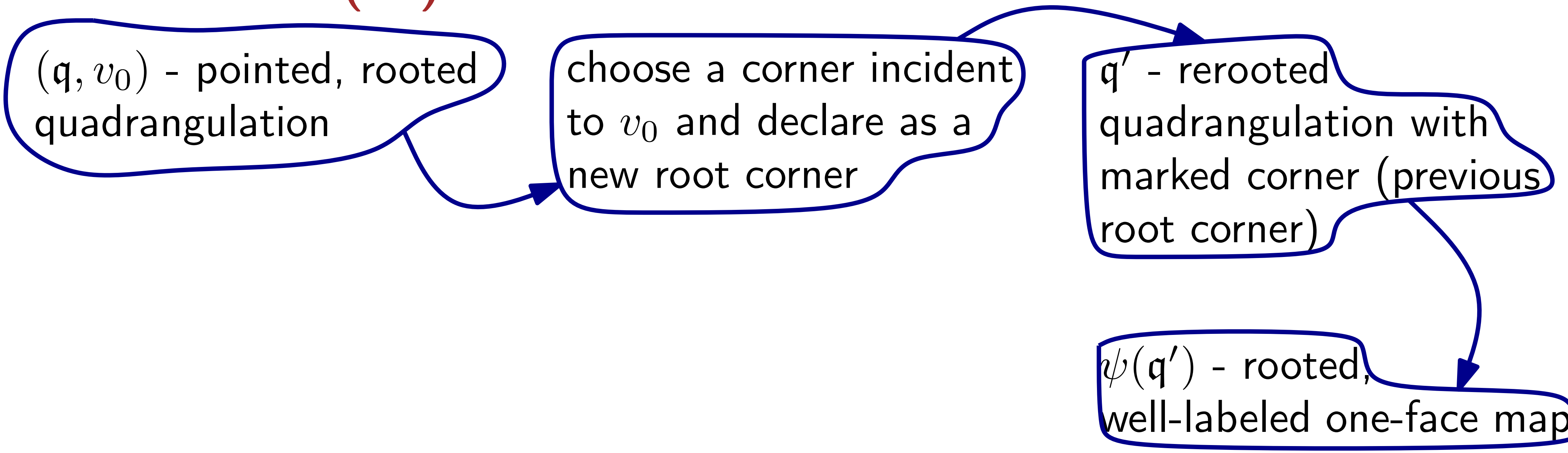

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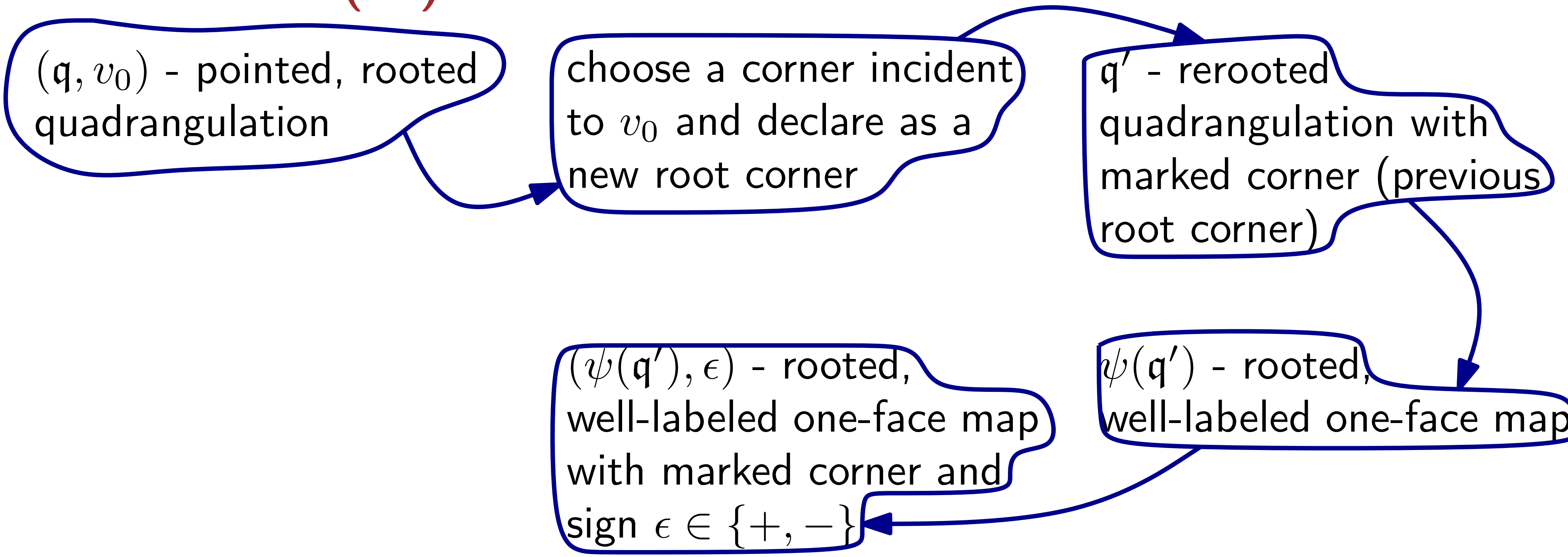

