GT Combinatoire — March 11, 2015

Reflections in Persistence and Quiver Theory

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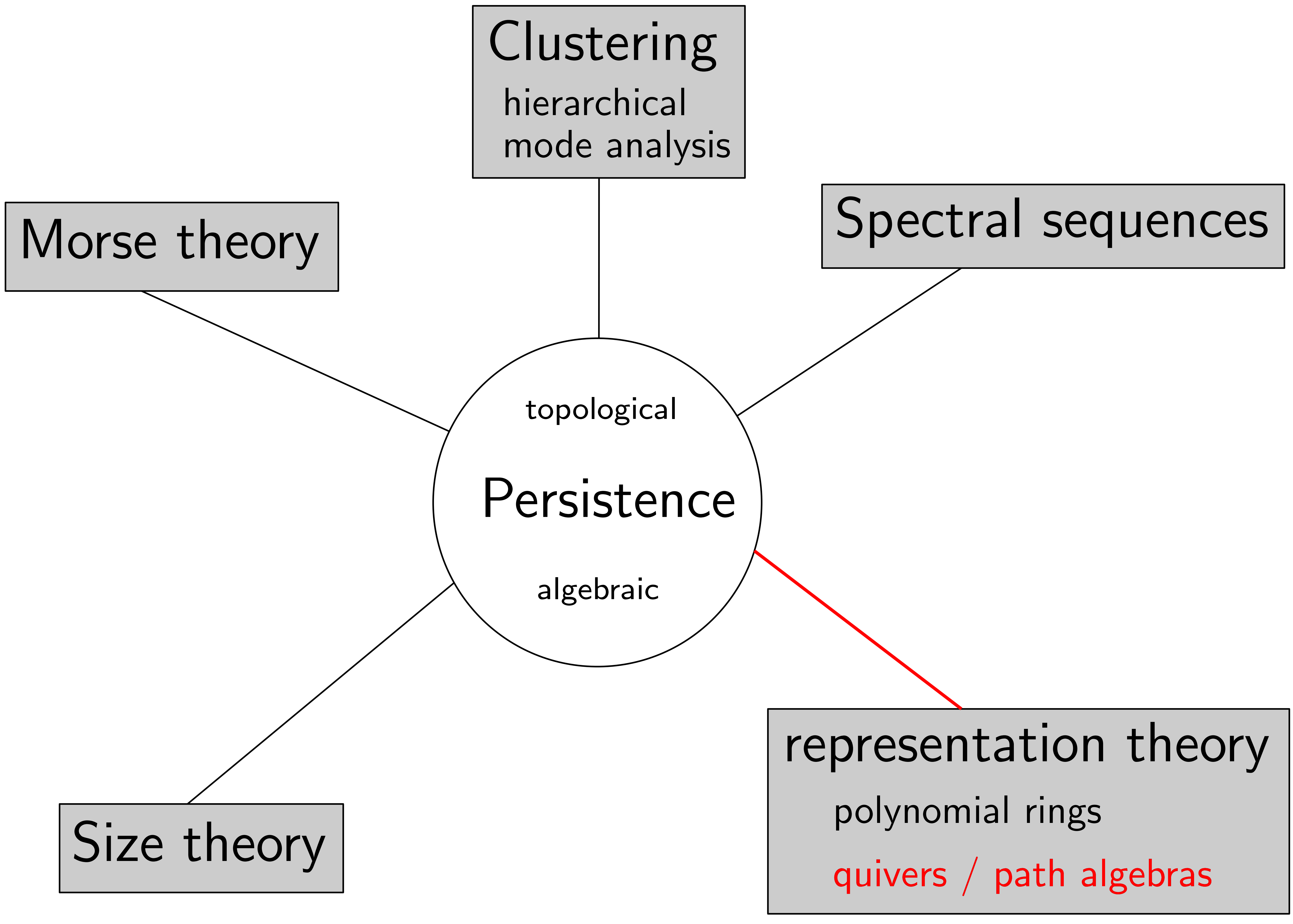
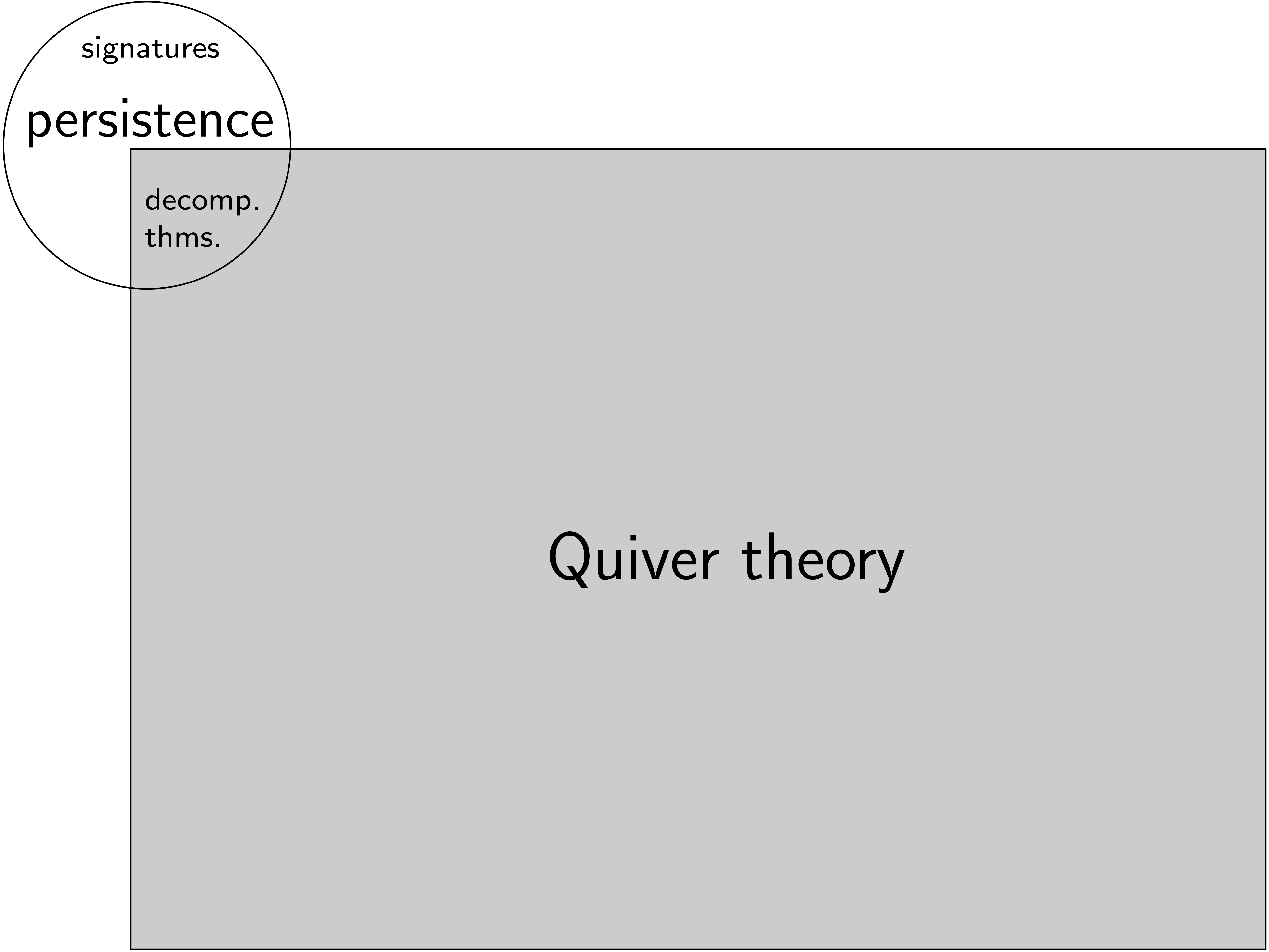
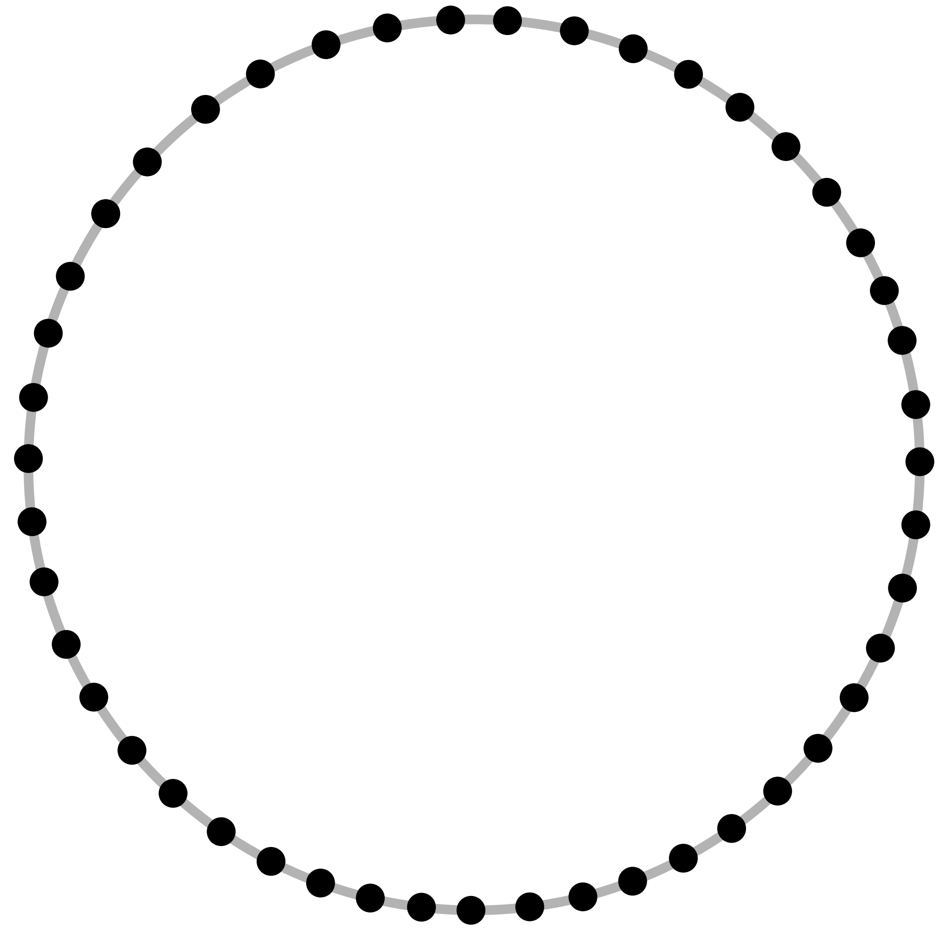

Clustering
hierarchical
mode analysis
Morse theory
Spectral sequences
topological
Persistence
algebraic
Size theory
representation theory
polynomial rings
quivers / path algebras


signatures
persistence
decomp.
thms.
Quiver theory

Exploratory data analysis

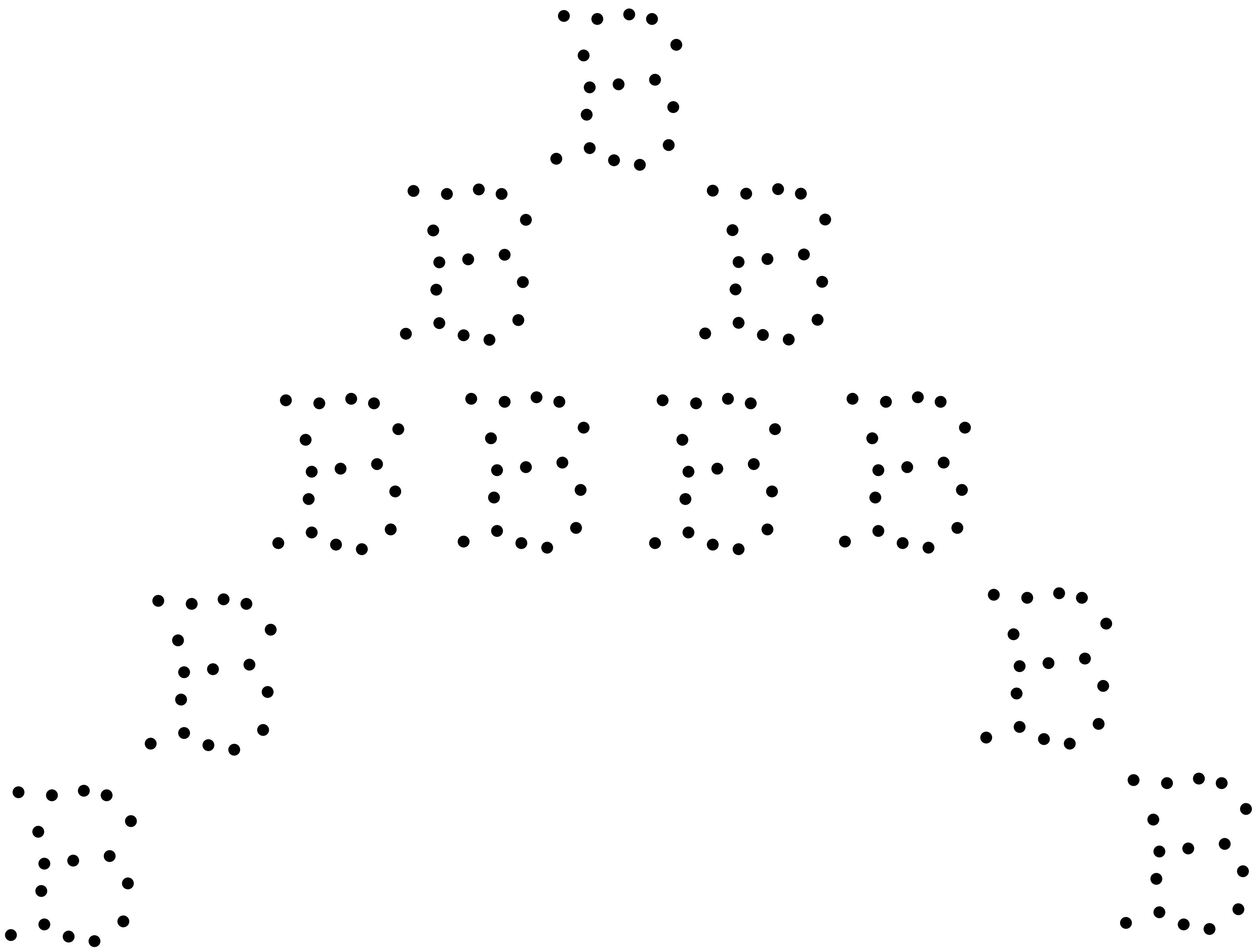


Setup: $K \subset \mathbb{R}^d$ a compact set, $p_1, \cdots, p_n$ data points sampled along (or close to) K

Goal: recover structural information about K, knowing only $p_1, \cdots, p_n$

Challenges in data analysis

①Scale



Challenges in data analysis

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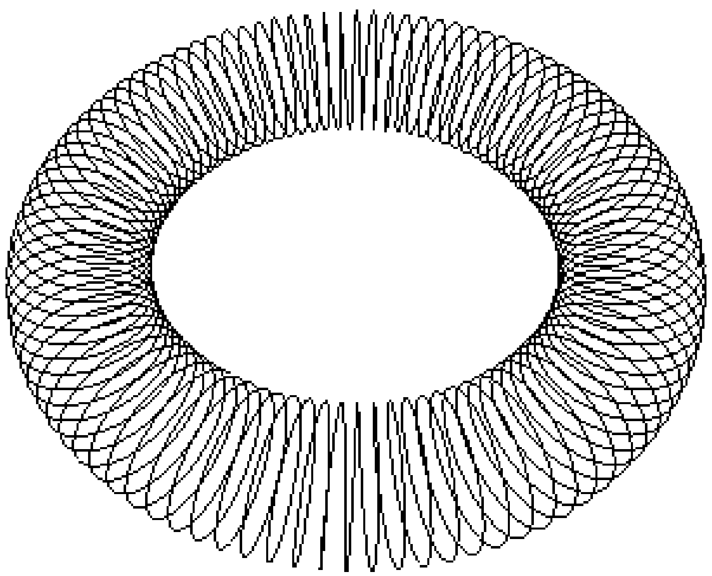
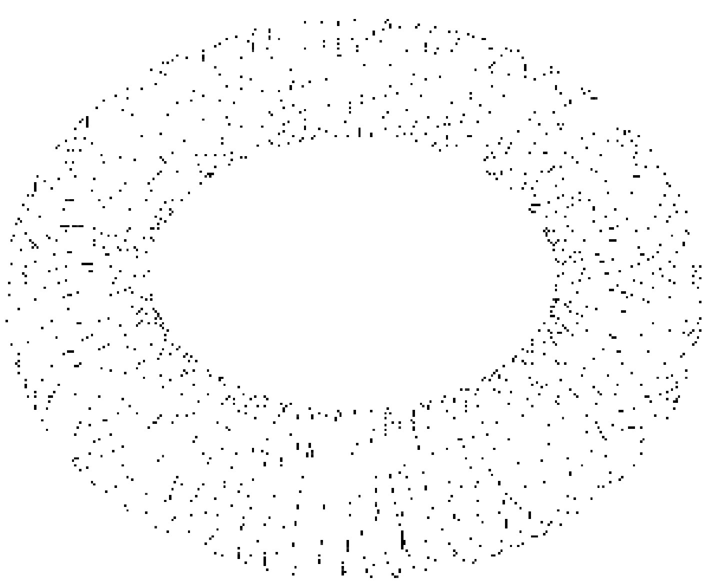
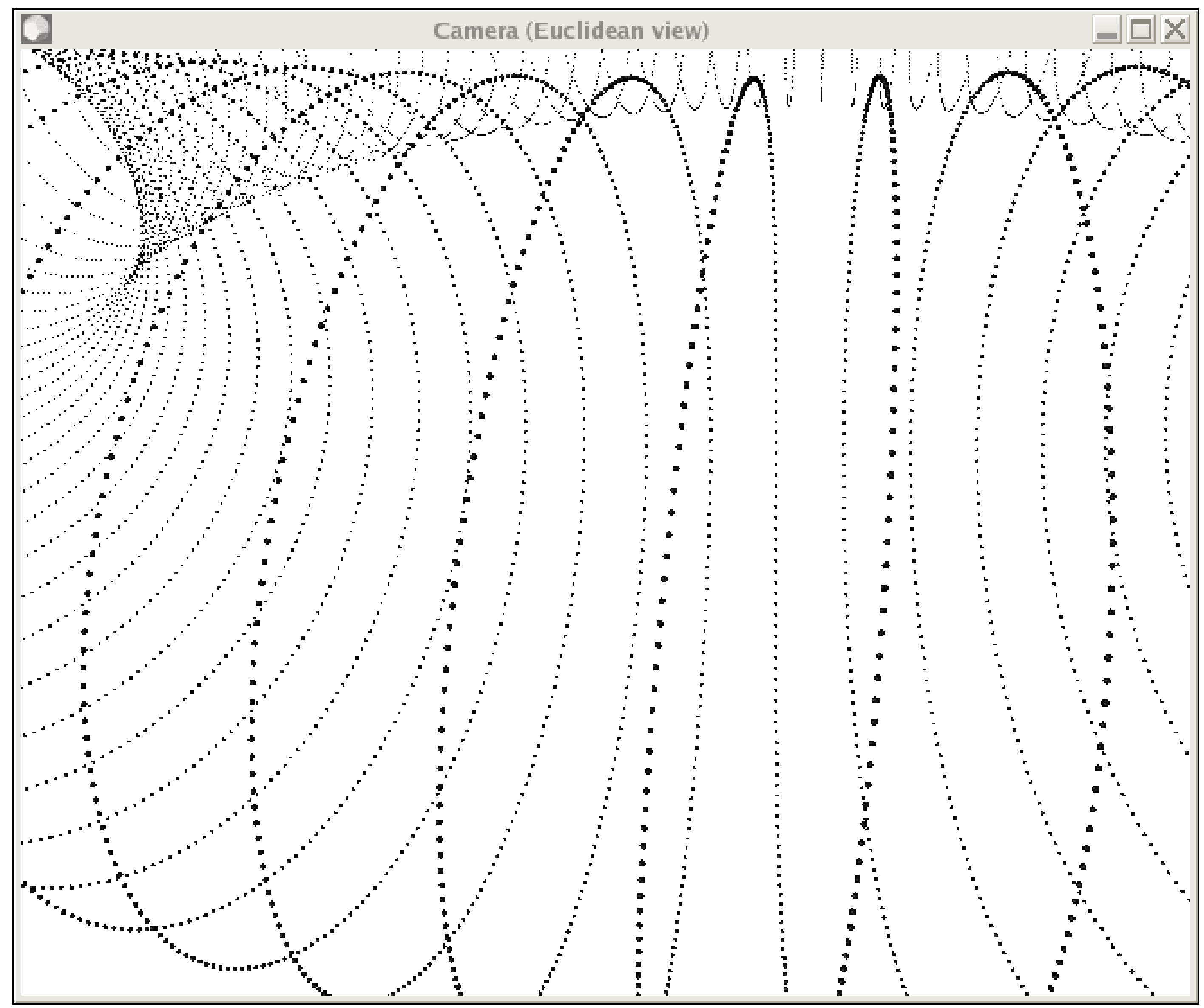
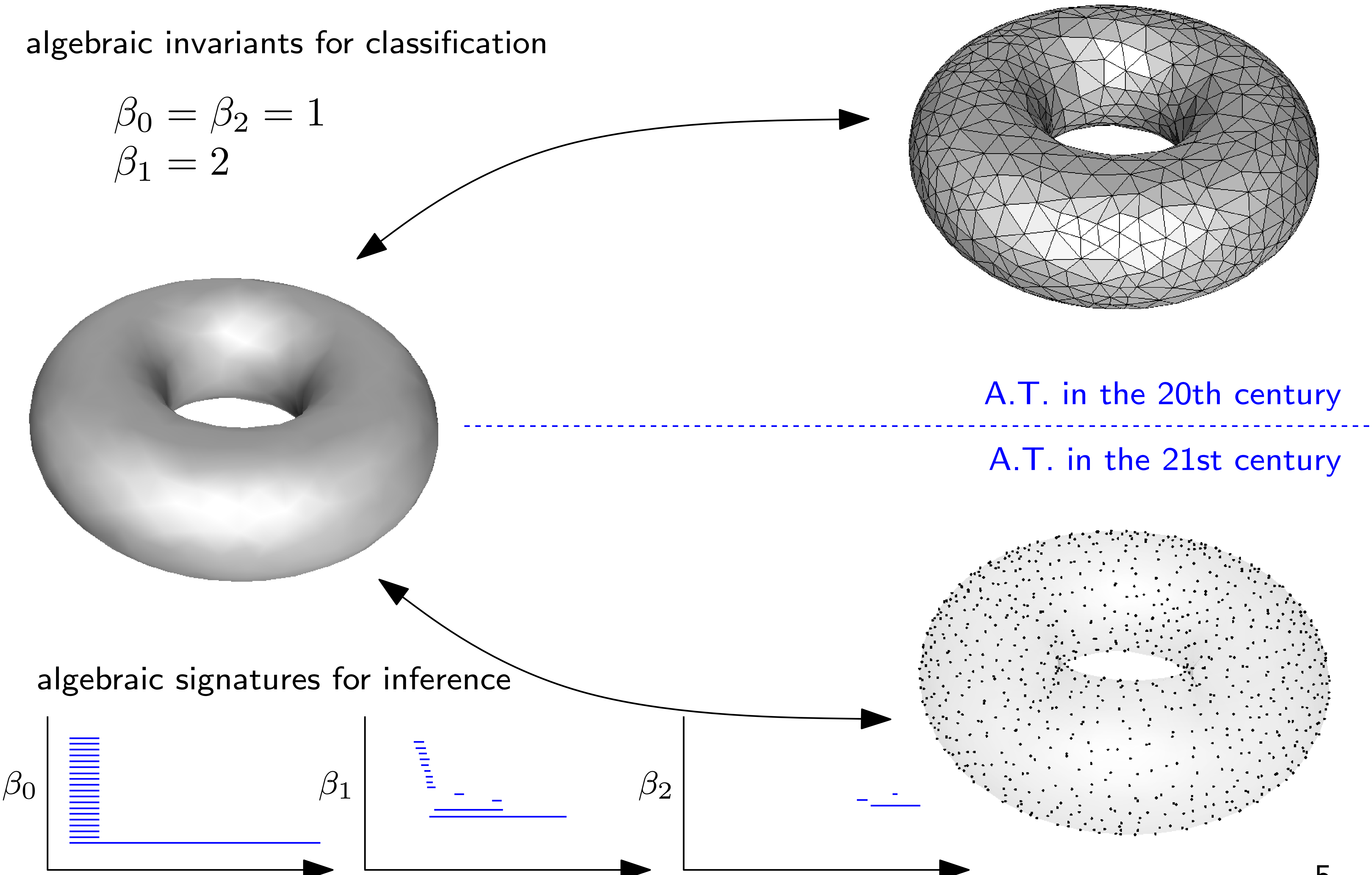


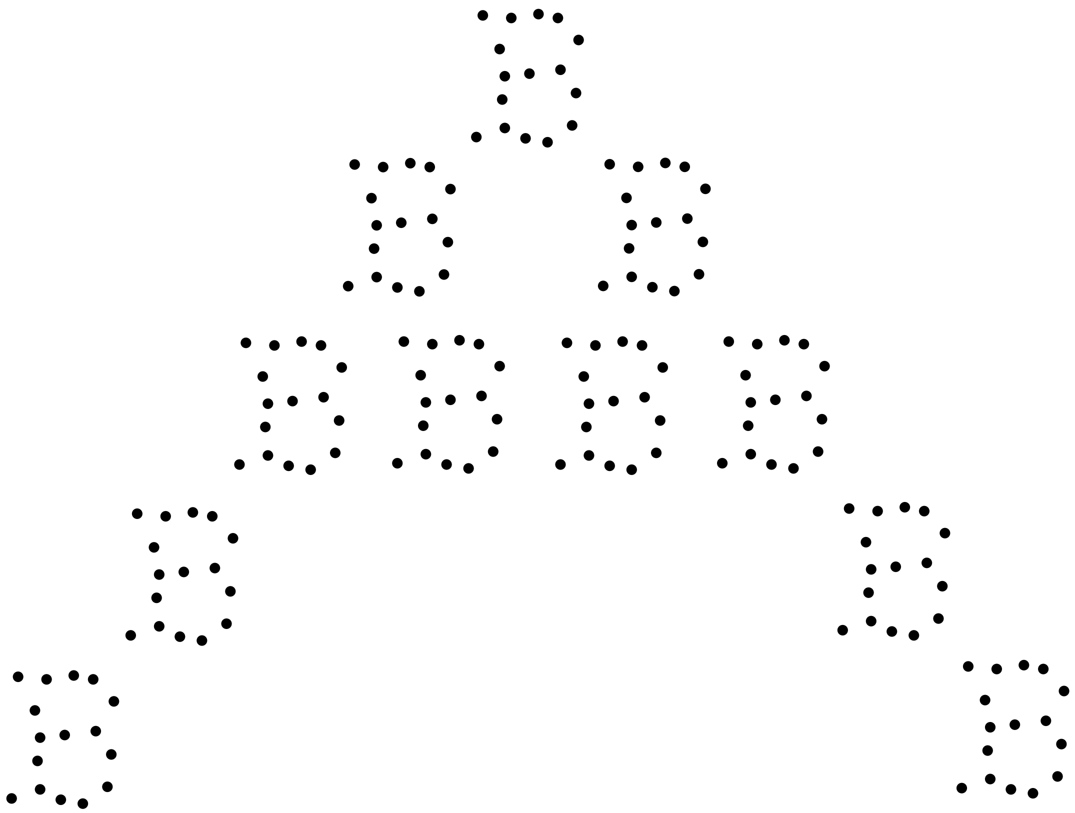


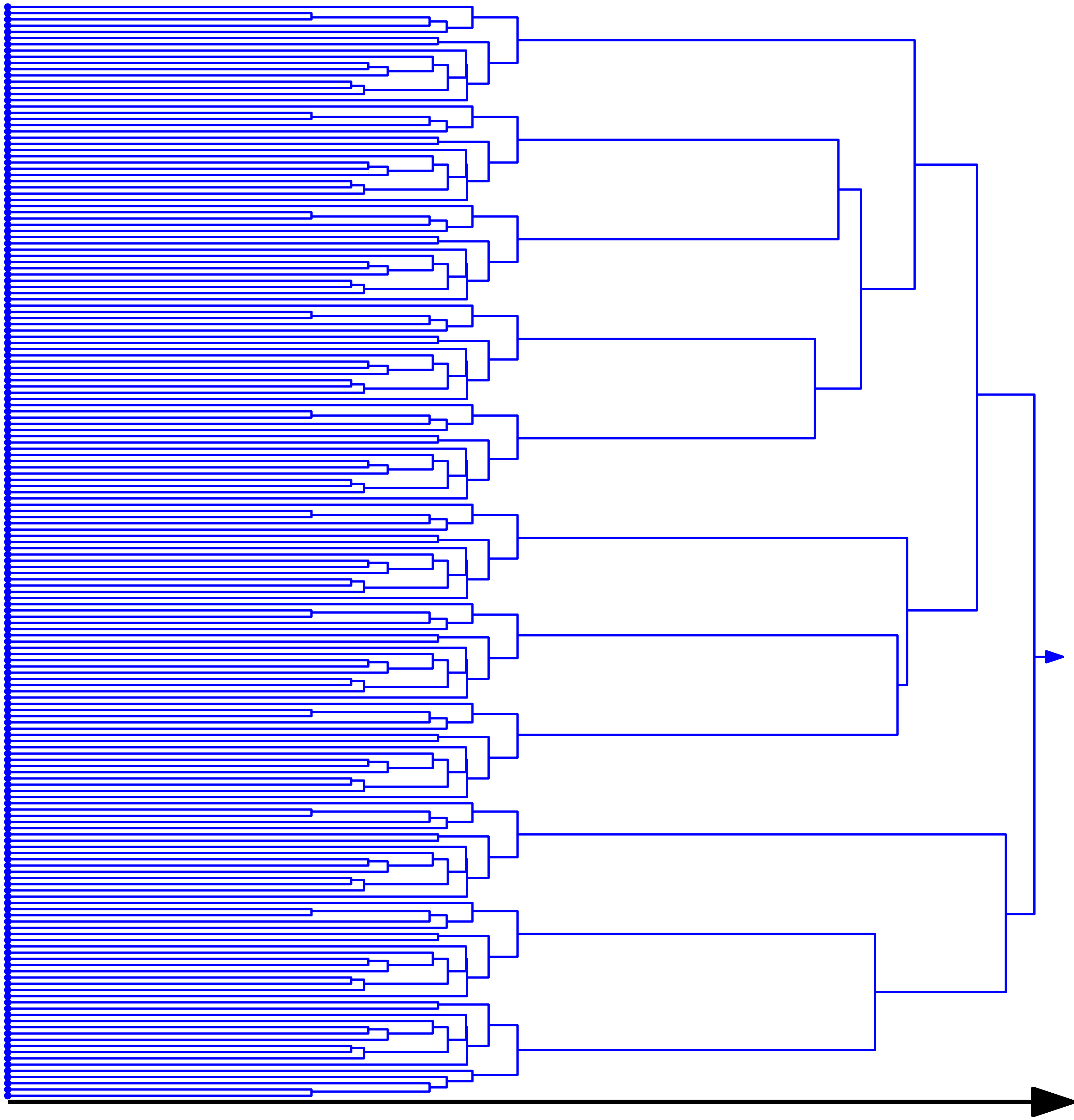


Inferring the topology of data



Intuitive viewpoint: hierarchical clustering





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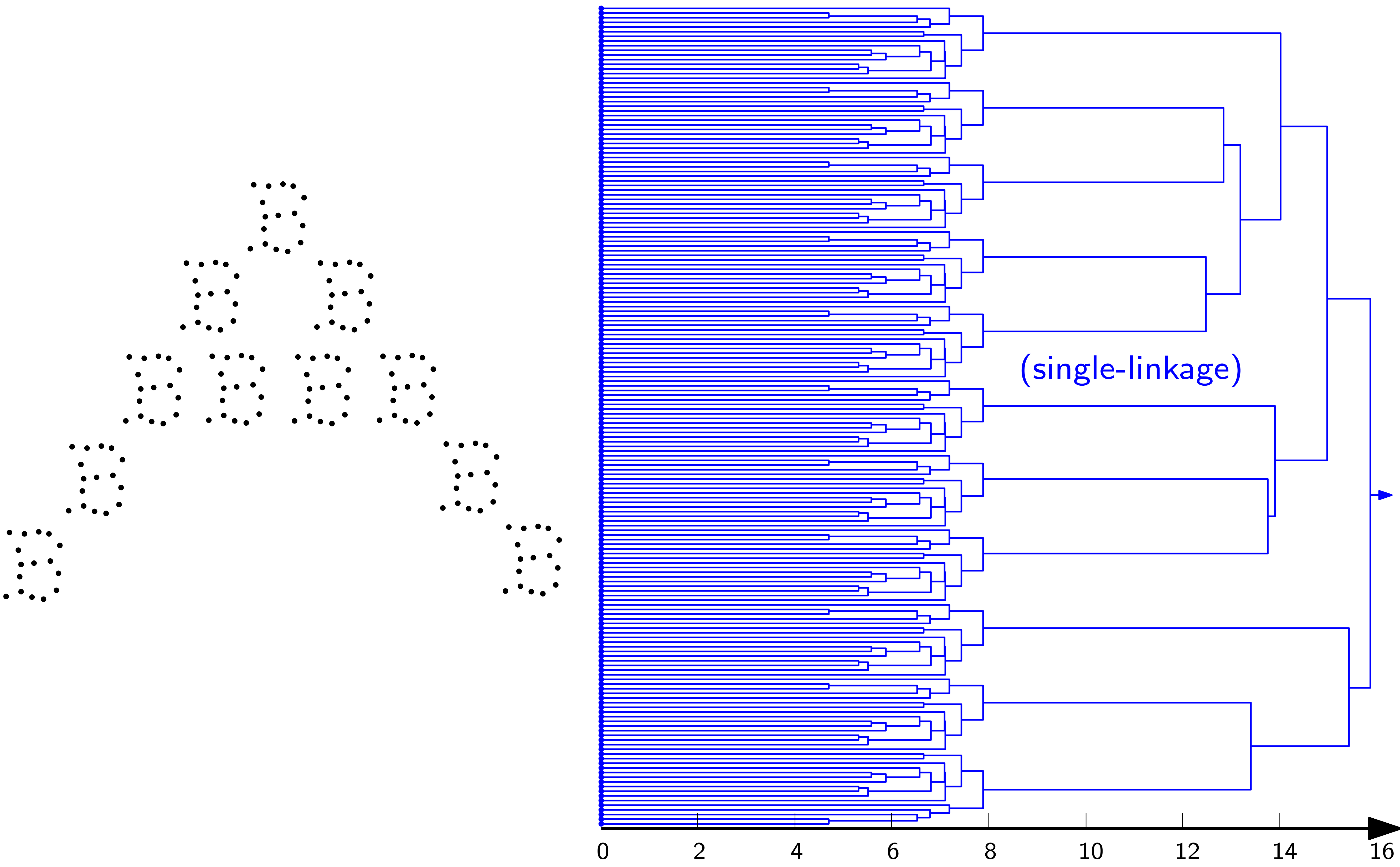

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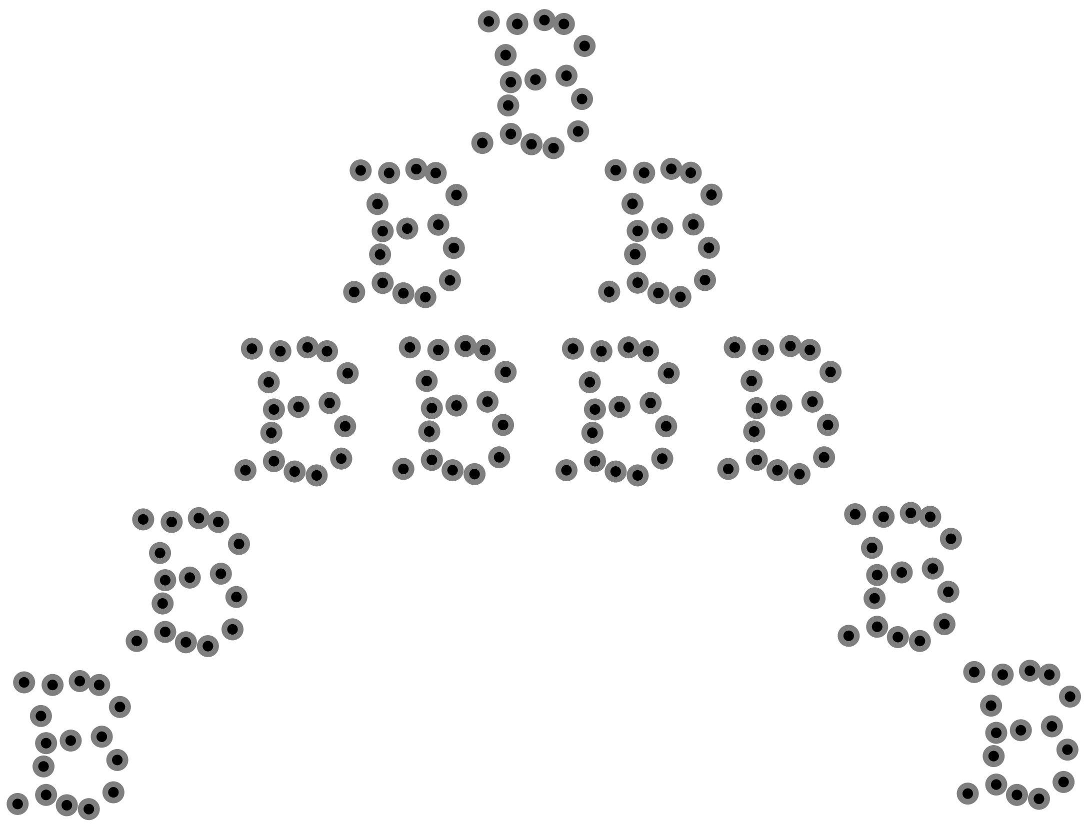

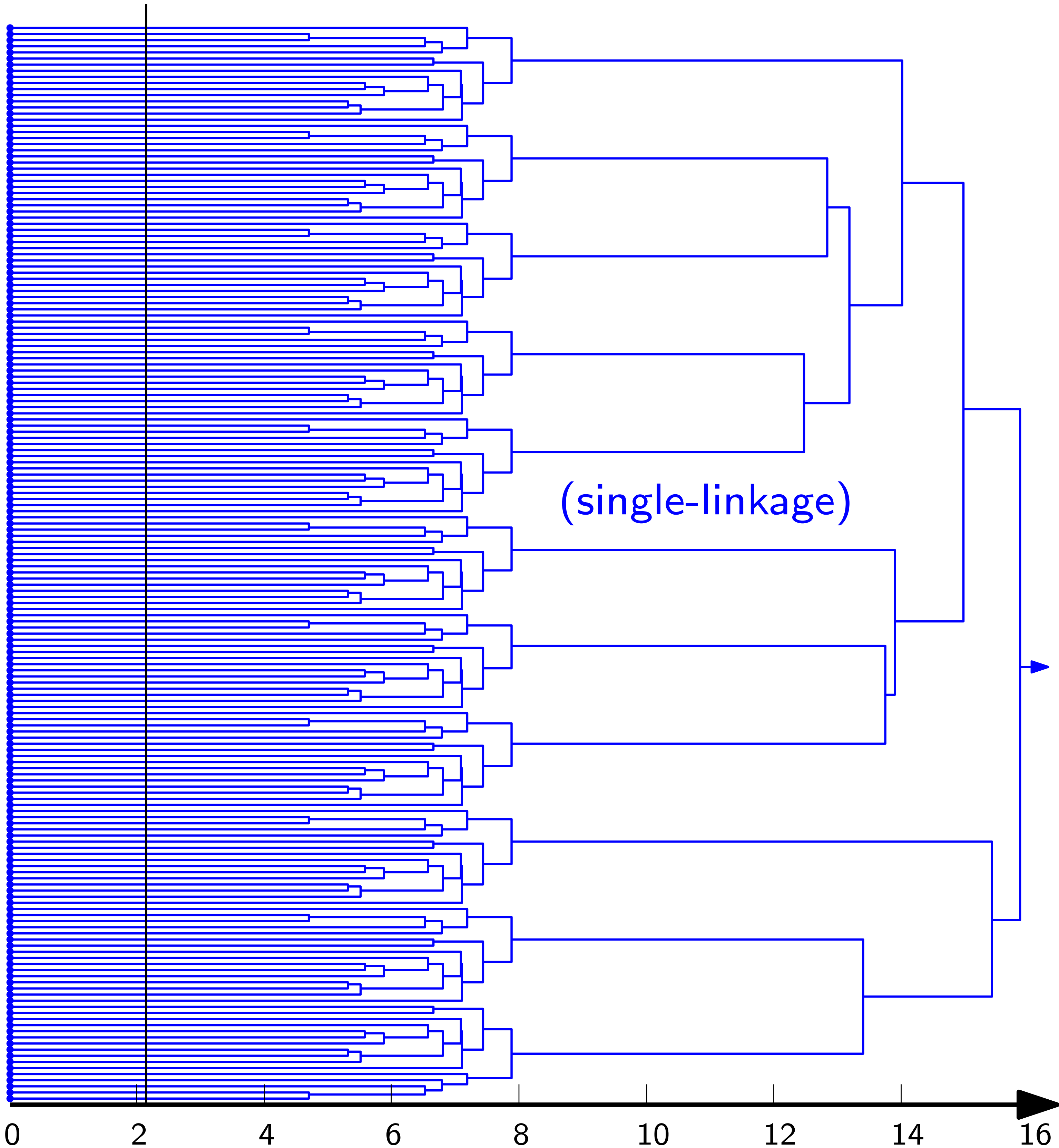

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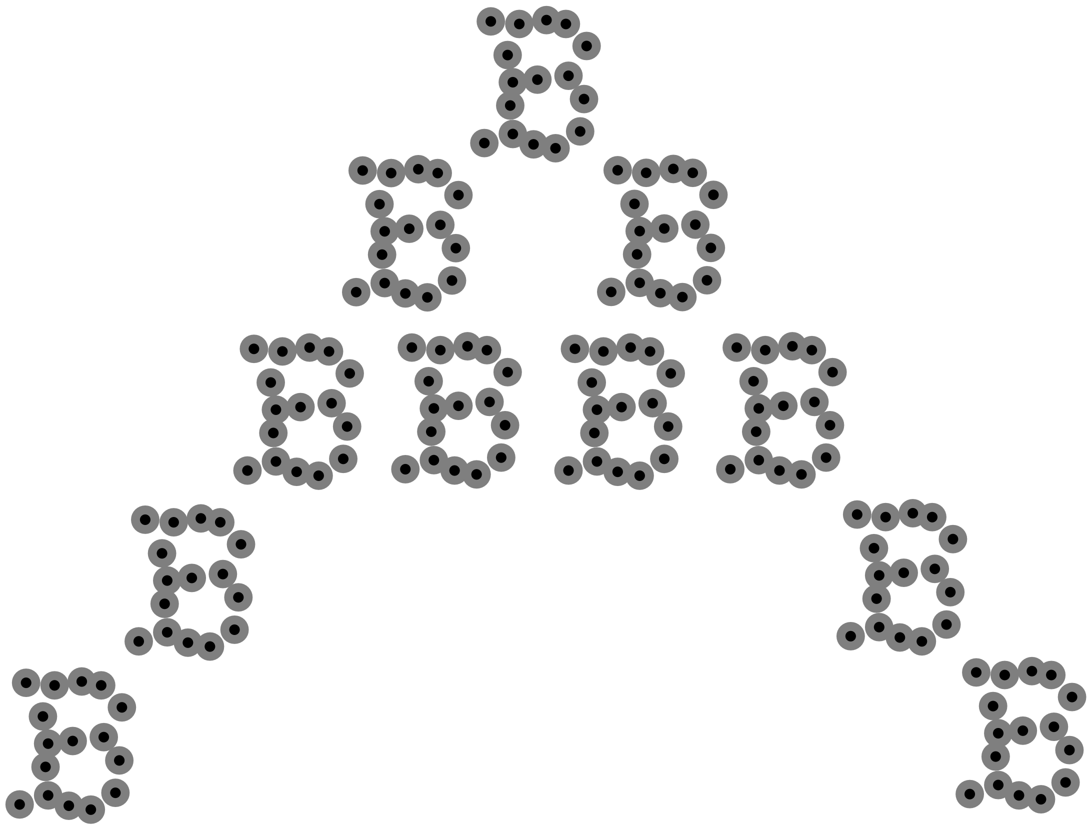

