SB-Labelings, Distributivity and Bruhat Order on Sortable Elements

Henri Mühle

LIAFA
Université Paris Diderot

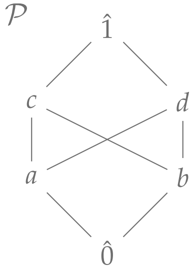
October 08, 2014

Outline

Outline

Basic Notions

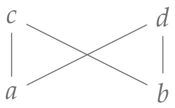
- **bounded poset**



Basic Notions

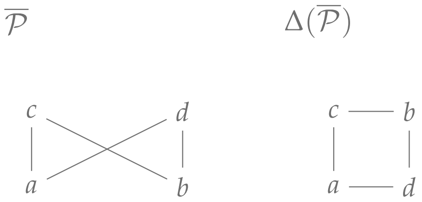
- **proper part**

$\overline{\mathcal{P}}$



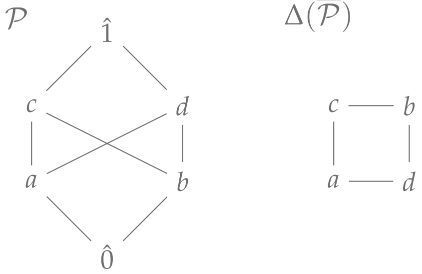
Basic Notions

- **order complex**



Basic Notions

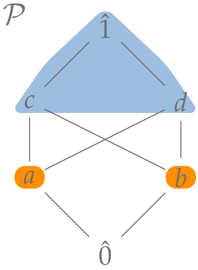
- Philip Hall's Theorem: $\mu_{\mathcal{P}}(\hat{0},\hat{1}) = \tilde{\chi}\big(\Delta(\overline{\mathcal{P}})\big)$ [Hall 1936]



$\mu_{\mathcal{P}}(\hat{0},\hat{1}) = -1$ $\tilde{\chi}\big(\Delta(\overline{\mathcal{P}})\big) = -1 + 4 - 4 = -1$

Basic Notions

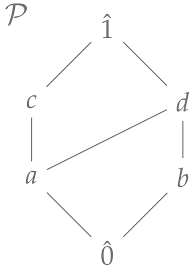
- **lattice**



not a lattice!

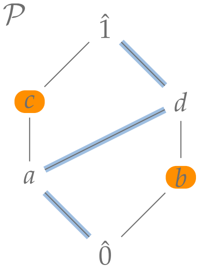
Basic Notions

- **lattice**



Basic Notions

- **crosscut**



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Basic Notions

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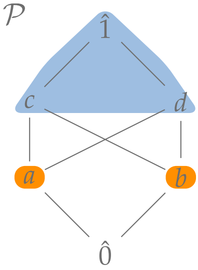

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Basic Notions

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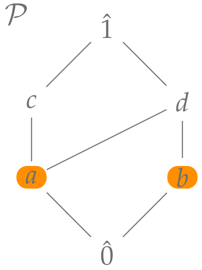
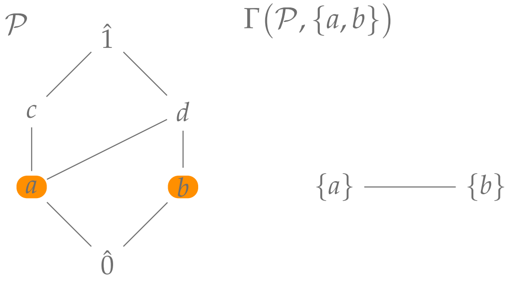


Basic Notions

- **crosscut complex**

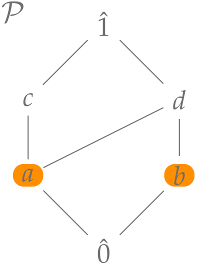


Basic Notions

- Crosscut Theorem: $\Gamma(\mathcal{P}, C) \cong \Delta(\mathcal{P})$

[Rota 1964, Folkman 1966, Björner 1981]