General case (IV)

$(\mathfrak{q}, v_0)$ - pointed, rooted quadrangulation

choose a corner incident to v_0 and declare as a new root corner

$\mathfrak{q}'$ - rerooted quadrangulation with marked corner (previous root corner)

$\psi(\mathfrak{q}')$ - rooted, well-labeled one-face map

$(\psi(\mathfrak{q}'), \epsilon)$ - rooted, well-labeled one-face map with marked corner and sign $\epsilon \in \{+, -\}$

$\mathfrak{q}'$ has $4n$ corners and map $\psi(\mathfrak{q}')$ has $2n$ corners - use some canonical way to choosa a corner in $\psi(\mathfrak{q}')$ and a sign $\epsilon \in \{+, -\}$ (for example - unique corner u incident to the root e of $\mathfrak{q}$ and sign depending on whether the label of u is smaller or greater than the other extremity of e)

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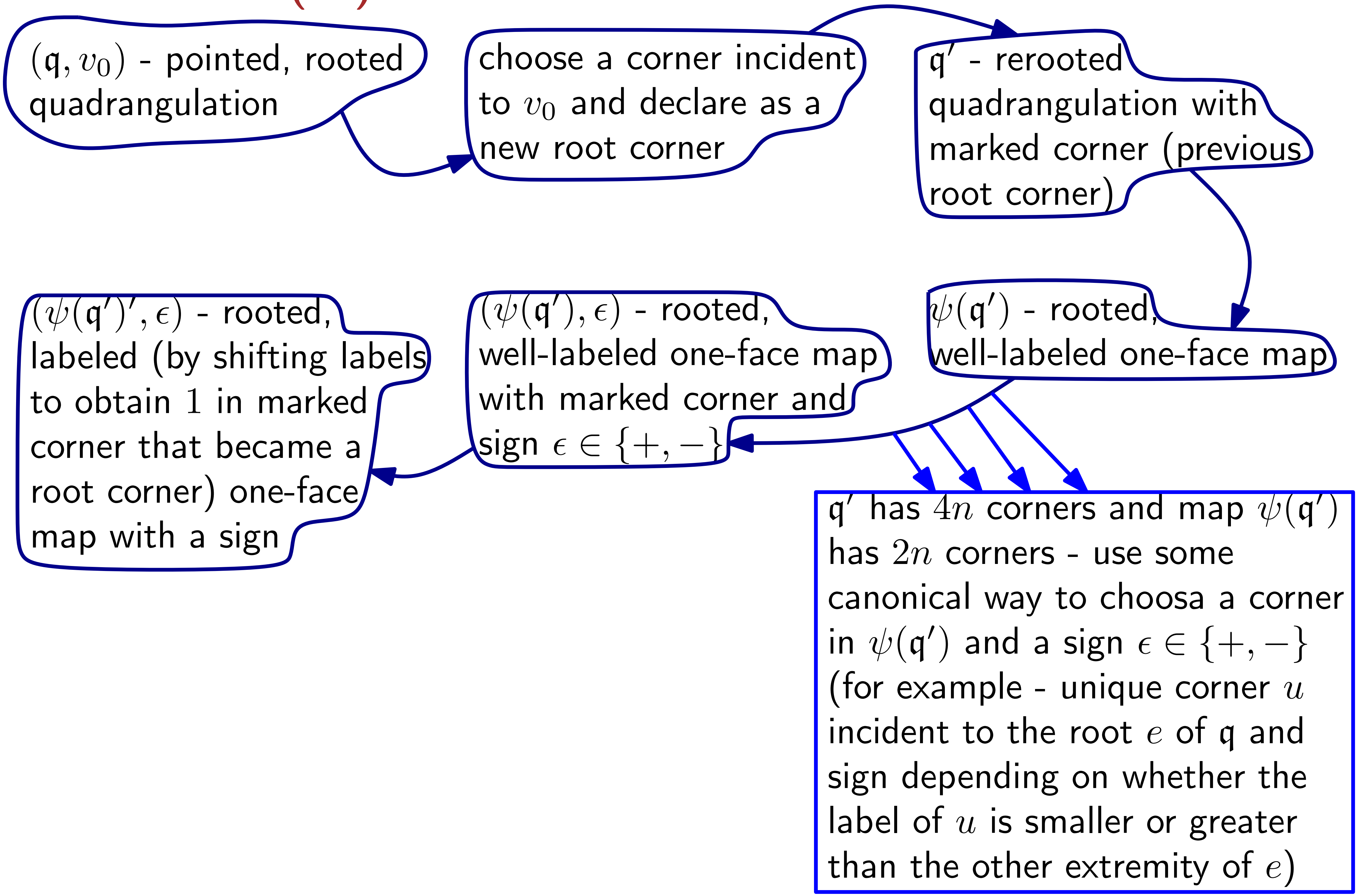