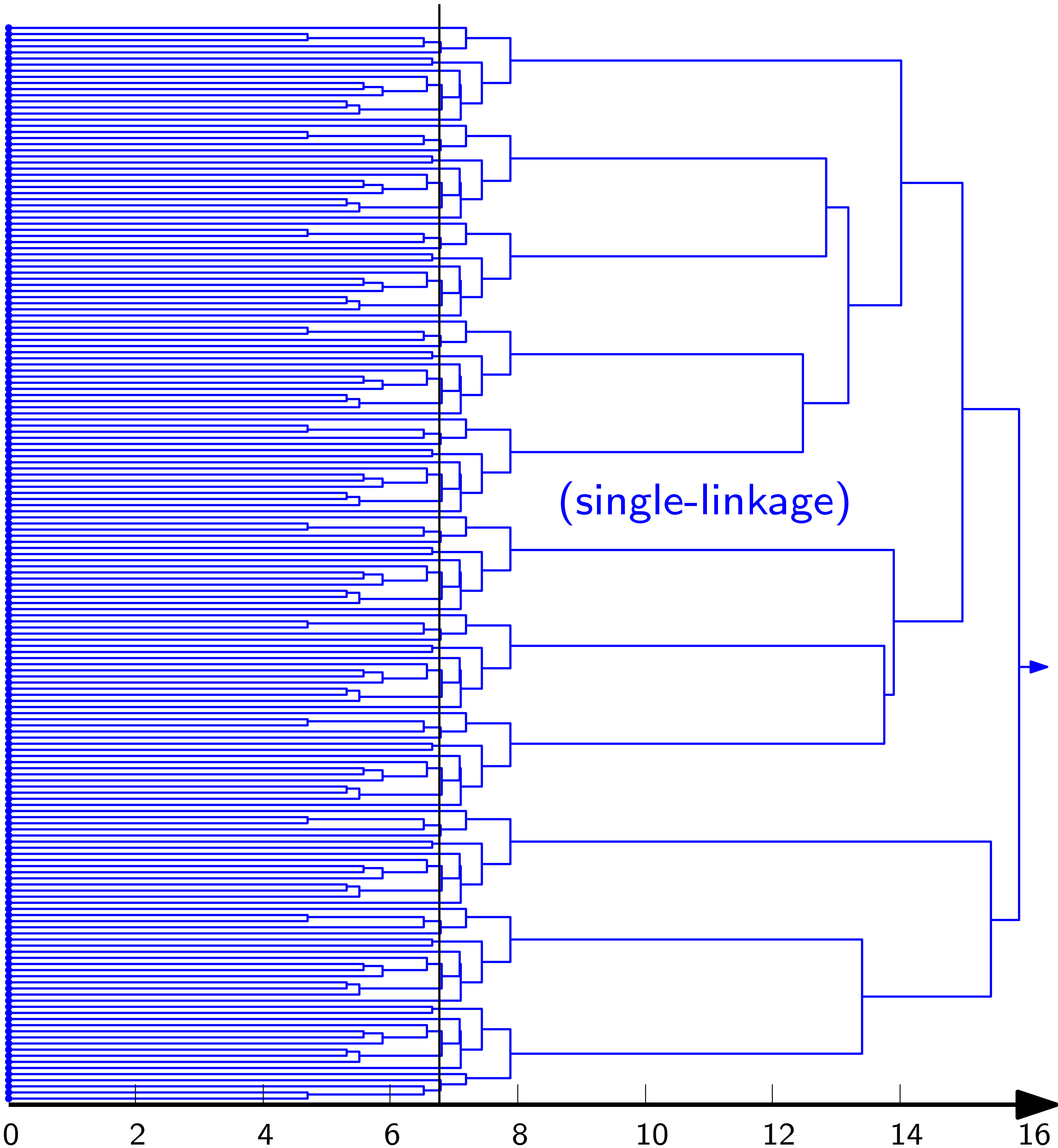

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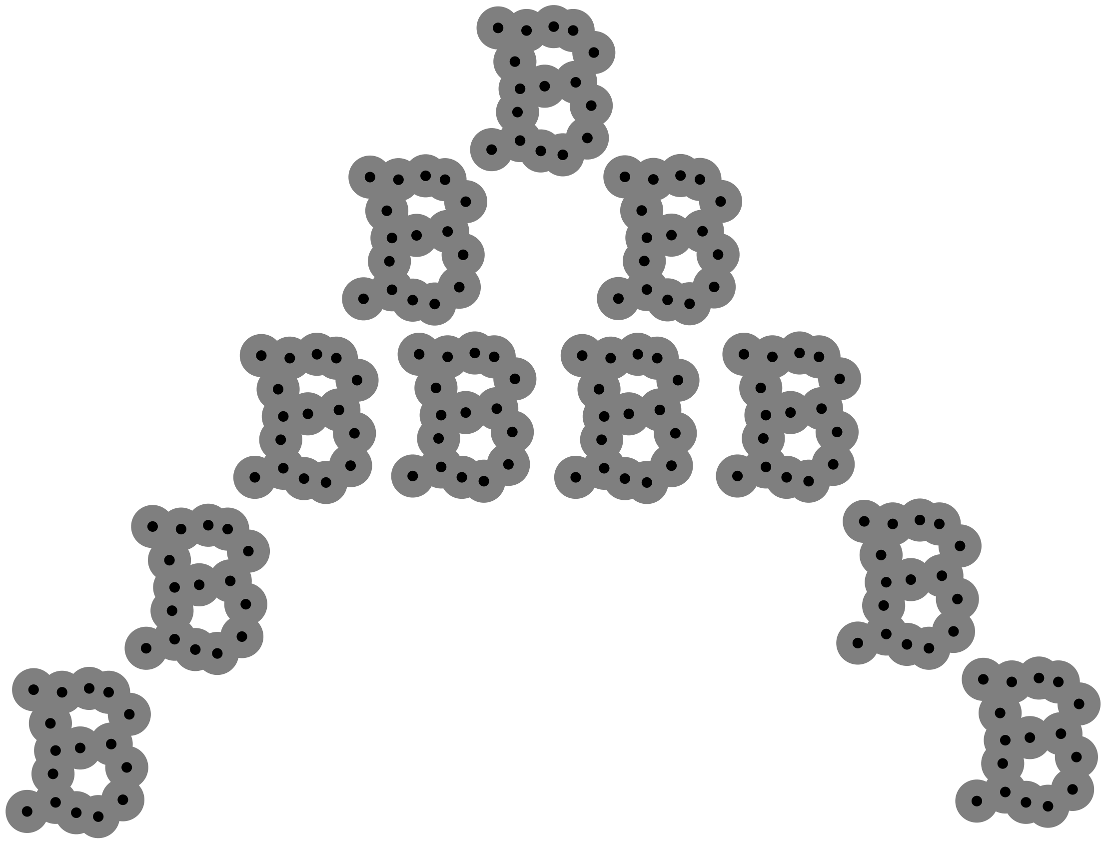

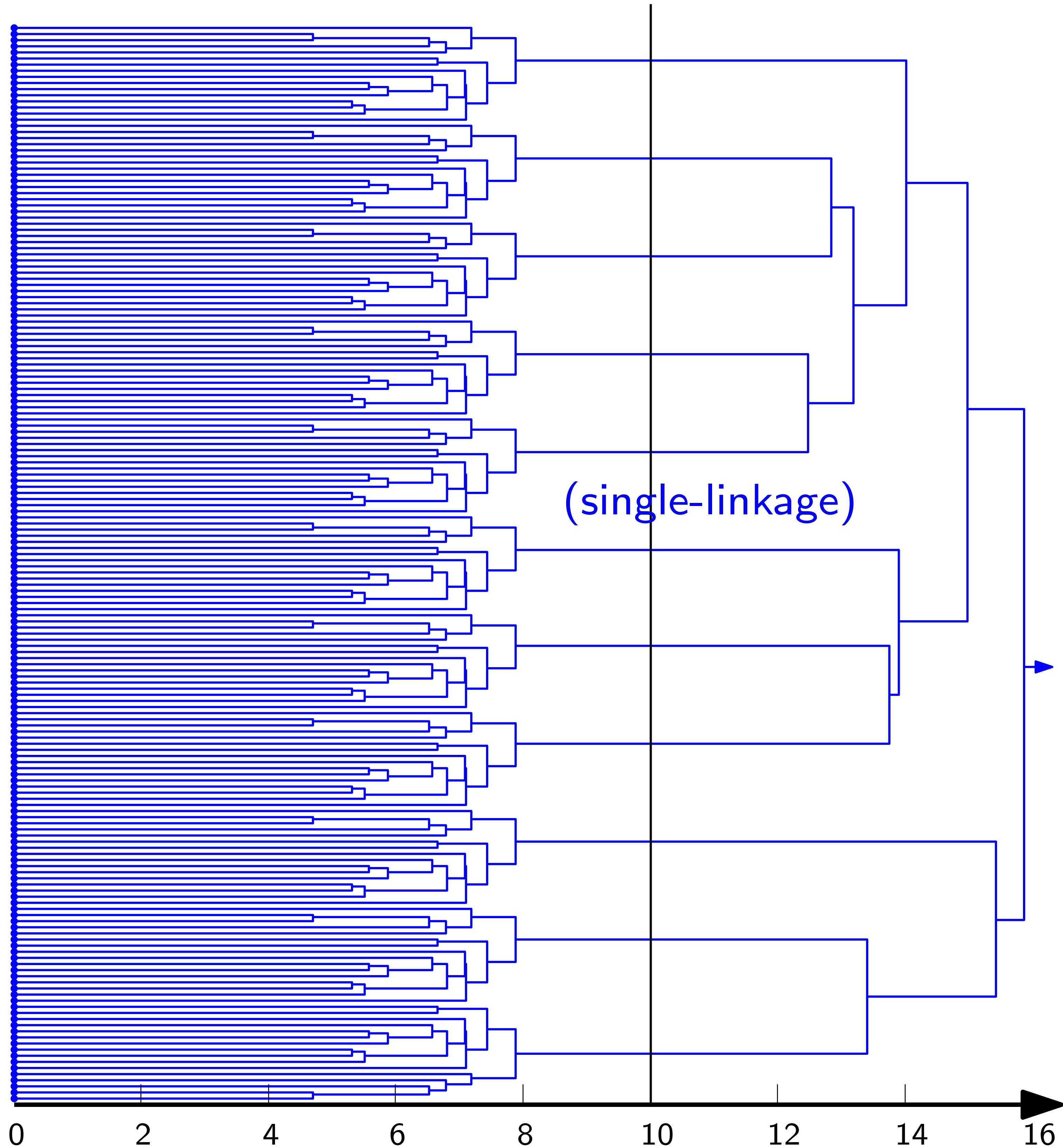

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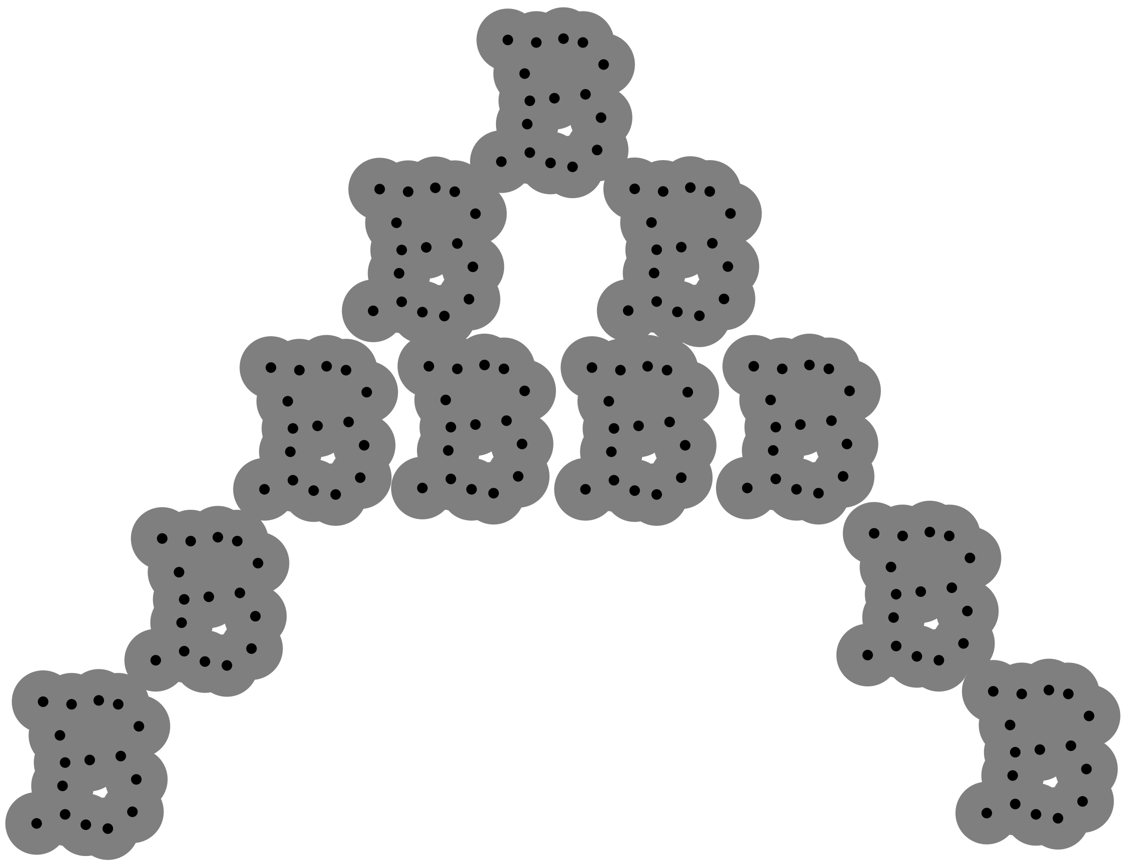

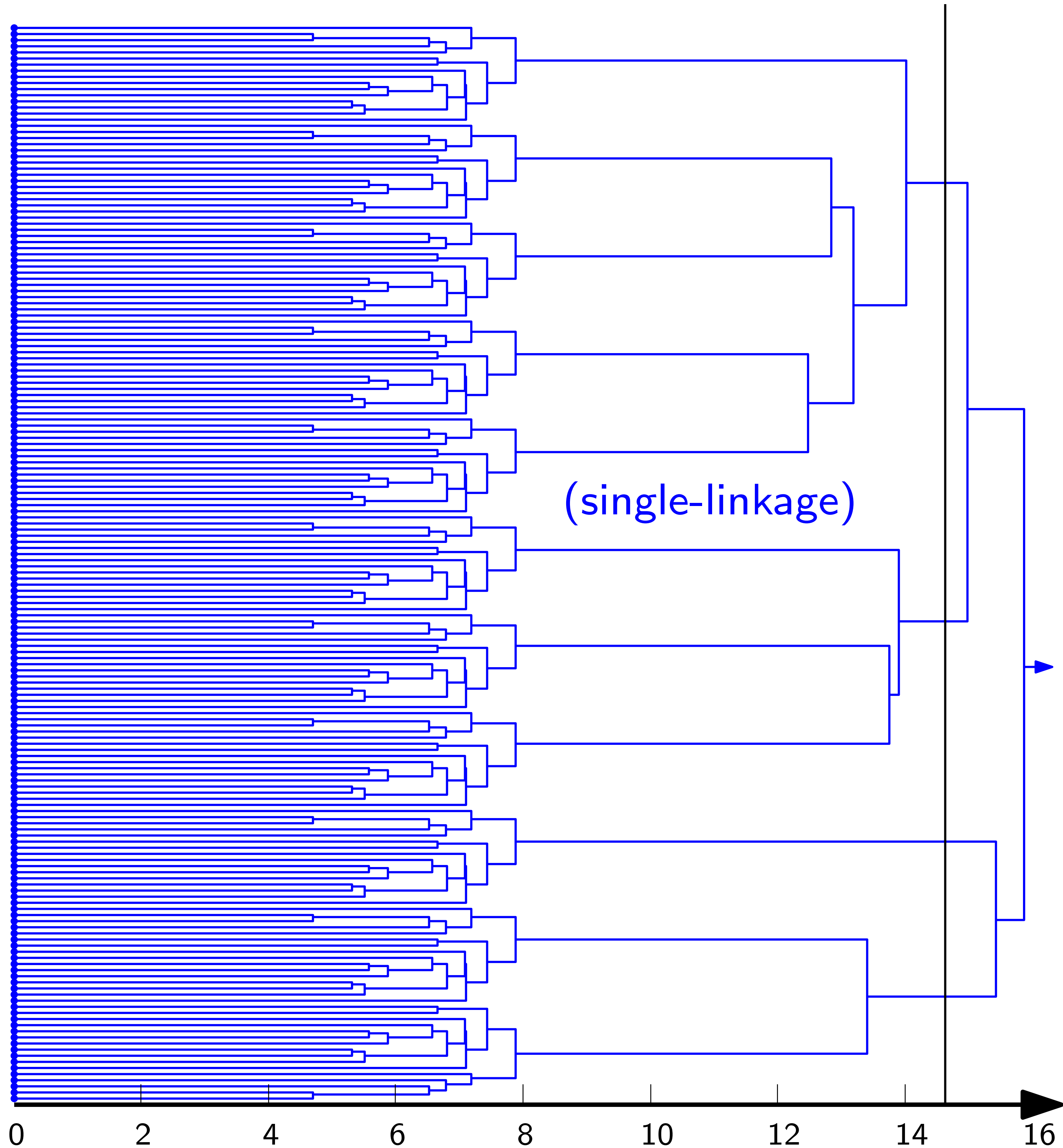

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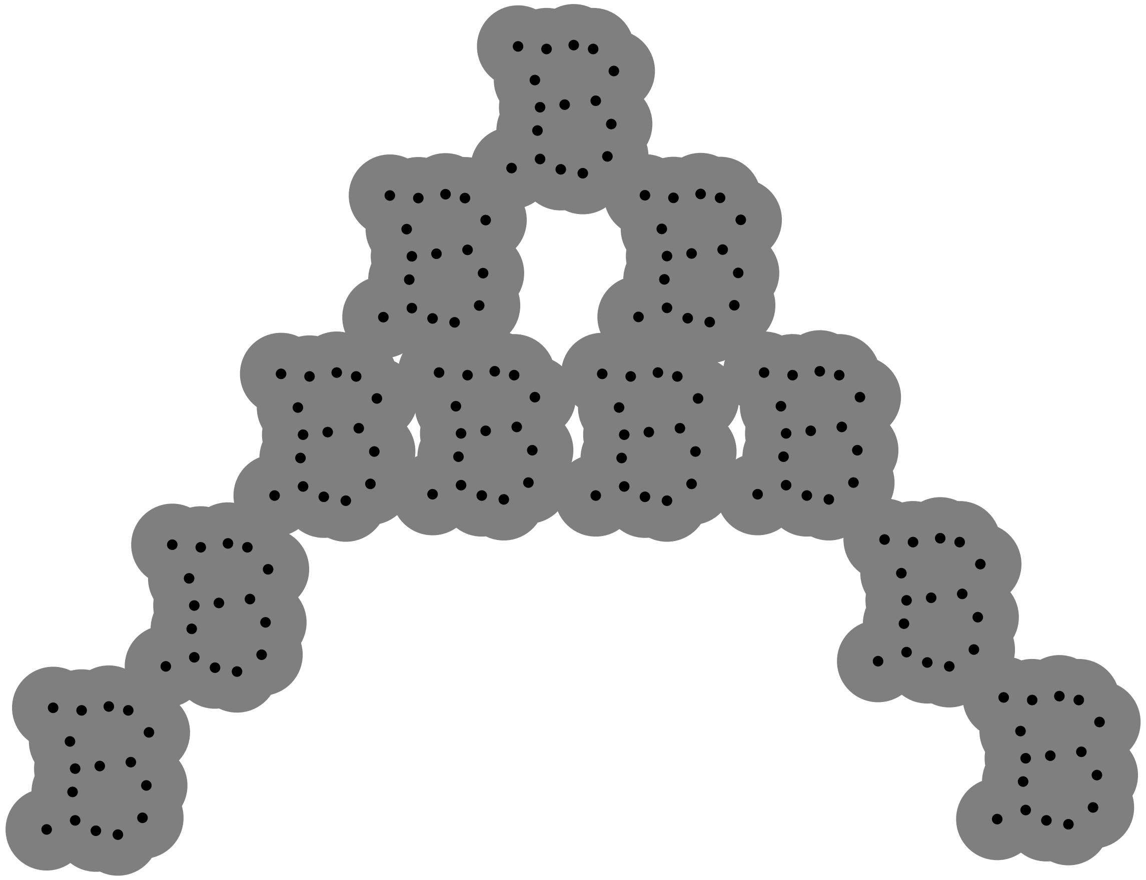

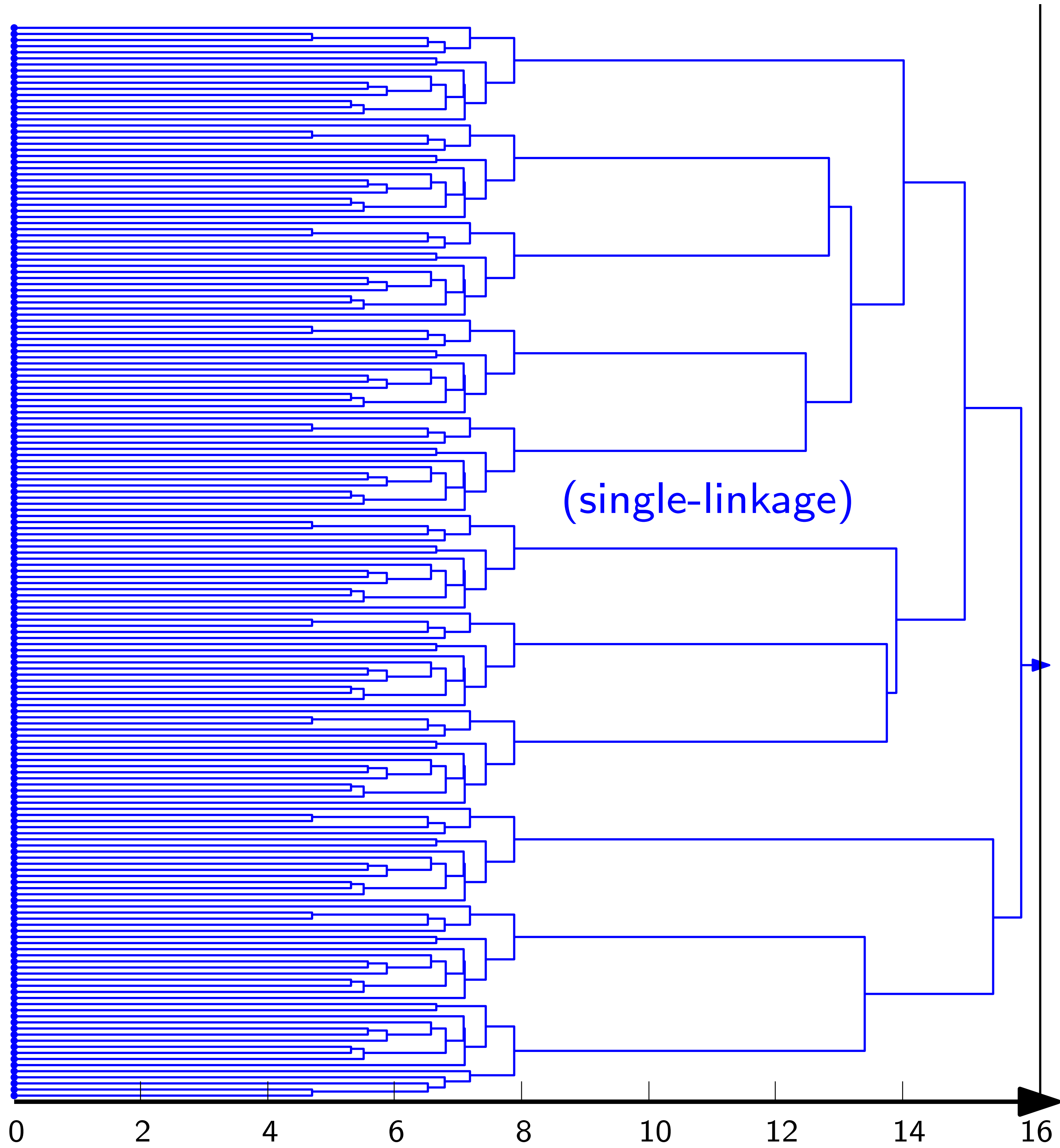

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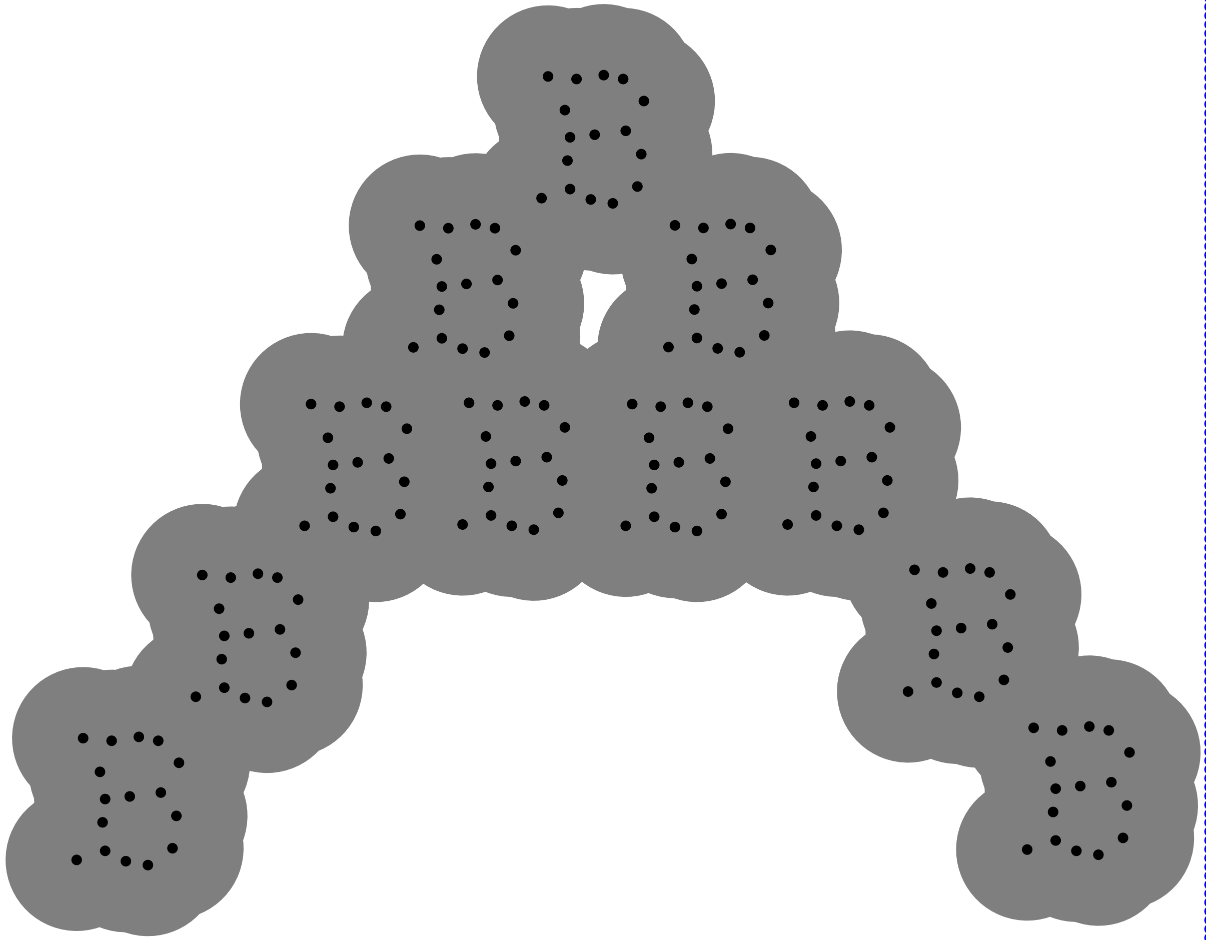

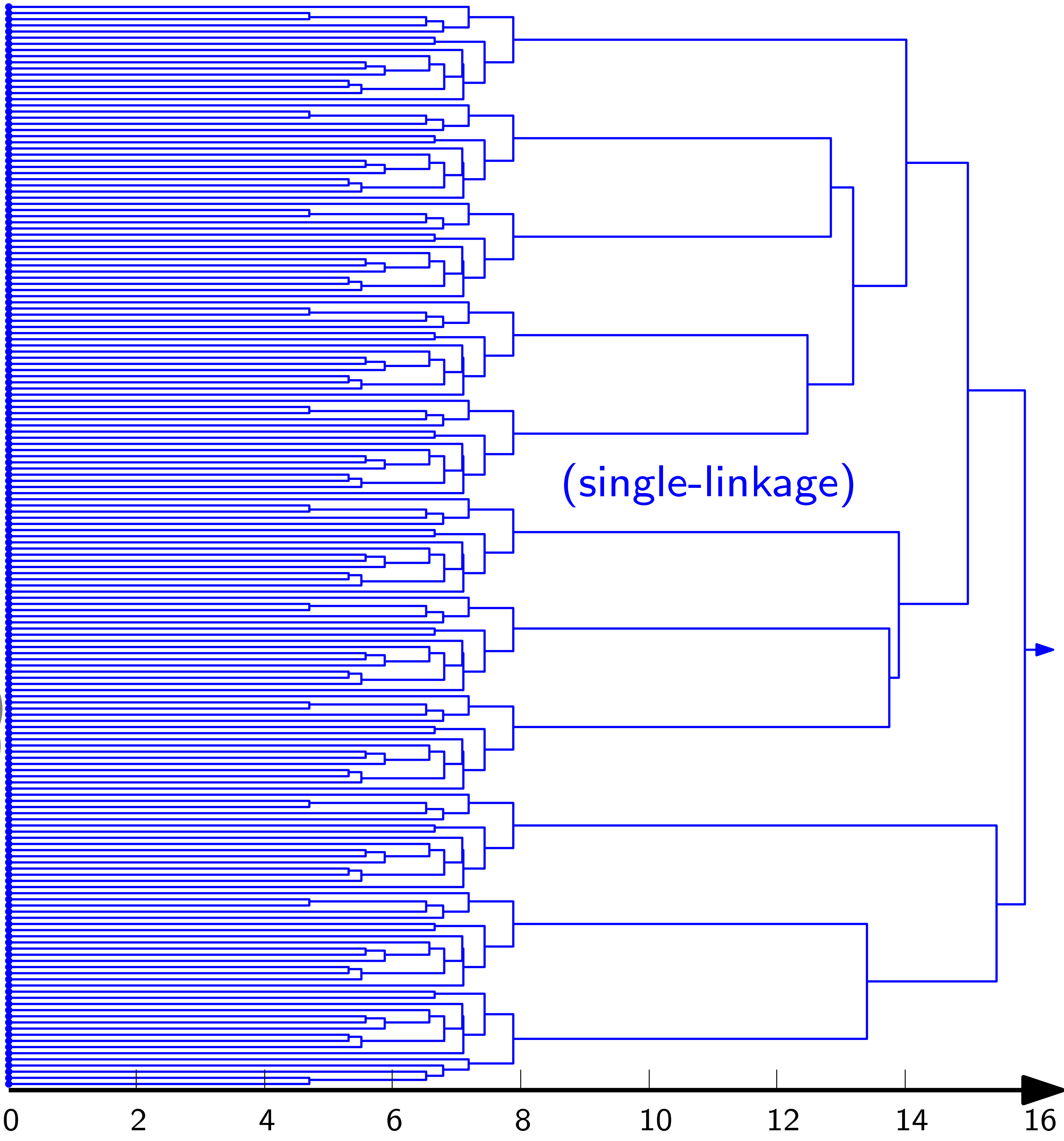

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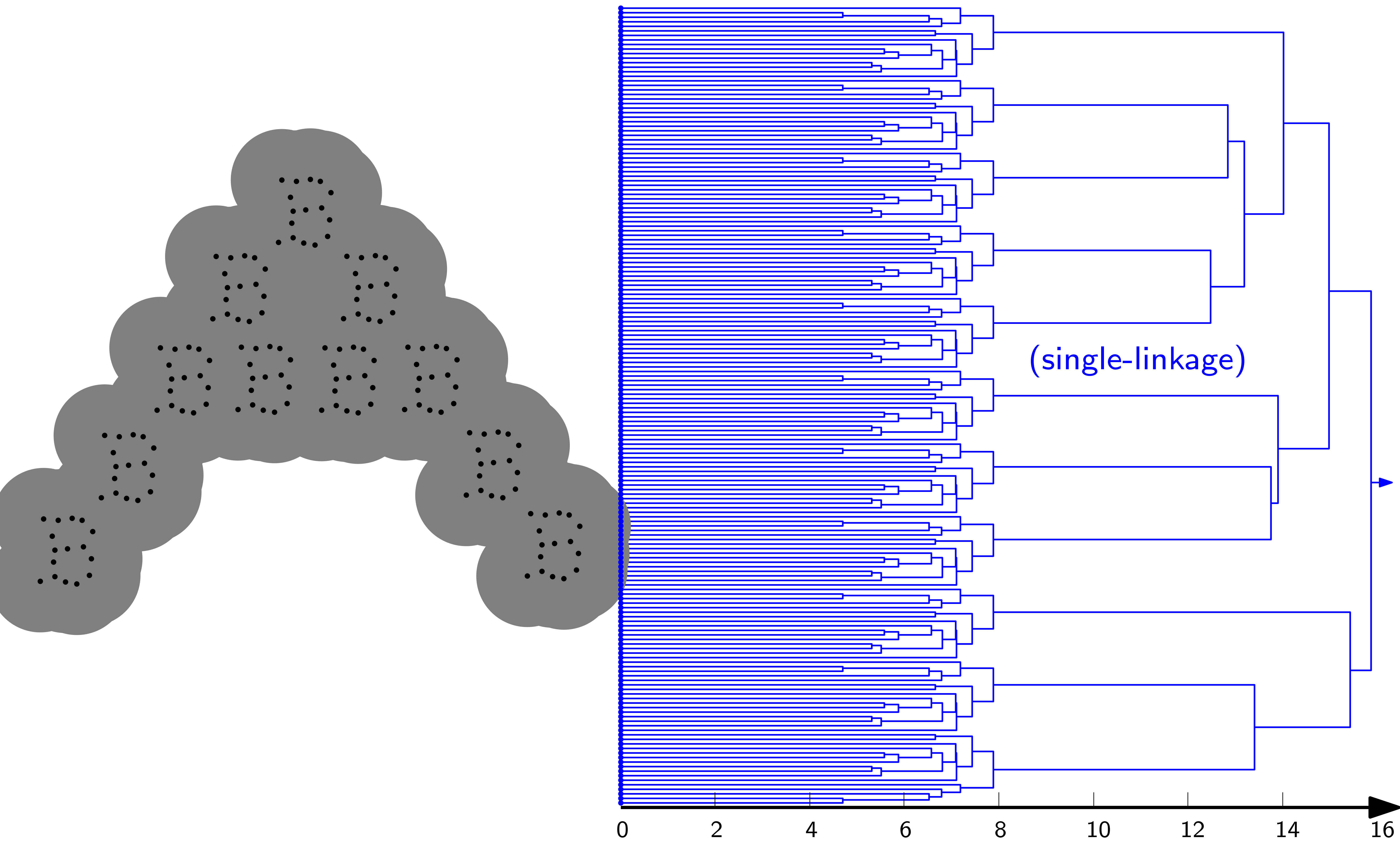

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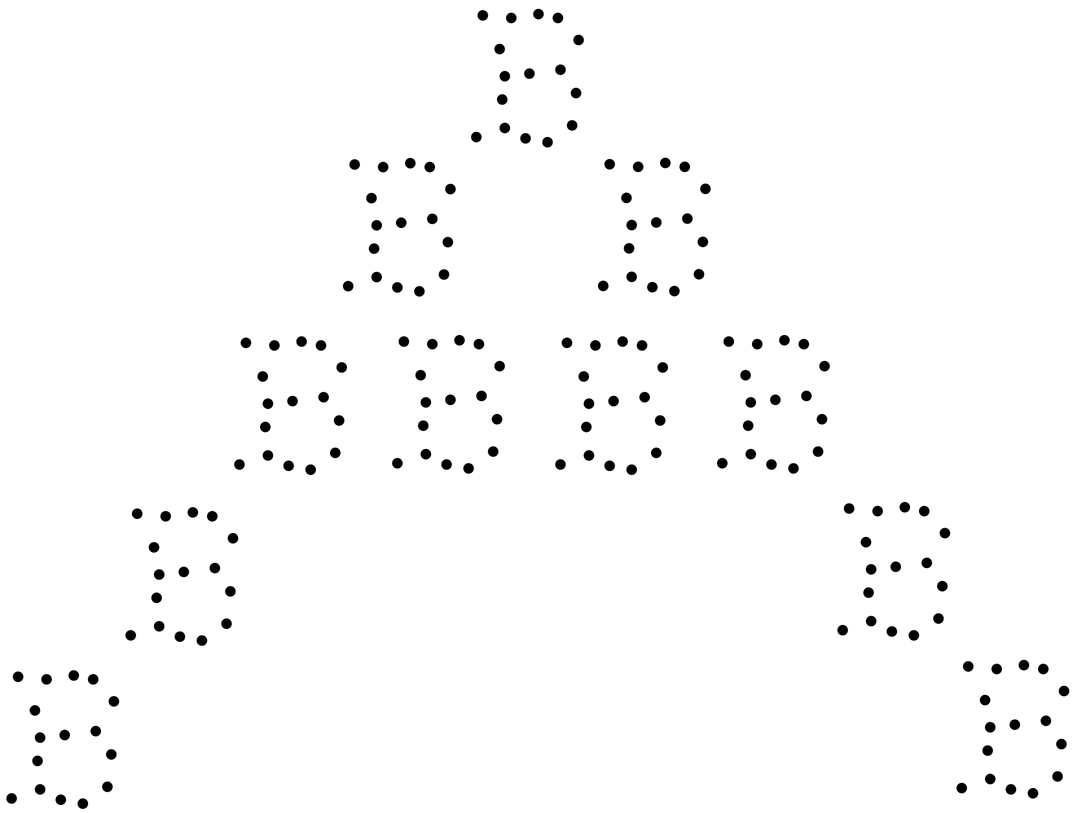
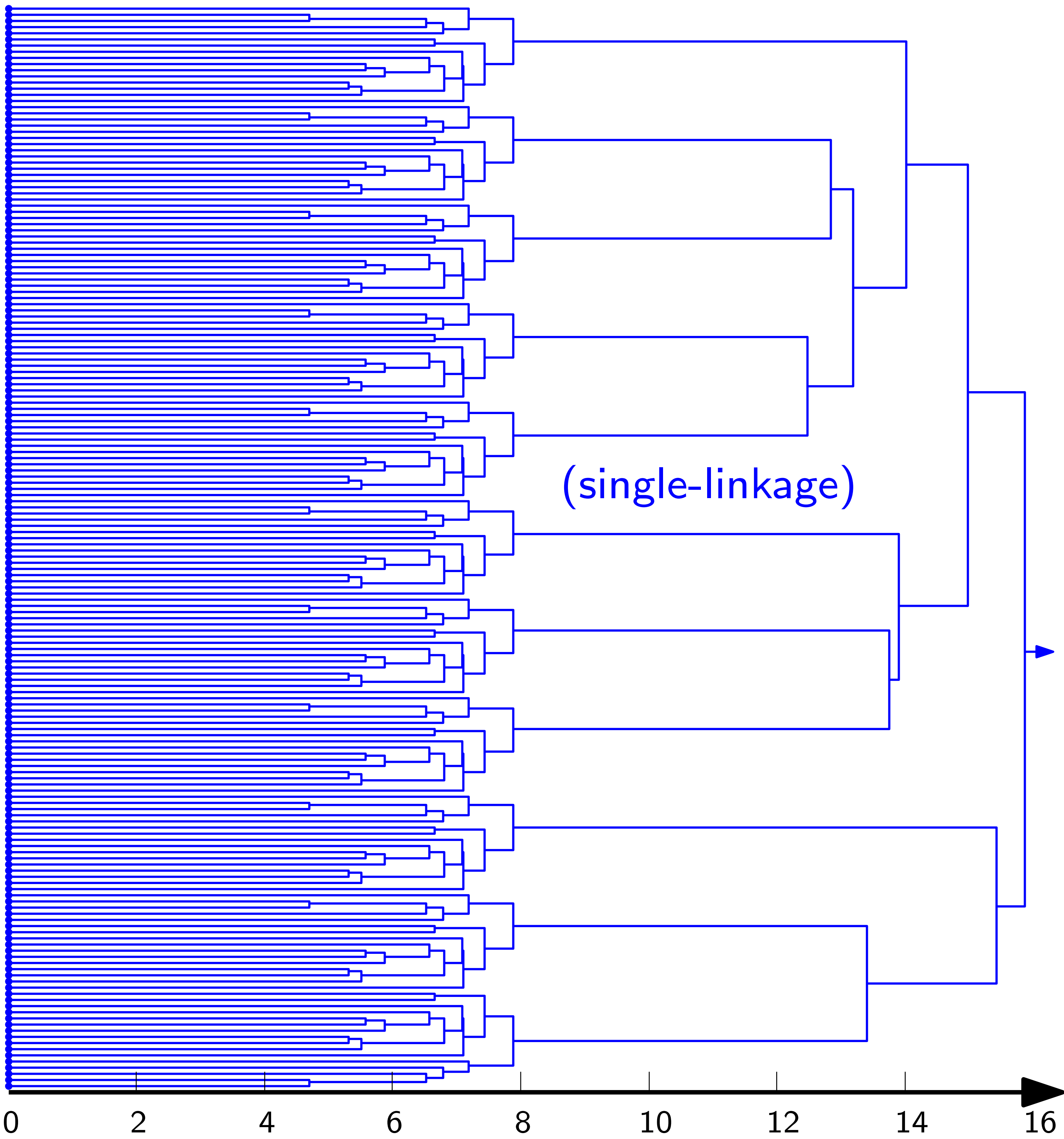


dendrogram is:

- informative
- unstable



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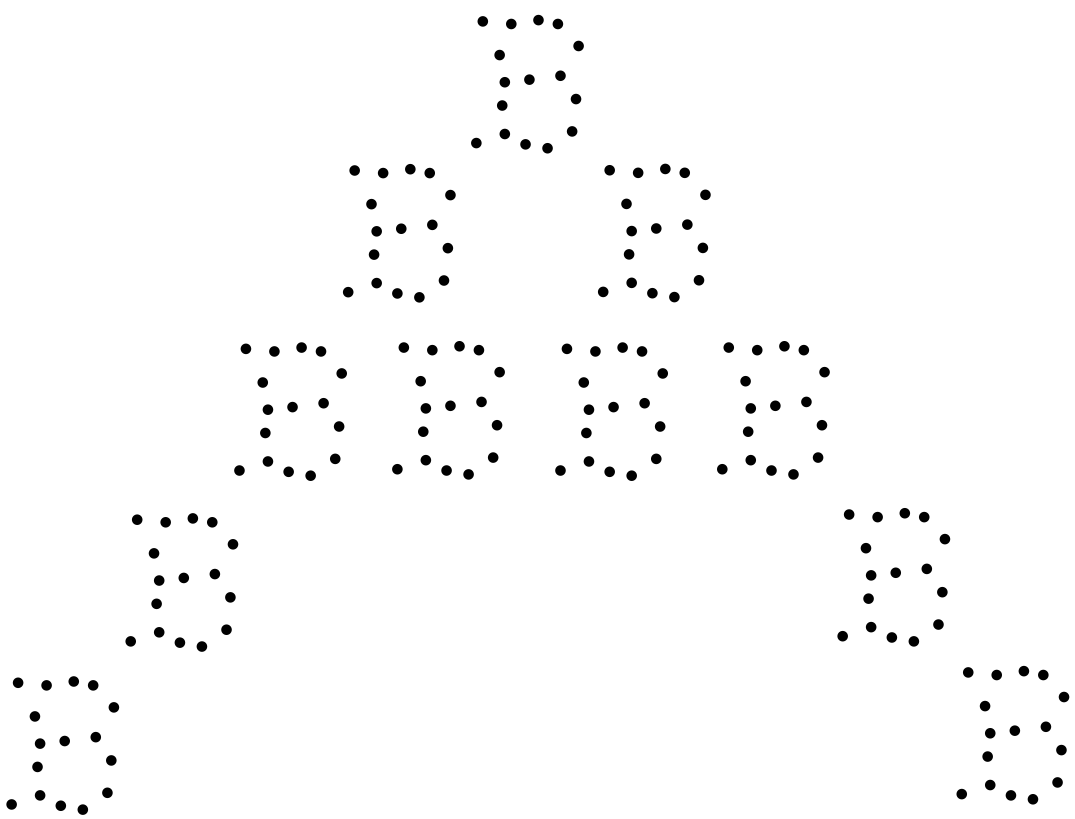
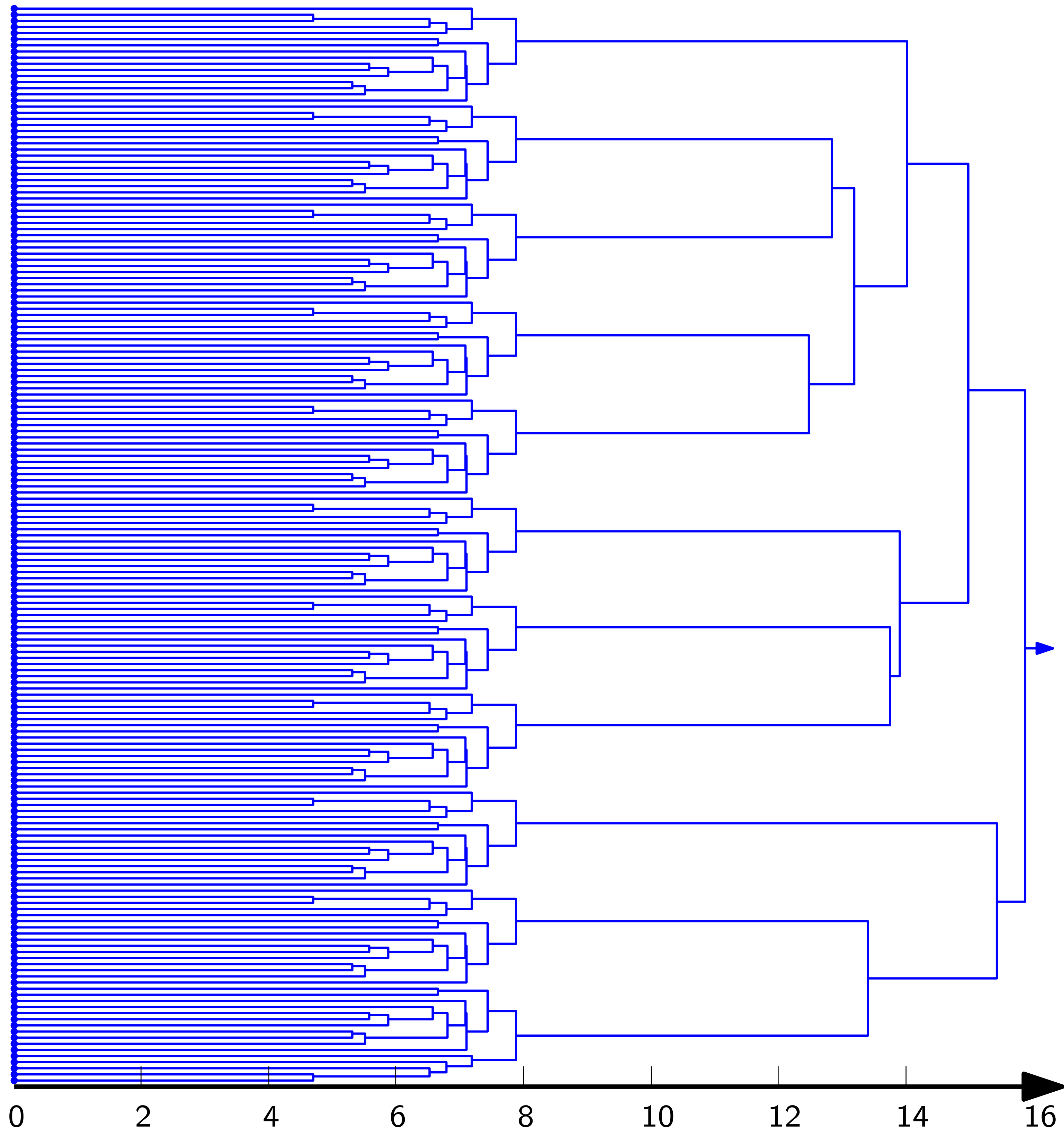


dendrogram → barcode



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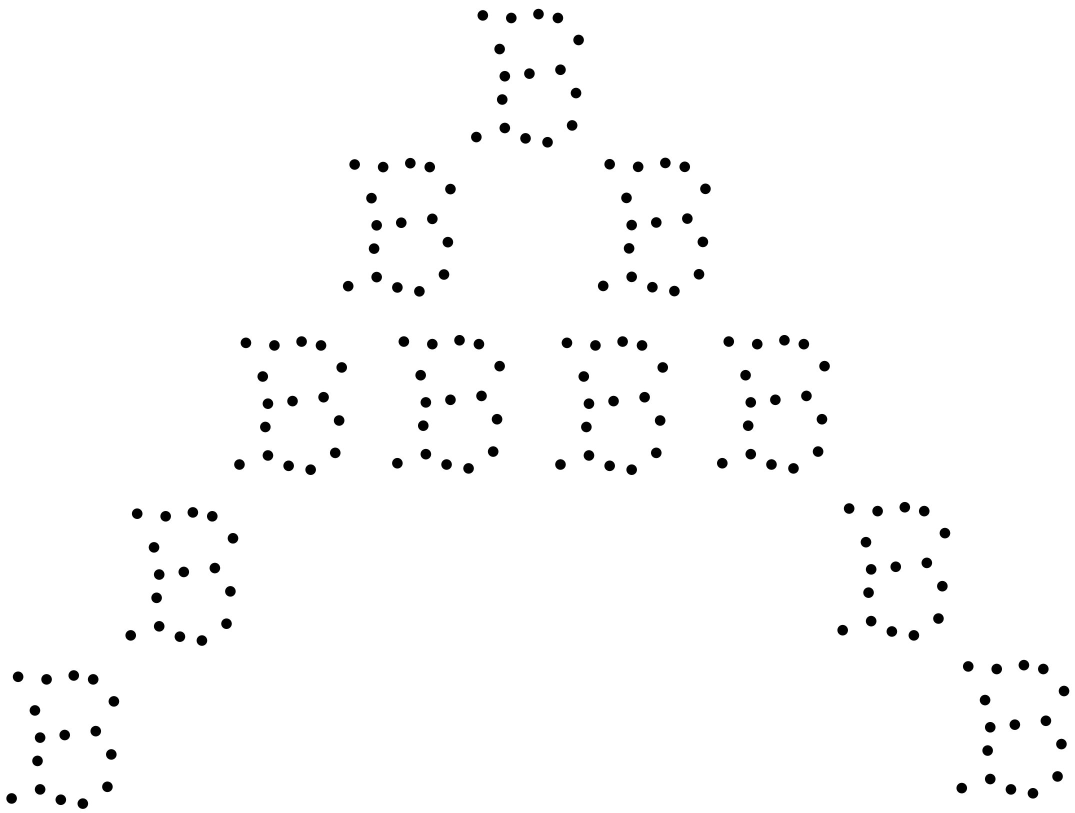