$\mathcal{P}$



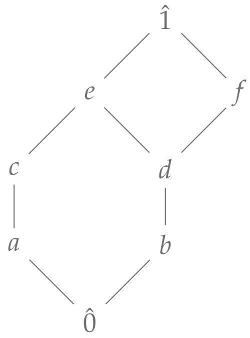
$\Gamma\big(\mathcal{P}, \{a, b\}\big)$

$\mu_{\mathcal{P}}(\hat{0}, \hat{1}) = 0$ $\tilde{\chi}\big(\Gamma(\mathcal{P}, C)\big) = -1 + 2 - 1 = 0$

Outline

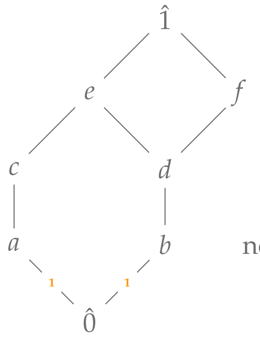
SB-Labelings

- **lower SB-labeling**



SB-Labelings

- **lower SB-labeling**



not allowed!

SB-Labelings

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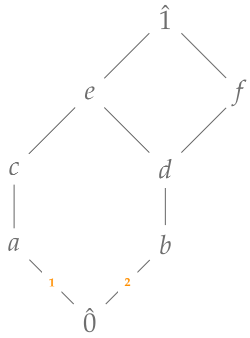

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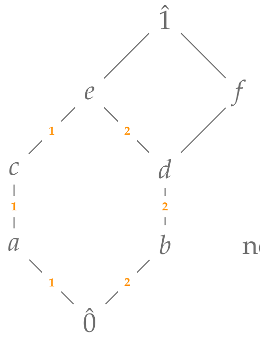

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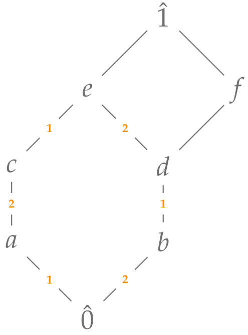
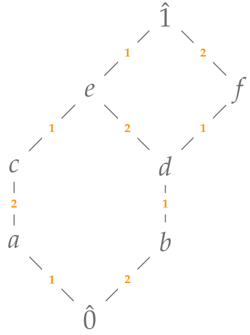
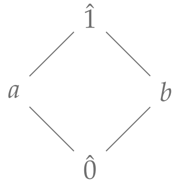


- **SB-labeling**



SB-Labelings

- label forcing on polygonal intervals



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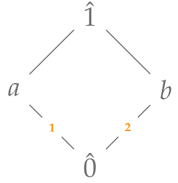

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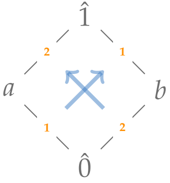

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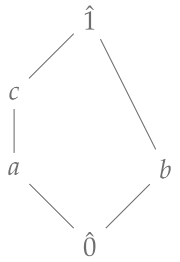

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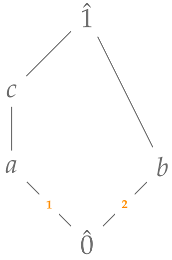

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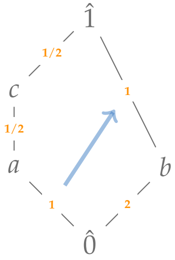


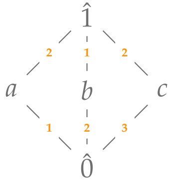
What is it good for?

Theorem (Hersh & Mészáros, 2014)

If $\mathcal{P} = (P, \leq)$ is a lattice that admits a lower SB-labeling, then we have $\mu_{\mathcal{P}}(\hat{0}, p) \in \{-1, 0, 1\}$ for every $p \in P$. Moreover, we have $\mu_{\mathcal{P}}(\hat{0}, p) = (-1)^d$ if and only if p can be expressed as a join of d atoms.

Sketch of Proof

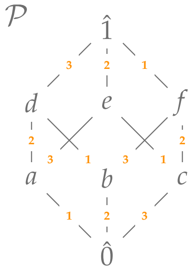
- different subsets of atoms have different joins



No SB-labeling!