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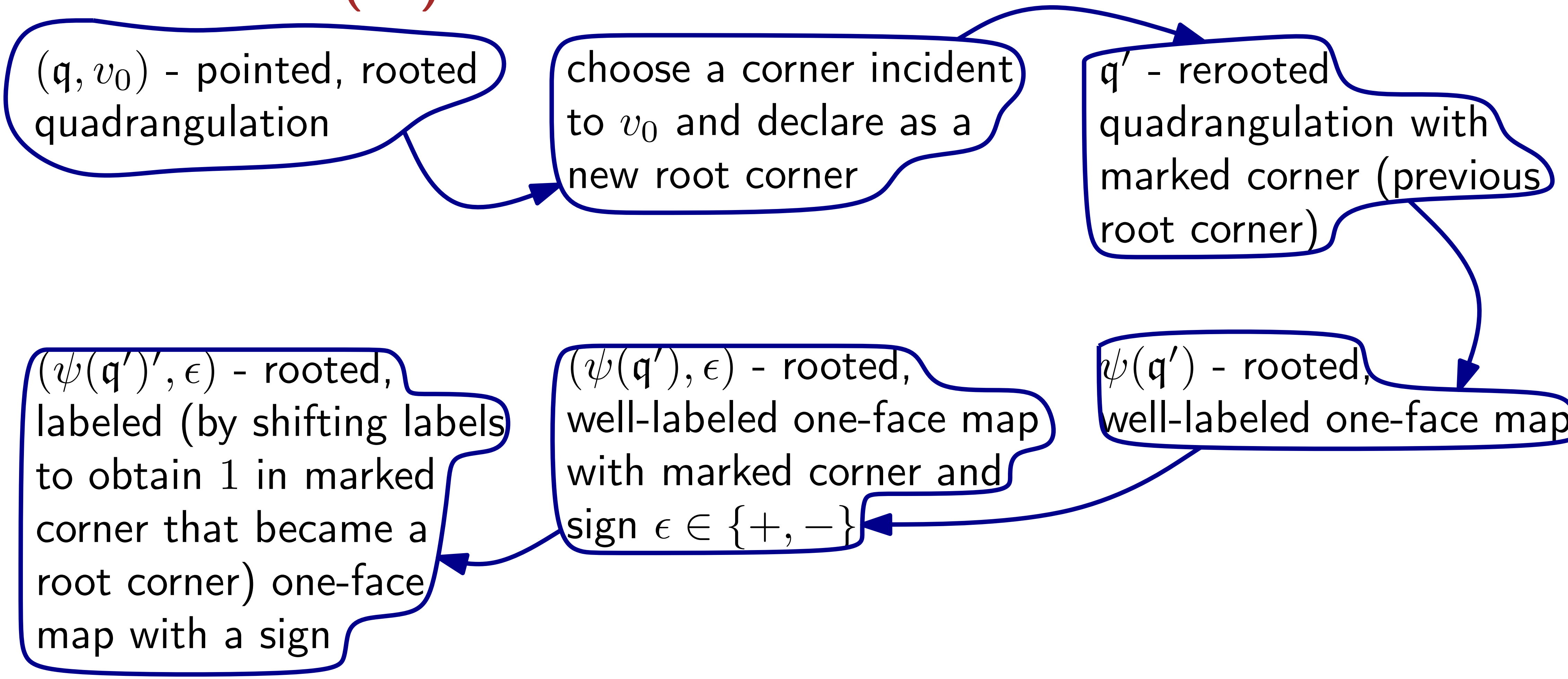