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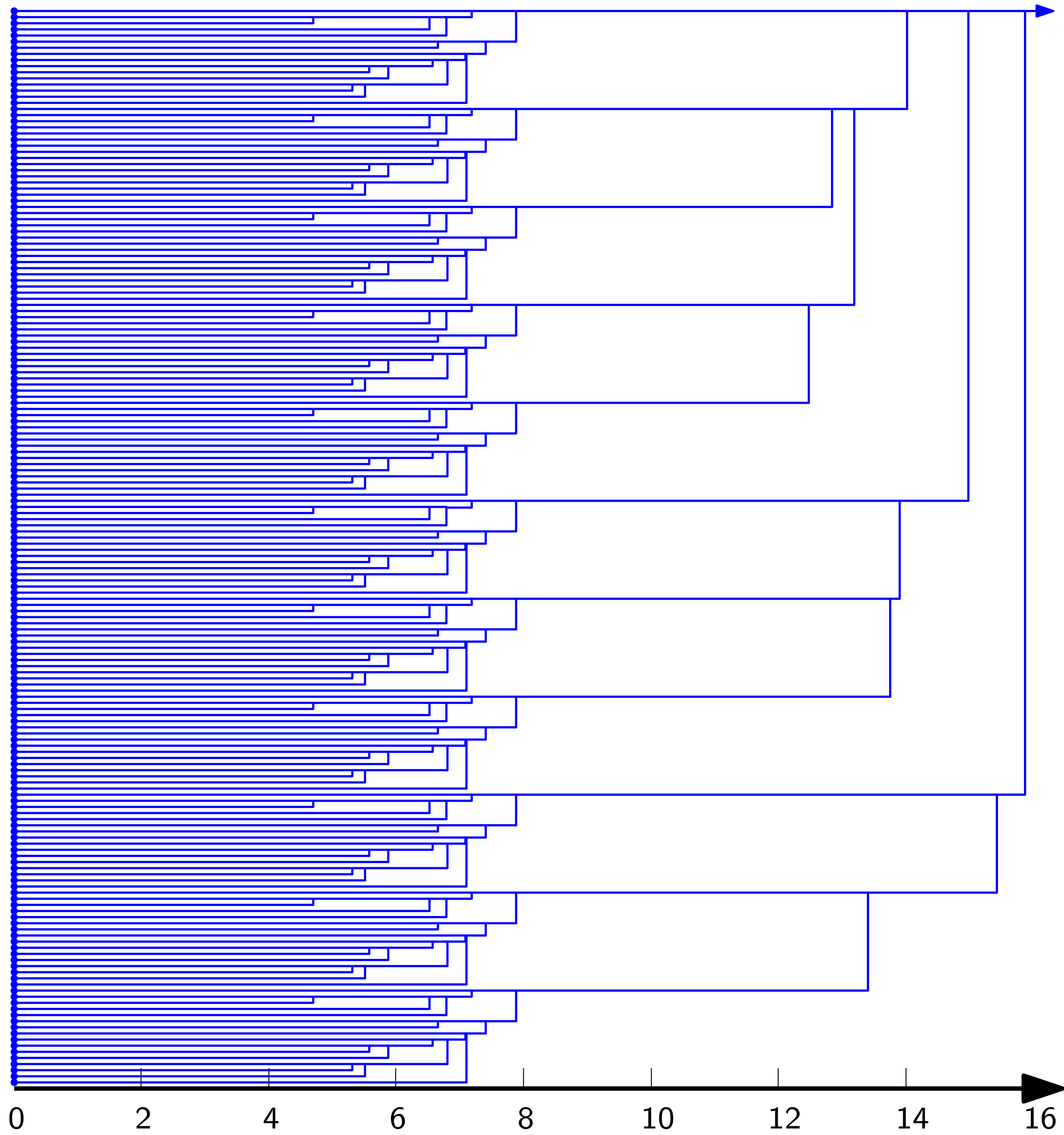

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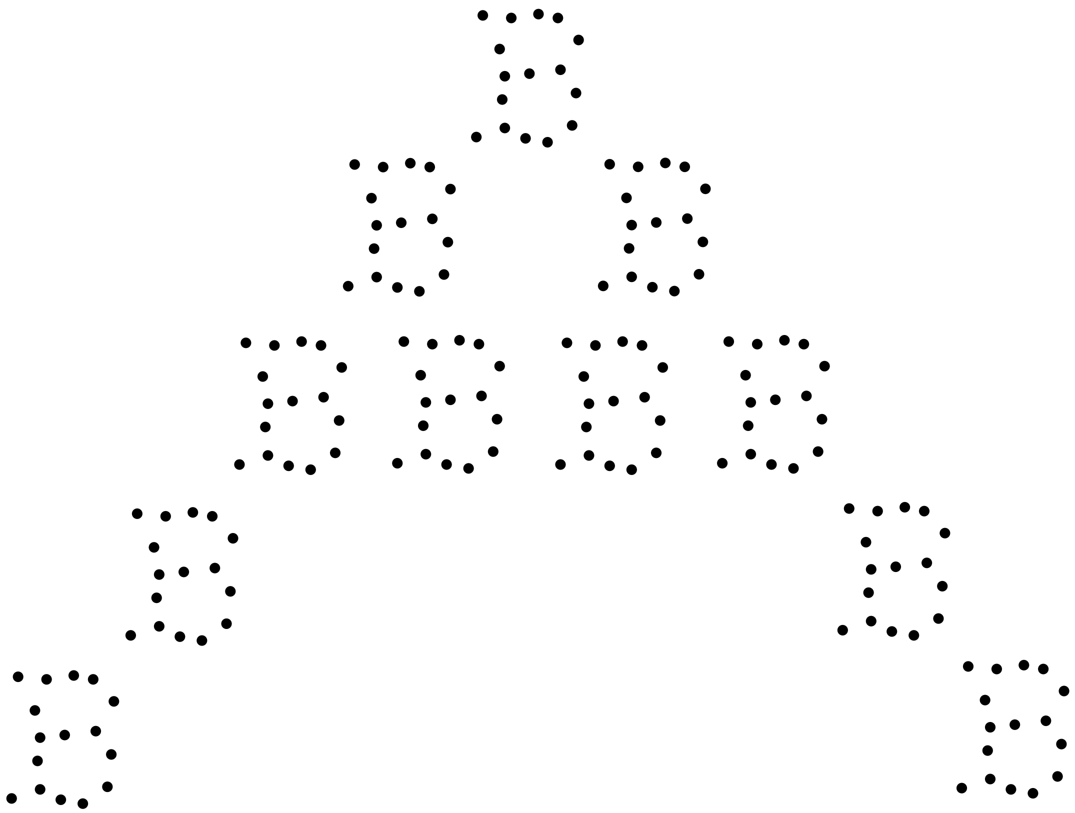

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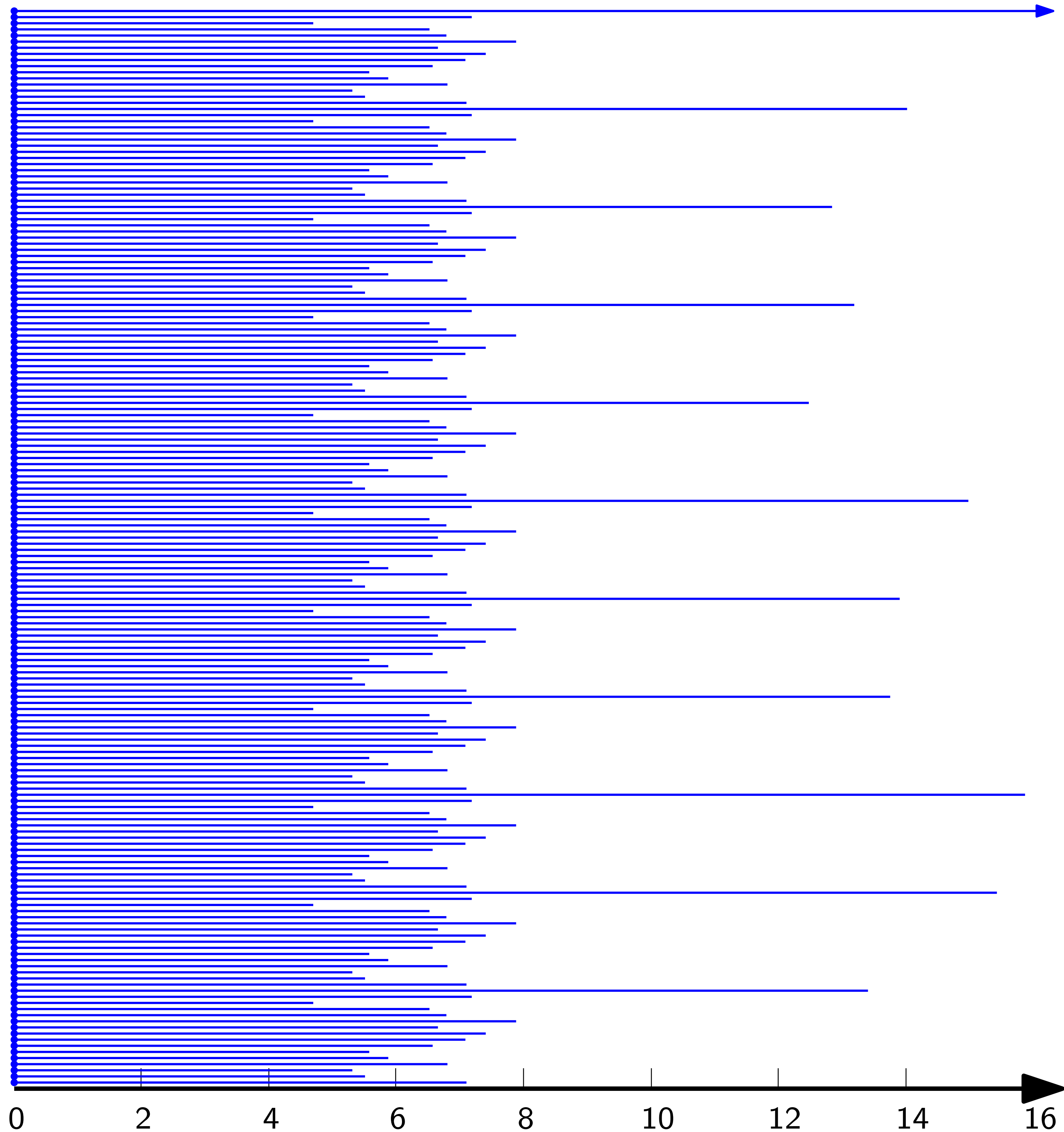

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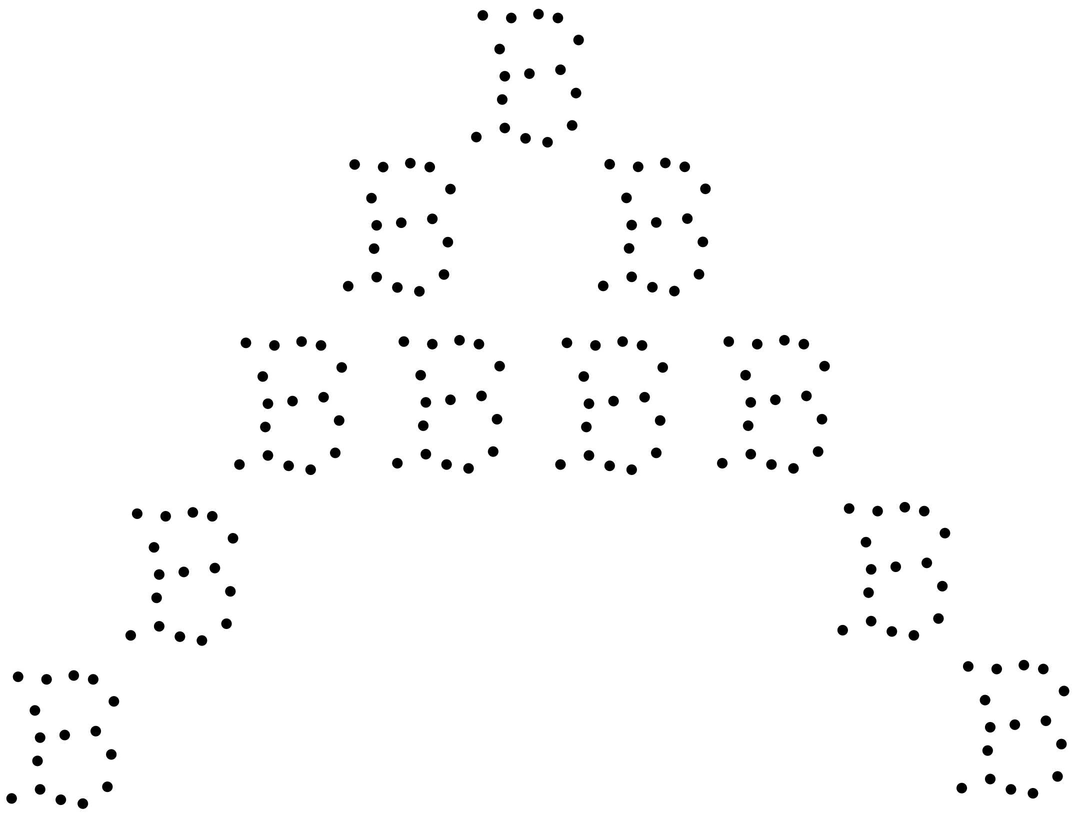
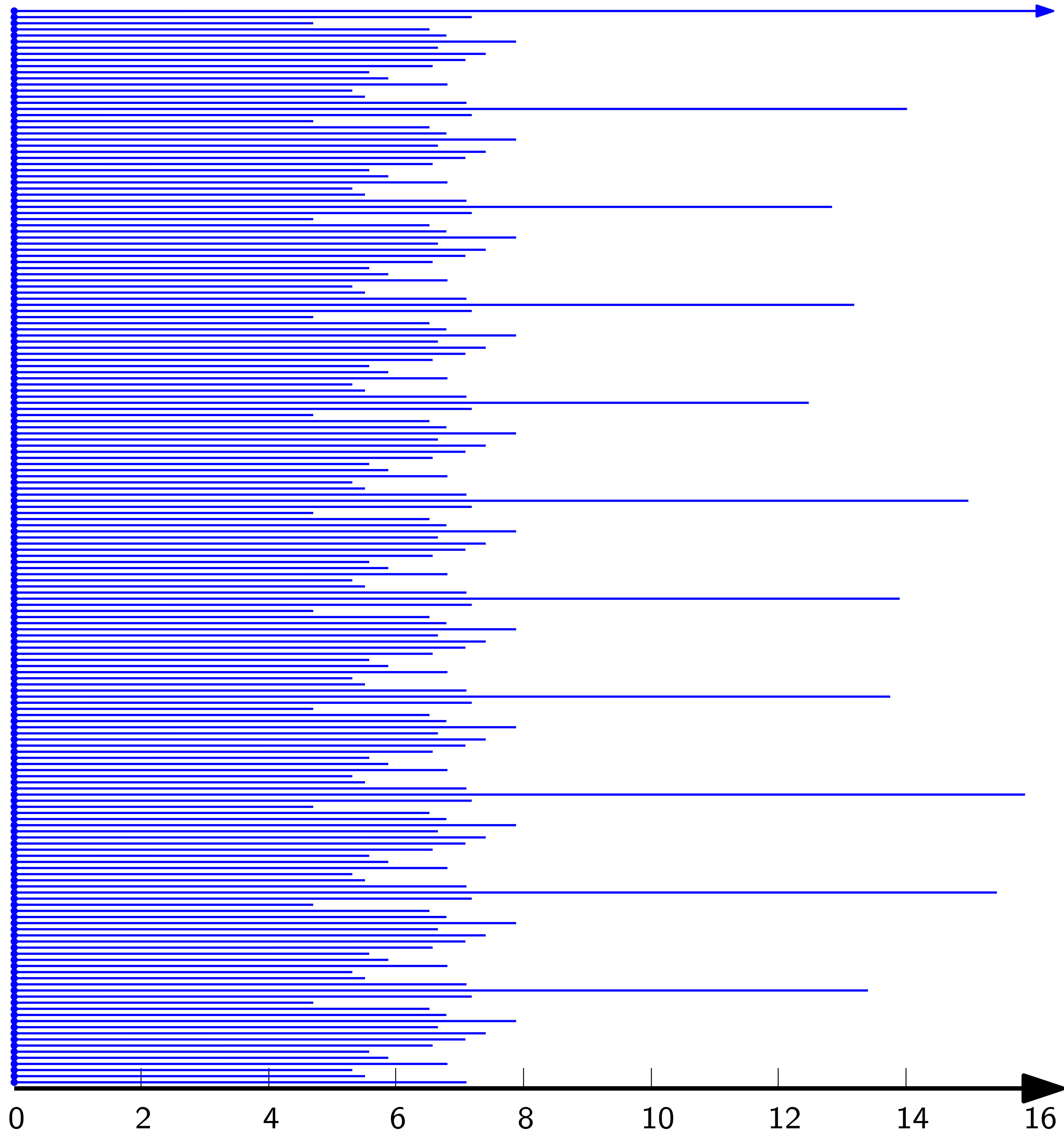


barcode is:

- less (but still) informative
- more stable



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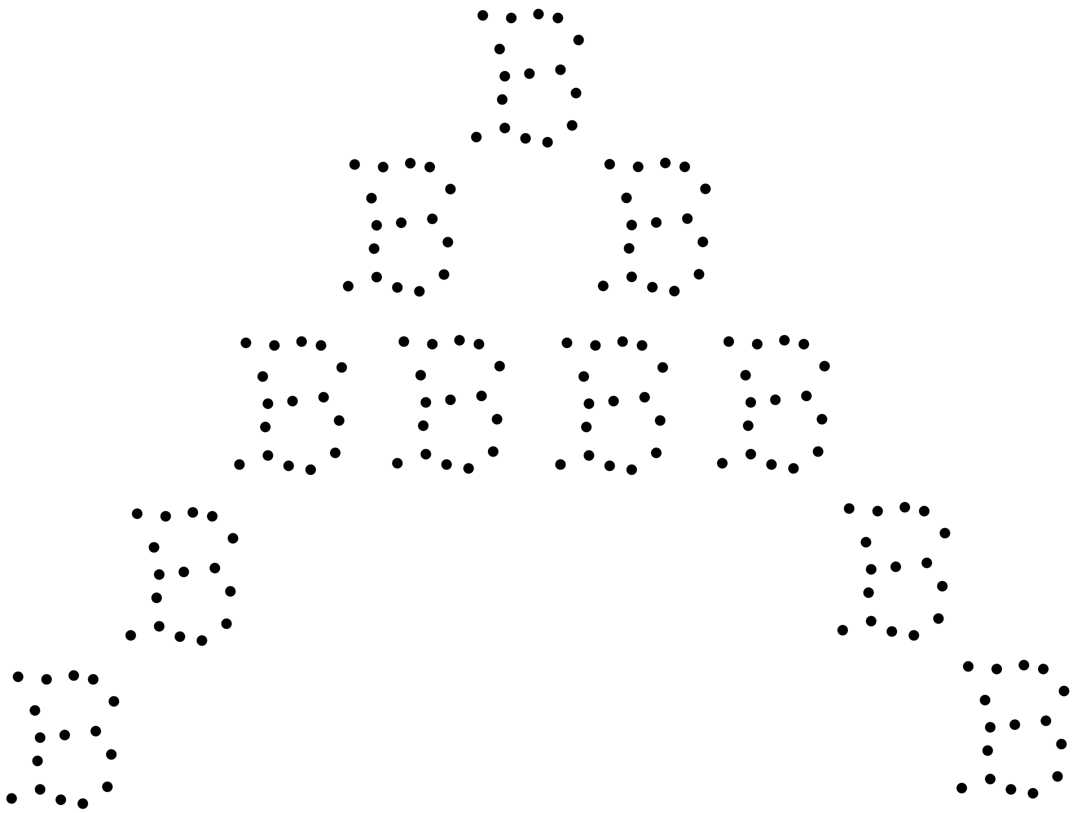

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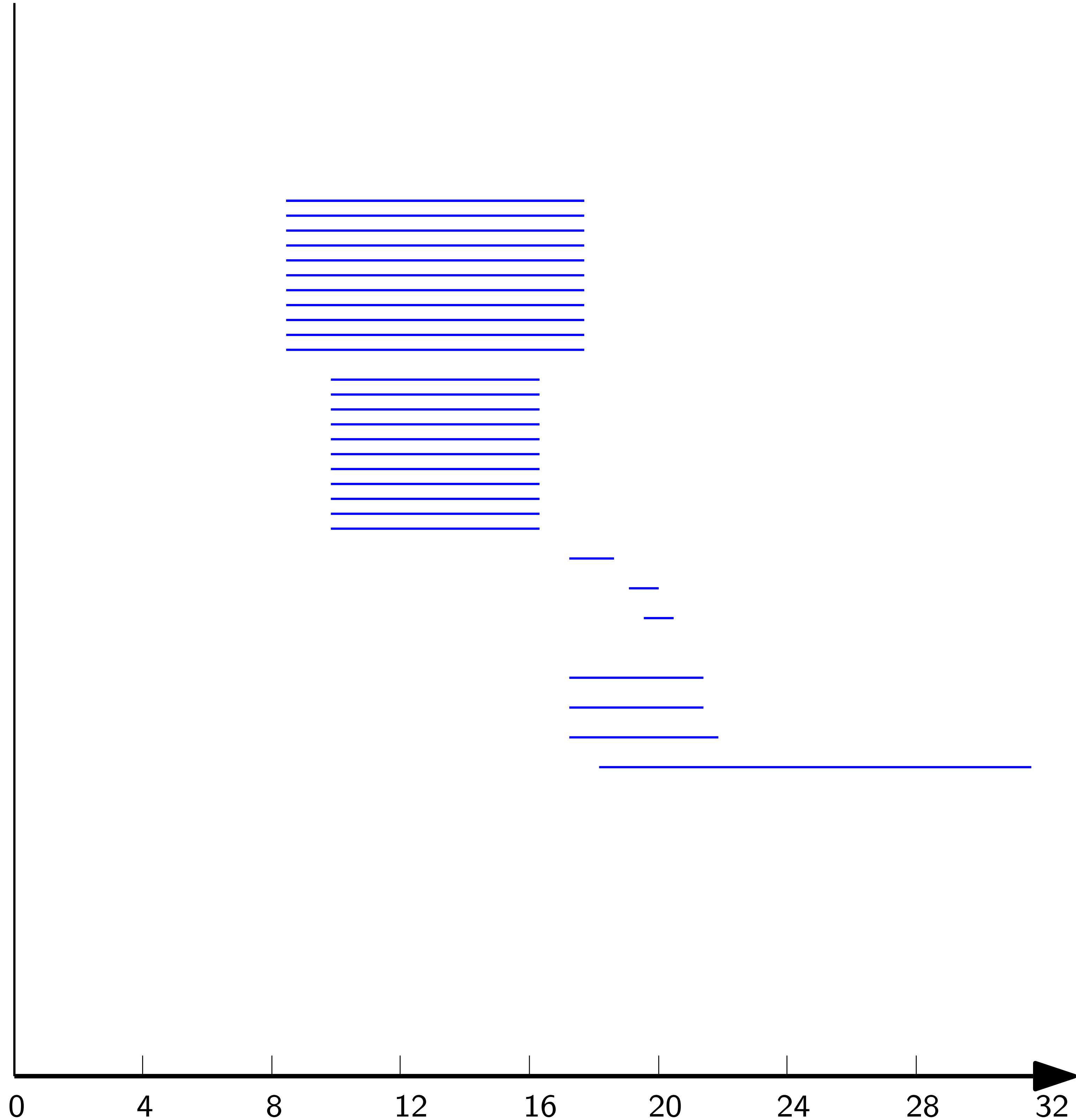
- generalizable



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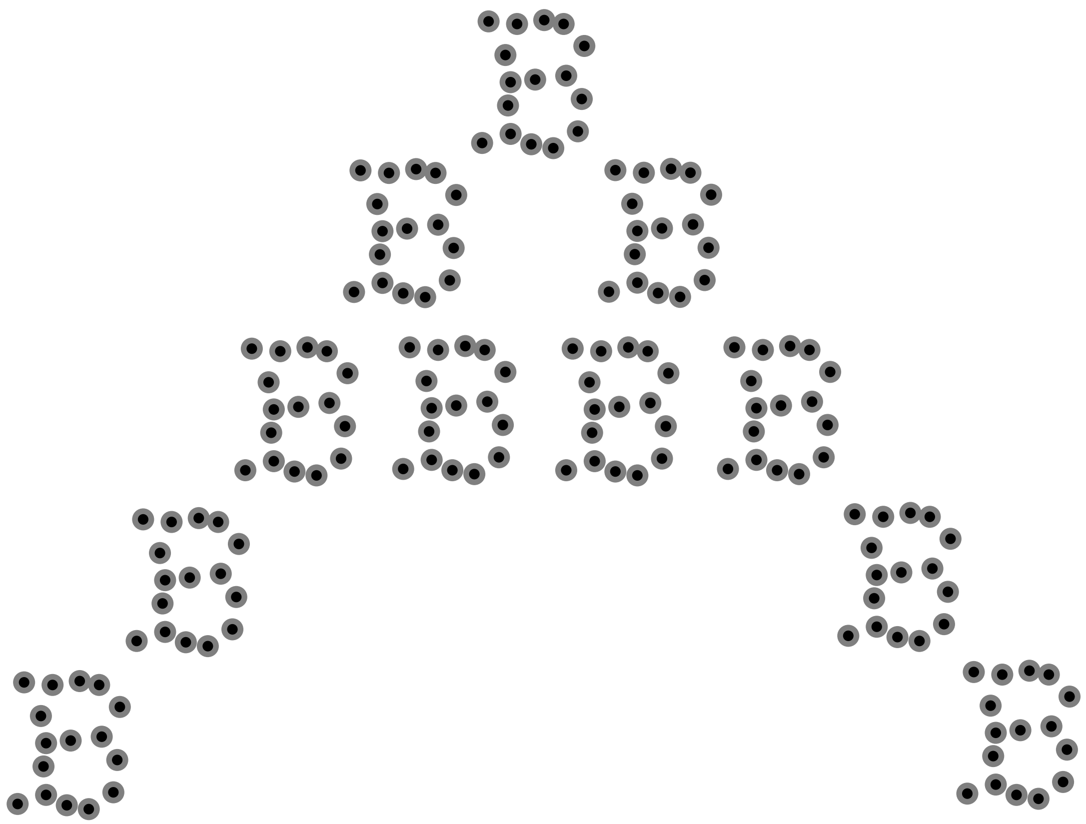

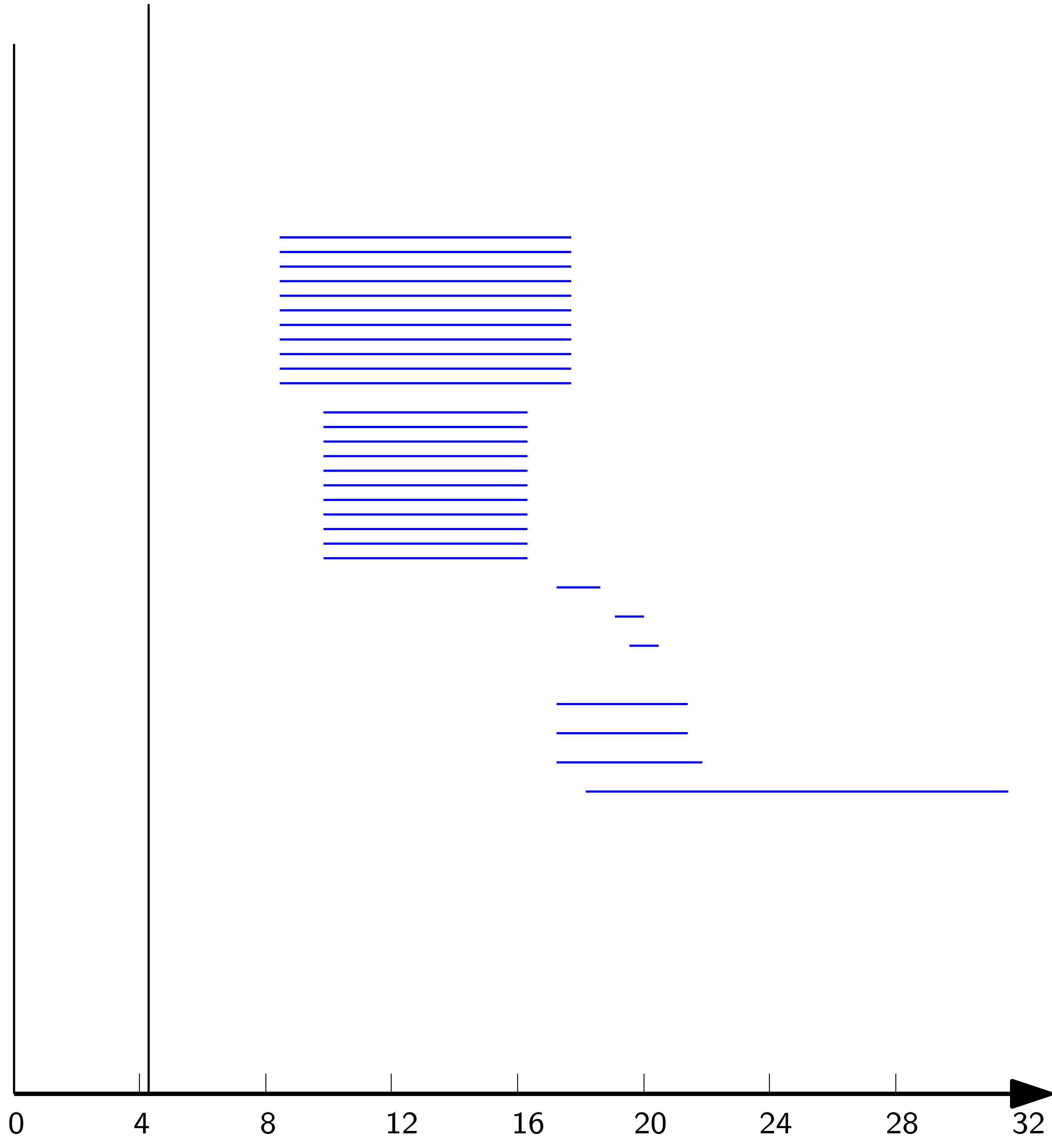

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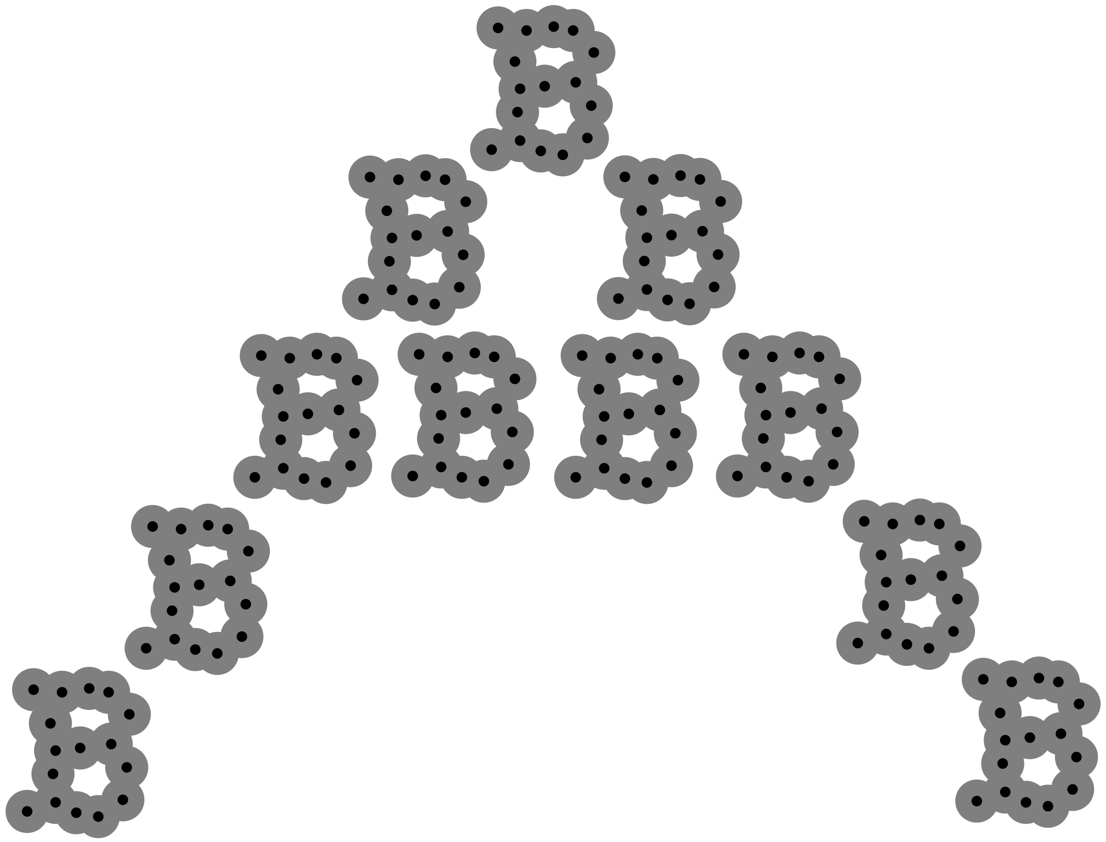

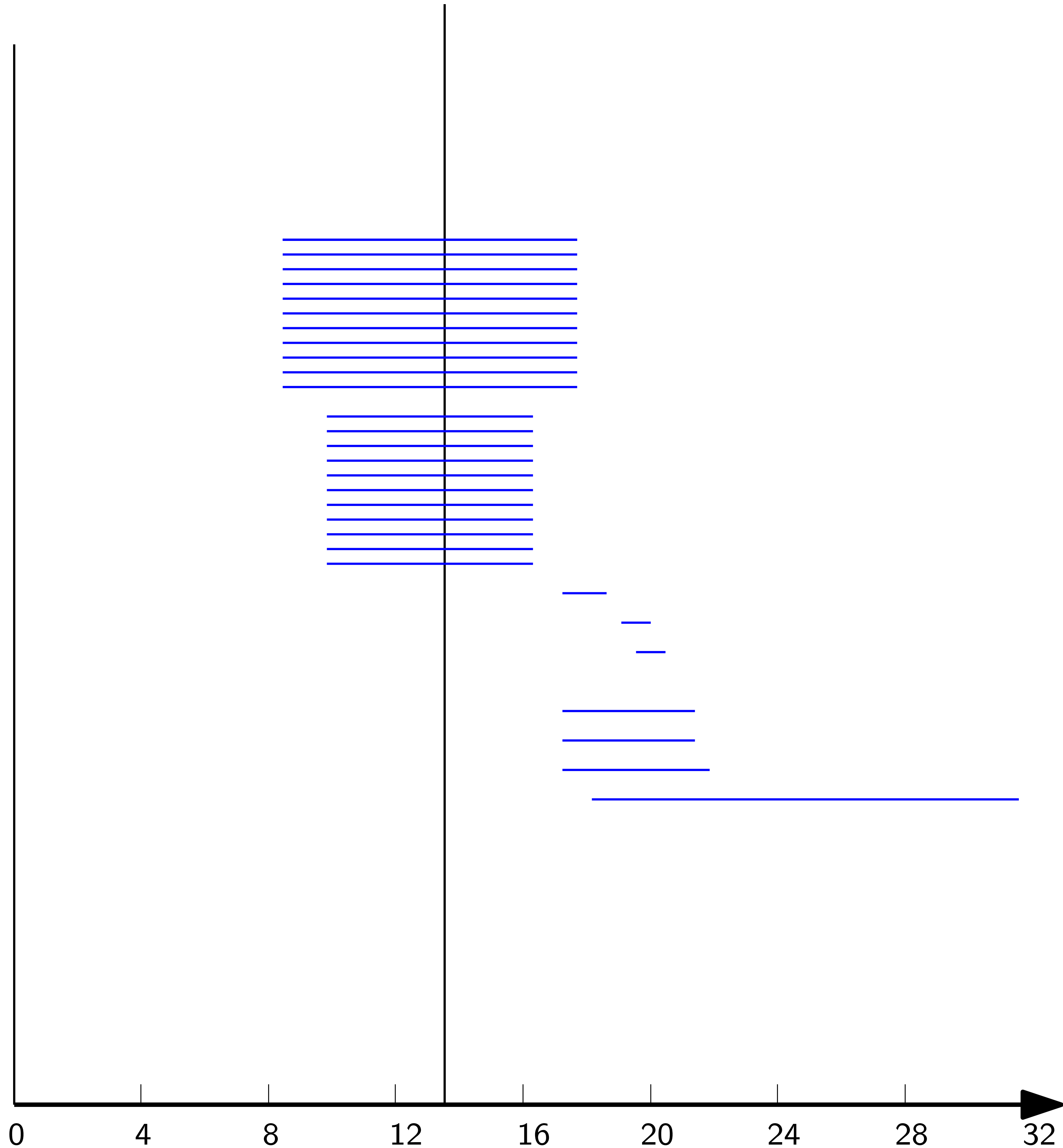

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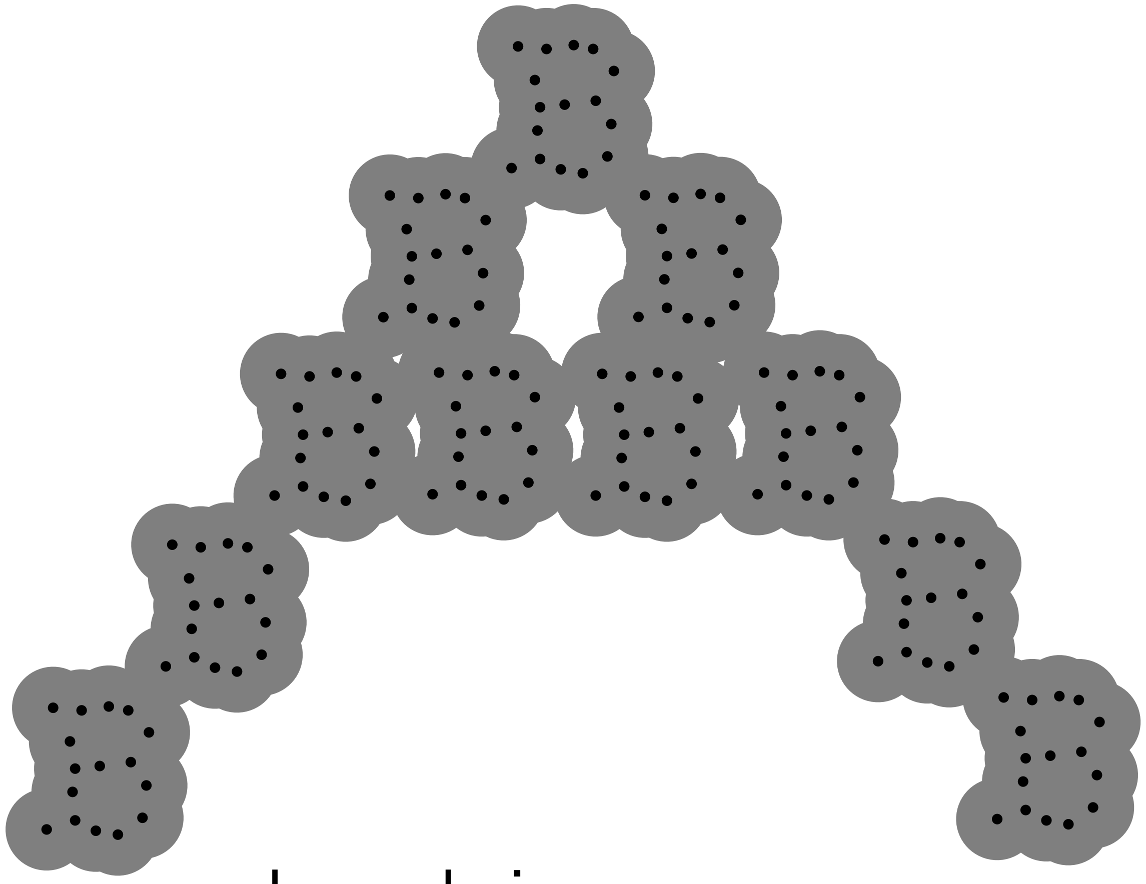

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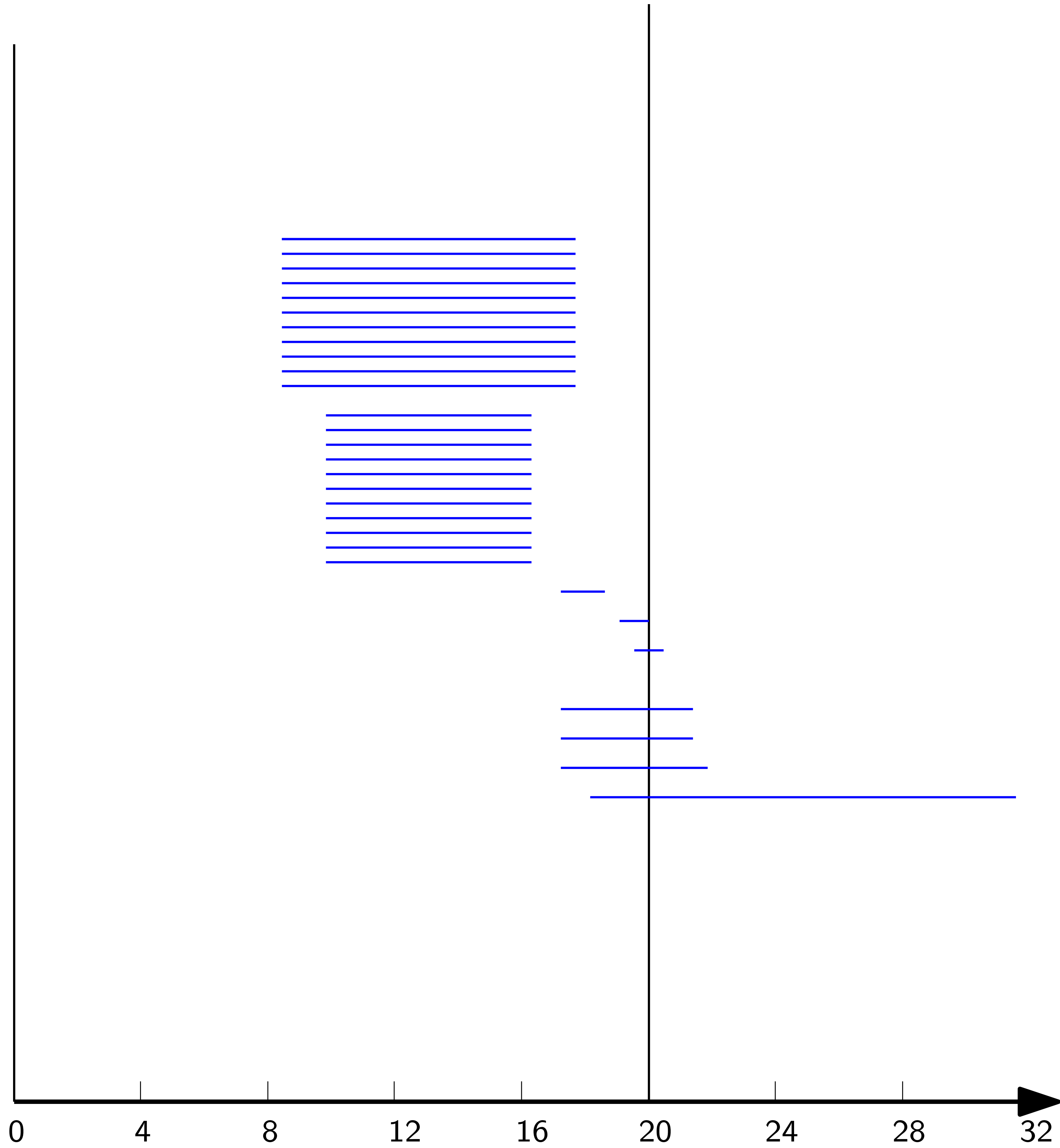

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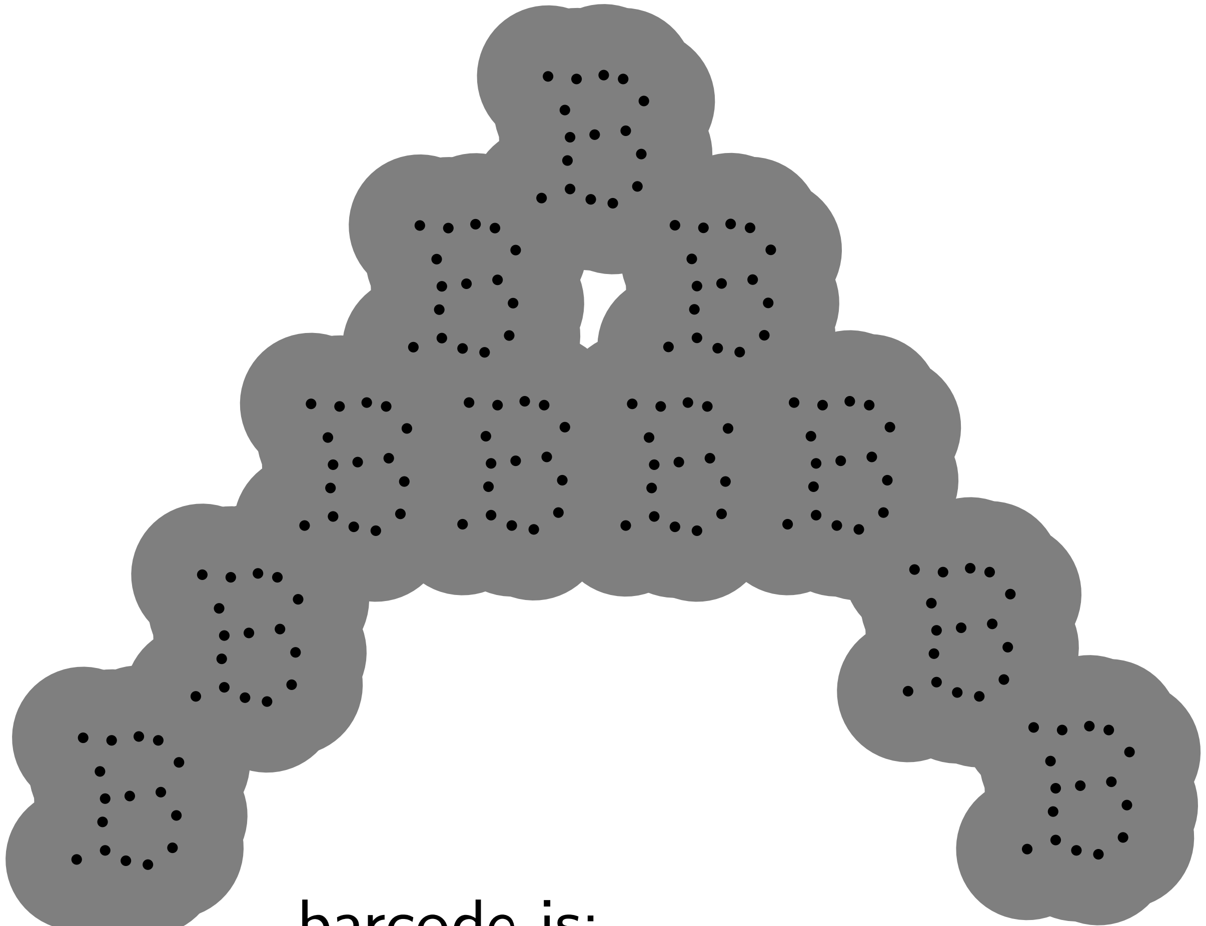