Sketch of Proof

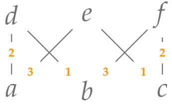
- apply crosscut theorem to atoms



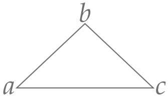
Sketch of Proof

- apply crosscut theorem to atoms

$\overline{\mathcal{P}}$



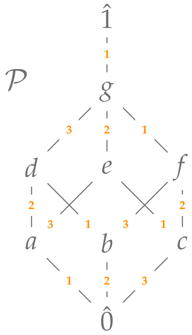
$\Gamma\big(\overline{\mathcal{P}}, \{a,b,c\}\big)$



$$\mu_{\mathcal{P}}(\hat{0},\hat{1}) = \tilde{\chi}\big(\Delta(\overline{\mathcal{P}})\big) = \tilde{\chi}\big(\Gamma(\overline{\mathcal{P}},\{a,b,c\})\big) = -1+3-3 = -1$$

Sketch of Proof

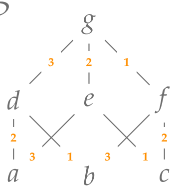
- apply crosscut theorem to atoms



Sketch of Proof

- apply crosscut theorem to atoms

$\overline{\mathcal{P}}$



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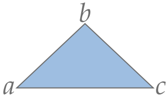


$$\mu_{\mathcal{P}}(\hat{0},\hat{1}) = \tilde{\chi}\big(\Delta(\overline{\mathcal{P}})\big) = \tilde{\chi}\big(\Gamma(\overline{\mathcal{P}}, \{a,b,c\})\big) = -1+3-3+1 = 0$$

The Consequence

Corollary (Hersh & Mészáros, 2014)

If $\mathcal{P}$ is a lattice that admits an SB-labeling, then every interval of $\mathcal{P}$ is homotopic to either a sphere or a ball.

Applications

Theorem (Hersh & Mészáros, 2014)

Distributive lattices admit an SB-labeling.

labeling: $(A, B) \mapsto B \setminus A$, where A, B are order ideals

Applications

Theorem (Hersh & Mészáros, 2014)

The weak order on a Coxeter group admits an SB-labeling.

labeling: $(u,v) \mapsto u^{-1}v$

Applications

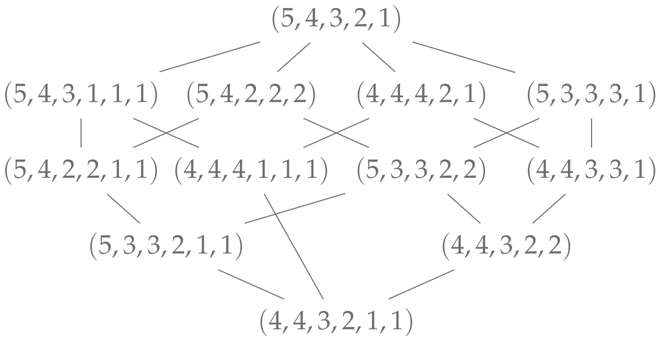
Theorem (Hersh & Mészáros, 2014)

The Tamari lattices admit an SB-labeling.

labeling: $\big((ab)c, a(bc)\big) \mapsto s$, where s is the rightmost letter in b

A Non-Example

- dominance order on integer partitions



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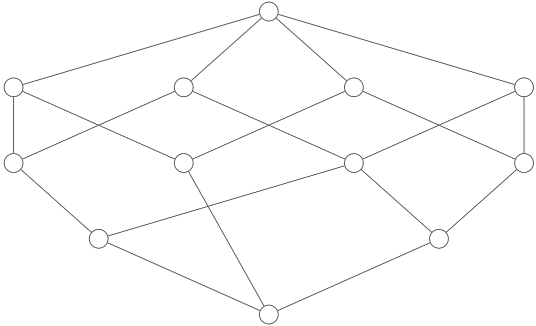