G - bipartite graph with the vertex set
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$\mathcal{Q}_{\mathbb{S},n,d}$ - set of all rooted quadrangulation on $\mathbb{S}$ with n faces and pointed vertex of degree d
$\mathcal{U}_{\mathbb{S},n,d}$ - set of all rooted labeled, one-face maps on $\mathbb{S}$ with n edges in which there are d corners with minimum label and equiped with a sign $\epsilon \in \{+, -\}$

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Hall's marriage theorem

Graph G has a perfect matching!

III. Applications

Enumeration - toy example

Let us try to enumerate maps with n edges on the projective plane:

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•number of rooted maps on the projective plane with n edges =

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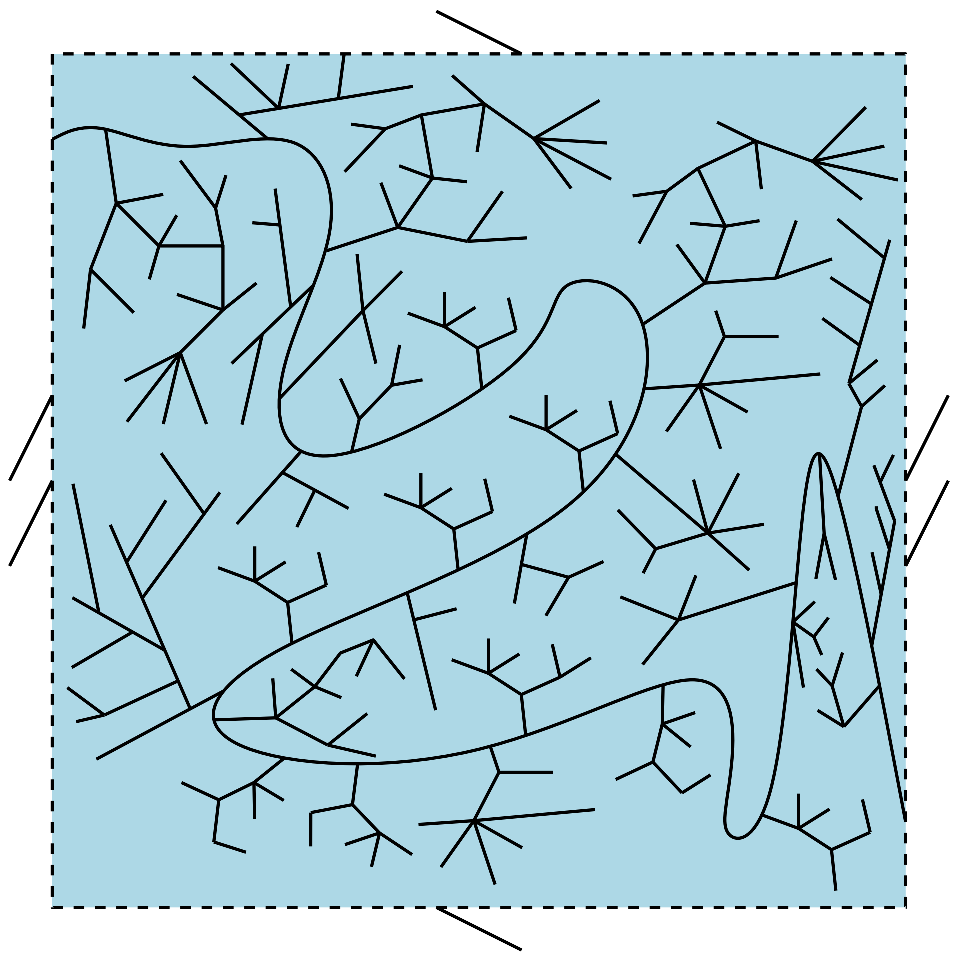