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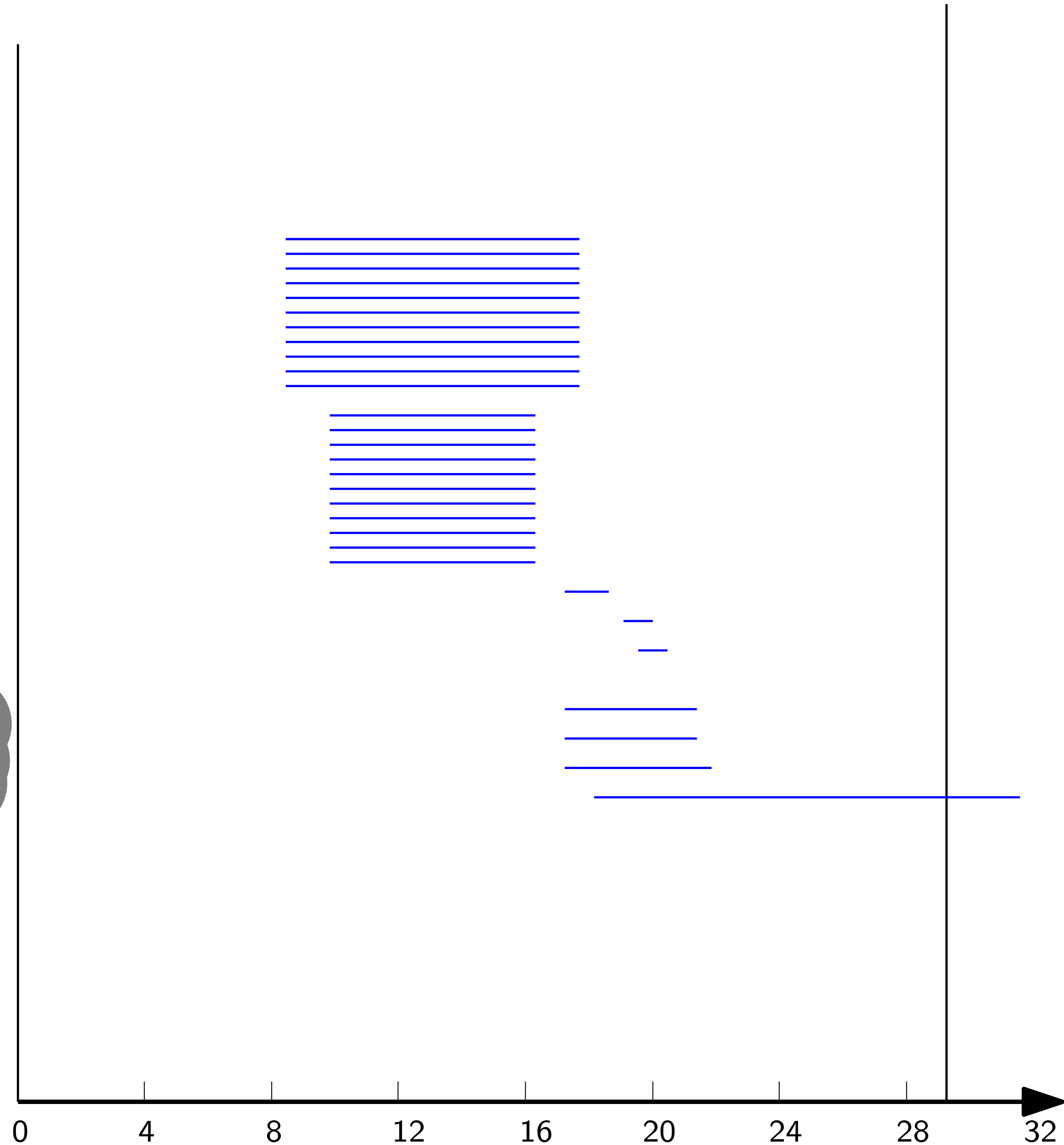

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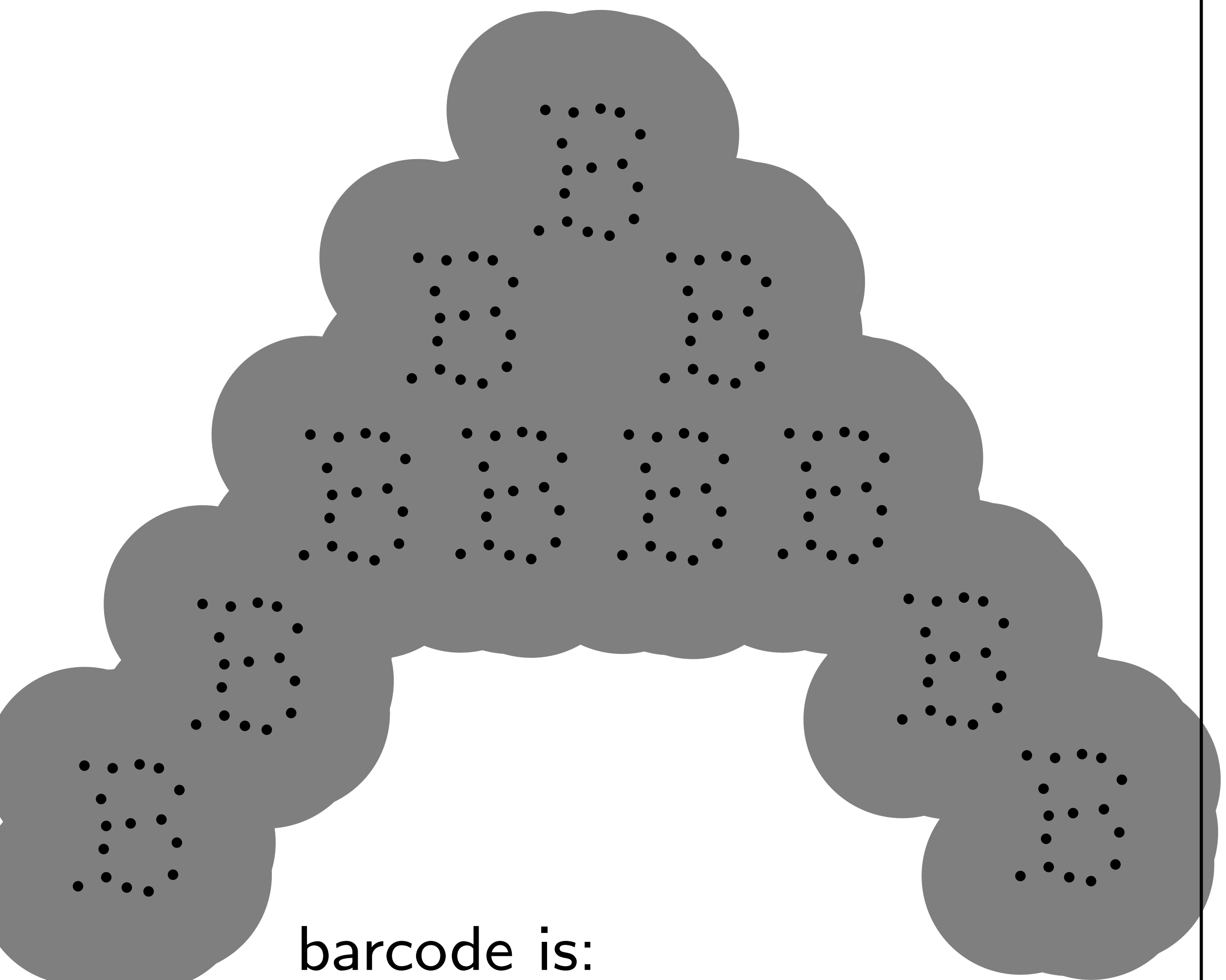

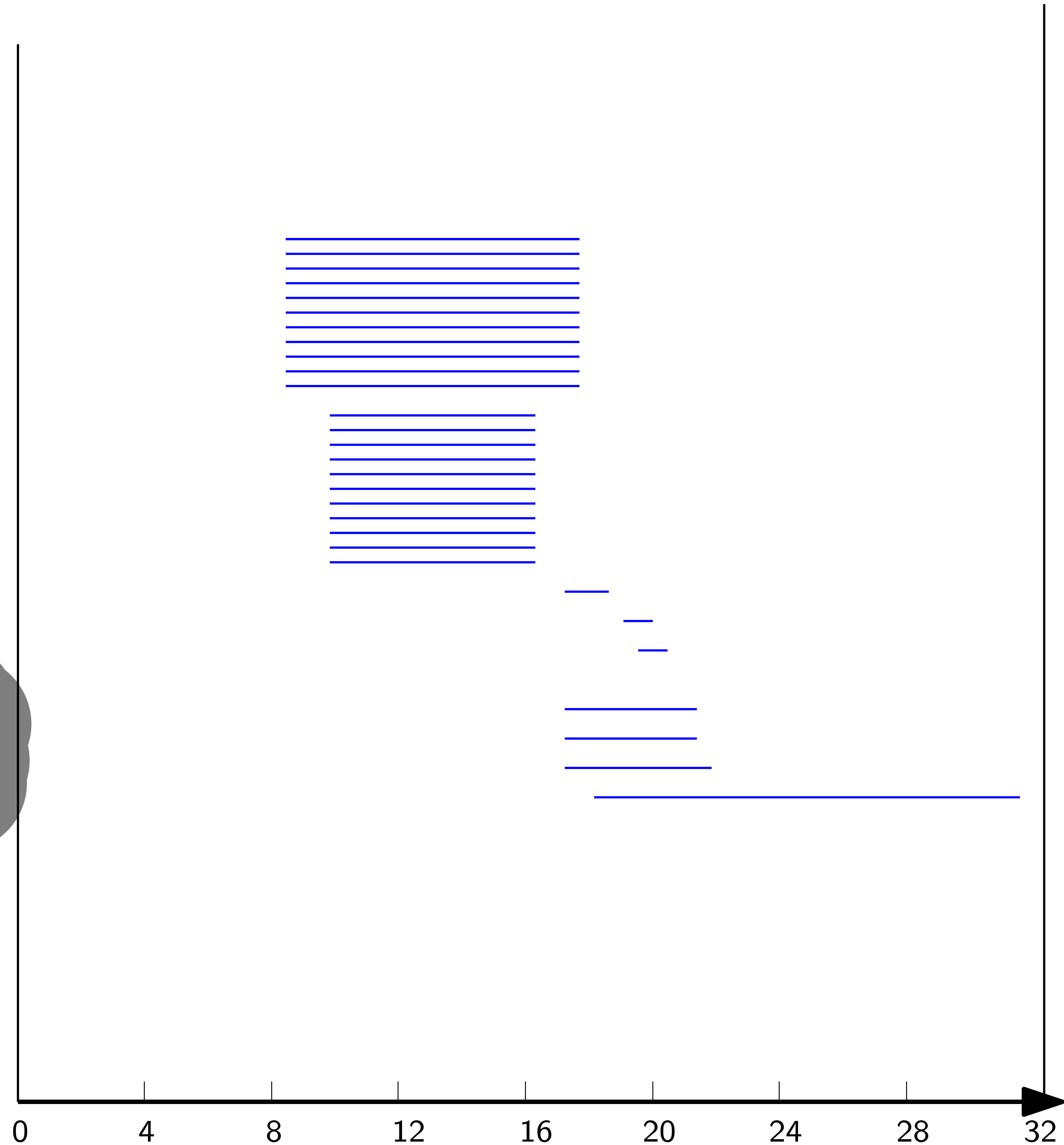

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Mathematical viewpoint

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

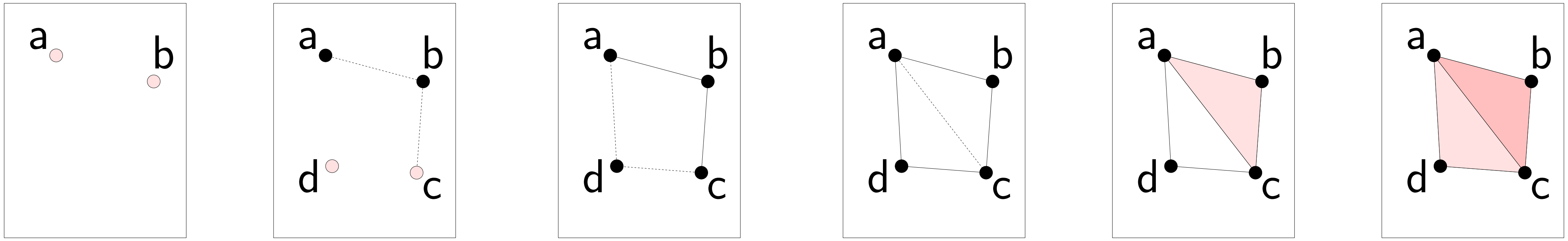
Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Mathematical viewpoint

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *simplicial filtration* (nested family of simplicial complexes)



F_1 F_2 F_3 F_4 F_5 F_6

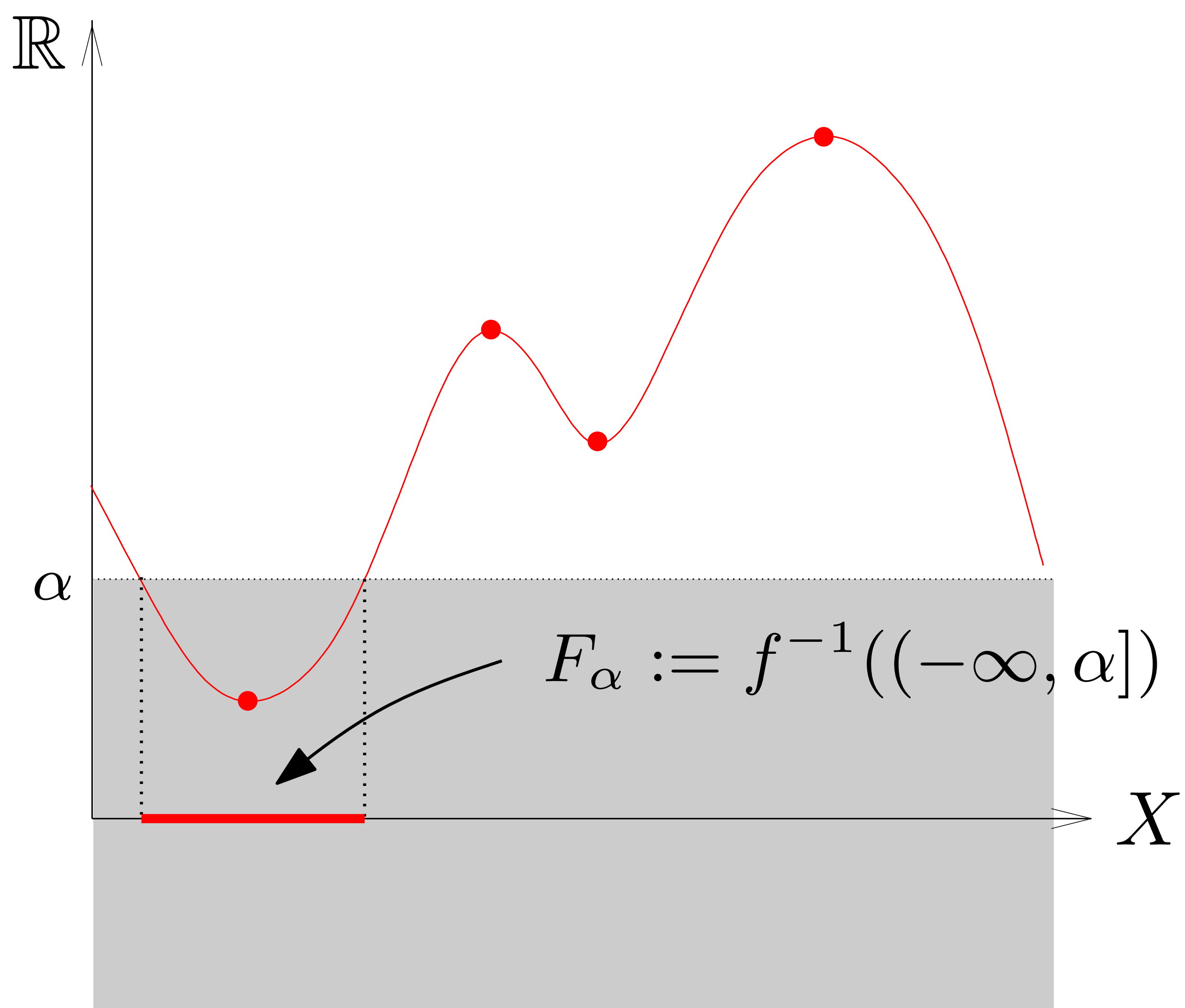
Mathematical viewpoint

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *simplicial filtration* (nested family of simplicial complexes)

Example 3: *sublevel-sets filtration* (family of sublevel sets of a function $f : X \to \mathbb{R}$)



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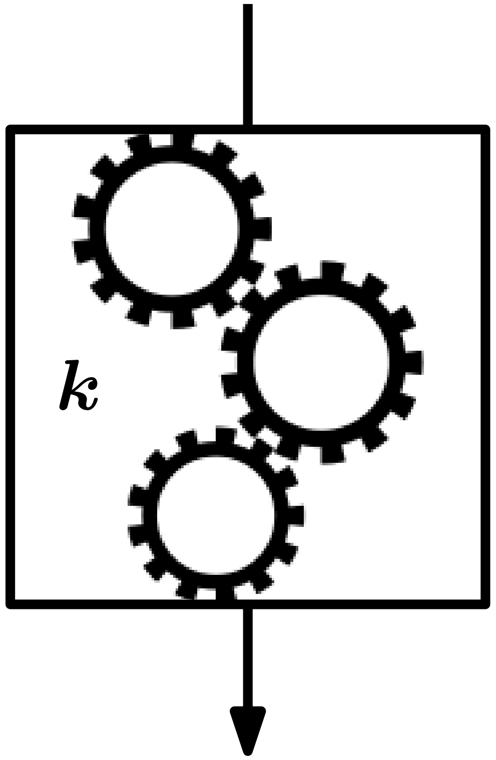
Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$



(homology functor)

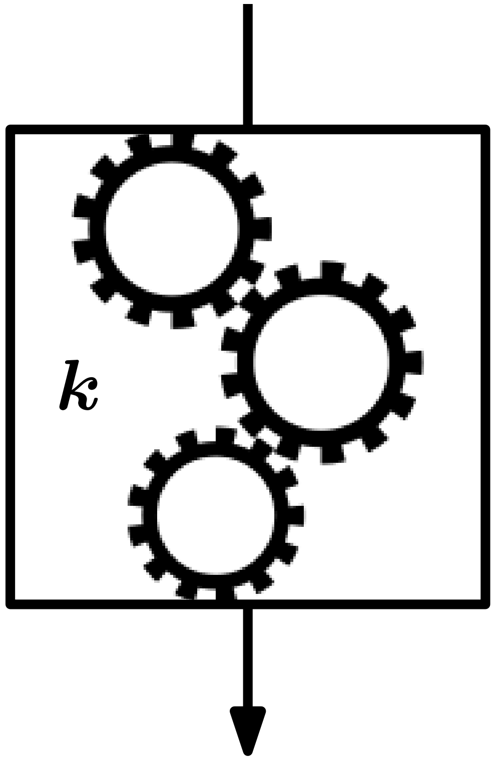
topological level

algebraic level

Persistence module: $H_*(F_1) \to H_*(F_2) \to H_*(F_3) \to H_*(F_4) \to H_*(F_5) \cdots$

Mathematical viewpoint

Zigzag: $F_1 \subseteq F_2 \supseteq F_3 \supseteq F_4 \subseteq F_5 \cdots$



(homology functor)

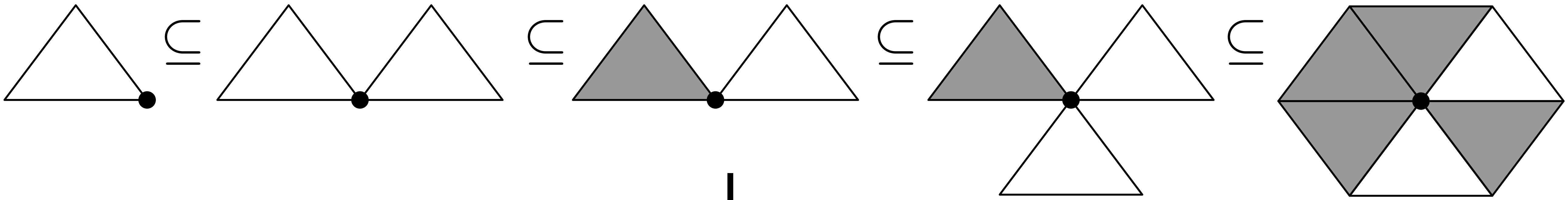
topological level

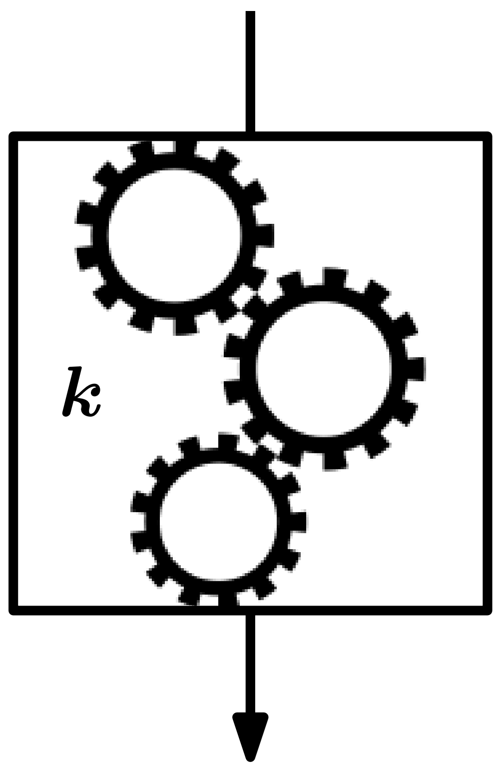
algebraic level

Zigzag module: $H_*(F_1) \rightarrow H_*(F_2) \leftarrow H_*(F_3) \leftarrow H_*(F_4) \rightarrow H_*(F_5) \cdots$

Mathematical viewpoint

Example:



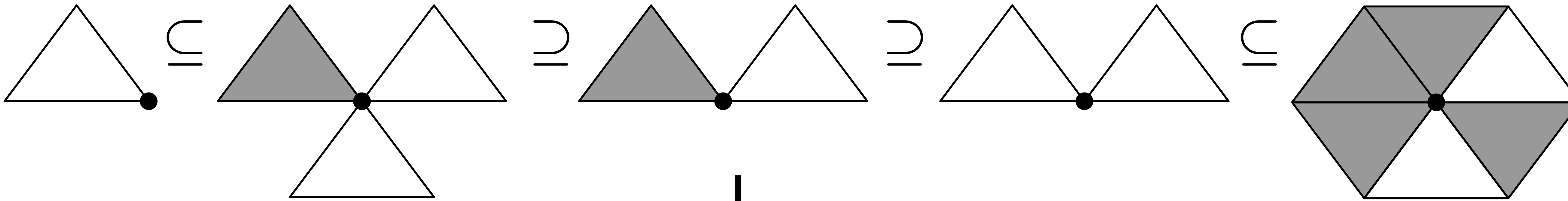


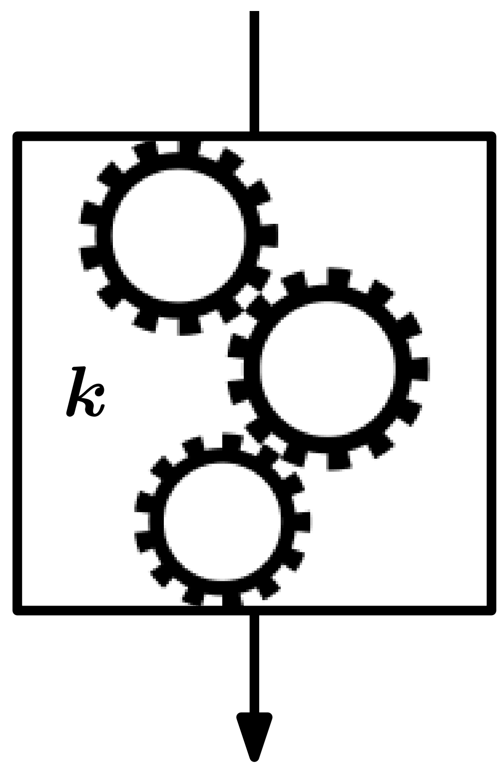
(1-homology functor)

$$\boldsymbol{k} \xrightarrow{\binom{1}{0}} \boldsymbol{k}^2 \xrightarrow{(\,0\ \ 1\,)} \boldsymbol{k} \xrightarrow{\binom{0}{1}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \boldsymbol{k}^2 \ \cdots$$

Mathematical viewpoint

Example:





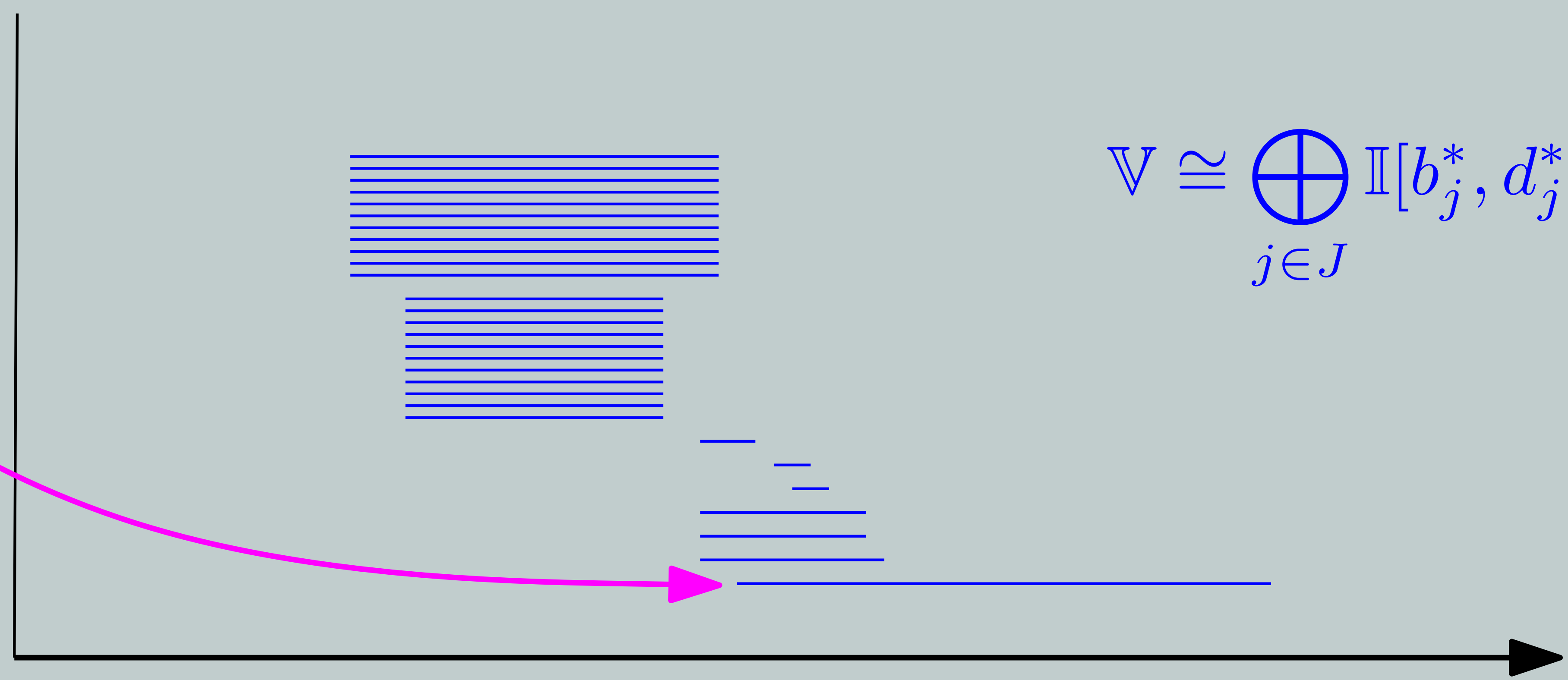
(1-homology functor)

$$\boldsymbol{k} \xrightarrow{\;0\;} \boldsymbol{k}^2 \xleftarrow{\begin{pmatrix}1\\0\end{pmatrix}} \boldsymbol{k} \xleftarrow{(0\ \ 1)} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}0&1\\0&0\end{pmatrix}} \boldsymbol{k}^2 \quad \cdots$$

Mathematical viewpoint

Theorem. Let $\mathbb{V}$ be a persistence/zigzag module over an index set $T \subseteq \mathbb{R}$. Then, $\mathbb{V}$ decomposes as a direct sum of **interval modules** $\mathbb{I}[b^*, d^*]$:

$$\underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{i<b^*} \xrightarrow{0} \underbrace{\boldsymbol{k} \xrightarrow{\mathbb{1}} \cdots \xrightarrow{\mathbb{1}} \boldsymbol{k}}_{[b^*,\, d^*]} \xrightarrow{0} \underbrace{0 \xrightarrow{0} \cdots \xrightarrow{0} 0}_{i>d^*}$$



$$\mathbb{V} \simeq \bigoplus_{j \in J} \mathbb{I}[b_j^*, d_j^*]$$

(the barcode is a complete descriptor of the algebraic structure of $\mathbb{V}$)

Mathematical viewpoint

Theorem. Let $\mathbb{V}$ be a persistence/zigzag module over an index set $T \subseteq \mathbb{R}$. Then, $\mathbb{V}$ decomposes as a direct sum of **interval modules** $\mathbb{I}[b^*, d^*]$:

$$\underbrace{0 \overset{0}{\longrightarrow} \cdots \overset{0}{\longrightarrow} 0}_{i<b^*} \overset{0}{\longrightarrow} \underbrace{\boldsymbol{k} \overset{\mathbb{1}}{\longrightarrow} \cdots \overset{\mathbb{1}}{\longrightarrow} \boldsymbol{k}}_{[b^*,\, d^*]} \overset{0}{\longrightarrow} \underbrace{0 \overset{0}{\longrightarrow} \cdots \overset{0}{\longrightarrow} 0}_{i>d^*}$$

in the following cases:

- T is finite [Gabriel 1972] [Auslander 1974],
- all arrows are forward and $\mathbb{V}$ is *pointwise finite-dimensional* (i.e. every space V_t has finite dimension) [Webb 1985] [Crawley-Boevey 2012].

Moreover, when it exists, the decomposition is **unique** up to isomorphism and permutation of the terms [Azumaya 1950].

(Note: this is independent of the choice of field $\boldsymbol{k}$.)

Persistence Modules vs. Quiver Representations

$\boldsymbol{k}$: field of coefficients

persistence/zigzag module: $$\boldsymbol{k} \xrightarrow{0} \boldsymbol{k}^2 \xleftarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \boldsymbol{k} \xleftarrow{\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)} \boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}0 & 1\\0 & 0\end{smallmatrix}\right)} \boldsymbol{k}^2$$

module type: $$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xleftarrow{b} \underset{3}{\bullet} \xleftarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet}$$

Persistence Modules vs. Quiver Representations

$\boldsymbol{k}$: field of coefficients

quiver representation: $$\boldsymbol{k} \xrightarrow{0} \boldsymbol{k}^2 \xleftarrow{\begin{pmatrix}1\\0\end{pmatrix}} \boldsymbol{k} \xleftarrow{(0\ 1)} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}0&1\\0&0\end{pmatrix}} \boldsymbol{k}^2$$

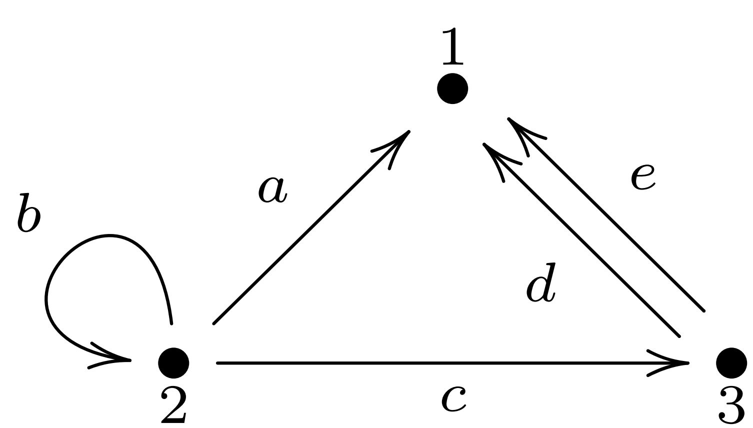
(path) quiver: $$\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xleftarrow{b} \underset{3}{\bullet} \xleftarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet}$$

Outline

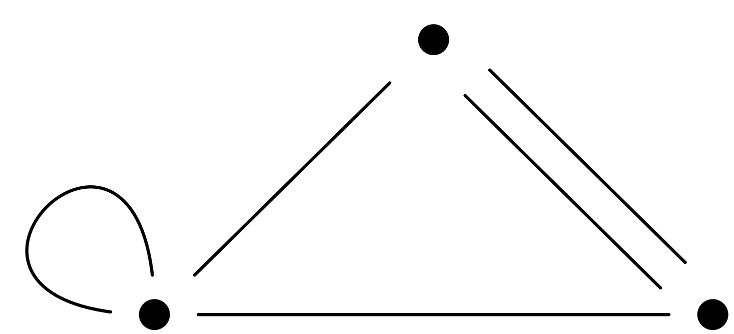
- quivers and representations, classification, Gabriel's theorem
- reflection functors, proof of Gabriel's theorem (A_n case)
- application 1: computing persistence for zigzags
- application 2: zigzags for topological inference

Quivers and Representations

Definition: A *quiver* $\mathtt{Q}$ consists of two sets Q_0, Q_1 and two maps $s, t : Q_1 \to Q_0$. The elements in Q_0 are called the *vertices* of $\mathtt{Q}$, while those of Q_1 are called the *arrows*. The *source map* s assigns a source s_a to every arrow $a \in Q_1$, while the *target map* t assigns a target t_a.



$\mathtt{Q}$



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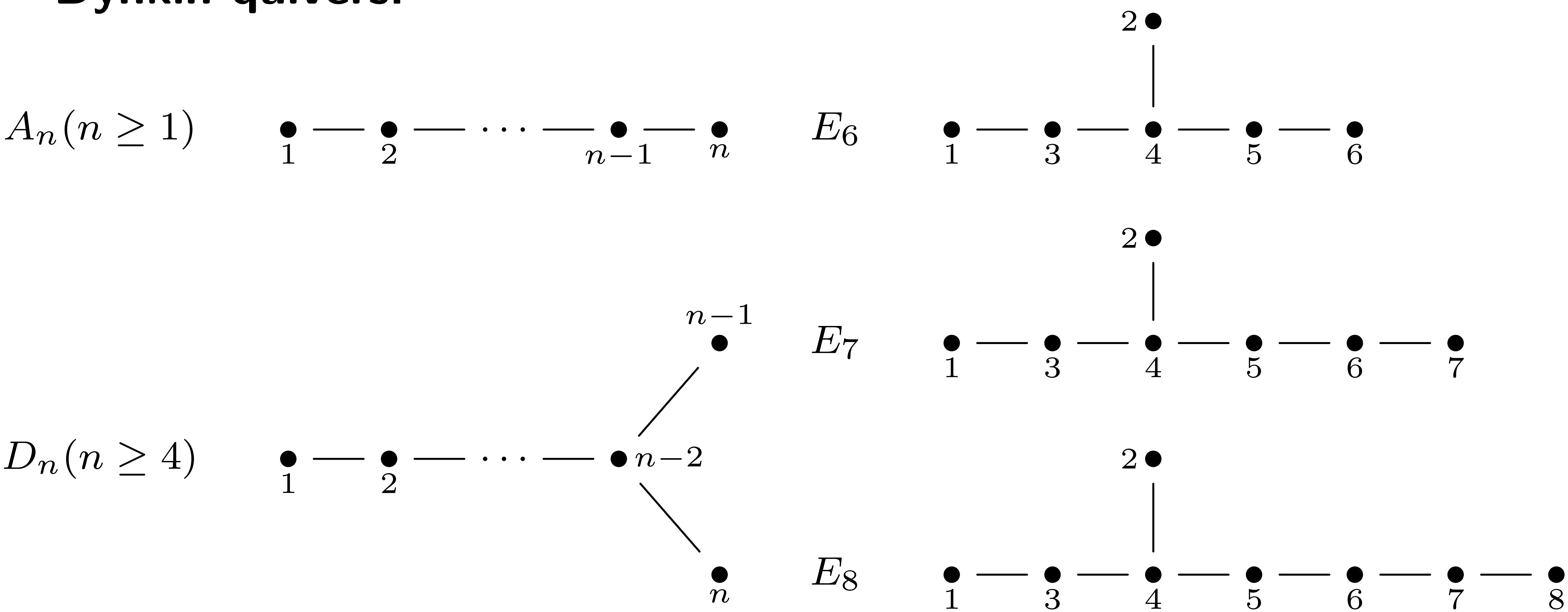

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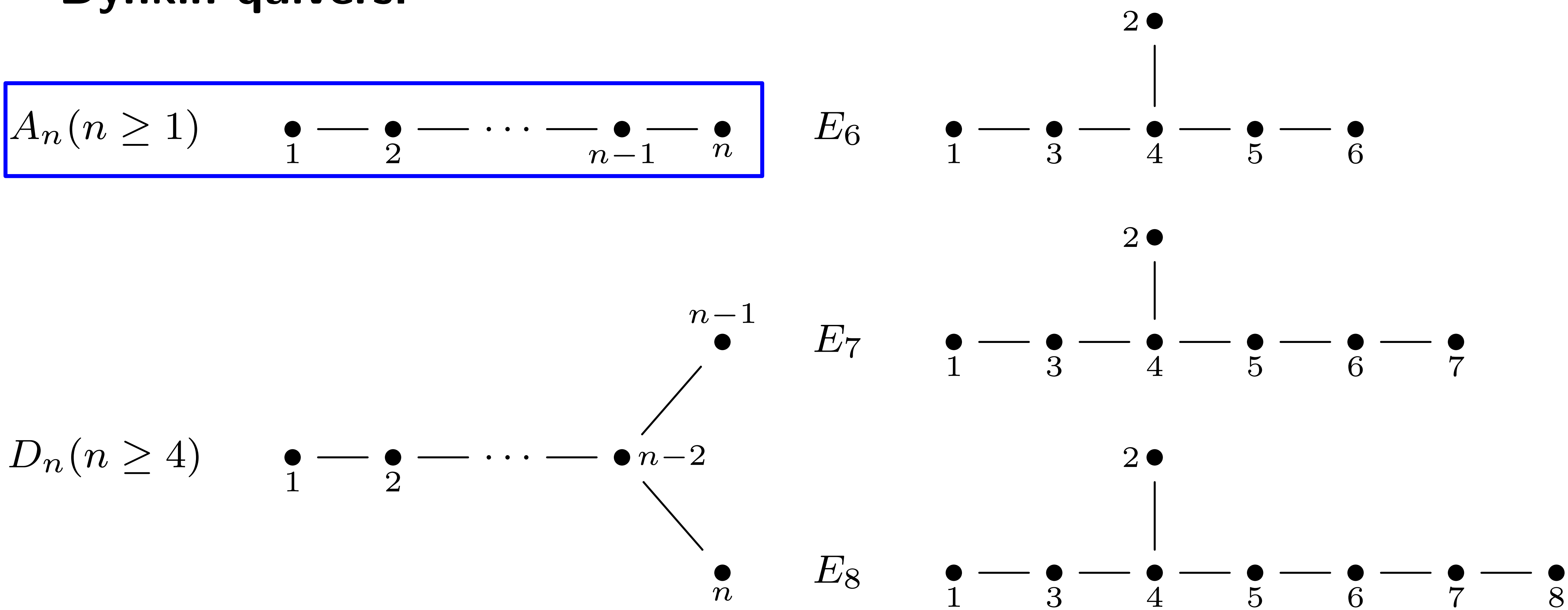

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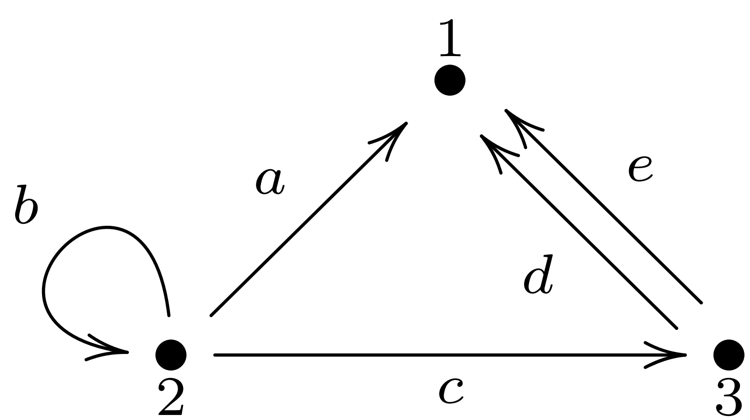
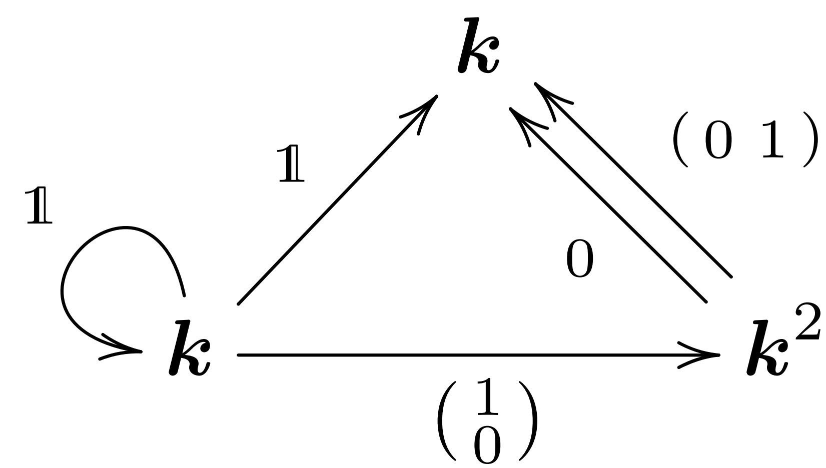


Q



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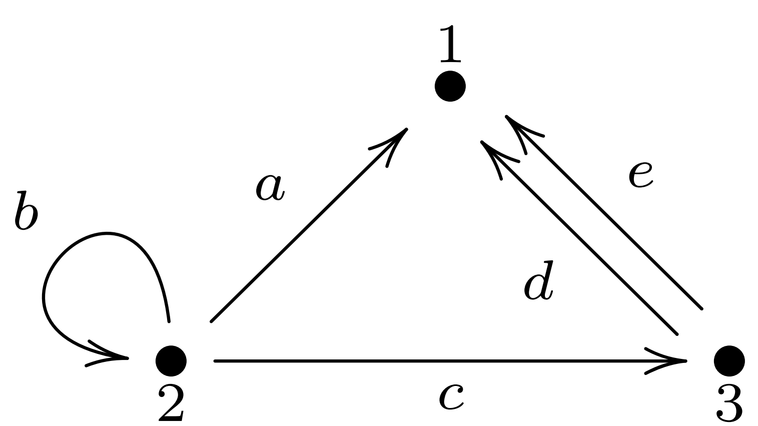

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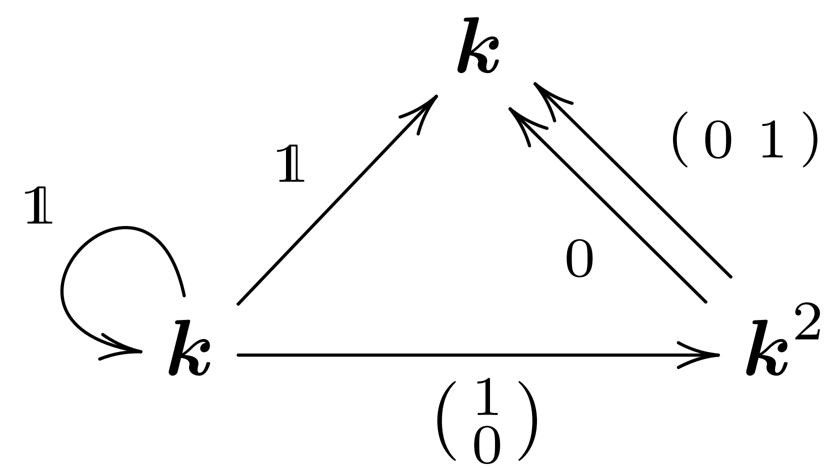

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Note: diagram commutativity is not required

Quivers and Representations

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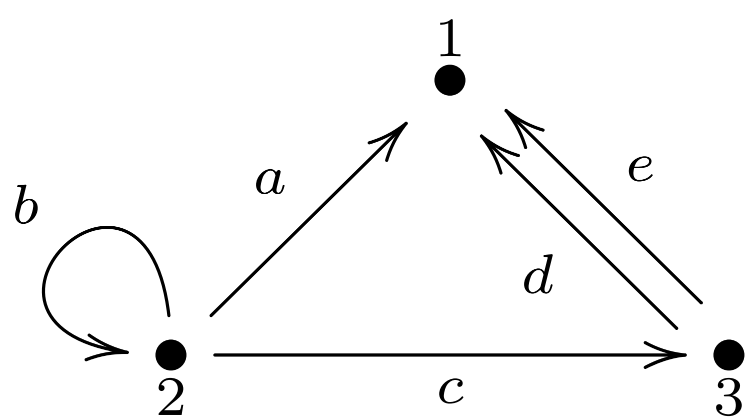

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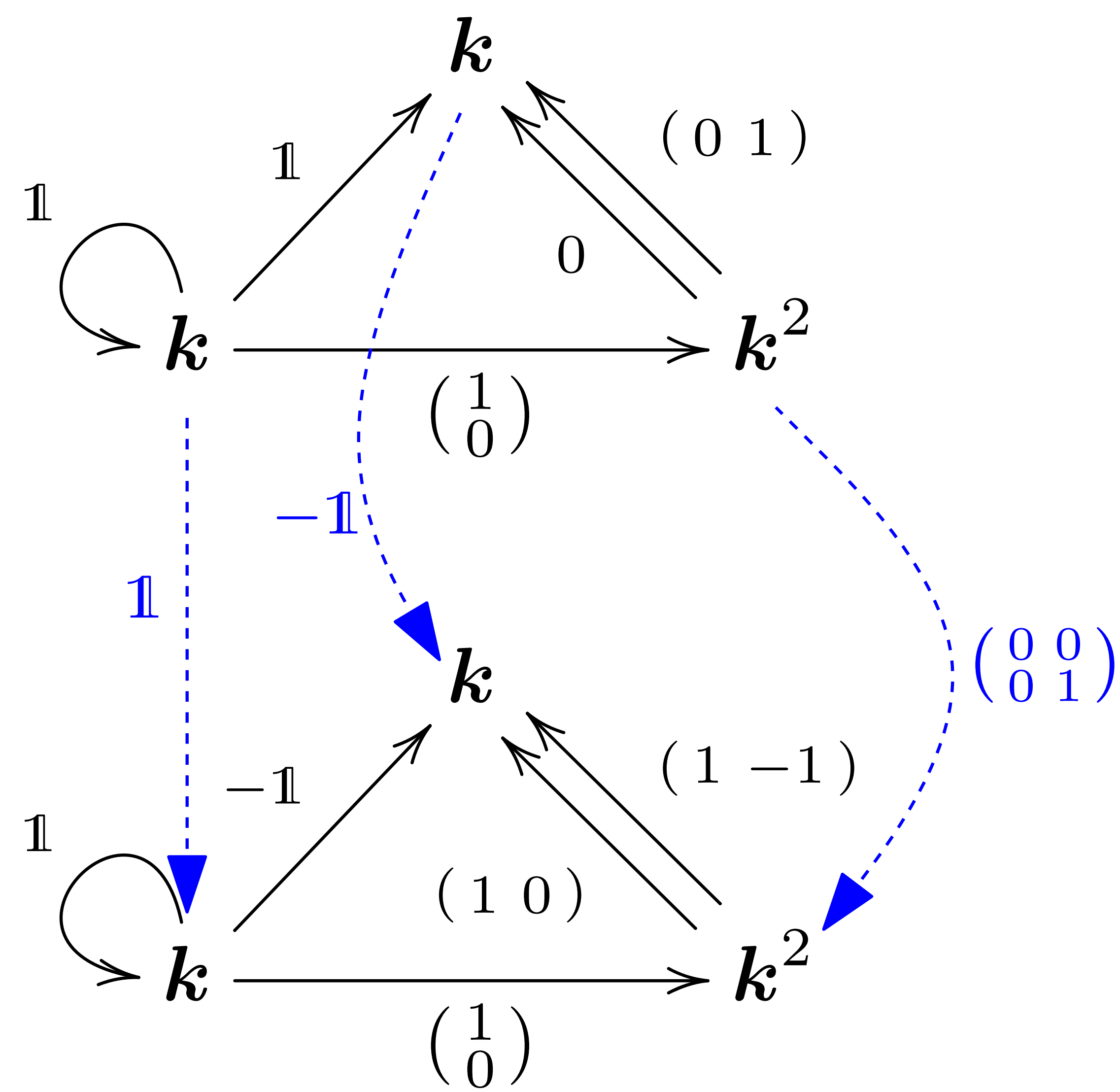

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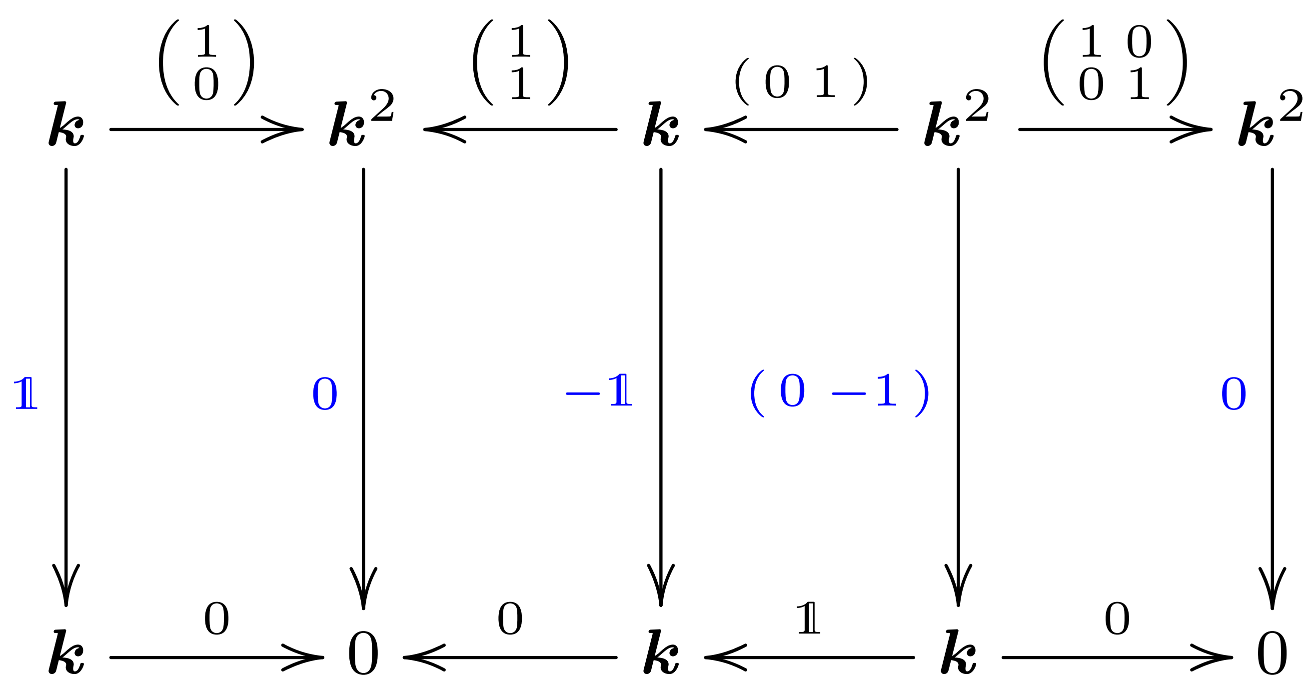

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{\scriptstyle \mathbb{1}}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle -\mathbb{1}}\big\downarrow & & {\scriptstyle (\,0\ -1\,)}\big\downarrow & & {\scriptstyle 0}\big\downarrow \\
\boldsymbol{k} & \xrightarrow{0} & 0 & \xleftarrow{0} & \boldsymbol{k} & \xleftarrow{\mathbb{1}} & \boldsymbol{k} & \xrightarrow{0} & 0
\end{array}
$$

every quadrangle commutes

Note: ϕ isomorphism iff every ϕ_i isomorphism

The Category of Representations

The representations of a quiver $\mathtt{Q} = (Q_0, Q_1)$, together with their morphisms, form a category called $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$. This category is **abelian**:

- it contains a zero object, namely the *trivial representation*

$$0 \longrightarrow 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow 0$$

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$$\bigoplus$$

$$\boldsymbol{k} \xrightarrow{0} 0 \xleftarrow{0} \boldsymbol{k} \xleftarrow{\mathbb{1}} \boldsymbol{k} \xrightarrow{0} 0$$

$$=$$

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WARNING: no semisimplicity (subrepresentations may not be summands)

$$\mathbb{V} = \; \boldsymbol{k} \xrightarrow{\;\mathbb{1}\;} \boldsymbol{k}$$

$$\mathbb{W} = \; 0 \xrightarrow{\;0\;} \boldsymbol{k}$$

The Classification Problem

Goal: Classify the representations of a given quiver $\mathtt{Q} = (Q_0, Q_1)$ up to isomorphism.