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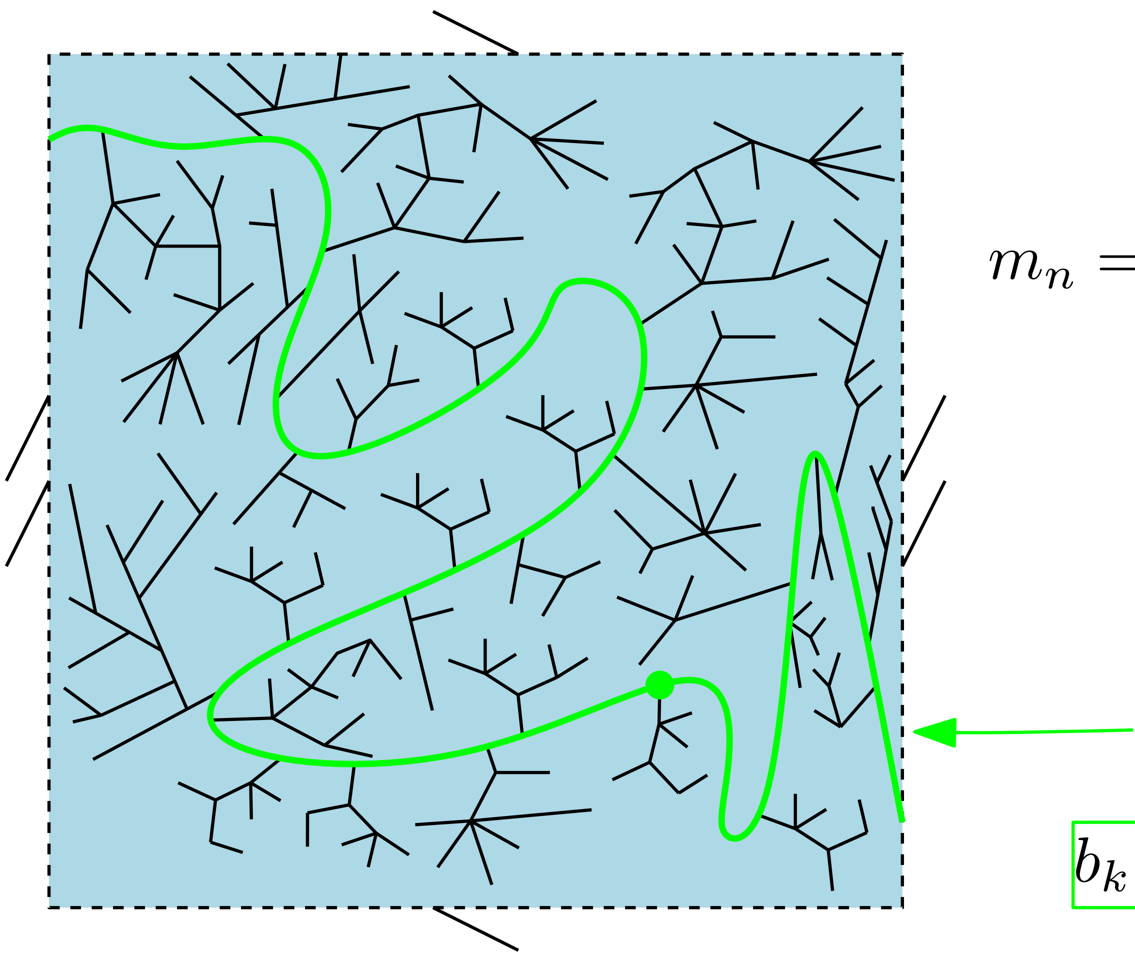

$$m_n = \frac{2}{n+1} \sum_{k \geq 1} b_k$$

Labeled cycle with k edges.

$$b_k = \sum_{a+2b=k} \binom{k}{a,b,b}$$

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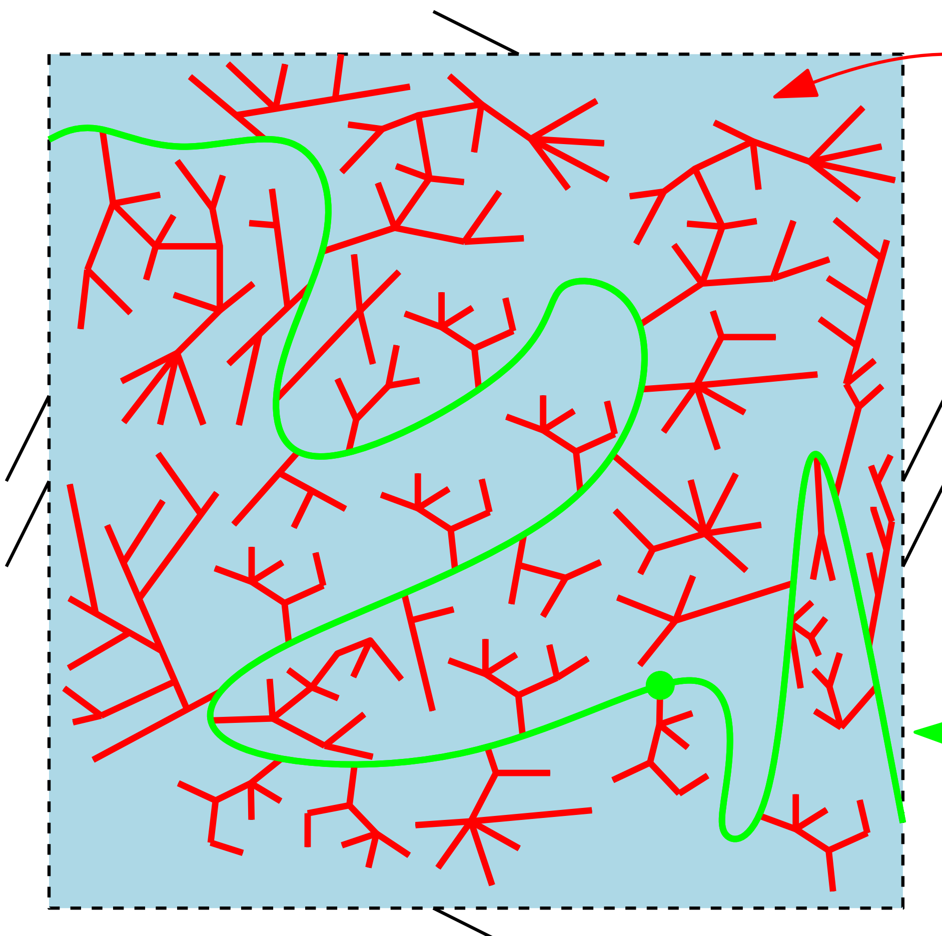

plane forests with $n-k$ edges and $2k$ components and labels.

$$m_n = \frac{2}{n+1} \sum_{k\geq 1} b_k \frac{2k}{n+k} \binom{2n-1}{n-k} \times 3^{n-k}$$

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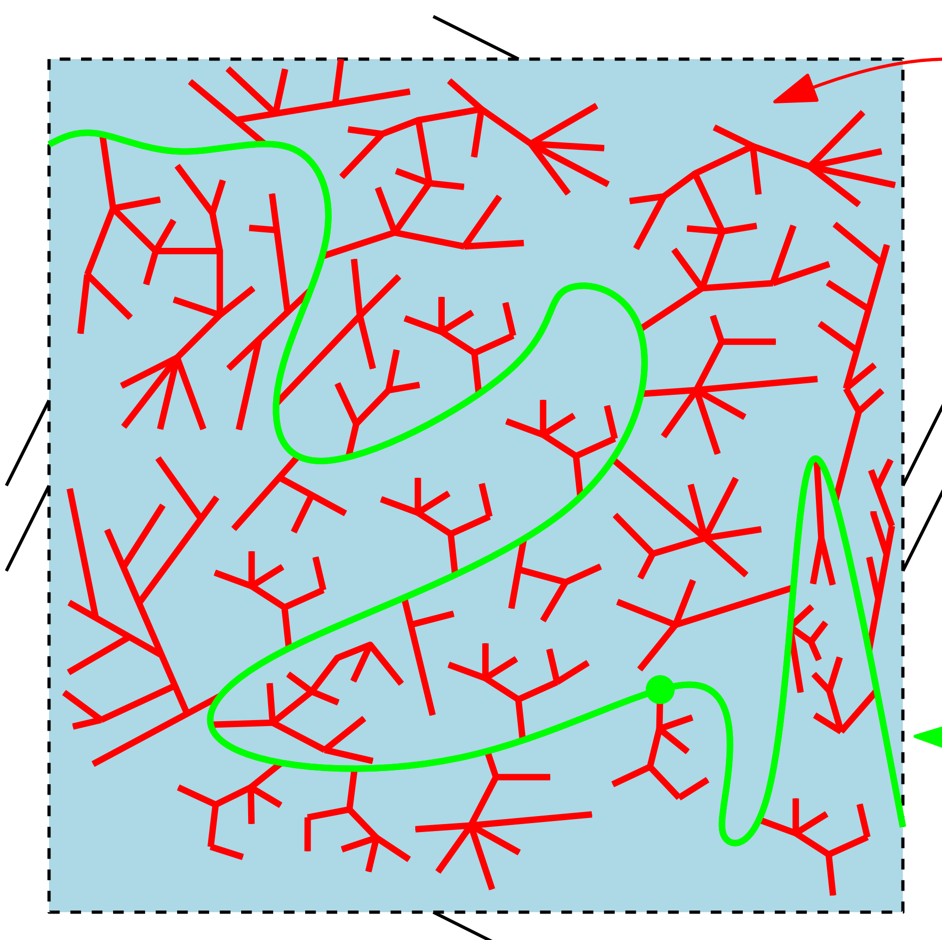

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$2n$ ways for choosing a root corner $\times$ 2 ways for choosing orientation.