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$\to$ simplifying assumptions:

- $\mathtt{Q}$ is finite and connected
- study the subcategory $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ of *finite-dimensional* representations

$$\underline{\dim}\, \mathbb{V} = (\dim V_1, \cdots, \dim V_n)^\top,$$
$$\dim \mathbb{V} = \|\underline{\dim}\, \mathbb{V}\|_1 = \sum_{i=1}^{n} \dim V_i.$$

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Theorem: [Krull-Remak-Schmidt-Azumaya]
$\forall \mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$, $\exists\ \mathbb{V}_1, \cdots, \mathbb{V}_r$ *indecomposable* s.t. $\mathbb{V} \cong \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r$.
The decomposition is unique up to isomorphism and reordering.

note: $\mathbb{V}$ indecomposable iff there are no $\mathbb{U}, \mathbb{W} \neq 0$ such that $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$

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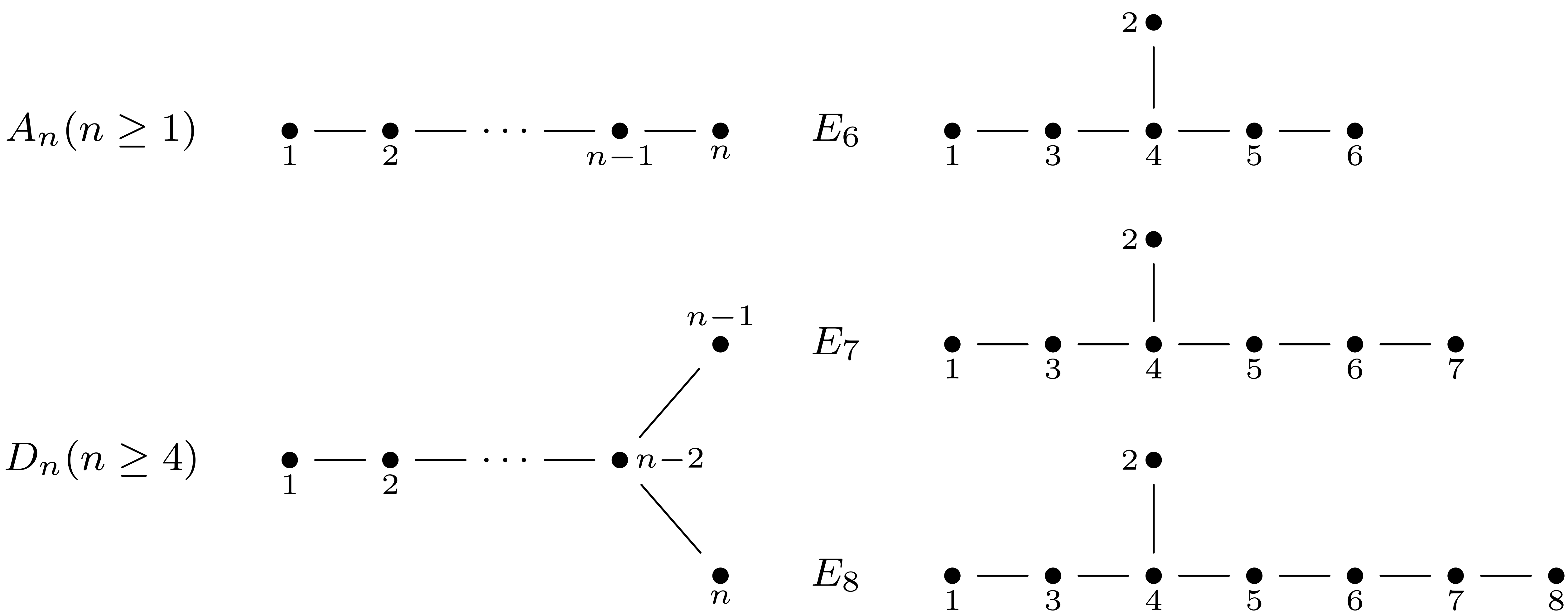
$\rightarrow$ problem becomes to identify the indecomposable representations of $\mathtt{Q}$

($\neq$ from identifying representations with no subrepresentations)

(no semisimplicity)

Gabriel's Theorem

Theorem: [Gabriel I]
Assuming Q is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ iff Q is Dynkin.



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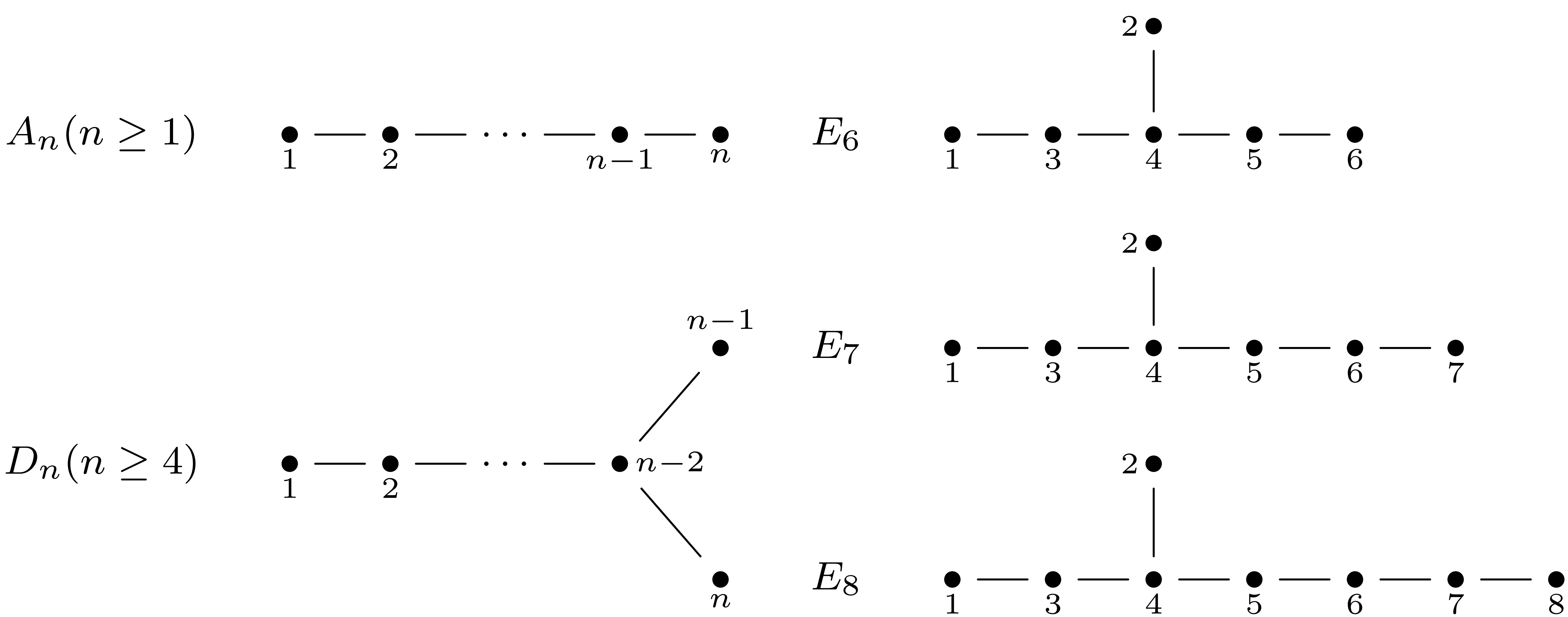

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(isom. classes of indecomposables are fully characterized by their dim. vectors)

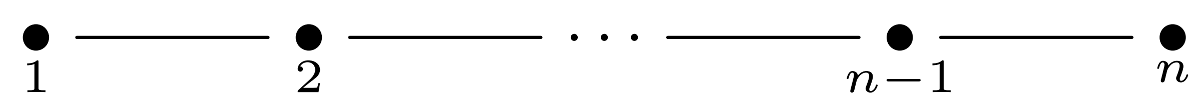
Gabriel's Theorem

Tits form: given $\mathtt{Q} = (Q_0, Q_1)$ with $|Q_0| = n$ and $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$,

$$q_{\mathtt{Q}}(x) = \textstyle\sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{s_a} x_{t_a}.$$

Proposition: $q_{\mathtt{Q}}$ is *positive definite* ($q_{\mathtt{Q}}(x) > 0 \ \forall x \neq 0$) iff $\mathtt{Q}$ is Dynkin.

example: $\mathtt{Q}$ of type A_n:



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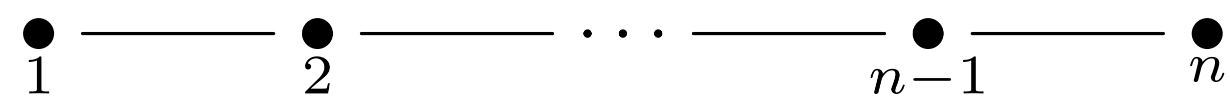

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the corresponding indecomp. representations are isomorphic to $\mathbb{I}_{\mathtt{Q}}[b, d]$:

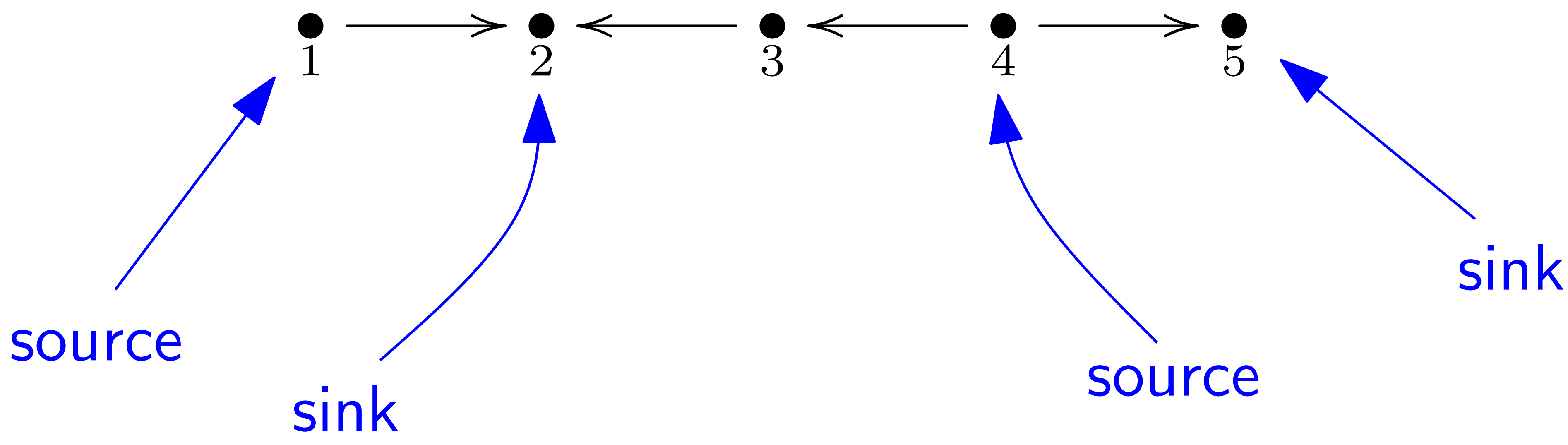
$$\underbrace{0 \xrightarrow{\;0\;} \cdots \xrightarrow{\;0\;} 0}_{[1,\, b-1]} \xrightarrow{\;0\;} \underbrace{\boldsymbol{k} \xrightarrow{\;\mathbb{1}\;} \cdots \xrightarrow{\;\mathbb{1}\;} \boldsymbol{k}}_{[b,\, d]} \xrightarrow{\;0\;} \underbrace{0 \xrightarrow{\;0\;} \cdots \xrightarrow{\;0\;} 0}_{[d+1,\, n]}$$

Reflection Functors

Advantage: explains the fact that only the dimension vectors play a role in the identification of indecomposable representations. In particular, arrow orientations are irrelevant.

idea: modify quivers by reversing arrows, and study the effect on their representations (peeling off summands).

Reflection Functors



Definition: sink = only incoming arrows; source = only outgoing arrows

Reflection Functors

$$1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5$$

$s_1\mathtt{Q}$: $$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5$$

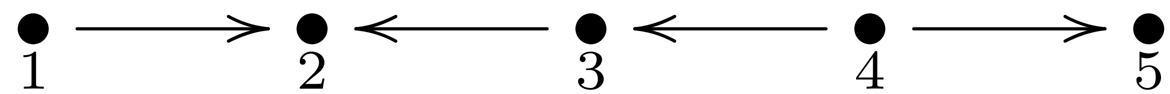
$s_2\mathtt{Q}$: $$1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5$$

$s_4\mathtt{Q}$: $$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$$

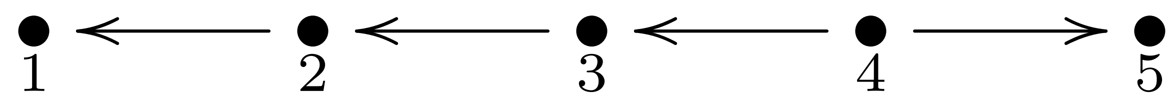
$s_5\mathtt{Q}$: $$1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$$

Definition: reflection s_i = reverse all arrows incident to sink/source i

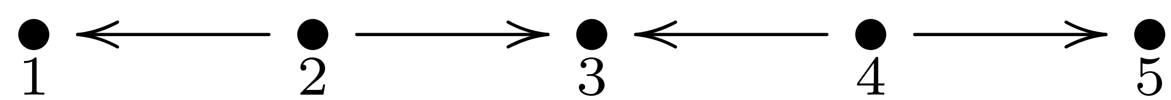
Reflection Functors



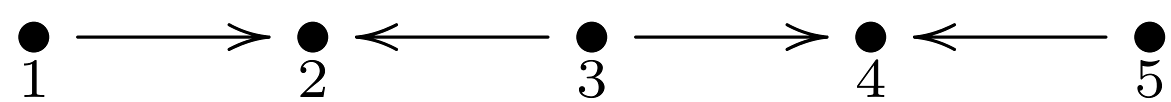
$s_1\mathsf{Q}$:



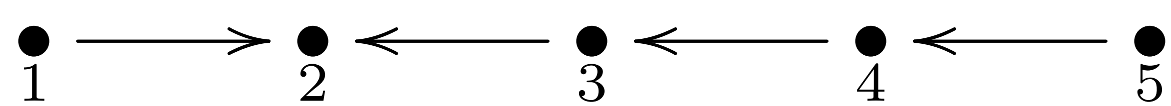
$s_2\mathsf{Q}$:



$s_4\mathsf{Q}$:



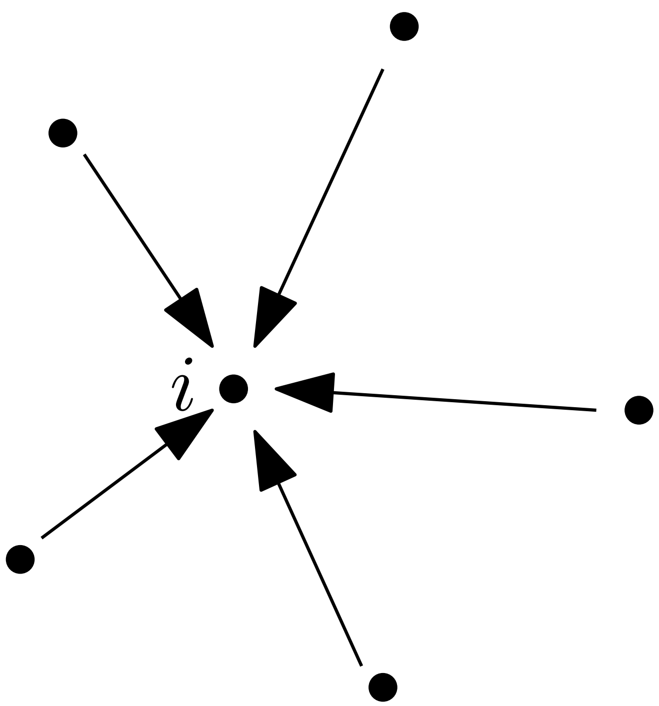
$s_5\mathsf{Q}$:



Definition: reflection functor $\mathcal{R}_i^{\pm}$ = functor $\mathrm{Rep}_{\boldsymbol{k}}(\mathsf{Q}) \to \mathrm{Rep}_{\boldsymbol{k}}(s_i\mathsf{Q})$

Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a sink



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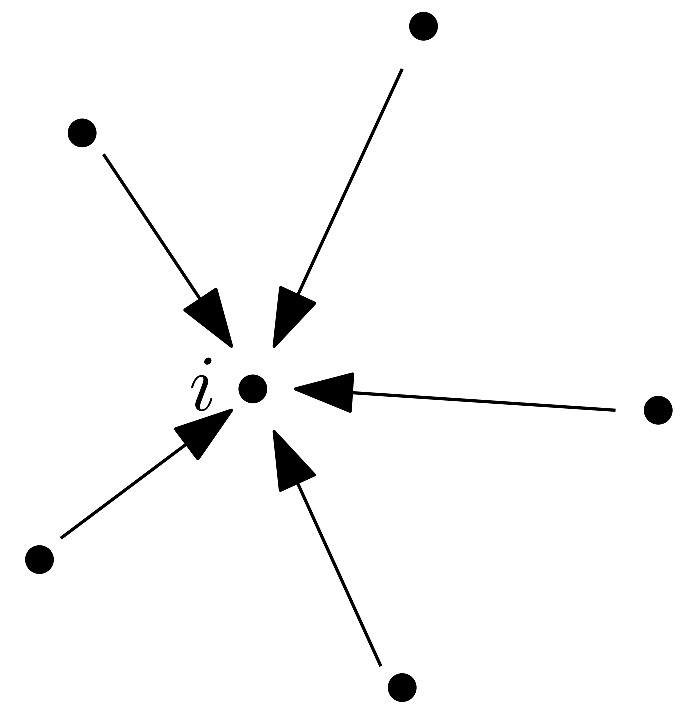
Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a sink

Definition: $\mathcal{R}_i^+\mathbb{V} = (W_i, w_a)$ is defined by :

- $W_j = V_j$ for all $j \neq i$
- $w_a = v_a$ for all $a \notin Q_1^i$ (arrows incident to i)



Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a sink

Definition: $\mathcal{R}_i^+\mathbb{V} = (W_i, w_a)$ is defined by :

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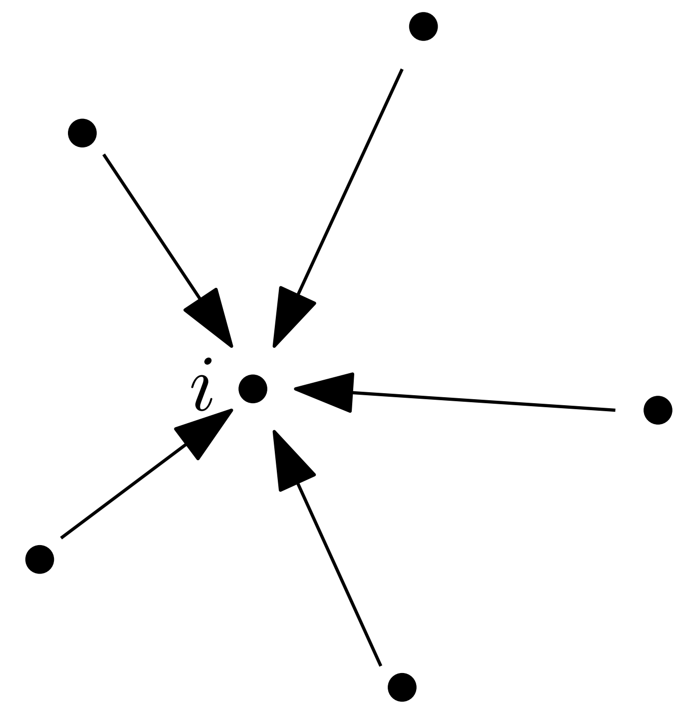

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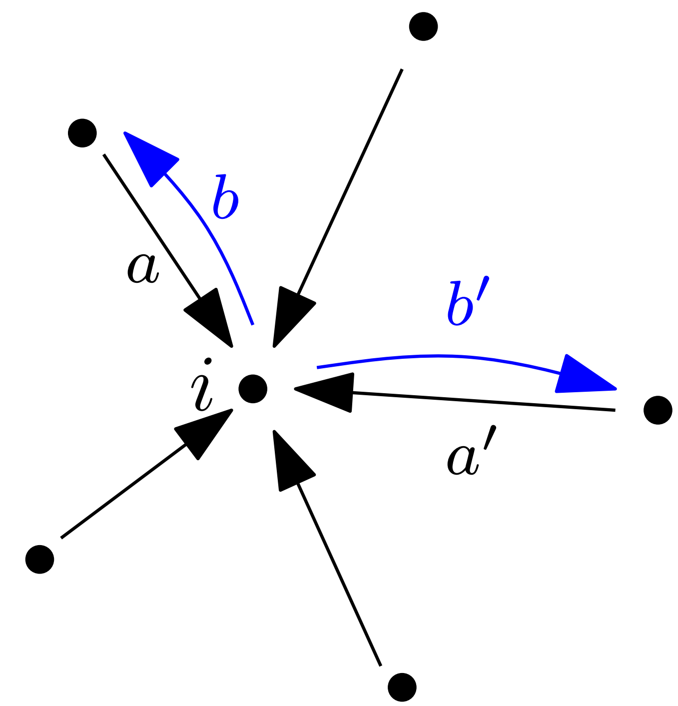

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- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

$$W_{s_b} = W_i = \ker \xi_i \hookrightarrow \bigoplus_{c \in Q_1^i} V_{s_c} \longrightarrow V_{s_a} = W_{s_a} = W_{t_b}$$

(canonical inclusion) (projection to component V_{s_a})

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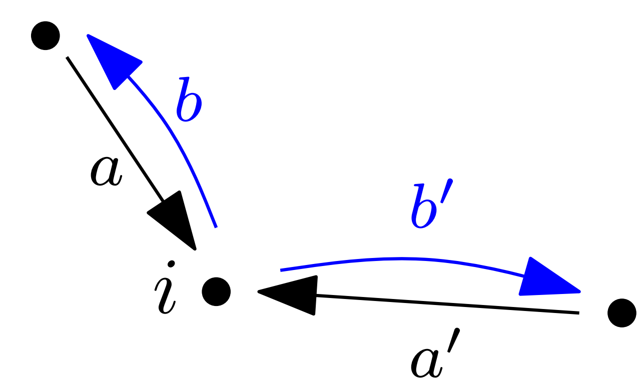

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intuition: W_i carries the information passing through V_i in $\mathbb{V}$

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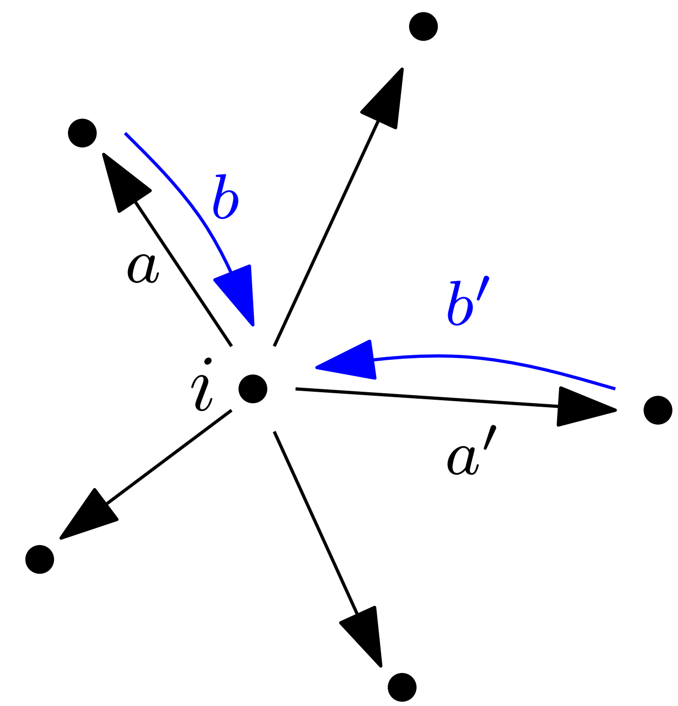

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  $x_i \longmapsto (v_a(x_i))_{a \in Q_1^i}$
- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

$$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \longrightarrow \mathrm{coker}\, \zeta_i = W_i = W_{t_b}$$

(canonical inclusion) (quotient modulo $\mathrm{im}\, \zeta_i$)

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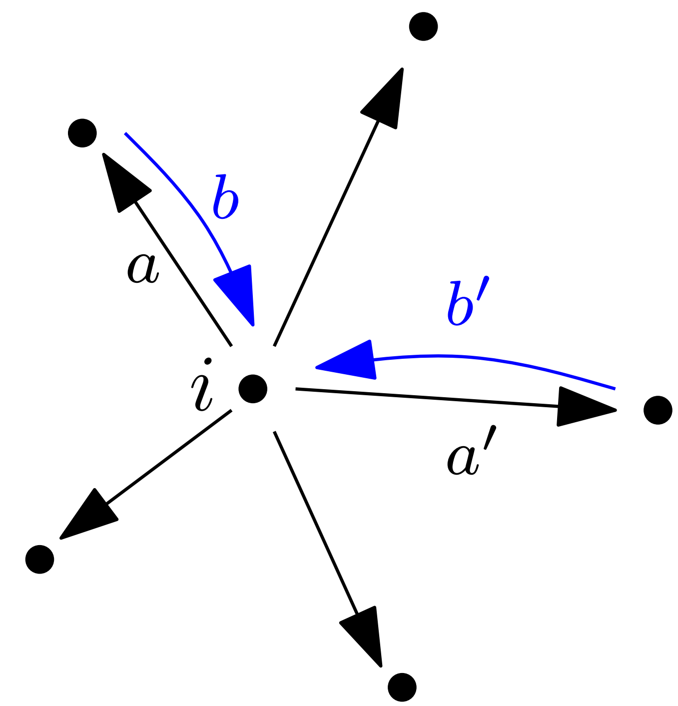

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intuition: this is the operation dual to the previous one (take $V_i = \ker \xi_i$)

- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

$$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \longrightarrow \mathrm{coker}\, \zeta_i = W_i = W_{t_b}$$

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Reflection Functors

$$\mathbb{V}: \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5$$

$$\mathcal{R}_5^+\mathbb{V}: \quad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \hookleftarrow \ker v_d$$

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$\cong \operatorname{im} v_d$

$\mathbb{V} \cong \mathcal{R}_5^-\mathcal{R}_5^+\mathbb{V} \oplus \mathbb{S}_5^r$, where $r = \dim \operatorname{coker} v_d$

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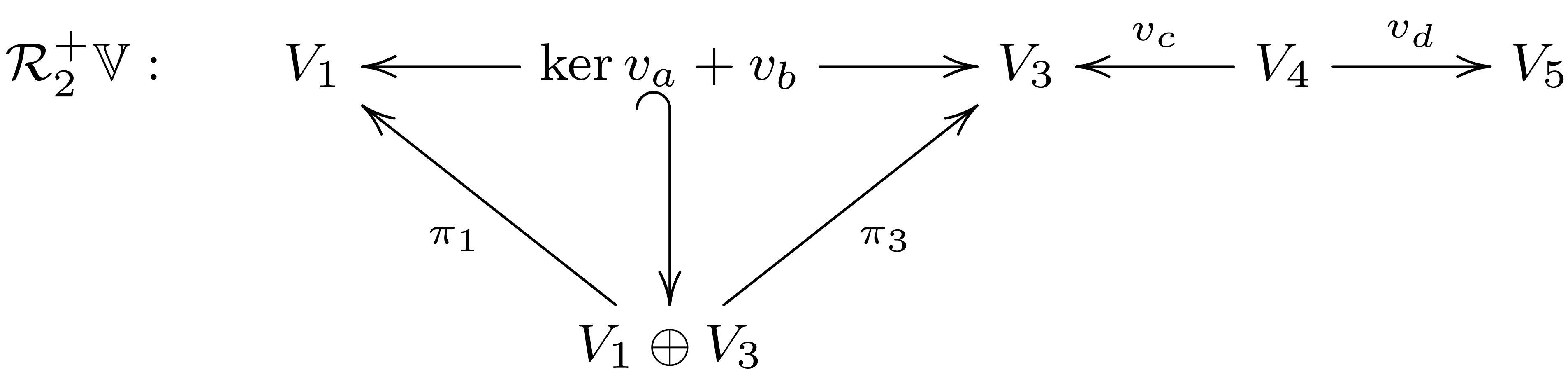

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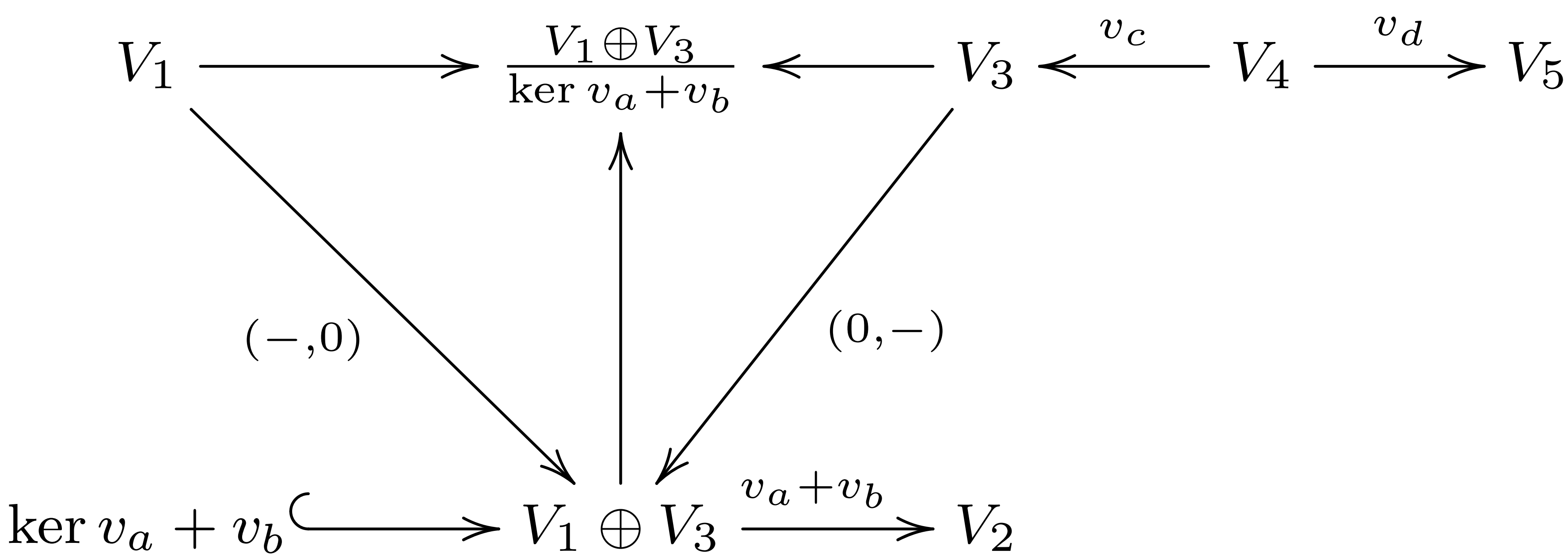
$\mathcal{R}_2^- \mathcal{R}_2^+ \mathbb{V}:$



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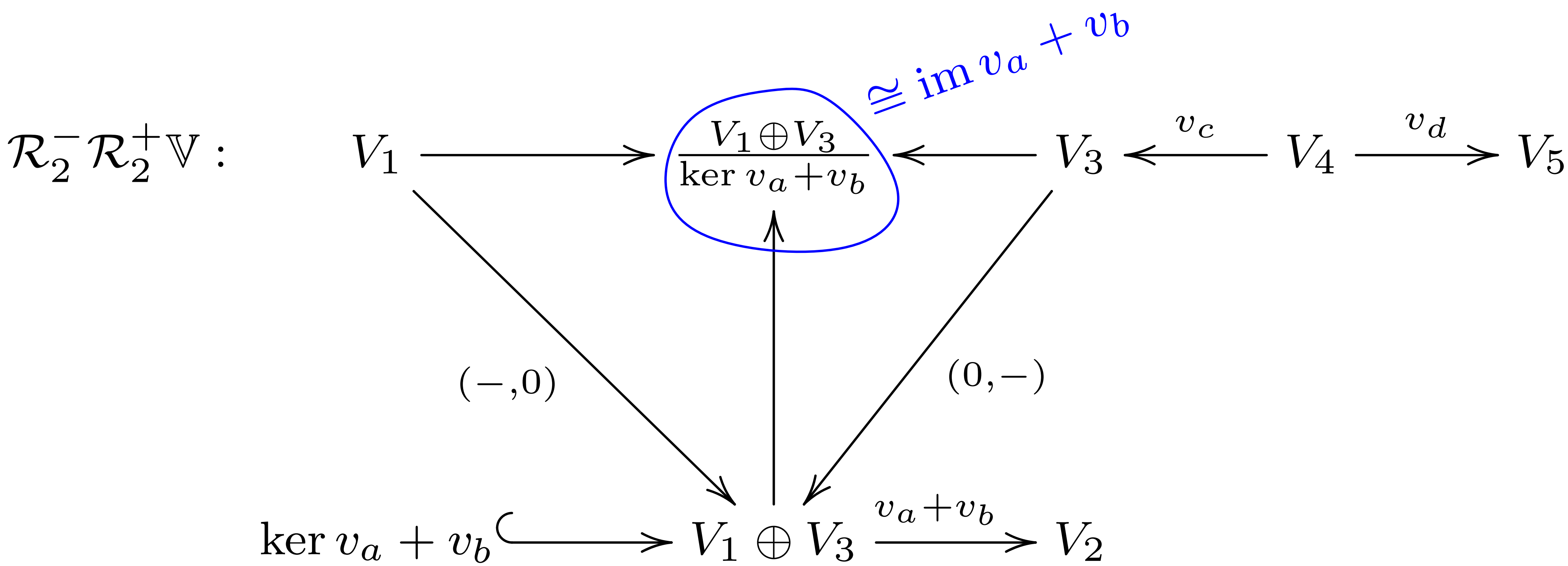

$\mathbb{V} \cong \mathcal{R}_2^-\mathcal{R}_2^+\mathbb{V} \oplus \mathbb{S}_2^r$, where $r = \dim \operatorname{coker} v_a + v_b$

Reflection Functors

Theorem: [Bernstein, Gelfand, Ponomarev]
Let `Q` be a finite connected quiver and let $\mathbb{V}$ be a representation of `Q`. If $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$, then for any source or sink $i \in Q_0$, $\mathcal{R}_i^{\pm}\mathbb{V} \cong \mathcal{R}_i^{\pm}\mathbb{U} \oplus \mathcal{R}_i^{\pm}\mathbb{W}$.

If now $\mathbb{V}$ is indecomposable:
1. If $i \in Q_0$ is a sink, then two cases are possible:

- $\mathbb{V} \cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{+}\mathbb{V} = 0$.
- $\mathbb{V} \not\cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{+}\mathbb{V}$ is nonzero and indecomposable, $\mathcal{R}_i^{-}\mathcal{R}_i^{+}\mathbb{V} \cong \mathbb{V}$, and the dimension vectors x of $\mathbb{V}$ and y of $\mathcal{R}_i^{+}\mathbb{V}$ are related to each other by the following formula:

$$y_j = \begin{cases} x_j & \text{if } j \neq i; \\ -x_i + \sum_{\substack{a \in Q_1 \\ t_a = i}} x_{s_a} & \text{if } j = i. \end{cases}$$

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If now $\mathbb{V}$ is indecomposable:
2. If $i \in Q_0$ is a source, then two cases are possible:

- $\mathbb{V} \cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{-}\mathbb{V} = 0$.
- $\mathbb{V} \ncong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{-}\mathbb{V}$ is nonzero and indecomposable, $\mathcal{R}_i^{+}\mathcal{R}_i^{-}\mathbb{V} \cong \mathbb{V}$, and the dimension vectors x of $\mathbb{V}$ and y of $\mathcal{R}_i^{-}\mathbb{V}$ are related to each other by the following formula:

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[...]

Corollary: Reflection Functors preserve the Tits form values except at simple representations:

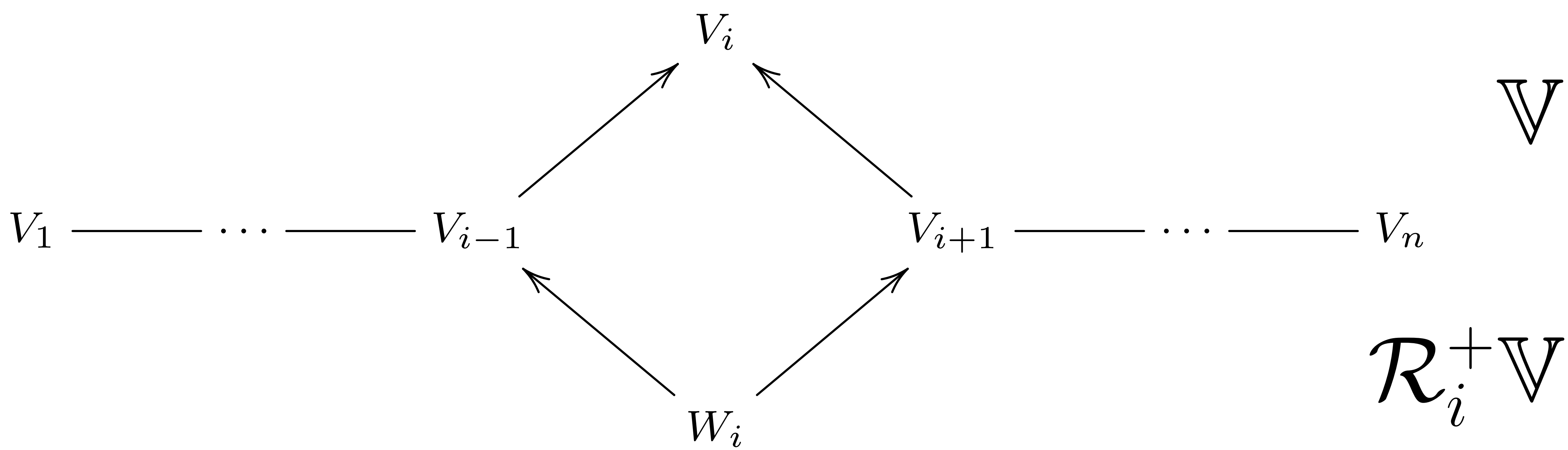
For i source/sink and $\mathbb{V}$ indecomposable,

- either $\mathbb{V} \cong \mathbb{S}_i$, in which case $q_{s_i\mathtt{Q}}(\underline{\dim}\, \mathcal{R}_i^{\pm}\mathbb{V}) = 0$,
- or $q_{s_i\mathtt{Q}}(\underline{\dim}\, \mathcal{R}_i^{\pm}\mathbb{V}) = q_{\mathtt{Q}}(\mathbb{V})$.