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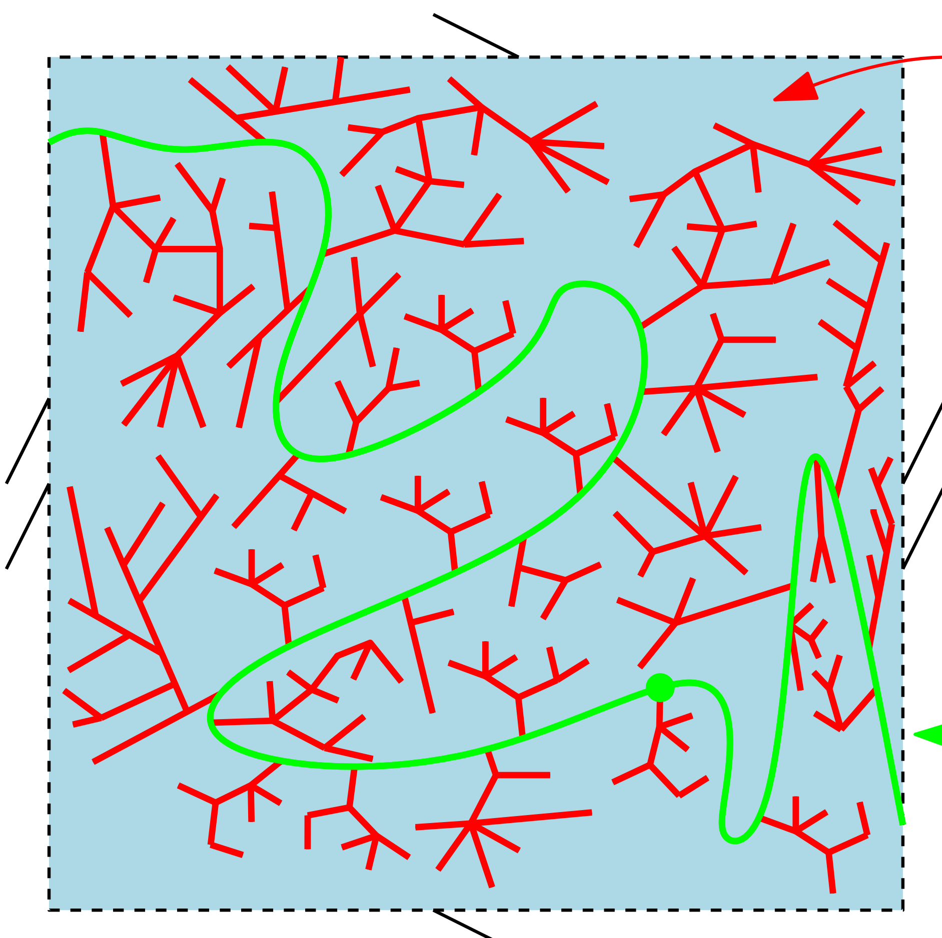

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$2n$ ways for choosing a root corner $\times$ 2 ways for choosing orientation.

$4k$ ways for deleting a second root.

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Enumeration

Theorem [Bender, Canfield 1986]

Let

$$Q_{\mathbb{S}}(t) := \sum_{n\geq 0} \vec{q}_{\mathbb{S},\bullet} t^n = \sum_{n\geq 0}(n+2-2h)\vec{q}_{\mathbb{S}}(n)t^n$$

be the generating function of rooted maps of type g pointed at a vertex or a face, by the number of edges. Moreover let $U \equiv U(t)$ and $T \equiv T(t)$ be the two formal power series defined by: $T = 1 + 3tT^2, \qquad U = tT^2(1 + U + U^2)$.
Then $Q_{\mathbb{S}}(t)$ is a rational function in U.

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Corollary [Bender, Canfield 1986]

For each $g \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$, there exists a constant p_g such that the number of rooted maps with n edges on the non-orientable surface of type g satisfies:

$$m_g(n) \sim p_g 12^n n^{\frac{5(g-1)}{2}}.$$

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Remark
Our main theorem allows us to recover Bender and Canfield results. In particular we can give some explicit (but very complicated) formula for the constant p_g.

Random maps

Let $(\mathcal{M}, v)$ be a map with distinguished vertex v. We define:

- radius of a map $\mathcal{M}$ centered at v by the quantity
$$R(\mathcal{M}, v) = \max_{u \in V(\mathcal{M})} d_{\mathcal{M}}(v, u);$$
- profile of distances from the distinguished point v (for any $r > 0$) by:
$$I_{(\mathcal{M},v)}(r) = \#\{u \in V(\mathcal{M}) : d_{\mathcal{M}}(v, u) = r\}.$$

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Theorem [Chapuy, D. 2015]
Let q_n be uniformly distributed over the set of rooted, bipartite quadrangulations with n faces on $\mathbb{S}$, let v_0 be a root vertex of q_n and let v_* be uniformly chosen vertex of q_n. Then, there exists a continuous, stochastic process $L^{\mathbb{S}} = (L_t^{\mathbb{S}}, 0 \leq t \leq 1)$ such that:

- $\frac{9}{8n}^{1/4} R(q_n, v_*) \to \sup L^{\mathbb{S}} - \inf L^{\mathbb{S}}$;
- $\frac{9}{8n}^{1/4} d_{q_n}(v_0, v_*) \to \sup L^{\mathbb{S}}$;
- $\frac{I_{(q_n,v_*)}\left((8n/9)^{1/4}\cdot\right)}{n+2-2h} \to \mathcal{I}^{\mathbb{S}}$,

where $\mathcal{I}^{\mathbb{S}}$ is defined as follows: for every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}_+$,
$$\langle \mathcal{I}^{\mathbb{S}}, g \rangle = \int_0^1 dt g(L_t^{\mathbb{S}} - \inf L^{\mathbb{S}}).$$

Random maps (II)

Few words about the process $L^{\mathbb{S}}$ ($\mathbb{S}$ = sphere for simplicity).

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Few words about the process $L^{\mathbb{S}}$ ($\mathbb{S}$ = sphere for simplicity).

• Contour process $c_n : [0, 2n] \to \mathbb{R}$ of the rooted, pointed quadrangulation $\mathfrak{q}_n$ with n faces: $c_n(i) = d_{\psi(\mathfrak{q}_n)}(v_i, v_0)$, where v_0 - root vertex of $\psi(\mathfrak{q}_n)$, v_i - vertex visited in the i-th step during the walk along the boundary of $\psi(\mathfrak{q}_n)$.

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• after proper normalization, the contour of uniformly chosen random rooted tree with n edges converges in distribution to the co-called normalized Brownian excursion $c^{\mathbb{S}}$ (informally - standard Brownian motion conditioned to remain non-negative on $[0, 1]$ and to take value 0 at the time 1).

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• Label process $L_n : [0, 2n] \to \mathbb{R}$ of the rooted, pointed quadrangulation $\mathfrak{q}_n$ with n faces: $L_n(i) = \ell(c_i)$, where c_0 - root corner of $\psi(\mathfrak{q}_n)$, c_i - corner visited in the i-th step during the walk along the boundary of $\psi(\mathfrak{q}_n)$.

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• after normalization by $\frac{9}{8n}^{1/4}$, label process of uniformly chosen pointed, rooted, planar quadrangulation with n faces converges to the so-called head of the Brownian snake $L^{\mathbb{S}} = (L^{\mathbb{S}}_t, 0 \leq t \leq 1)$ which is, conditionally on $c^{\mathbb{S}}$, continuous Gaussian process with covariance:

$$Cov(L^{\mathbb{S}}_s, L^{\mathbb{S}}_t) = \inf\{c^{\mathbb{S}}_u : \min(s, t,) \leq u \leq \max(s, t)\}.$$

IV. Further directions

• Generalization of the Bouttier-Di Francesco-Guitter bijection for non-orientable maps (bijection between bipartite $2p$-angulations, or, more generally bipartite maps with n faces of prescribed degrees and some kind of non-orientable mobiles?)

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• Studying random maps on ANY surface in Gromov-Hausdorff topology (using our bijection and already established methods we (Bettinelli, Chapuy, D.) can prove a convergence of bipartite quadrangulations up to extraction of SUBSEQUENCE - what about full convergance)?).

THANK
YOU!