For $\mathbb{V}$ arbitrary,
$\mathbb{V} \cong \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r \oplus \mathbb{S}_i^s \implies q_{s_i\mathtt{Q}}(\underline{\dim}\, \mathcal{R}_i^{\pm}\mathbb{V}) = q_{\mathtt{Q}}(\underline{\dim}\, \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r)$

Reflection Functors

Example: $\mathtt{Q}$ of type A_n, i sink, $\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_{\mathtt{Q}}[b_j, d_j] \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$:



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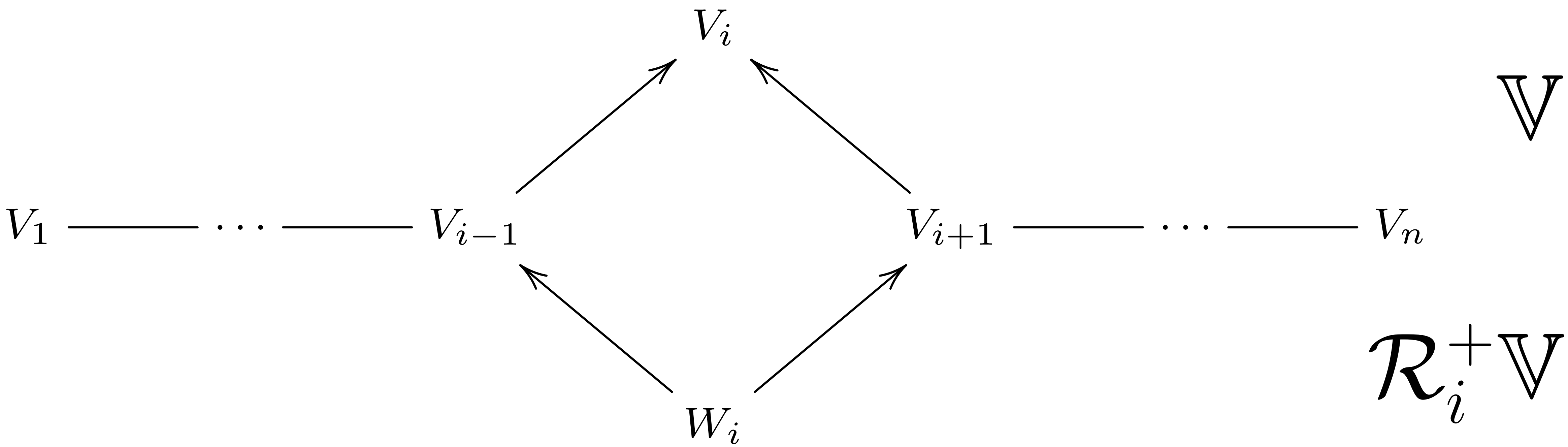

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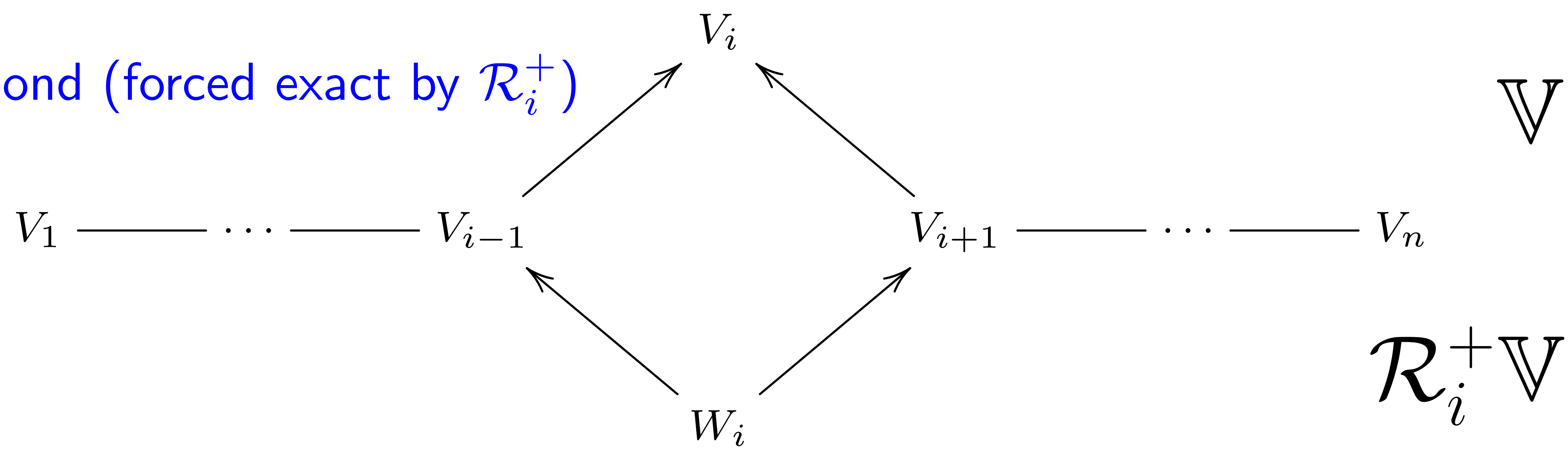
Diamond (forced exact by $\mathcal{R}_i^+$)



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Diamond Principle [Carlsson, de Silva]

Proof of Gabriel's Theorem (A_n case)

Theorem: [Gabriel I, A_n type]
Assuming $\mathtt{Q}$ is of type A_n, every isomorphism class of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ contains $\mathbb{I}_{\mathtt{Q}}[b,d]$ for some $1 \leq b \leq d \leq n$.

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What we are currently able to do:

- turn indecomposable representations of Q into indecomposable representations of reflections of Q (or zero)
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What we are currently able to do:

- turn indecomposable representations of $\mathtt{Q}$ into indecomposable representations of reflections of $\mathtt{Q}$ (or zero)
- while doing so, preserve the value of the Tits form (or zero)

$\rightarrow$ idea: turn $\mathtt{Q}$ into itself via sequences of reflections, and observe the evolution of the indecomposables and their Tits form values

Proof of Gabriel's Theorem (A_n case)

Special case: linear quiver $\mathtt{L}_n$: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$\rightarrow$ apply reflections $s_1 s_2 \cdots s_{n-1} s_n \mathtt{L}_n$ and observe evolution of $\underline{\dim}\,\mathbb{V}$

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$$\underline{\dim}\,\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-1}, x_{n-1} - x_n)^\top$$

$$\underline{\dim}\,\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\cdots$$

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$$\cdots$$

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≤ 0

$$\underline{\dim}\, \mathcal{R}_1^+ \mathcal{R}_2^+ \cdots \mathcal{R}_{n-1}^+ \mathcal{R}_n^+ \mathbb{V} = 0 \text{ or } (-x_n, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\Longrightarrow \mathcal{C}^+ \mathbb{V} = \mathcal{R}_1^+ \mathcal{R}_2^+ \cdots \mathcal{R}_{n-1}^+ \mathcal{R}_n^+ \mathbb{V} = 0 \text{ or } x_n = 0$$

Proof of Gabriel's Theorem (A_n case)

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$\Rightarrow \exists i_1, i_2, \cdots, i_{s-1}, i_s$ s.t. $\mathcal{R}^+_{i_s} \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2} \mathcal{R}^+_{i_1} \mathbb{V} = 0$

$\mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2} \mathcal{R}^+_{i_1} \mathbb{V} \neq 0$

$$\underline{\dim}\, \underbrace{\mathcal{C}^+ \cdots \mathcal{C}^+}_{n \text{ times}} \mathbb{V} = 0$$

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$\Longrightarrow \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V}$ is indecomposable and isomorphic to $\mathbb{S}_r$ for some $1 \leq r \leq n$

(Reflection Functor Thm)

Proof of Gabriel's Theorem (A_n case)

Special case: linear quiver $\mathtt{L}_n$: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{L}_n)$ indecomposable, $\underline{\dim}\, \mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

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$\Longrightarrow q_{\mathtt{L}_n}(\underline{\dim}\, \mathbb{V}) = q_{s_{i_{s-1}} \cdots s_{i_1} \mathtt{L}_n}(\underline{\dim}\, \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2} \mathcal{R}^+_{i_1} \mathbb{V}) = q_{s_{i_{s-1}} \cdots s_{i_1} \mathtt{L}_n}(\underline{\dim}\, \mathbb{S}_r) = 1$

(Corollary)

Proof of Gabriel's Theorem (A_n case)

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$\Longrightarrow \underline{\dim}\,\mathbb{V} = \underline{\dim}\,\mathbb{I}_{\mathtt{L}_n}[b,d]$ for some $1 \leq b \leq d \leq n \Longrightarrow \mathbb{V} \cong \mathbb{I}_{\mathtt{L}_n}[b,d]$ $\square$

(Example)

Proof of Gabriel's Theorem (A_n case)

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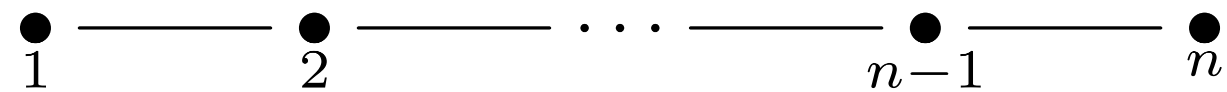
$\square$

Algo: apply Coxeter functor to peel off summands $\mathbb{I}_{\mathtt{L}_n}[b_i, n]$ and to shift other summands to the right. Repeat until all summands have been peeled off.

Proof of Gabriel's Theorem (A_n case)

A_n-type quiver $\mathtt{Q}$:



$\rightarrow$ goal: find a sequence of indices $i_1, i_2, \cdots, i_{s-1}, i_s$ s.t.

$$\mathcal{R}^+_{i_s}\mathcal{R}^+_{i_{s-1}}\cdots\mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V} = 0 \text{ for all } \mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$$

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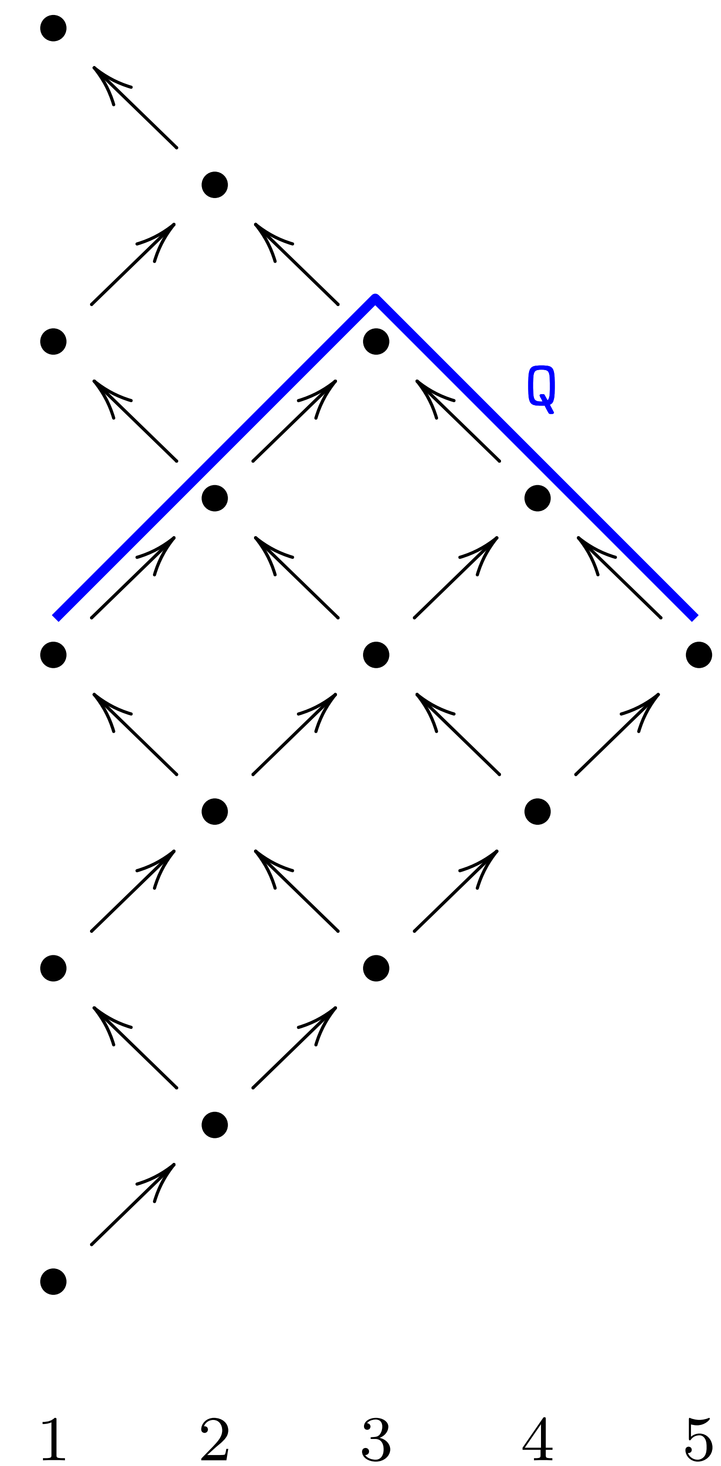
$\rightarrow$ idea: turn $\mathtt{Q}$ into $\mathtt{L}_n$, then use the same sequence a before

Proof of Gabriel's Theorem (A_n case)

A_n-type quiver Q:

- embed Q in a giant pyramid



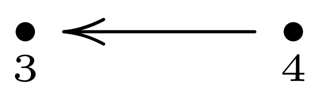
Proof of Gabriel's Theorem (A_n case)

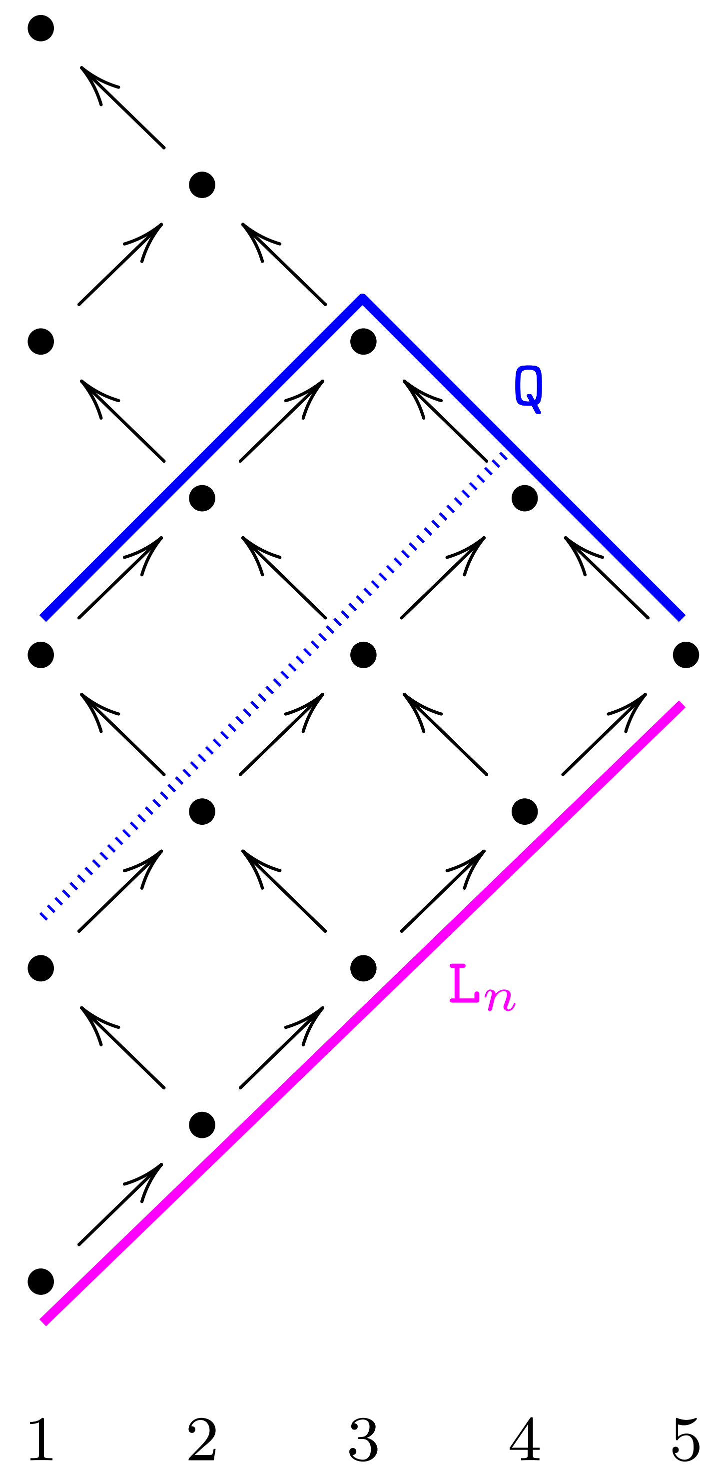
A_n-type quiver Q:

- embed Q in a giant pyramid

- travel down the pyramid to its bottom L_n

  $\longrightarrow$ travelling one level down reverses the leftmost backward arrow

  e.g. $s_1 s_2 s_3$ reverses 



Proof of Gabriel's Theorem (A_n case)

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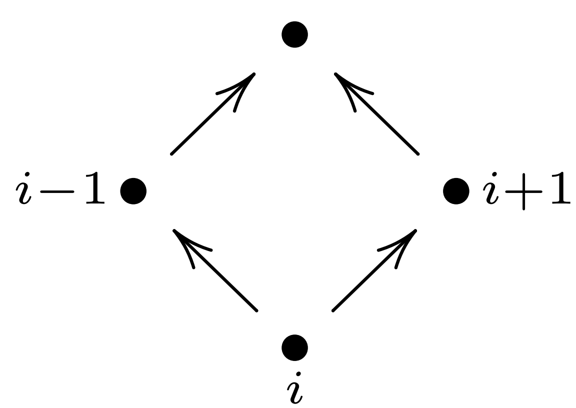
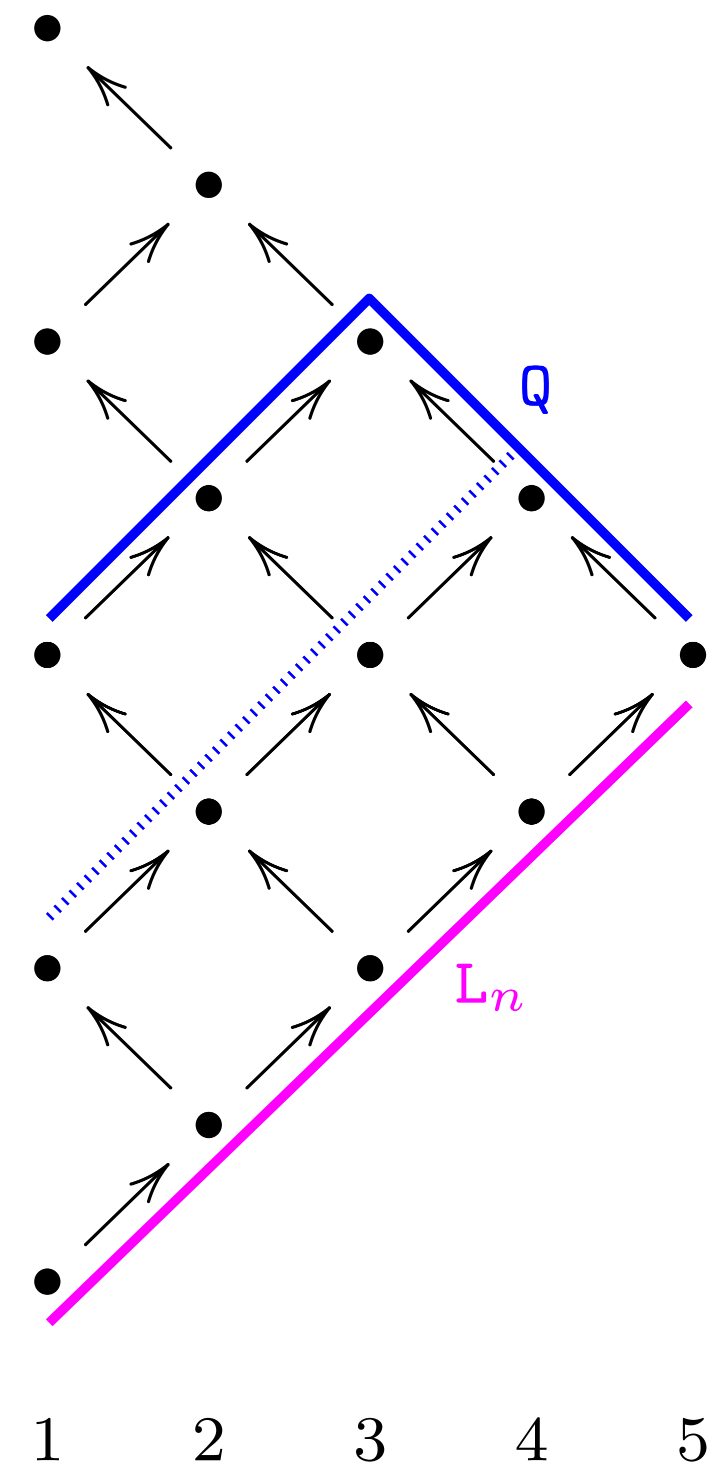
  → travelling one level down reverses the leftmost backward arrow

  e.g. $s_1 s_2 s_3$ reverses $\underset{3}{\bullet} \longleftarrow \underset{4}{\bullet}$

- each diamond



  is travelled down using $\mathcal{R}_i^+$



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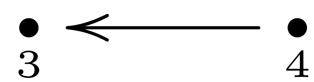
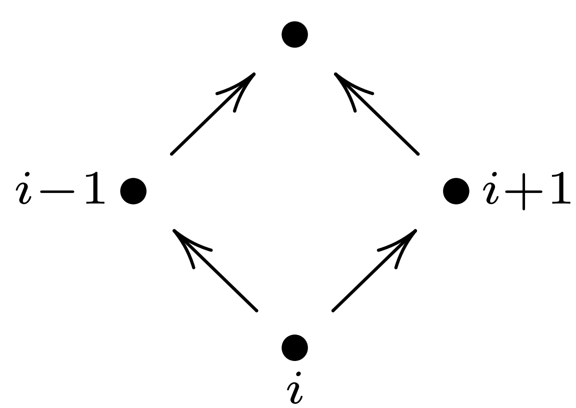
  e.g. $s_1 s_2 s_3$ reverses 

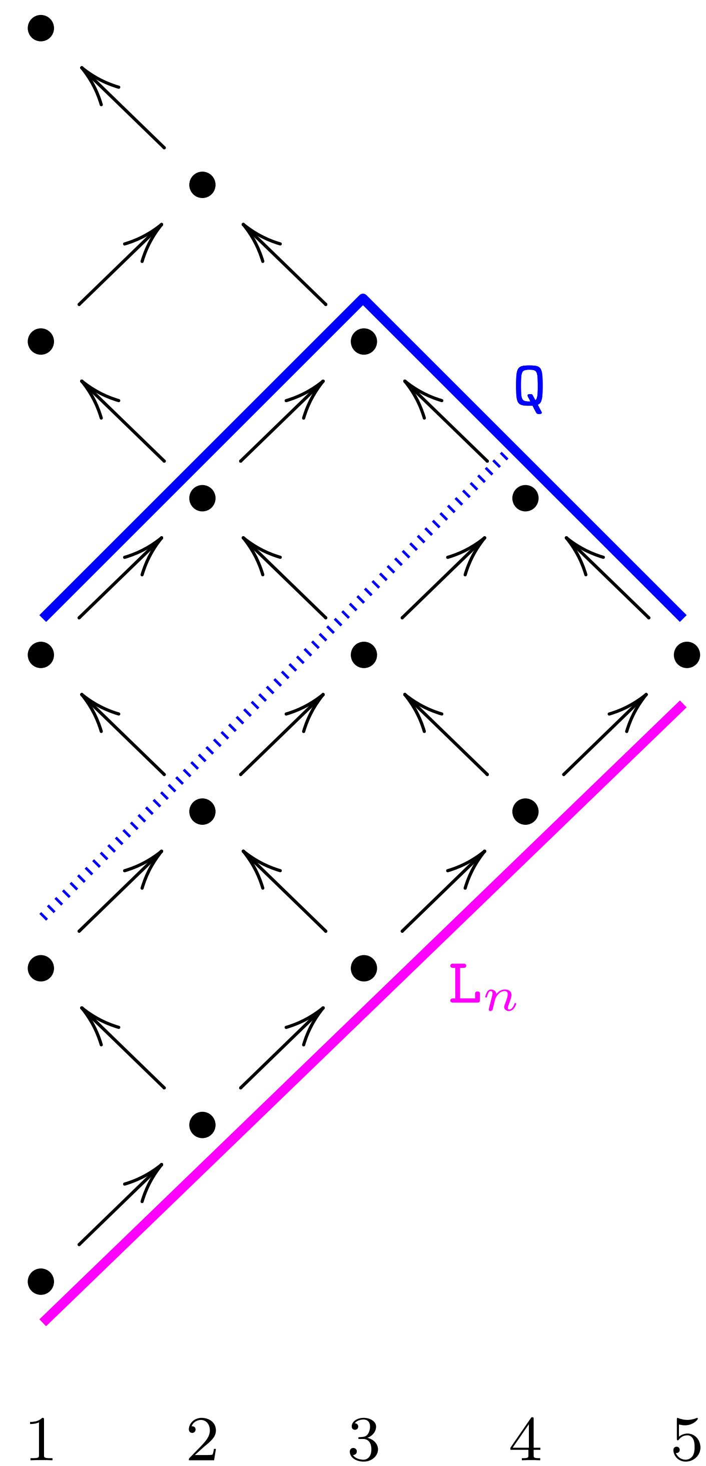
- each diamond



  is travelled down using $\mathcal{R}_i^+$

$\rightarrow$ algo. to compute zigzag persistence

(at the algebraic level $\rightarrow$ maintain bases)



Proof of Gabriel's Theorem (A_n case)

Theorem: [Gabriel II]
Assuming Q is Dynkin with n vertices, the map $\mathbb{V} \mapsto \underline{\dim}\,\mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of Q and the set of *positive roots* of the *Tits form* of Q.

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What we know:

- the positive roots of $q_{\mathtt{Q}}$ are the dimension vectors of interval modules $\mathbb{I}_{\mathtt{Q}}[b,d]$
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What we know:

- the positive roots of $q_{\mathtt{Q}}$ are the dimension vectors of interval modules $\mathbb{I}_{\mathtt{Q}}[b, d]$
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Additional observations:

- $\neq$ interval modules are $\ncong$, therefore each class C contains 1 interval module
- each interval module is indecomposable (endomorphism ring isom. to $\boldsymbol{k}$)

$\square$

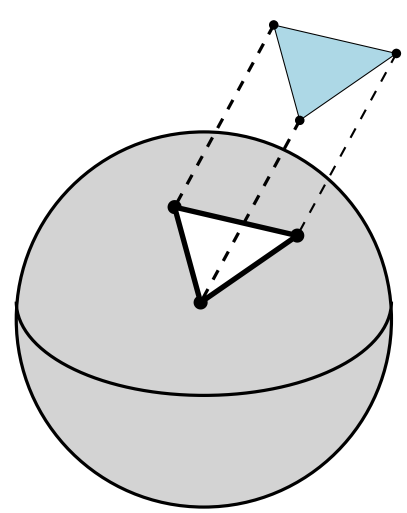
Application 1: Persistence Computation

$$
\begin{array}{ccccccccc}
K_1 & \text{———} & \cdots & \text{———} & K_i & \overset{\sigma}{\text{———}} & K_{i+1} & \text{———} & \cdots & \text{———} & K_n \\
\downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
\mathsf{H}(K_1) & \text{———} & \cdots & \text{———} & \mathsf{H}(K_i) & \overset{f}{\text{———}} & \mathsf{H}(K_{i+1}) & \text{———} & \cdots & \text{———} & \mathsf{H}(K_n)
\end{array}
$$

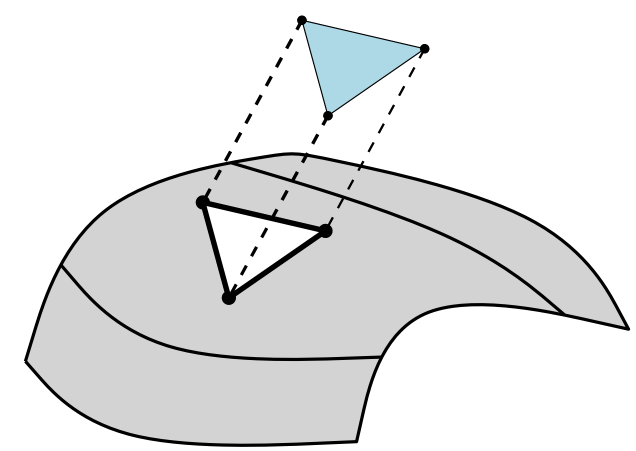
- every horizontal map is either forward or backward
- the K_i are simplicial complexes, the inclusions are *elementary*
- the $\mathsf{H}(K_i)$ are vector spaces connected by linear maps (quiver representation)

$$
\begin{array}{ccc}
K & \xrightarrow{\ \sigma\ } & K \cup \{\sigma\} \\
\downarrow & & \downarrow \\
\mathsf{H}(K) & \xrightarrow{\ f\ } & \mathsf{H}(K \cup \{\sigma\})
\end{array}
$$

$\ker f = [\partial\sigma]$

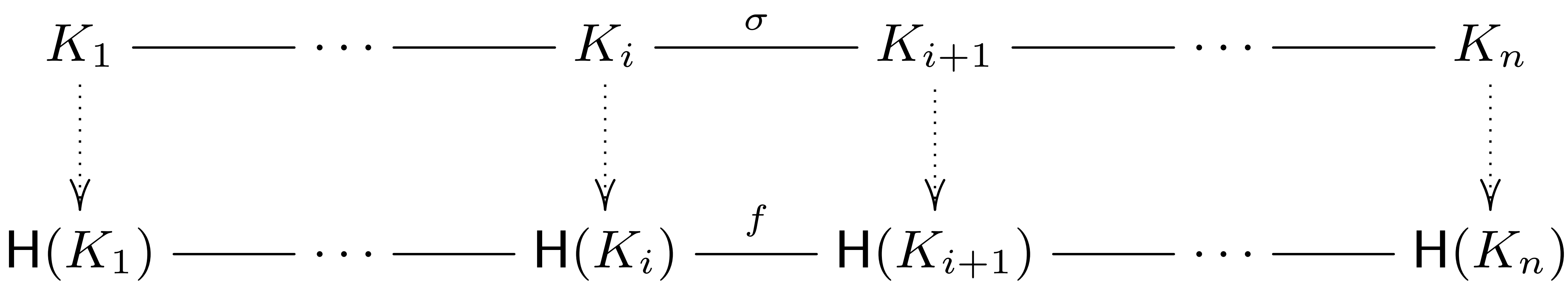


f inj. of corank 1



f surj. of nullity 1

Application 1: Persistence Computation



- every horizontal map is either forward or backward
- the K_i are simplicial complexes, the inclusions are *elementary*
- the $\mathsf{H}(K_i)$ are vector spaces connected by linear maps (quiver representation)

Algorithms for when all maps are forward:

- Gaussian elimination: worst-case $O(n^3)$, highly optimized in practice
- Fast matrix multiplication: worst-case $O(n^\omega)$, not implemented

Algorithms for when maps can be forward or backward:

- Gaussian elimination + *right filtration* functor: worst-case $O(n^3)$, not optimized

Application 1: Persistence Computation

We compute of the persistent homology of:

$$K_1 \text{ ——— } K_2 \text{ ——— } \cdots \text{ ——— } K_i \overset{\sigma}{\text{———}} K_{i+1} \text{ ——— } \cdots \text{ ——— } K_{n-1} \text{ ——— } K_n$$

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by maintaining a compatible homology basis for

$$\underbrace{K_1 \text{———} \cdots \text{———} K_i}_{\mathbb{K}[1;i]}$$

[Carlsson, de Silva '10],[C,deS, Morozov '09]

Application 1: Persistence Computation

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[Maria, O. '15]

$$\underbrace{K_1 \longleftrightarrow \cdots \longleftrightarrow K_i = K'_m}_{\mathbb{K}[1;i]} \xleftarrow{\tau_m} K'_{m-1} \xleftarrow{\tau_{m-1}} K'_{m-2} \xleftarrow{\tau_{m-2}} \cdots \xleftarrow{\tau_1} \emptyset$$

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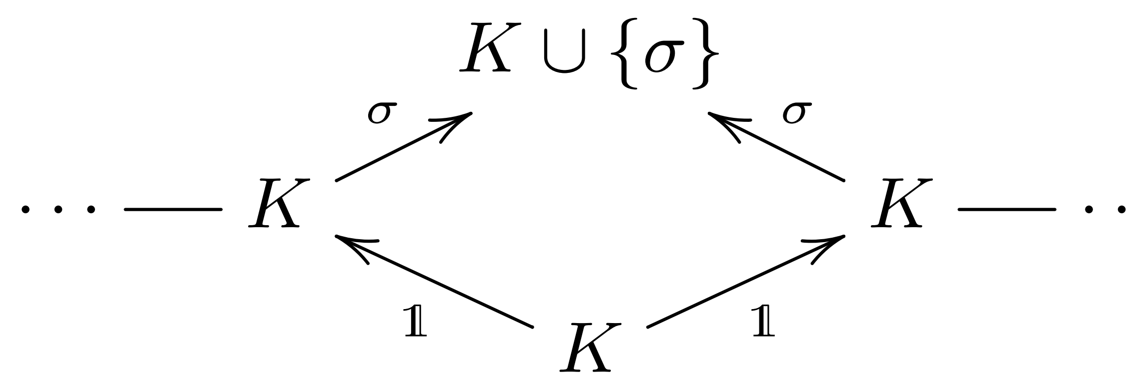
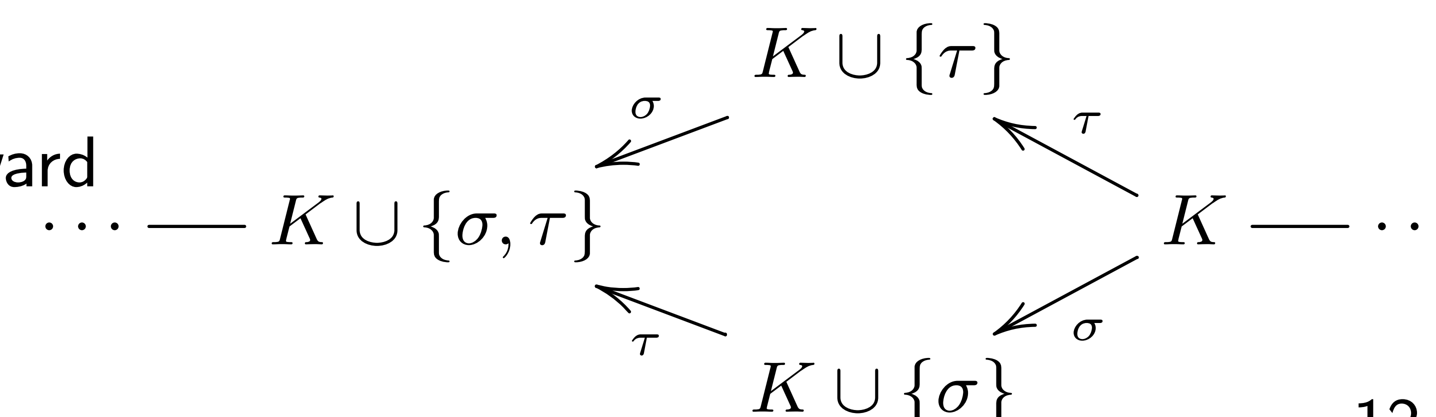
- arrow reflection if $\xrightarrow{\sigma}$ is forward



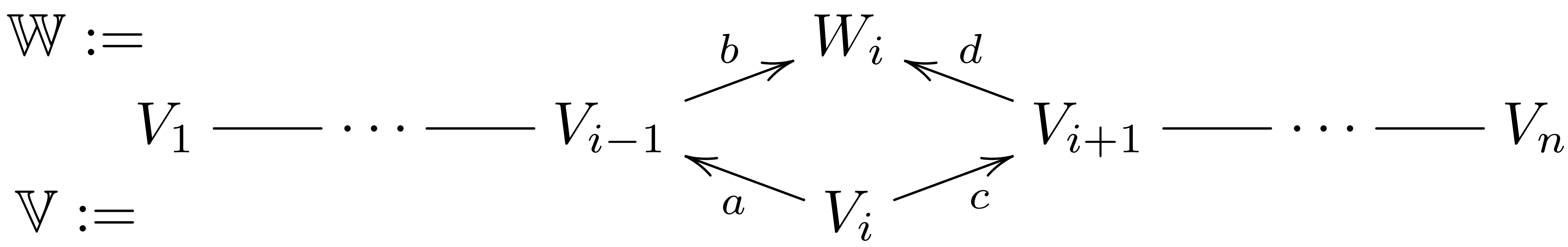
- arrow transposition if $\xleftarrow{\sigma}$ is backward



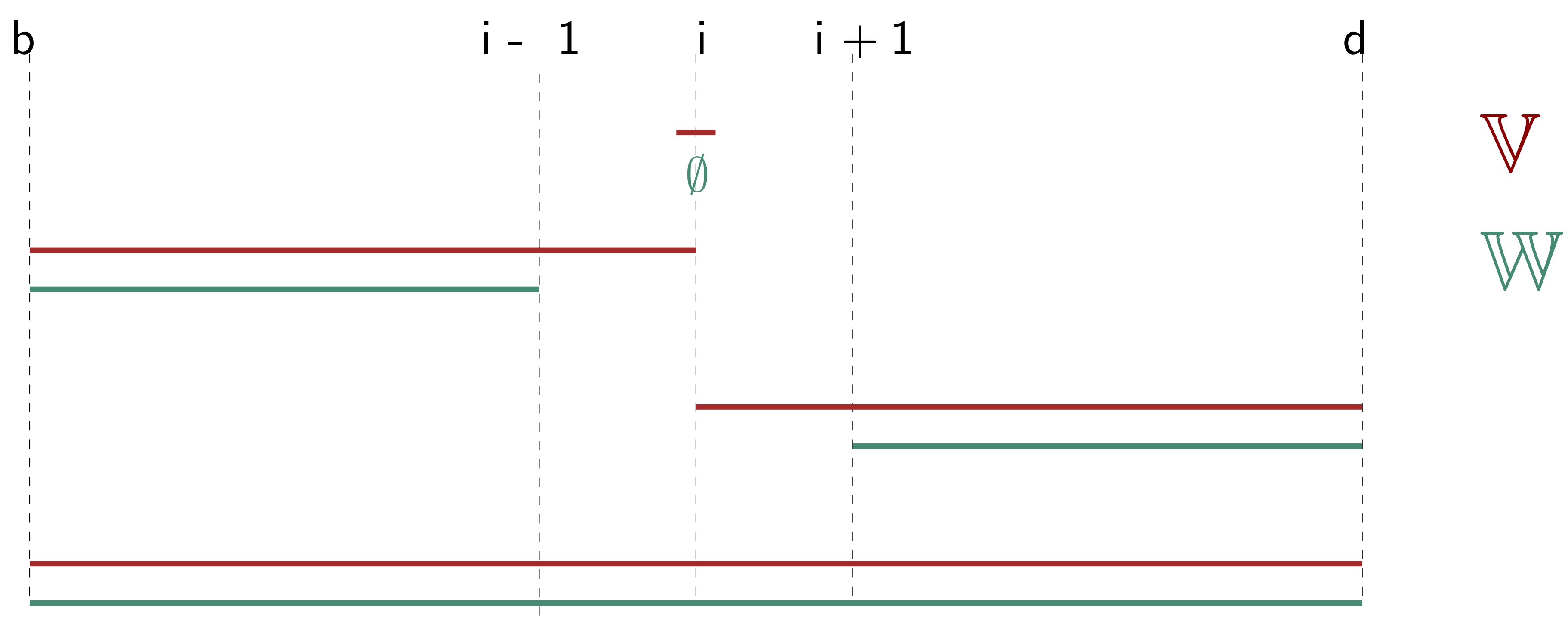
Application 1: Persistence Computation

Theorem: Exact Diamond Principle [Carlsson, de Silva '10]

Under the *exactness* hypothesis on the diamond:



Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related as follows:



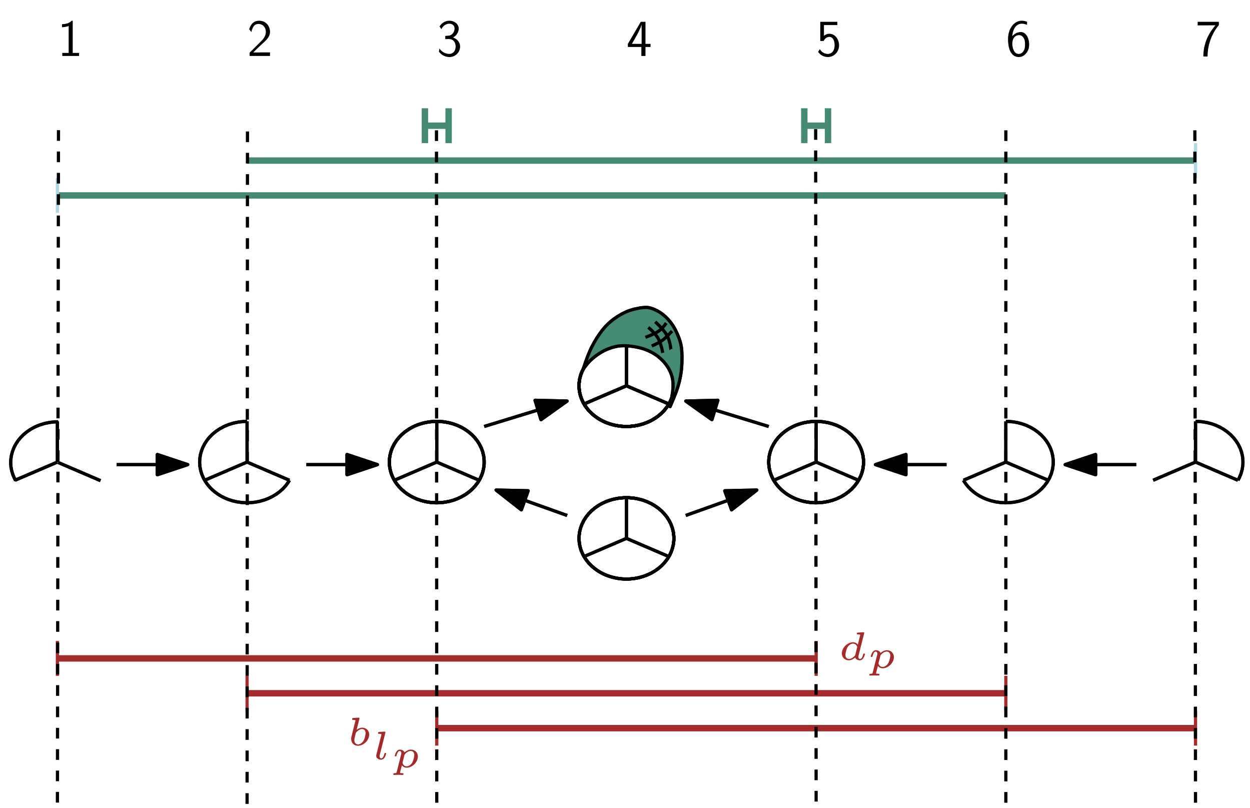
Application 1: Persistence Computation

Theorem: Injective/Surjective Diamond Principles [Maria, O. '15]
For f injective of corank 1 or surjective of nullity 1:

$$\mathbb{W} := \qquad V_1 \;\text{———}\; \cdots \;\text{———}\; V_{i-1} \xrightarrow{f} W_i \xleftarrow{f} V_{i+1} \;\text{———}\; \cdots \;\text{———}\; V_n$$

$$\mathbb{V} := \qquad V_1 \;\text{———}\; \cdots \;\text{———}\; V_{i-1} \xleftarrow{\mathbb{1}} V_i \xrightarrow{\mathbb{1}} V_{i+1} \;\text{———}\; \cdots \;\text{———}\; V_n$$

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related through *greedy rule*.



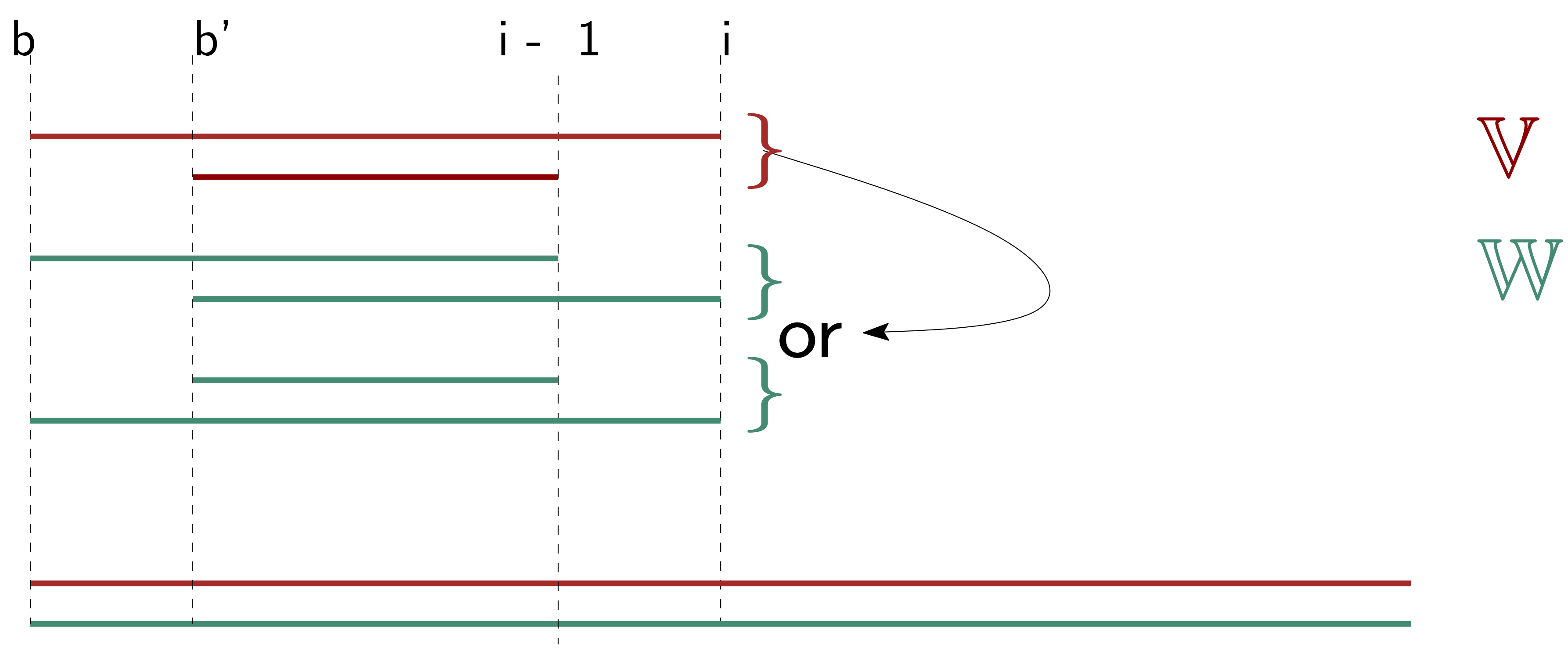
Application 1: Persistence Computation

Theorem: Transposition Diamond Principle [Maria, O. '15]

For an *exact* diamond + morphisms inj. of corank 1 or surj. of nullity 1:

$$\begin{array}{ccccccccc} \mathbb{W} := & & & & & W_i & & & \\ & & & & \overset{b}{\swarrow} & & \overset{d}{\nwarrow} & & \\ & V_1 & \text{———} \cdots \text{———} & V_{i-1} & & & & V_{i+1} & \text{———} \cdots \text{———} \; V_n \\ & & & & \underset{a}{\nwarrow} & & \underset{c}{\swarrow} & & \\ \mathbb{V} := & & & & & V_i & & & \end{array}$$

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related as follows:



Application 1: Persistence Computation

Concluding remarks on this application:

- extensions of Exact Diamond Principle / Reflection Functors

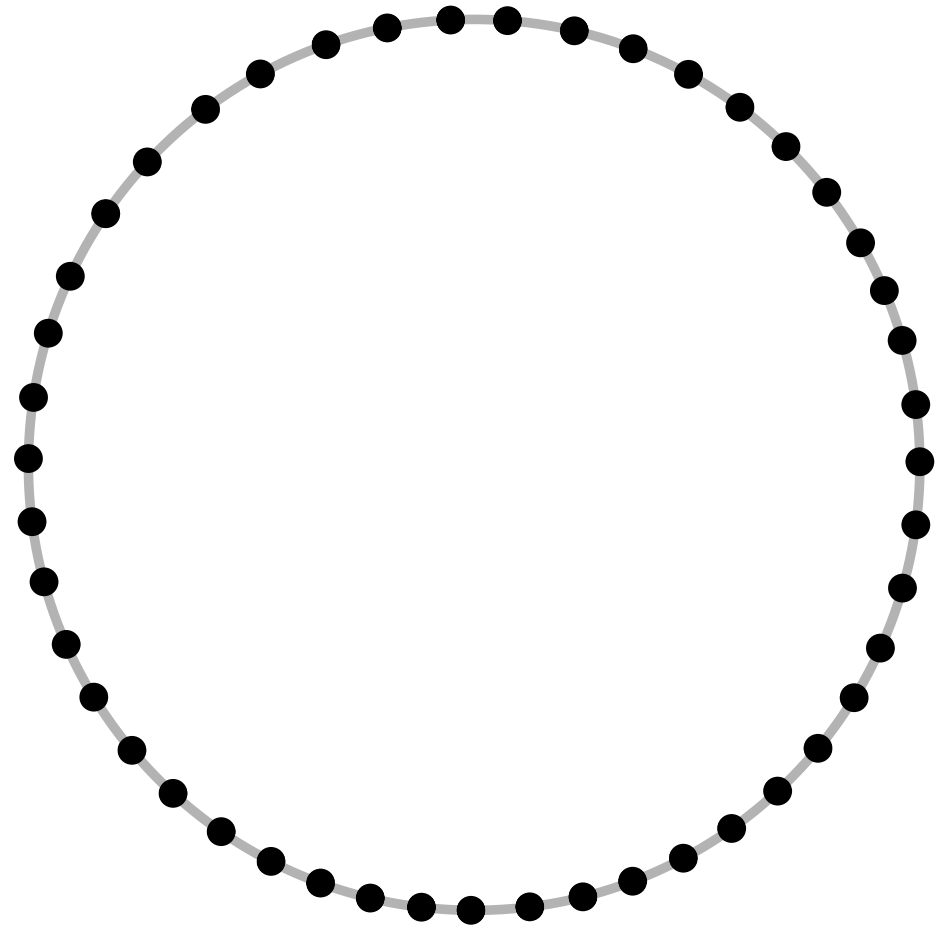
  (cf. injective/surjective diamonds and transposition diamonds)

- same asymptotic complexity: $O(n^3)$ in the worst case

- better performances than [CdSM'09] in practice $(\times 0.2)$

- extension to cohomology $\rightarrow$ significant improvement expected $(\times 0.01)$

Application 2: Zigzags for homology inference

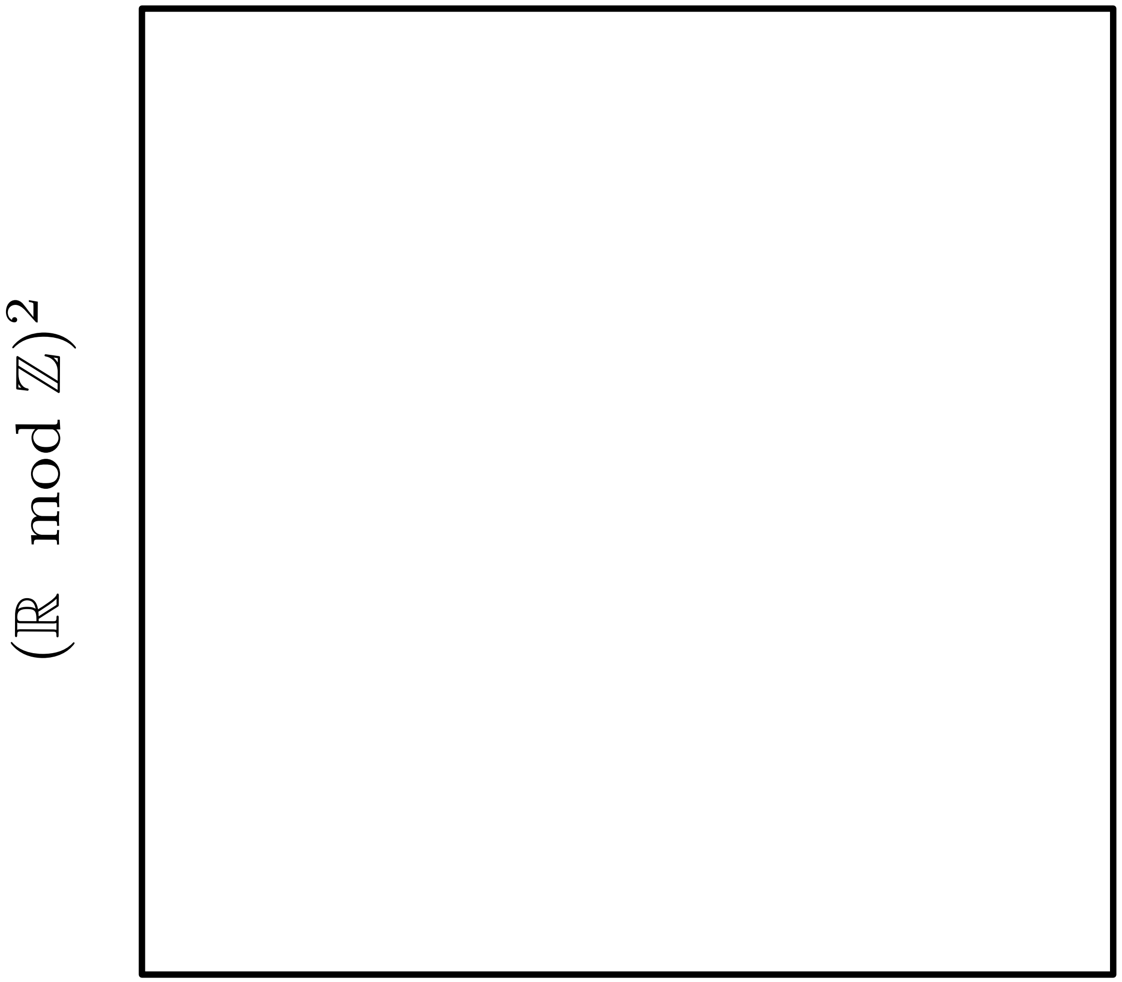


Setup: $K \subset \mathbb{R}^d$ a compact set, $p_1, \cdots, p_n$ data points sampled along (or close to) K

Goal: infer the topology (homology) of K, knowing only $p_1, \cdots, p_n$

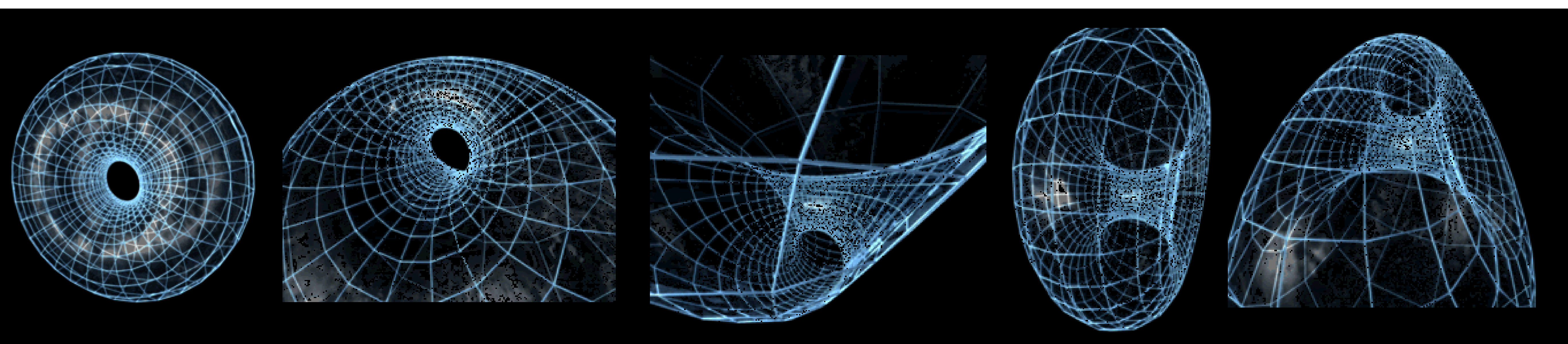
Application 2: Zigzags for homology inference

Manufactured Data Set



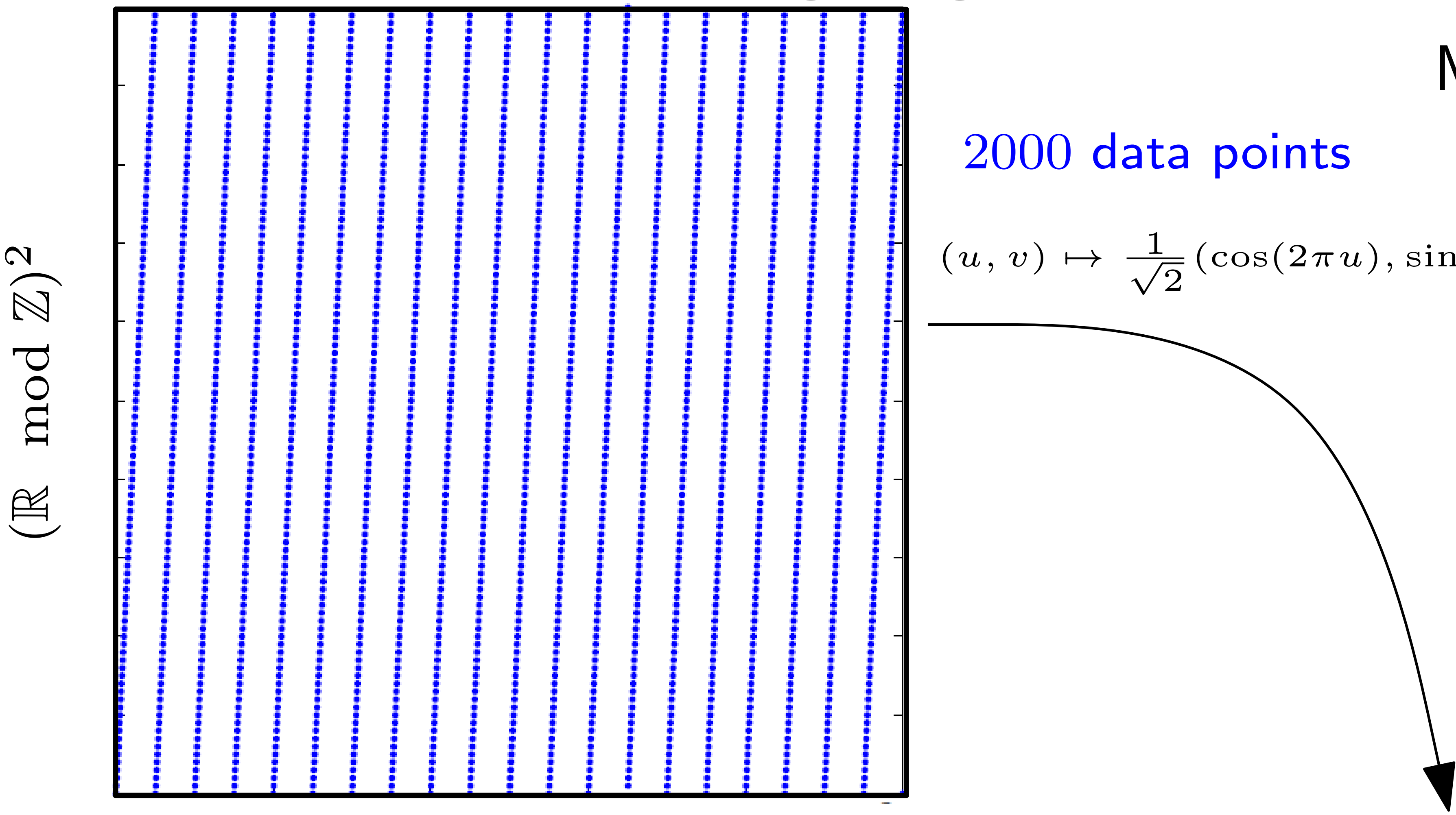
$$(u, v) \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi u), \sin(2\pi u), \cos(2\pi v), \sin(2\pi v))$$

$\subset \mathbb{S}3 \subset \mathbb{R}^4$



source: `http://http://en.wikipedia.org/wiki/Clifford_torus`

Application 2: Zigzags for homology inference



Manufactured Data Set

2000 data points

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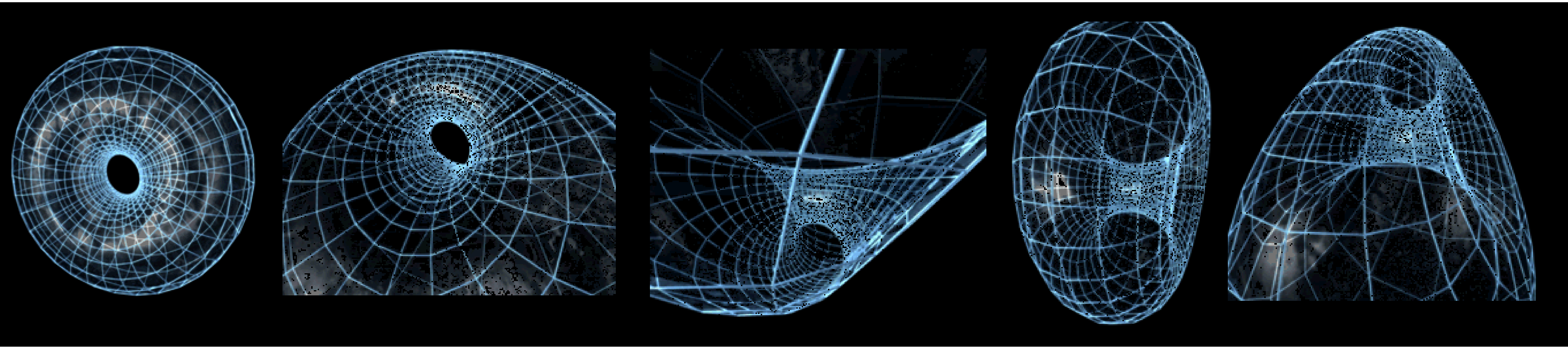

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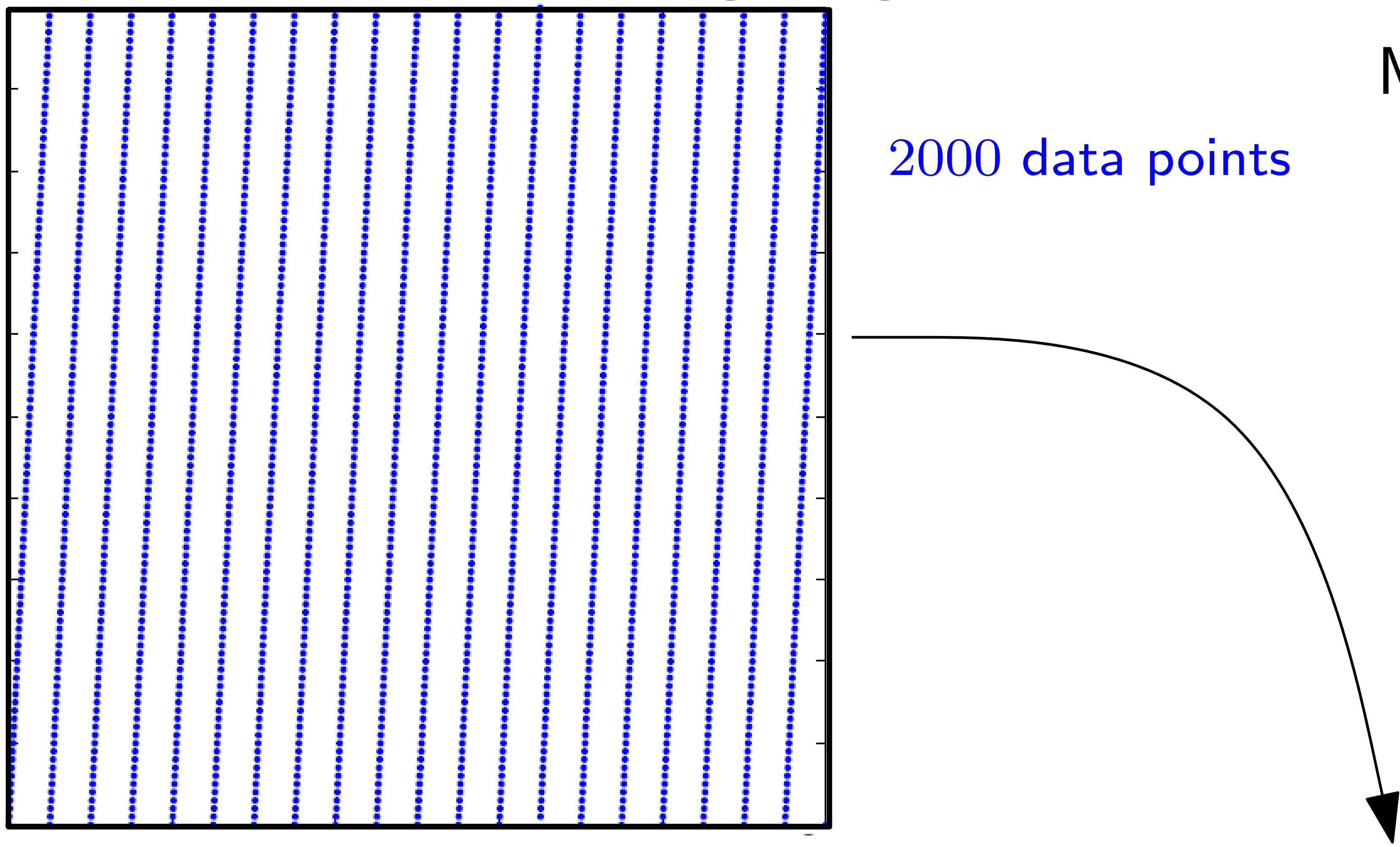
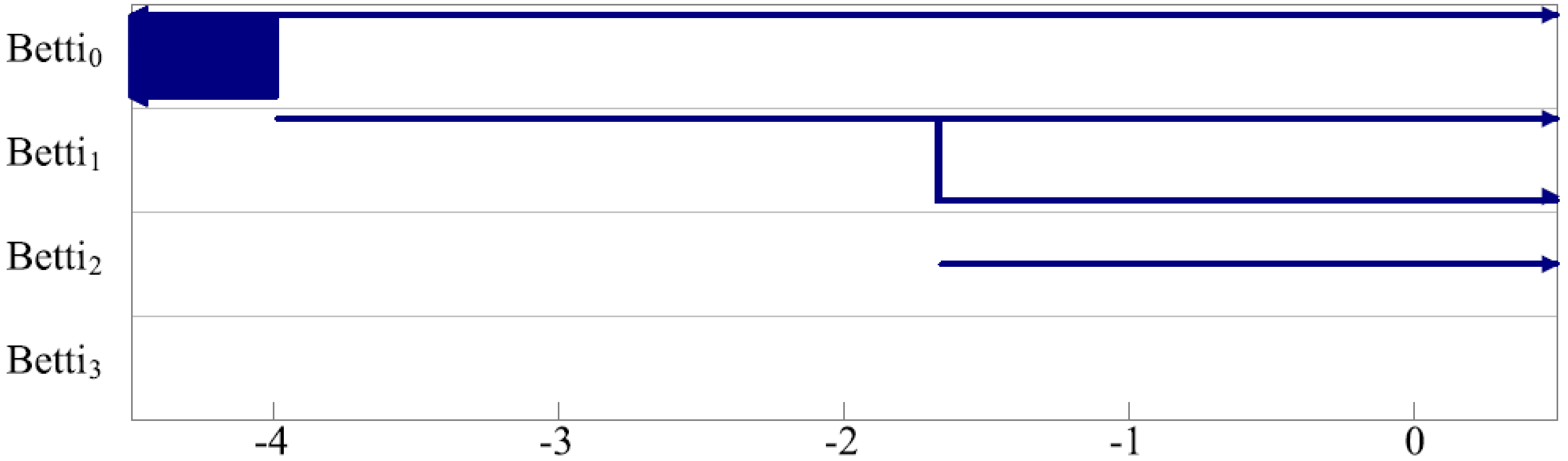
Application 2: Zigzags for homology inference

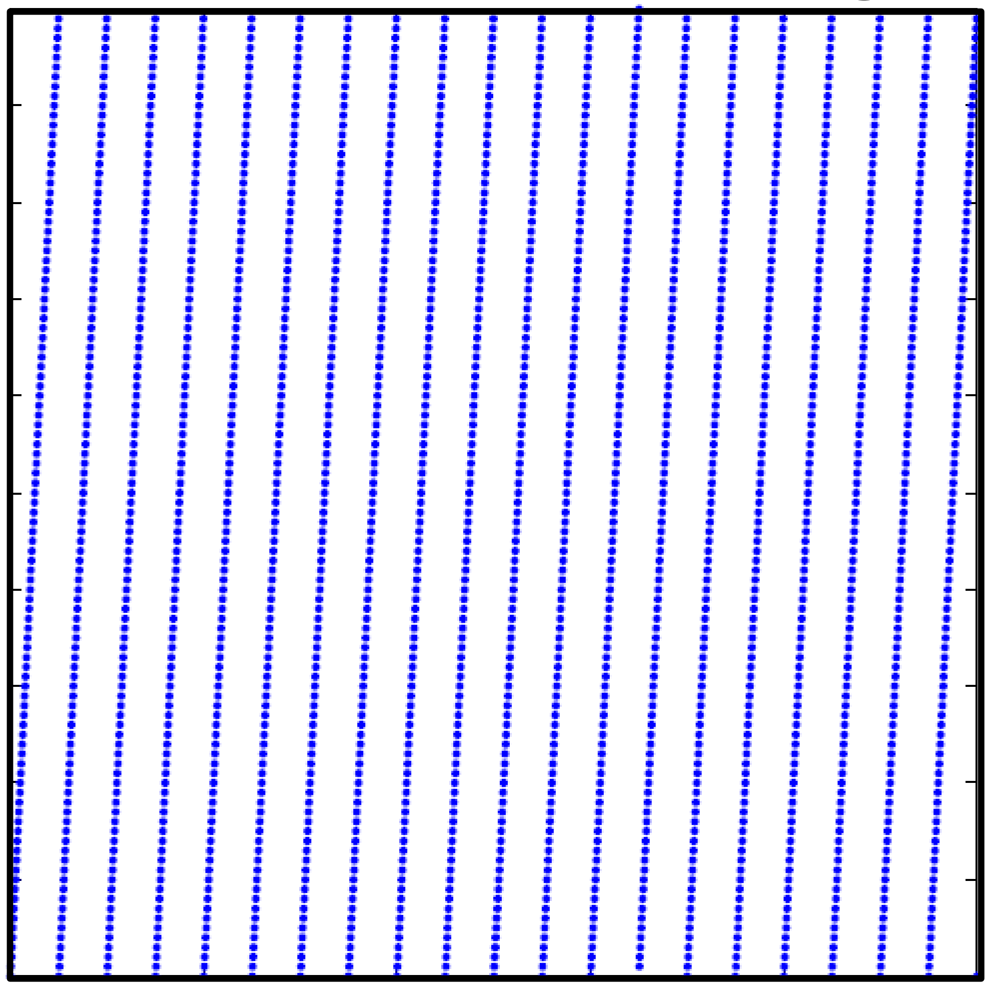
Manufactured Data Set





Rips

Application 2: Zigzags for homology inference

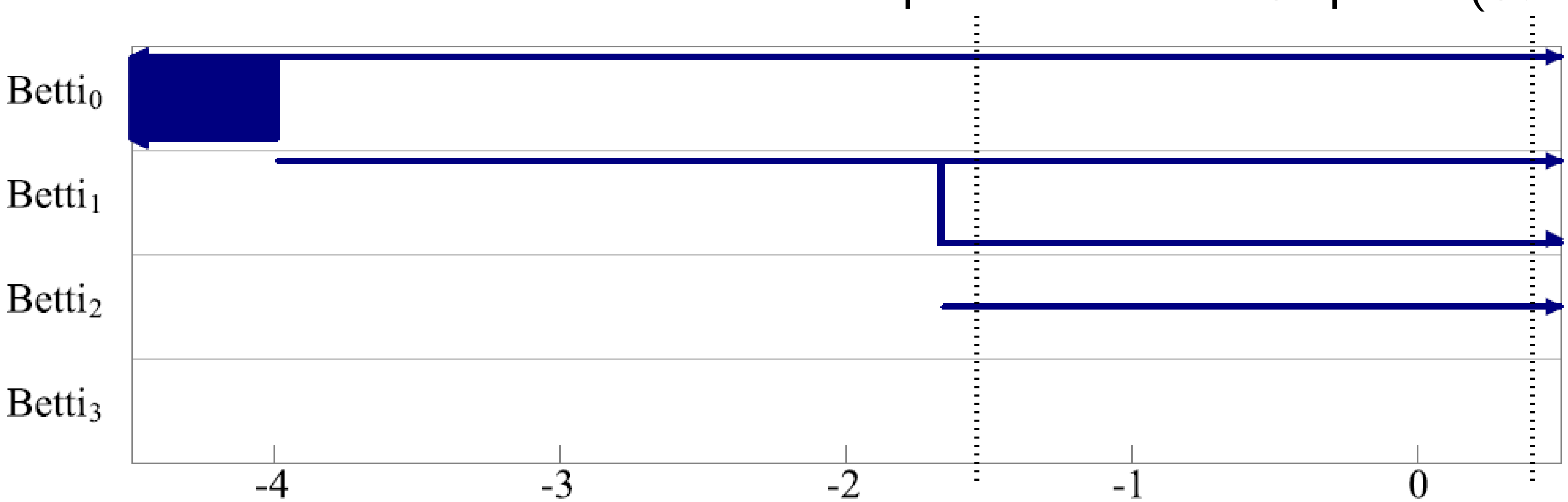


Manufactured Data Set

2000 data points

comput. limit

3-sphere ($37 \cdot 10^9$ simplices)



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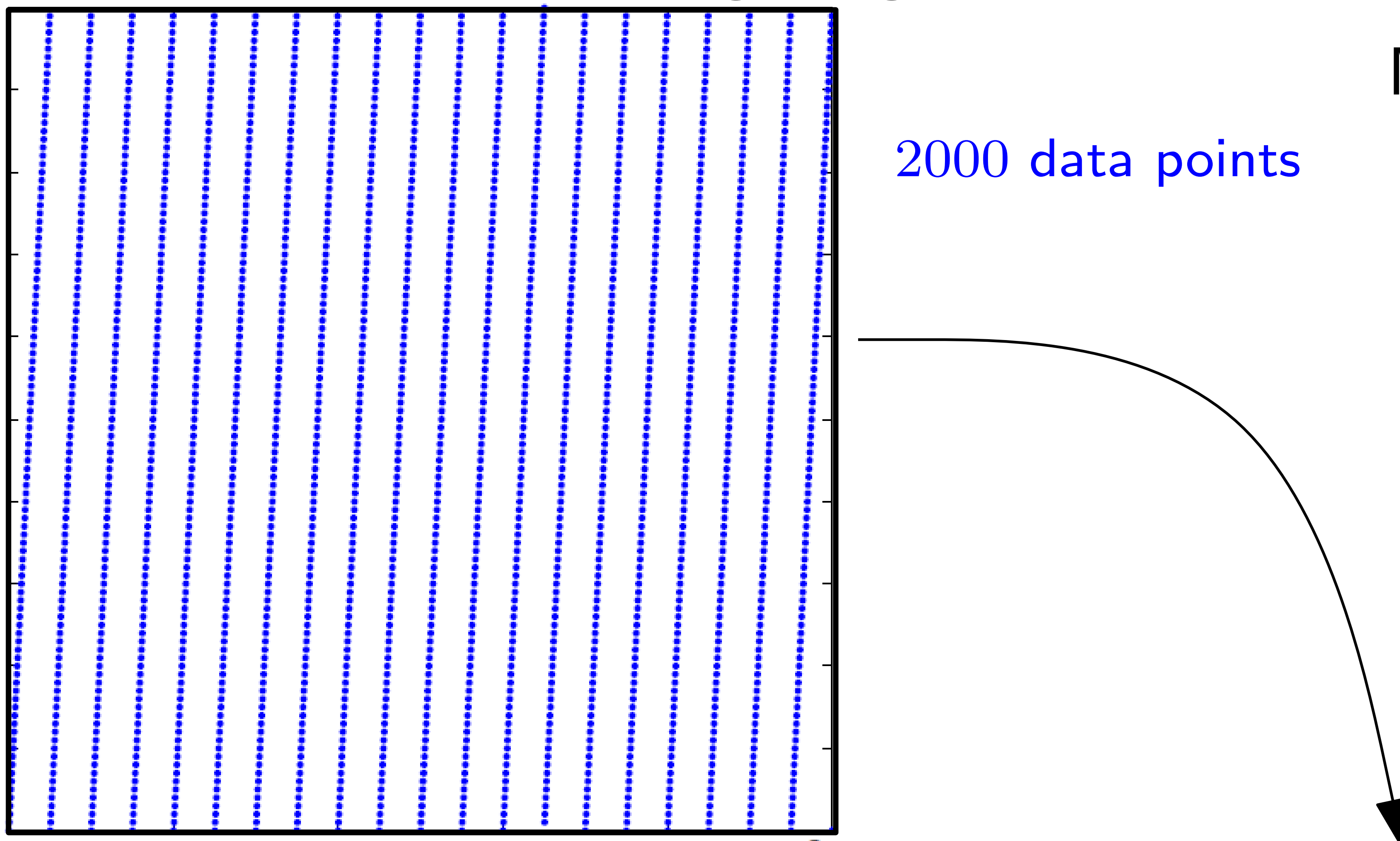
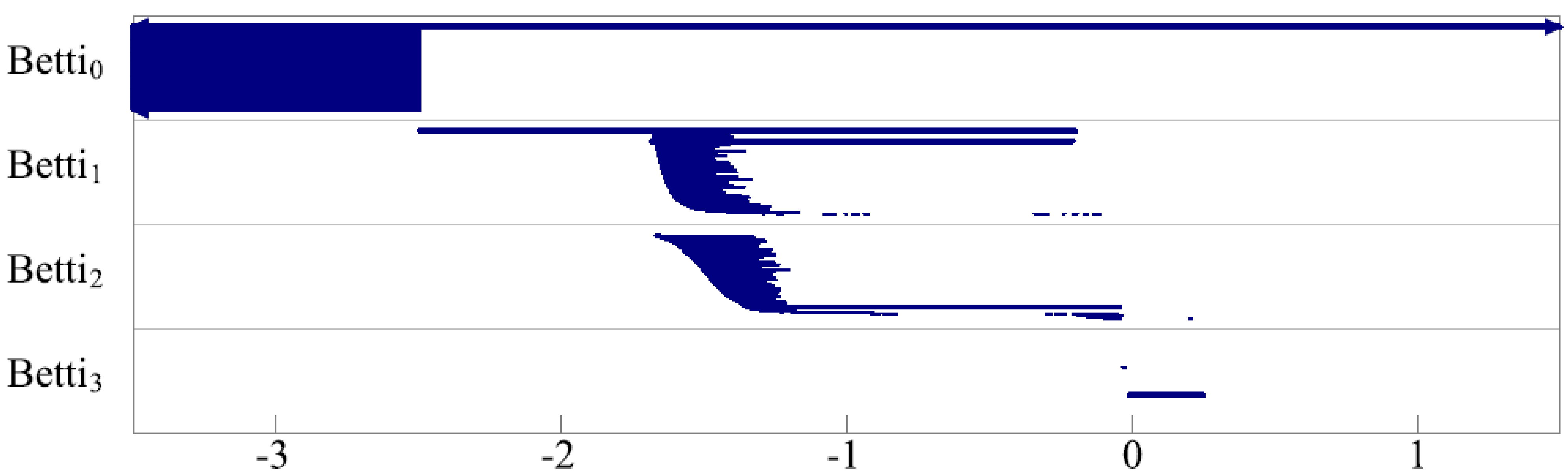


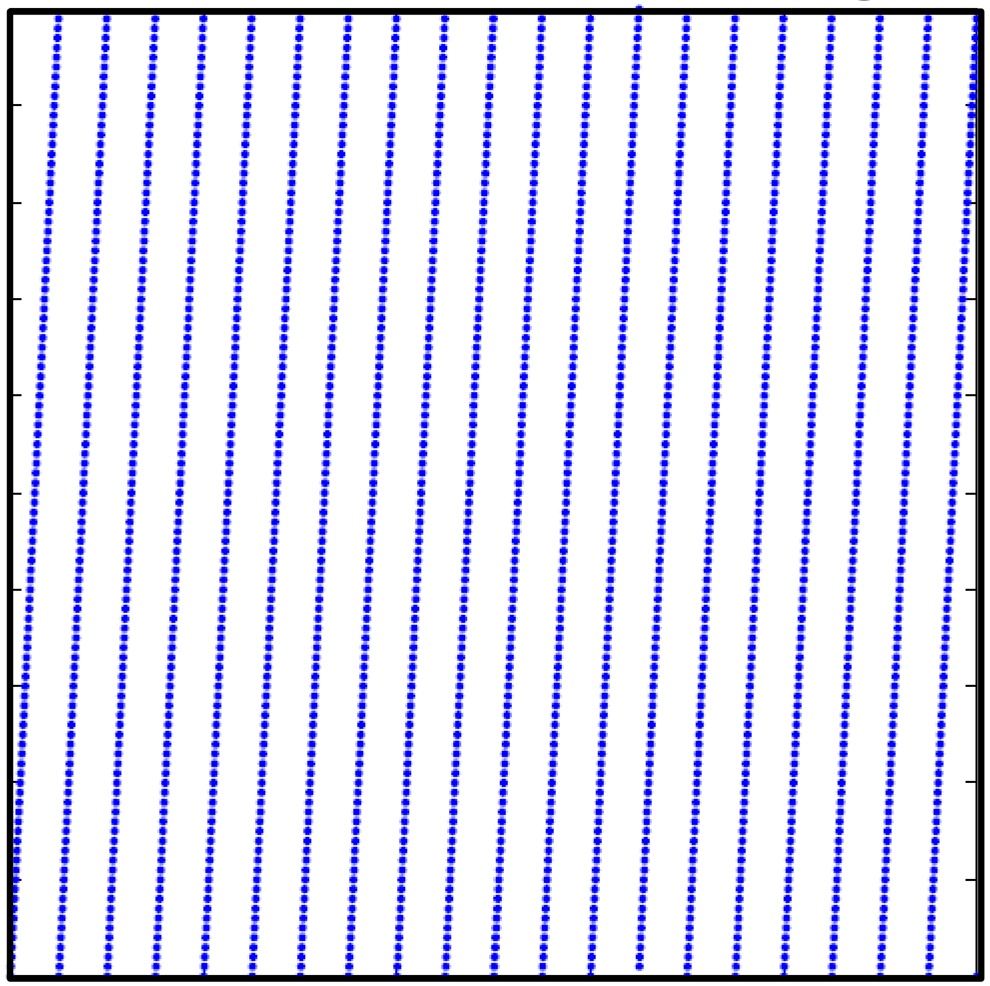
$(12 \cdot 10^6$ simplices)



mesh-based

[HMOS10]

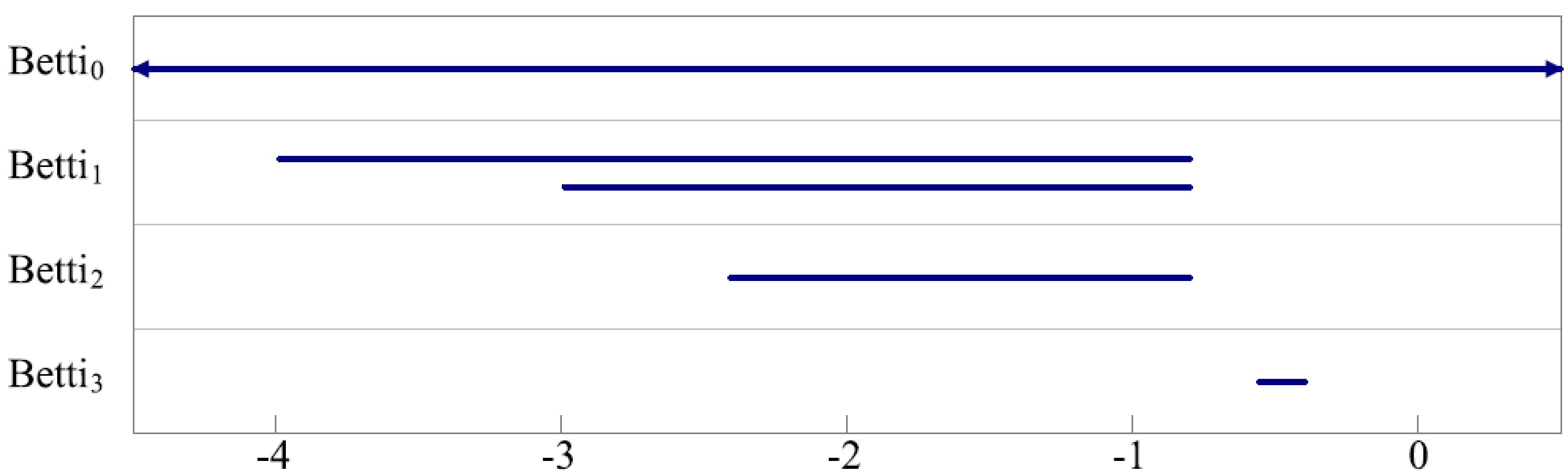
Application 2: Zigzags for homology inference



Manufactured Data Set

2000 data points

($200 \cdot 10^3$ simplices)



[OS14]

Application 2: Zigzags for homology inference

Manufactured Data Set

What we learn from this experiment:

- commonly used filtrations (Čech, Rips, alpha, witness, graph-induced) become **huge** at large scales and/or in high ambient dimensions: 2^n, $n^{\frac{d}{2}}$, etc.

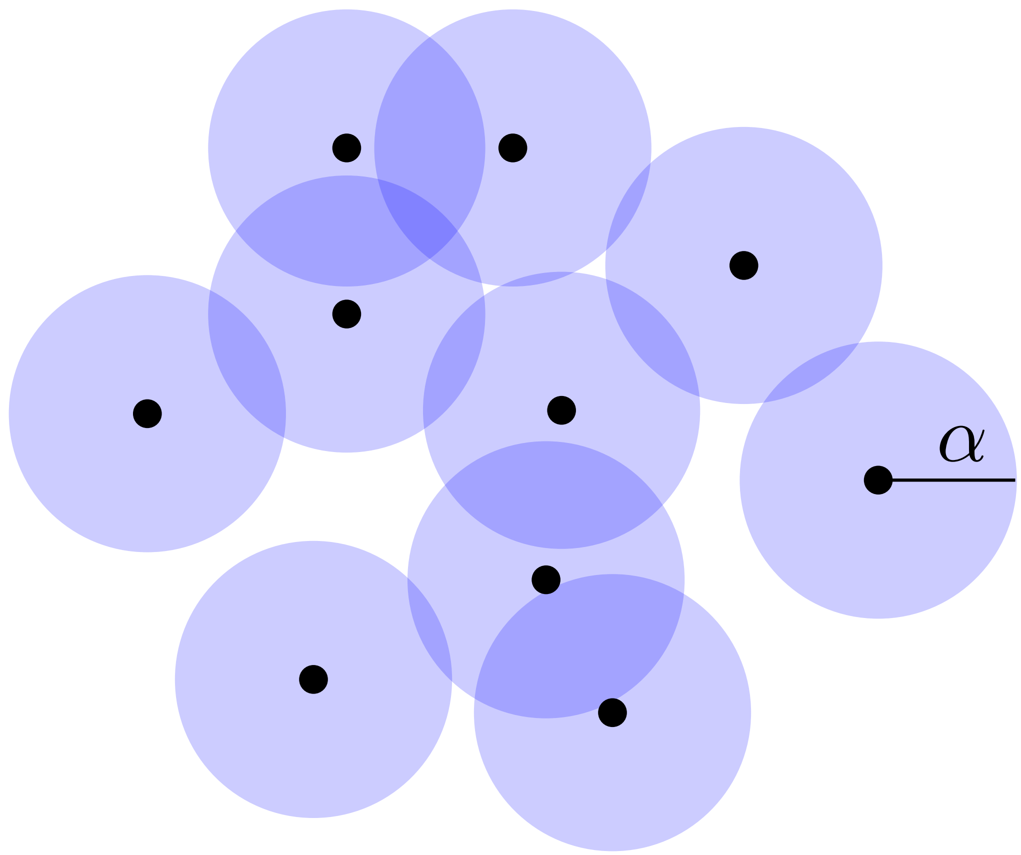
- approximations (mesh-based, sparse Rips, simplicial maps) may introduce defects in the barcodes: extra noise, over-simplification, etc.

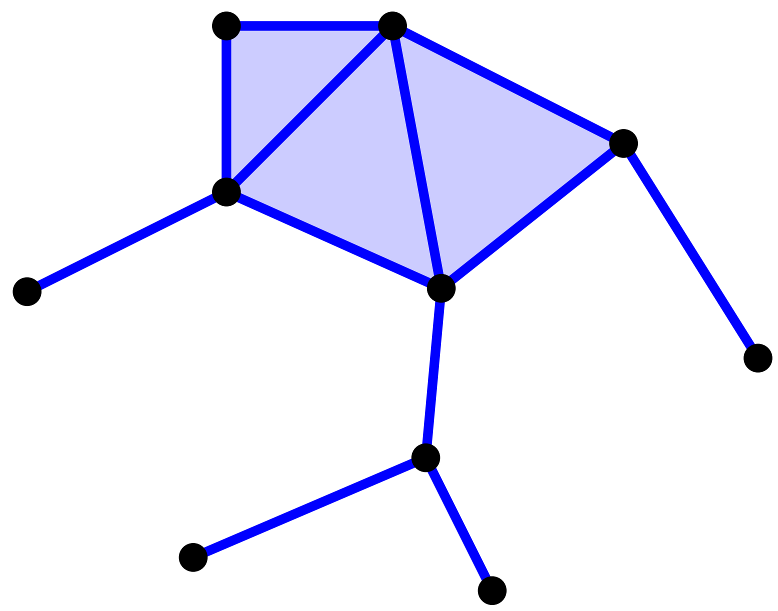
- it is possible to take advantage of both worlds...

Application 2: Zigzags for homology inference

Rips filtration

Input: $P \subset \mathbb{R}^d$ finite





$\mathcal{R}_\alpha(P)$ = clique complex of intersection graph of balls of radius α

$\neq$ *nerve* of union of balls of radius α (*Čech complex*)

Rips filtration: $\{\mathcal{R}_\alpha(P)\}_{\alpha=0}^{+\infty}$

Application 2: Zigzags for homology inference

Approach

Input: $P \subset \mathbb{R}^d$ finite

Params: $\rho \geq \eta \geq 0$, ordering $p_1, \cdots, p_n$ of P (e.g. furthest-point order)

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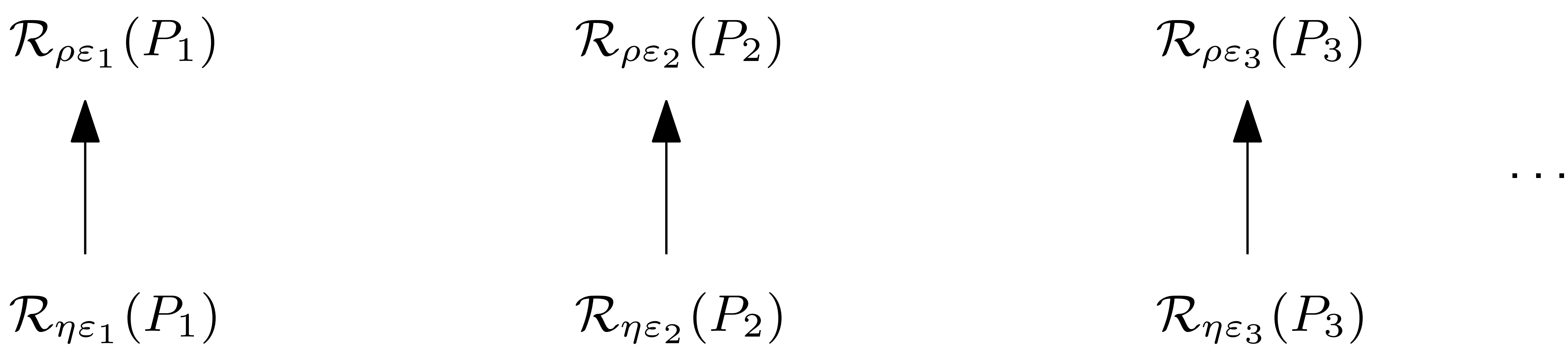
- let $P_i = \{p_1, \cdots, p_i\}$ be the i-th prefix, and $\varepsilon_i = \mathrm{d_H}(P_i, P)$ the i-th scale
- $\forall i$, compute $\mathcal{R}_{\eta\varepsilon_i}(P_i)$ and $\mathcal{R}_{\rho\varepsilon_i}(P_i)$



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- $\forall i$, compute $\mathcal{R}_{\eta\varepsilon_i}(P_i)$ and $\mathcal{R}_{\rho\varepsilon_i}(P_i)$
- [CO08] $\forall i$, compute $r_i^* = \operatorname{rank} \mathsf{H}\mathcal{R}_{\eta\varepsilon_i}(P_i) \to \mathsf{H}\mathcal{R}_{\rho\varepsilon_i}(P_i)$

Output: plot r_i^* against ε_i

$$\begin{array}{cccc} \mathcal{R}_{\rho\varepsilon_1}(P_1) & \mathcal{R}_{\rho\varepsilon_2}(P_2) & \mathcal{R}_{\rho\varepsilon_3}(P_3) & \\ \uparrow & \uparrow & \uparrow & \cdots \\ \mathcal{R}_{\eta\varepsilon_1}(P_1) & \mathcal{R}_{\eta\varepsilon_2}(P_2) & \mathcal{R}_{\eta\varepsilon_3}(P_3) & \end{array}$$

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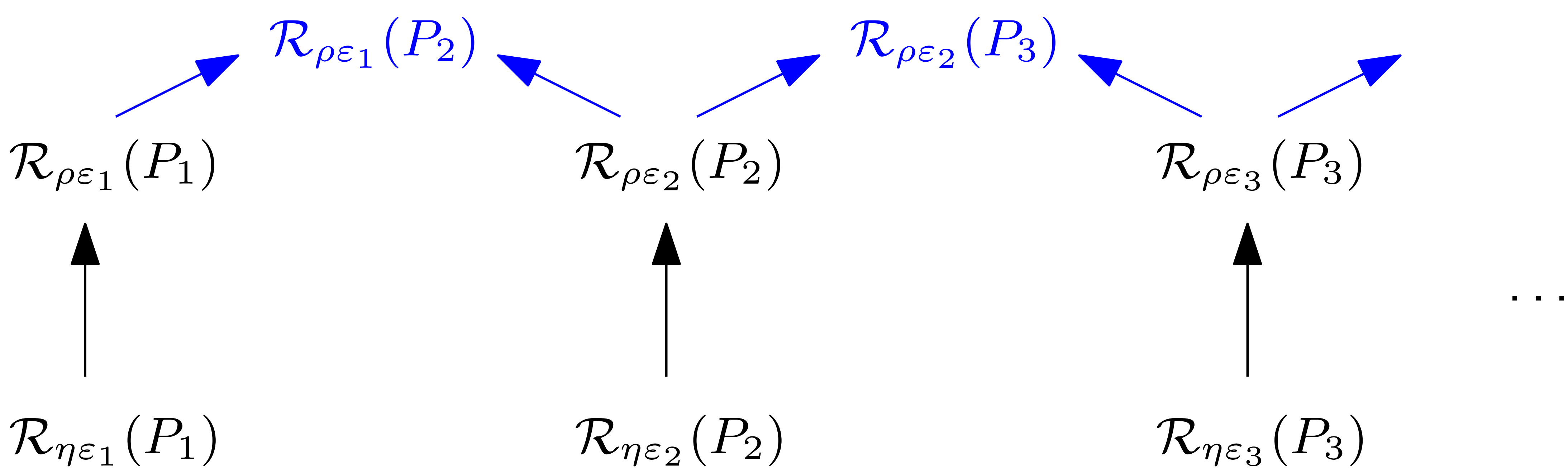
- [OS14] relate the $\mathcal{R}_{\eta\varepsilon_i}(P_i) \longrightarrow \mathcal{R}_{\rho\varepsilon_i}(P_i)$ through the following zigzag:



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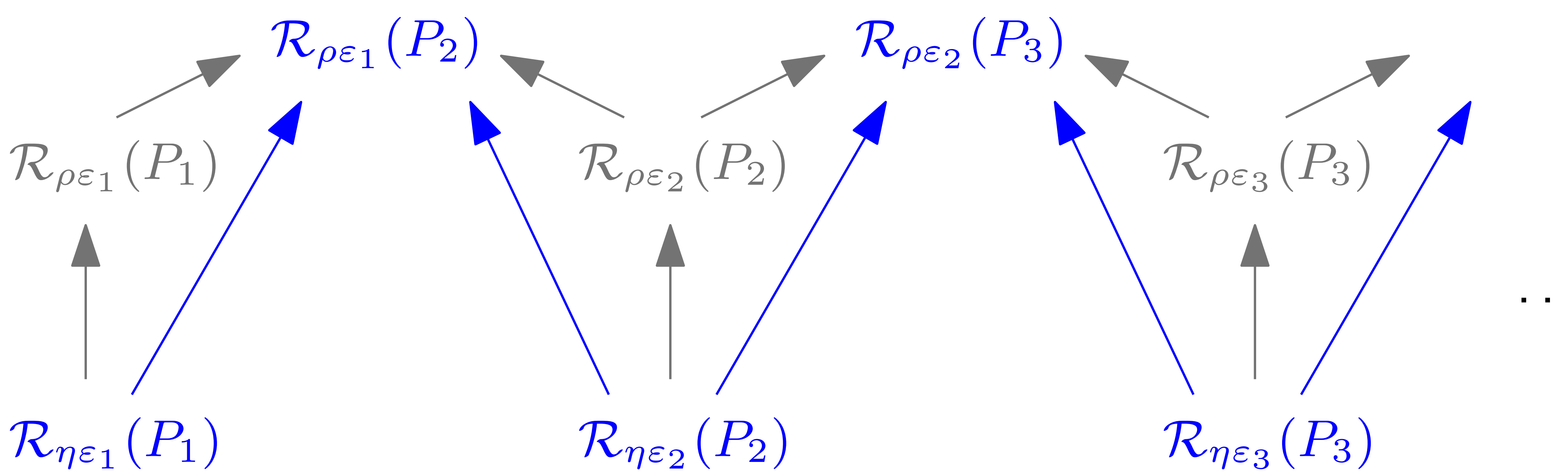

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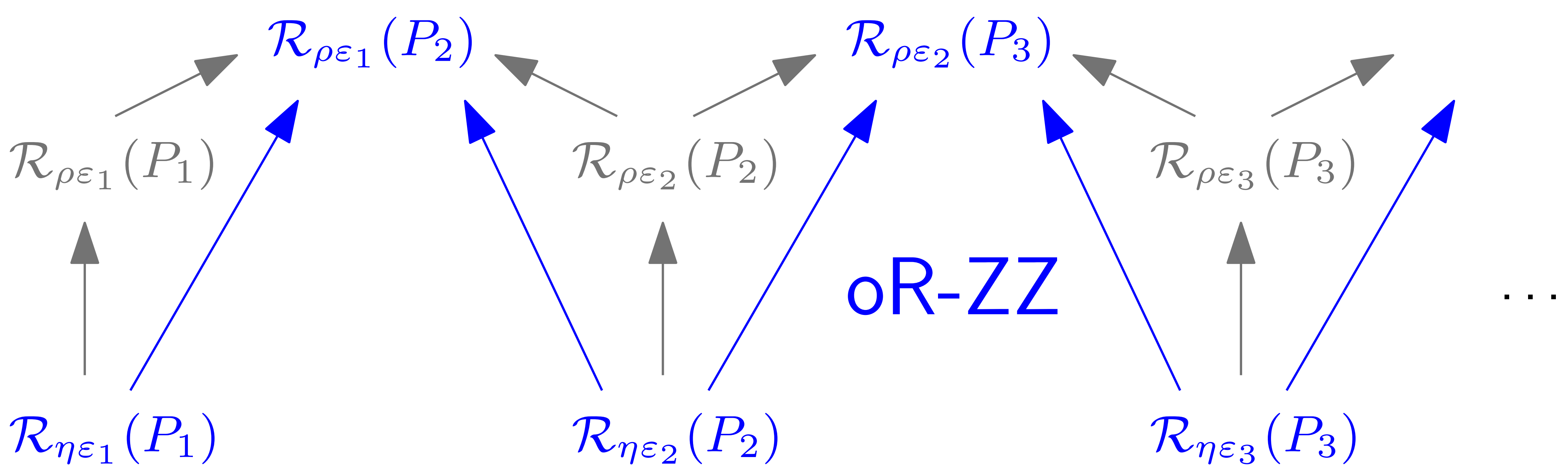
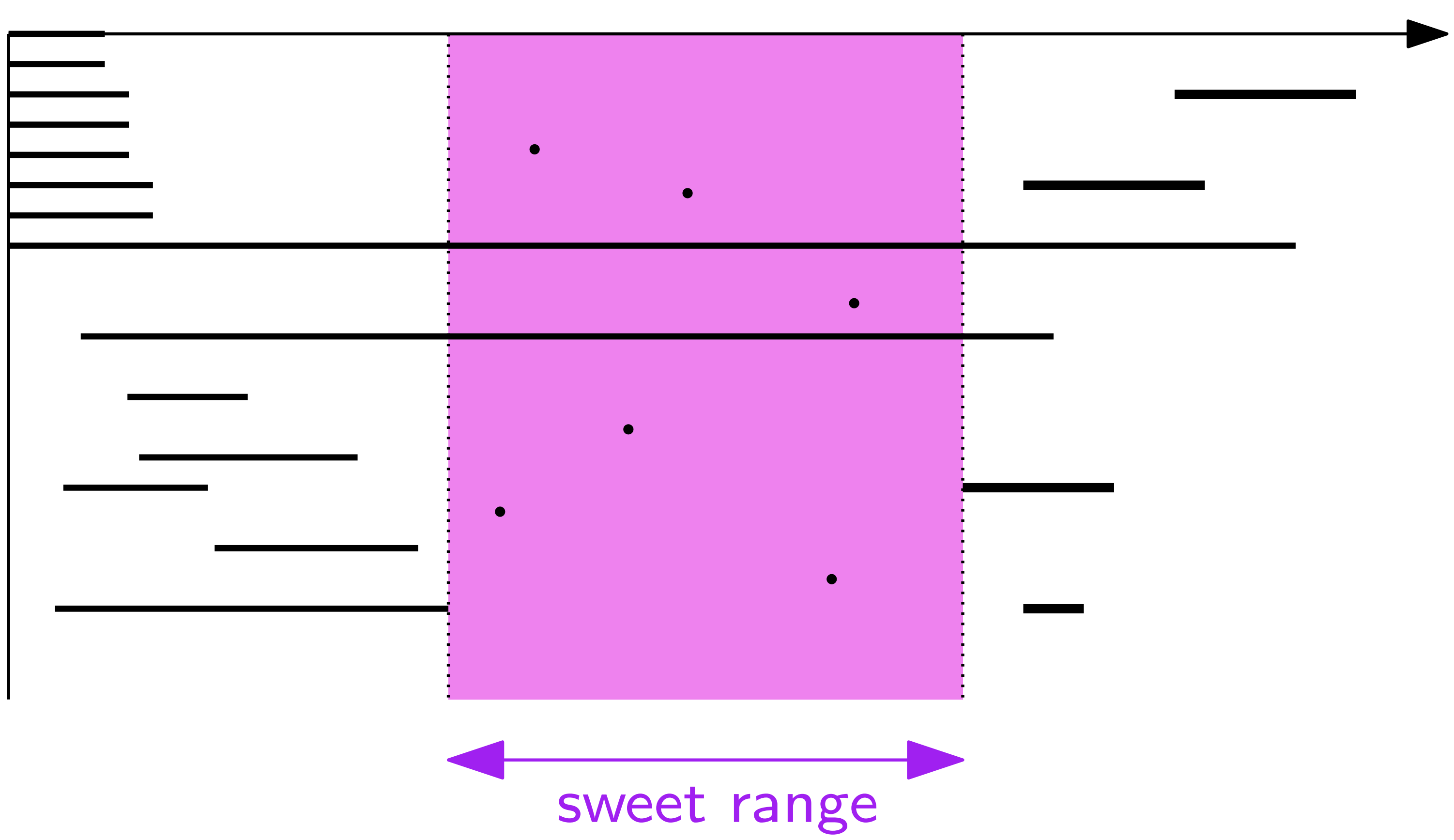


Application 2: Zigzags for homology inference

Thm: "If P is ε-close to X in the Hausdorff distance, with $\varepsilon < \Theta(1)\ \mathrm{wfs}(X)$, then there exists a sweet range of scales $[O(\varepsilon),\ \Omega(\mathrm{wfs}(X))]$ such that the oR-ZZ restricted to this range has a persistence barcode made only of full-length intervals, revealing the homology of X, and of ephemeral (length zero) intervals."



Application 2: Zigzags for homology inference

Thm: Let ρ and η be multipliers such that $\rho > 10$ and $\frac{3}{\vartheta_d} < \eta < \frac{\rho-4}{2\vartheta_d}$. Let $X \subset \mathbb{R}^d$ be a compact set and let $P \subset \mathbb{R}^d$ be such that $\mathrm{d_H}(P, X) < \varepsilon$ with

$$\varepsilon < \min\left\{\frac{\vartheta_d\eta - 3}{6\vartheta_d\eta},\ \frac{\eta - 3/\vartheta_d}{3\rho+\eta},\ \frac{\rho - 2\vartheta_d\eta - 4}{6(\rho - 2\vartheta_d\eta)},\ \frac{\rho - 2\vartheta_d\eta - 4}{(4\vartheta_d+1)\rho - 2\vartheta_d\eta}\right\}\mathrm{wfs}(X).$$

Then, for any $k < l$ such that

$$\max\left\{\frac{3\varepsilon}{\vartheta_d\eta - 3},\ \frac{4\varepsilon}{\rho - 2\vartheta_d\eta - 4}\right\} \leq \varepsilon_k, \varepsilon_l < \min\left\{\frac{1}{6}\mathrm{wfs}(X) - \varepsilon,\ \frac{1}{\vartheta_d\rho + 1}\left(\mathrm{wfs}(X) - \varepsilon\right)\right\},$$

the oR-ZZ restricted to $\mathcal{R}_{\rho\varepsilon_k}(P_{k+1}) \leftarrow \cdots \leftarrow \mathcal{R}_{\eta\varepsilon_l}(P_l)$ has a persistence barcode made only of full-length intervals and ephemeral (length zero) intervals, the number of full-length intervals being equal to the dimension of $H_*(X^\lambda)$ for any $\lambda \in (0, \mathrm{wfs}(X))$.

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$\Theta(1)$

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$O(\varepsilon)$

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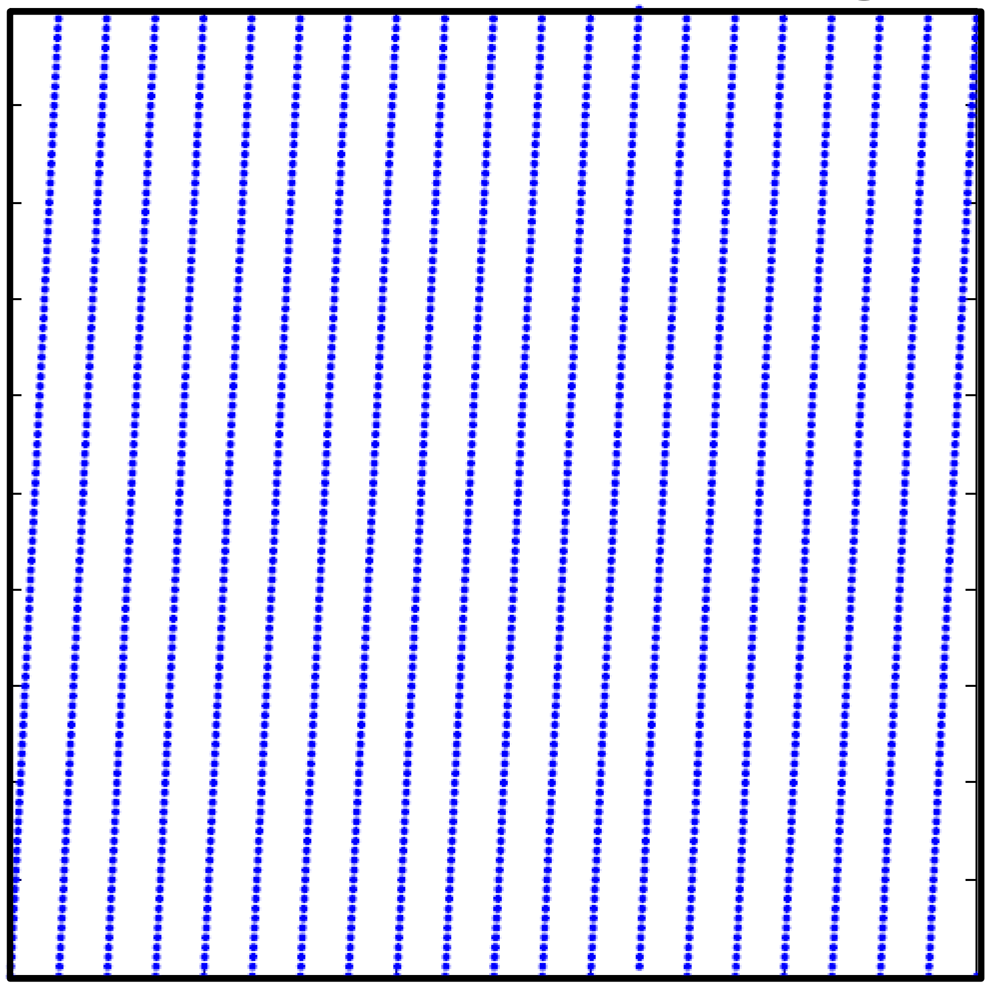
Application 2: Zigzags for homology inference



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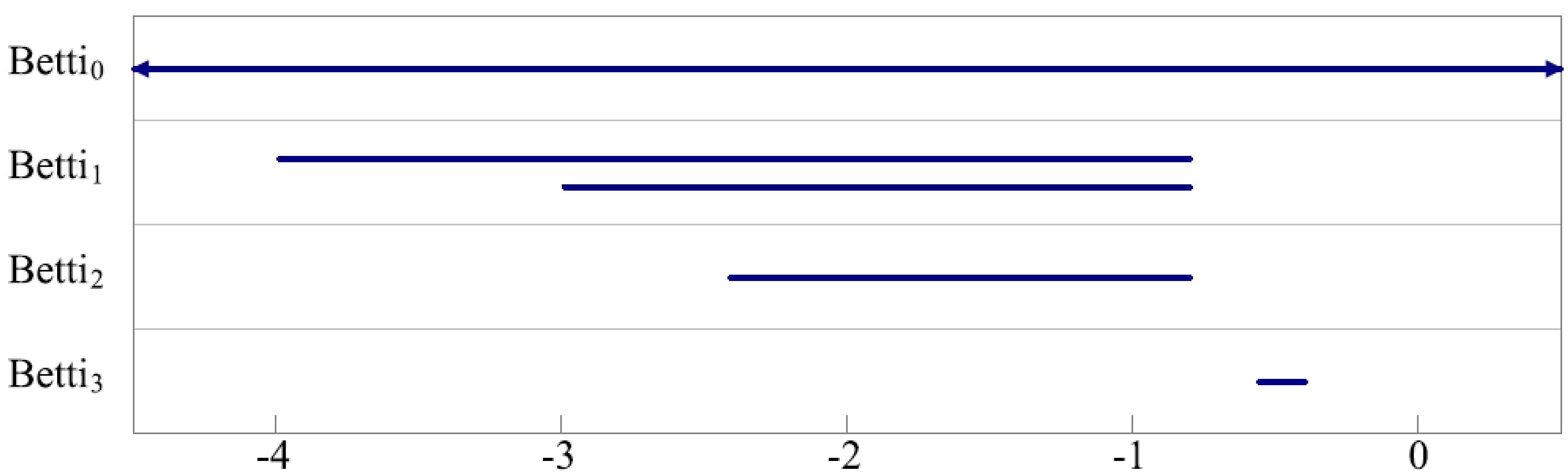
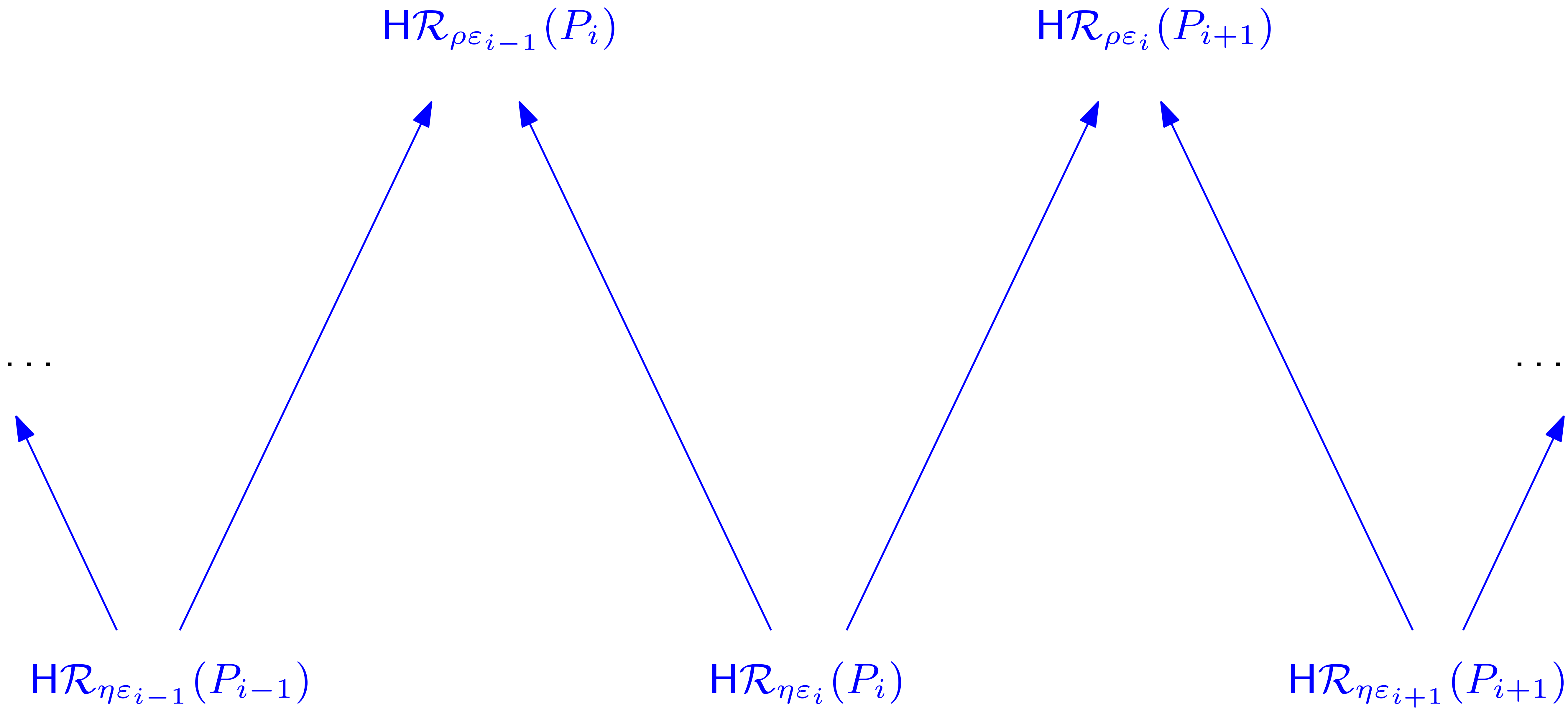


[OS13]

Application 2: Zigzags for homology inference

proof strategy:



Application 2: Zigzags for homology inference

proof strategy:

- not much control over the topological behavior of Rips complexes

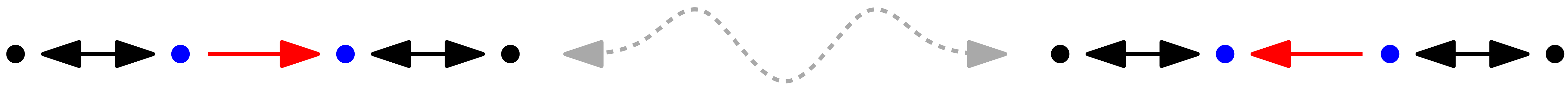
- exploit interleaving with Čech complexes (cf. standard persistence)

- turn oR-ZZ into some Čech-based zigzag while tracking changes in PD

- perform two types of low-level modifications:
    - arrow reversal
    - arrows composition / splitting

Application 2: Zigzags for homology inference

Thm (Arrow Reversal):
"Any arrow in a zigzag module can be reversed while preserving the persistence diagram. The properties of the reverse map also help preserve commutativity."



Thm (Arrow Composition/Splitting):
"Contiguous arrows with same orientation in a zigzag module can be composed, with the same effect on the persistence diagram as in standard persistence."

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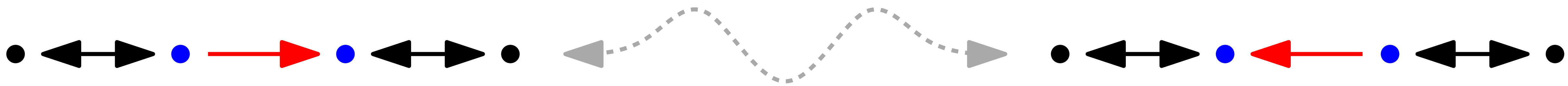

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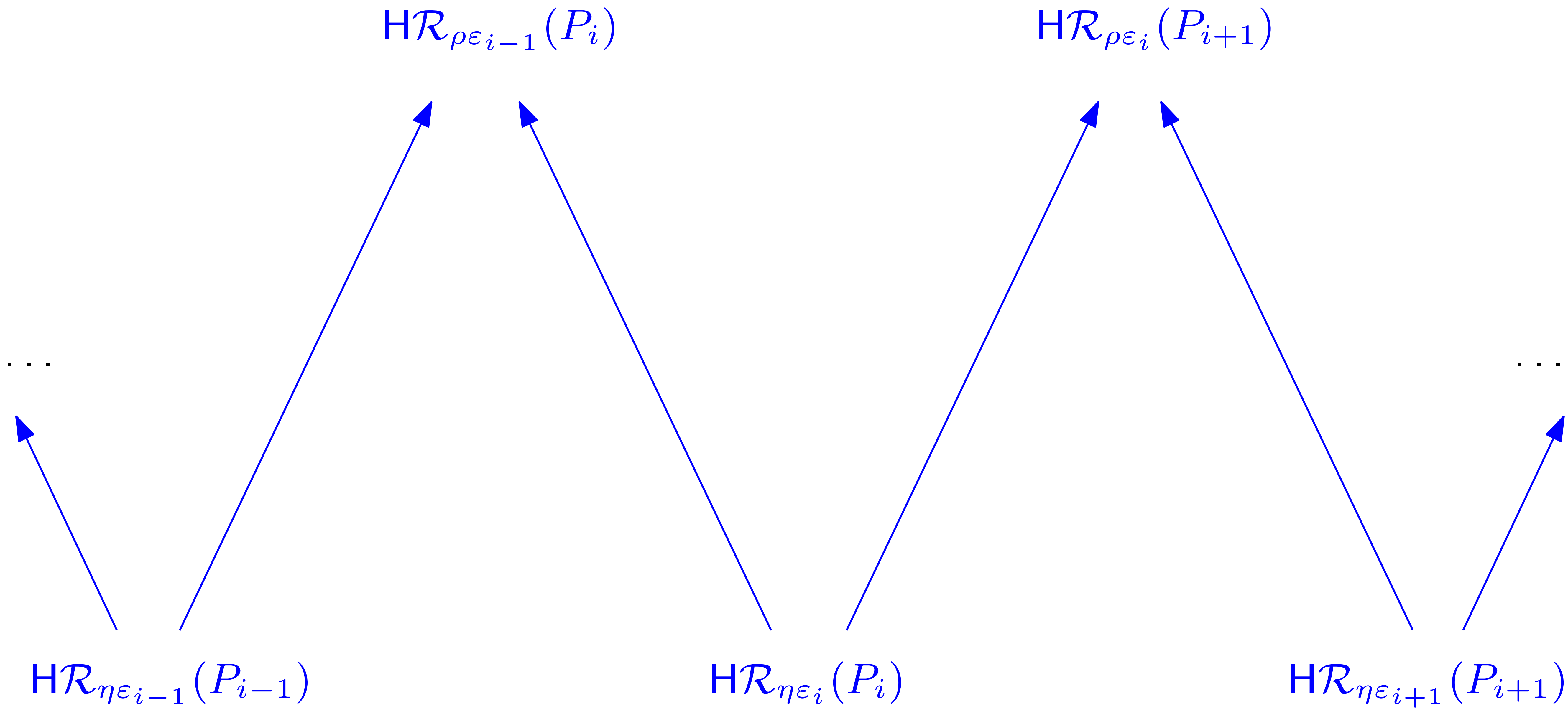
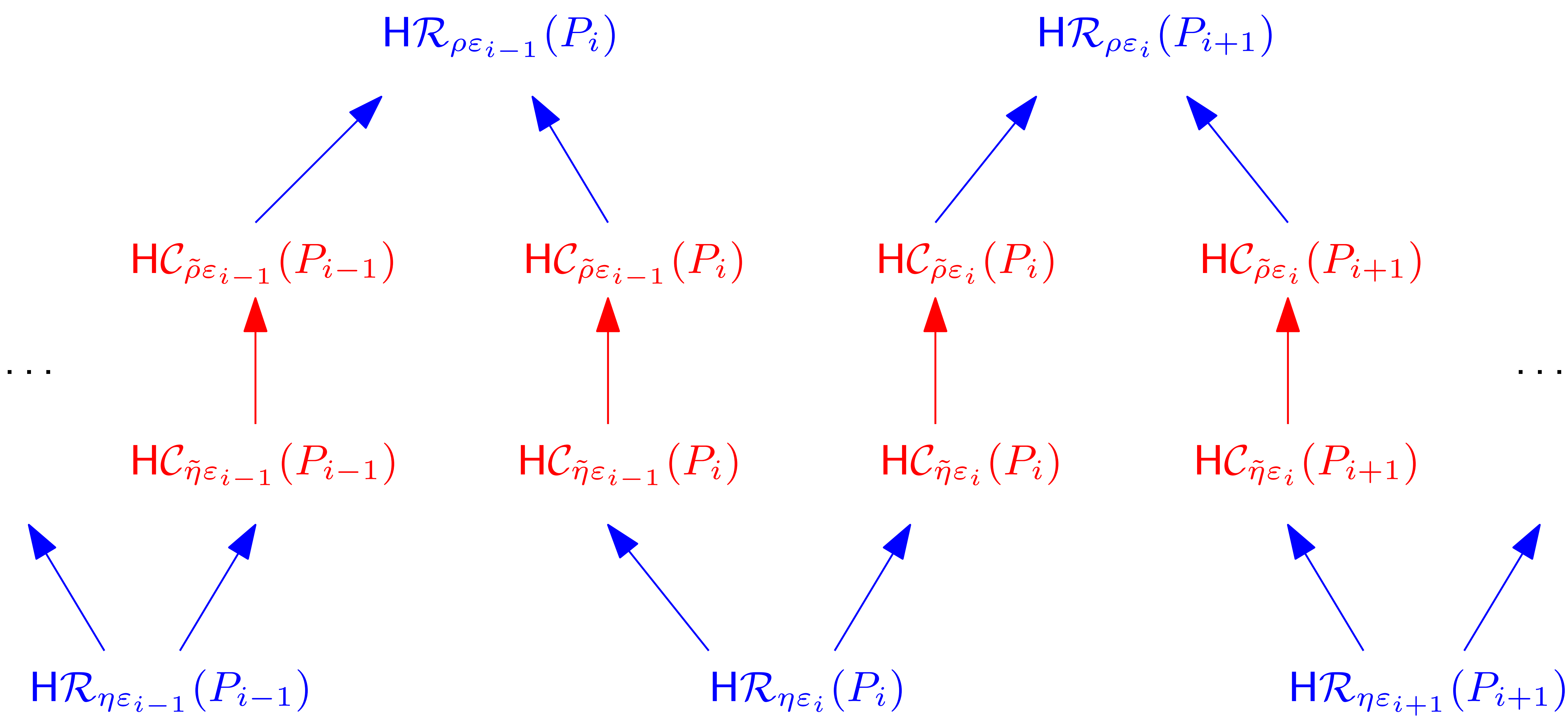
→ proofs by decomposition (use Gabriel's theorem)

Application 2: Zigzags for homology inference



Application 2: Zigzags for homology inference

$\tilde{\eta} = \vartheta_d \eta$ and $\tilde{\rho} = \frac{\rho}{2}$



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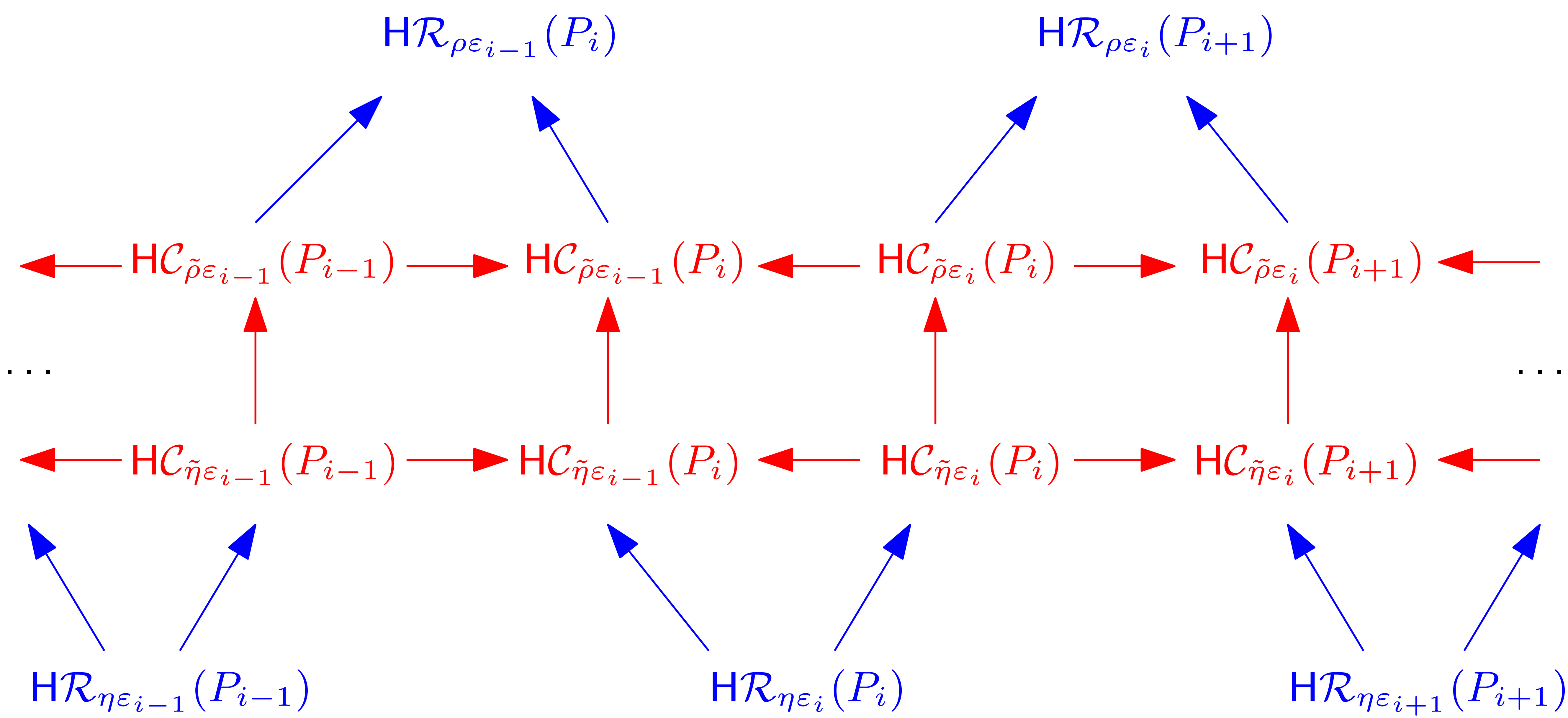

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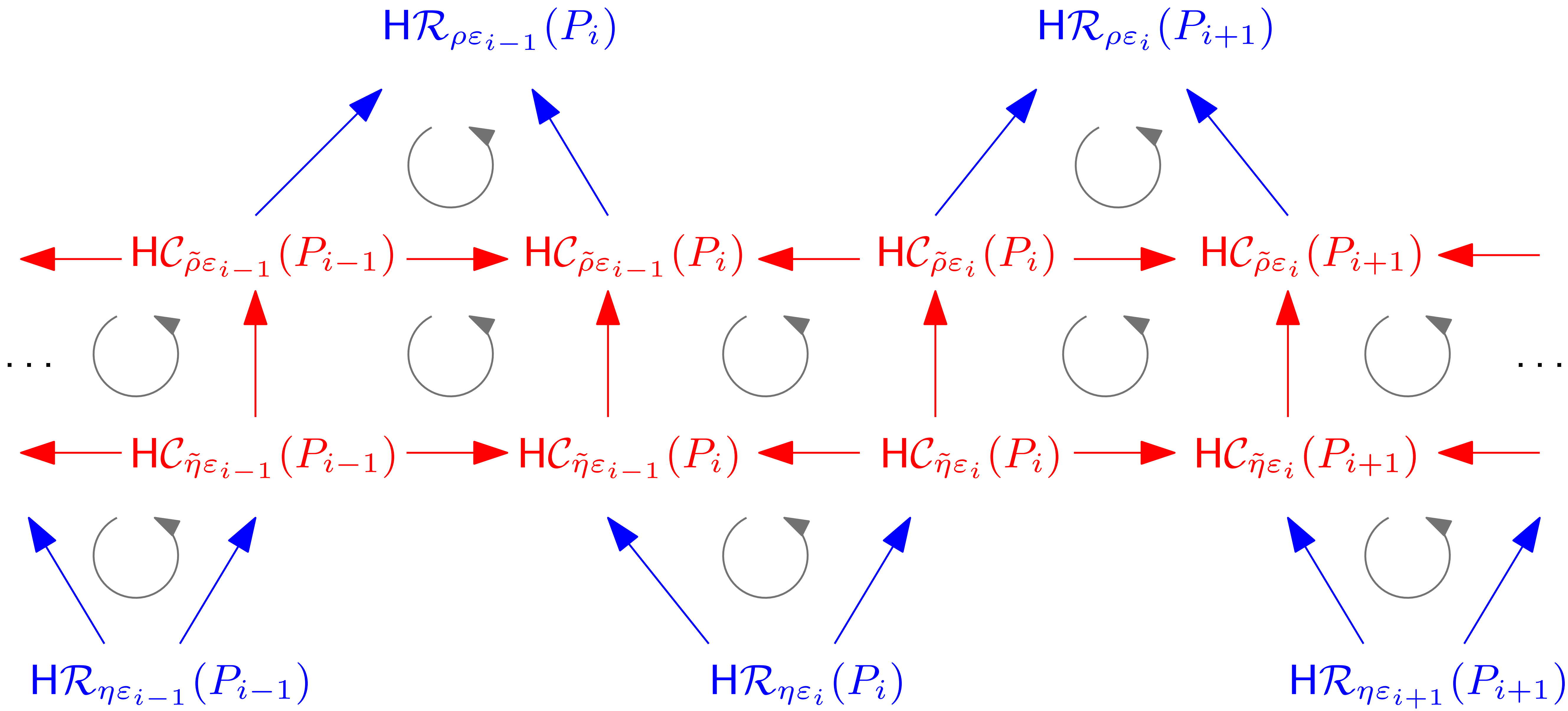

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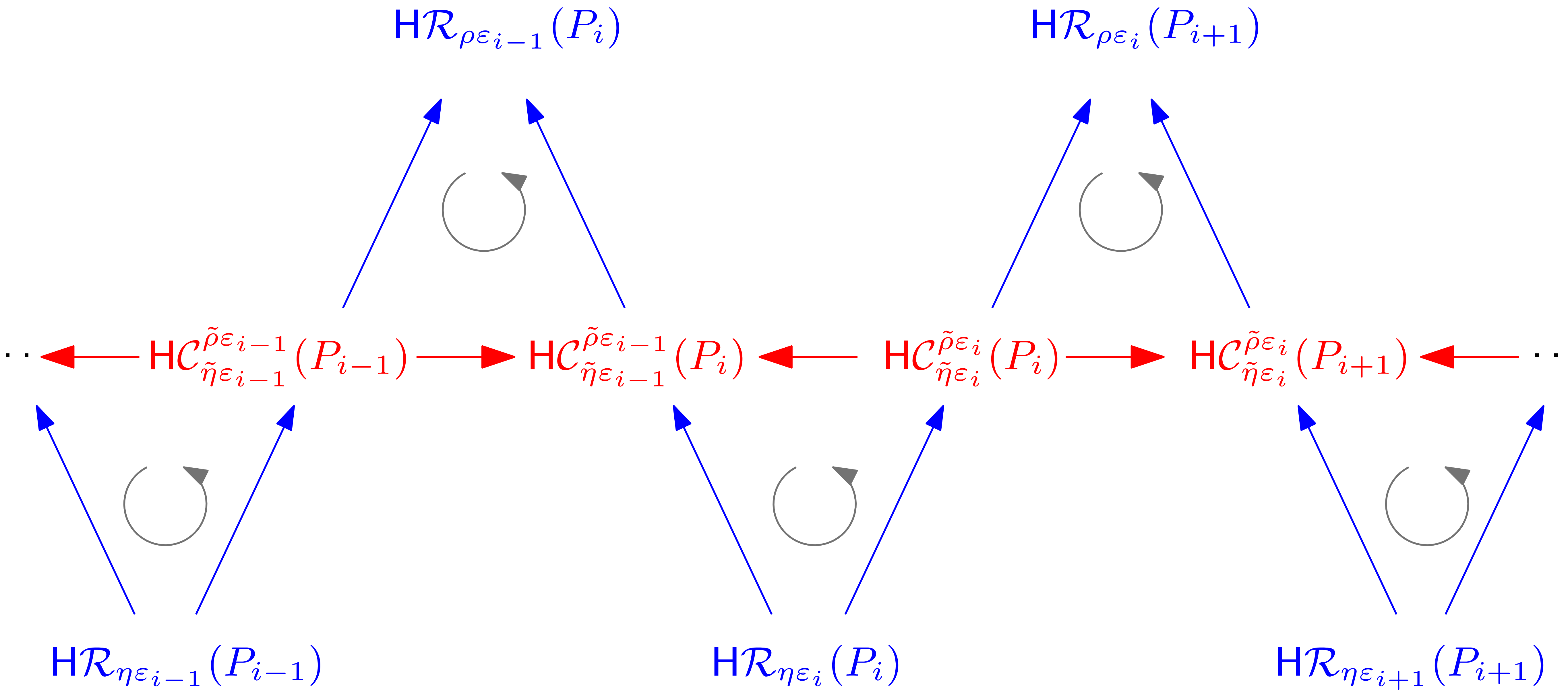

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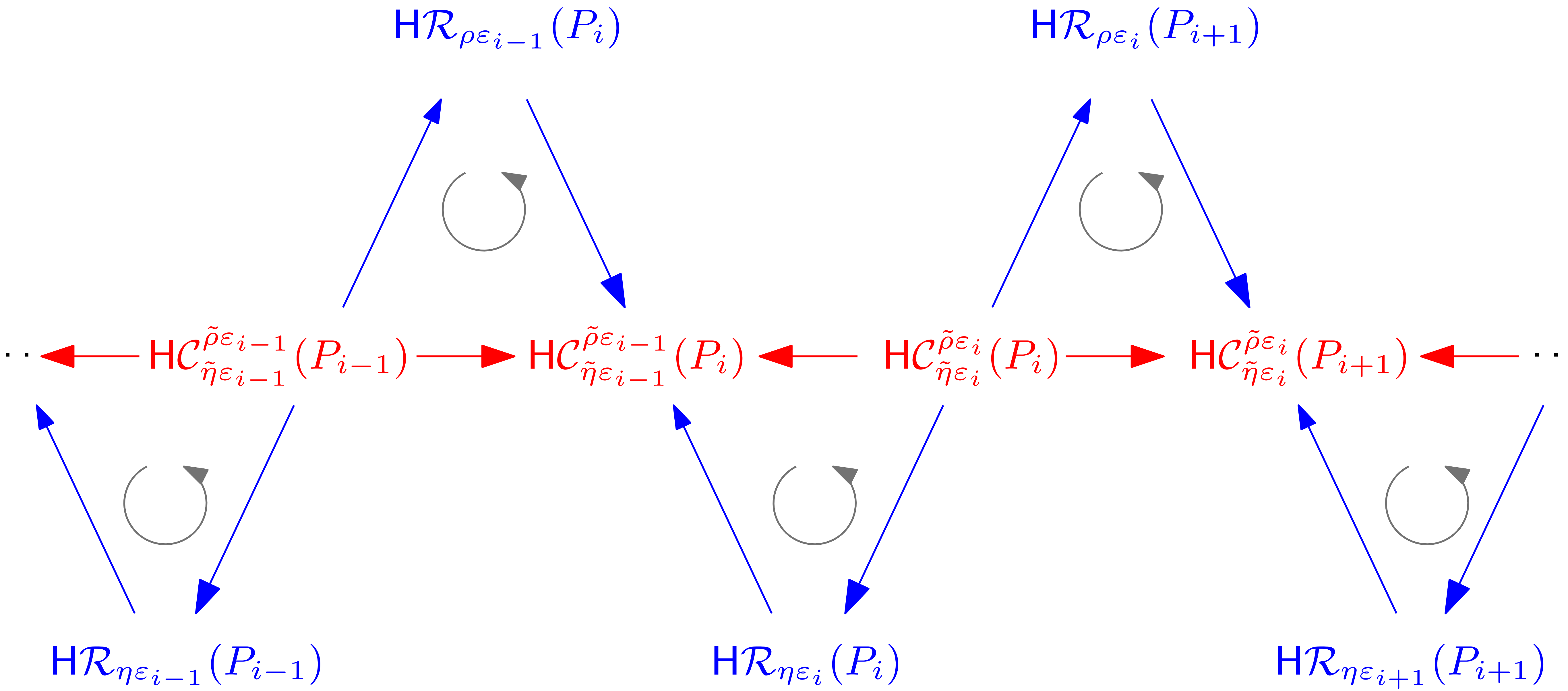

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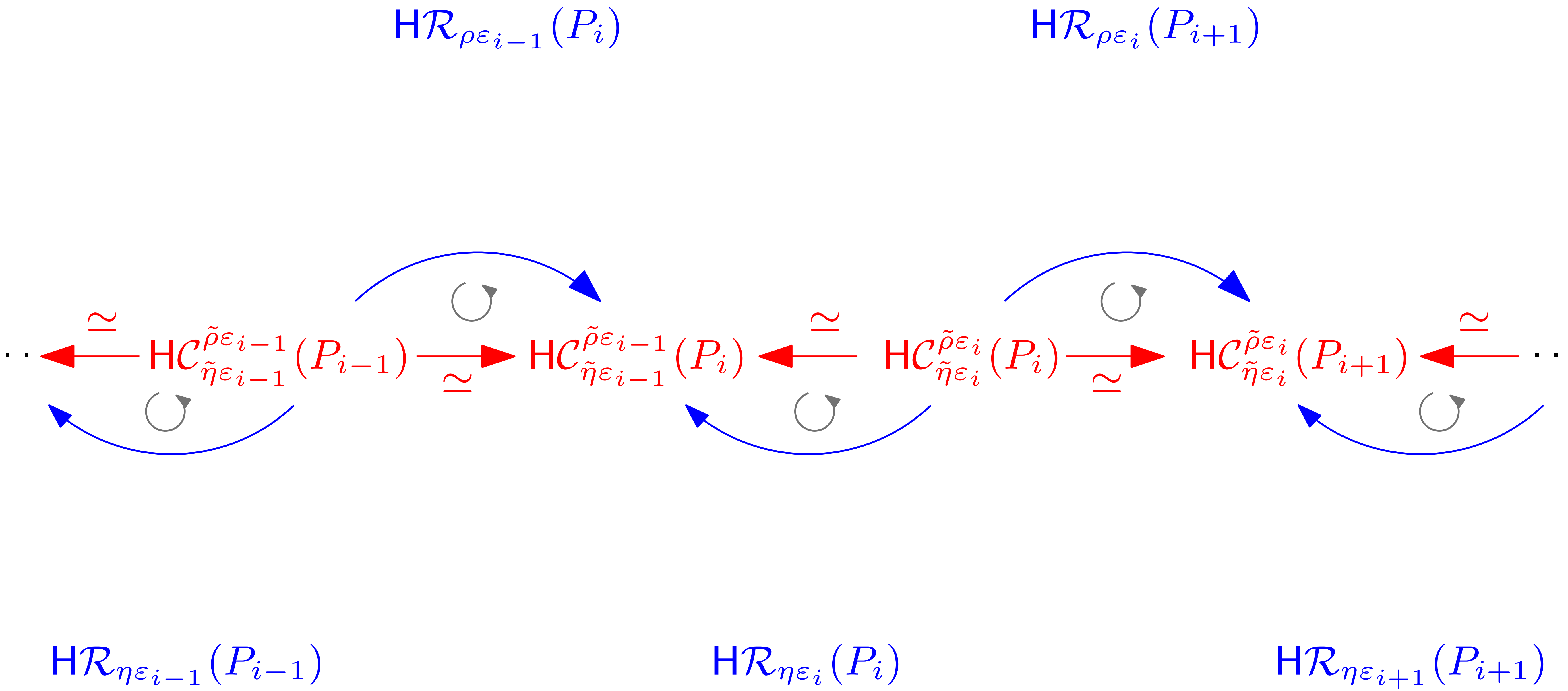

Application 2: Zigzags for homology inference

Complexity bounds

- Assume the ordering of P is by furthest point sampling.

Thm (size bound): Let m be the doubling dimension of (P, d). Then, at any time the number of k-simplices in the current complex is:
- $2^{O(kd \log \rho)}|P|$ for the M-ZZ, oR-ZZ and iR-ZZ of parameters $\rho \geq \eta$,
- $2^{O(kd \log \frac{\rho}{\zeta})}|P|$ for the dM-ZZ of parameters ρ, ζ.

Thm (running time bound): Let m be the doubling dimension of (P, d). Then, the total number of k-simplices inserted in the zigzag is:
- $2^{O(kd \log \rho)}|P|$ for the M-ZZ and iR-ZZ of parameters $\rho \geq \eta$,
- $2^{O(kd \log \frac{\rho}{\zeta})}|P|$ for the dM-ZZ of parameters ρ, ζ,
- $2^{O(kd \log \rho)}|P|^2$ for the oR-ZZ of parameters $\rho \geq \eta$.

Application 2: Zigzags for homology inference

Concluding remarks on this application:

- Rips-based zigzags with the following properties:
  - controlled size and running time
  - improved signal-to-noise ratio in the barcode
  - Analysis based on arrow reflections (similar but $\neq$ from reflection functors)

- Perspectives:
  - to be coupled with efficient algorithms of zigzag persistence: cf. application 1
  - zigzag with other complexes: witness complex, graph-induced complex, etc.