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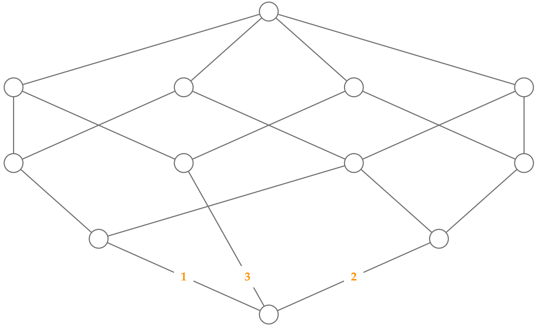

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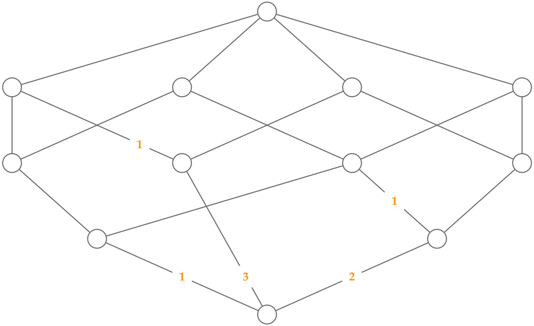

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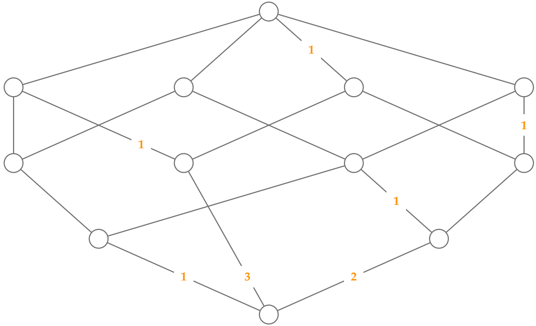

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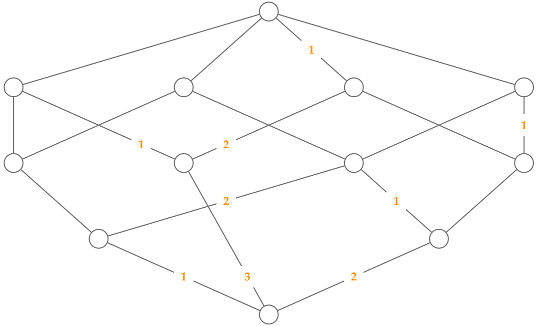

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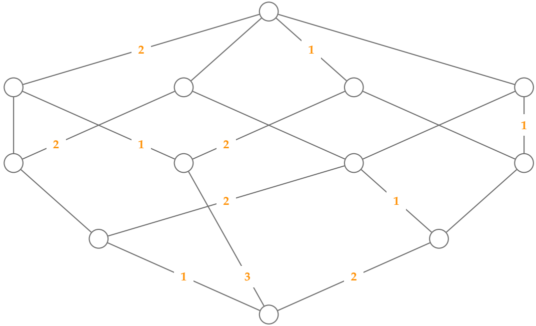

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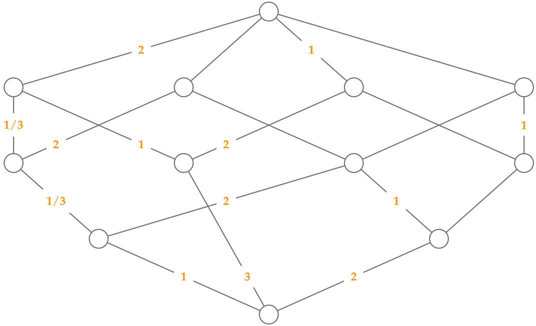

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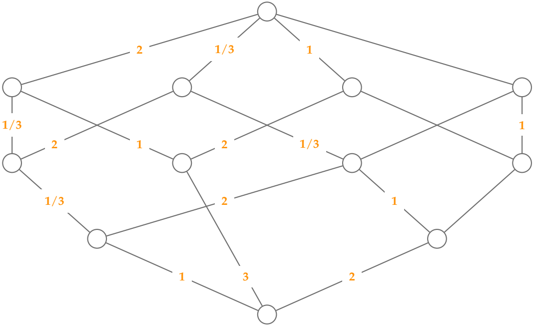

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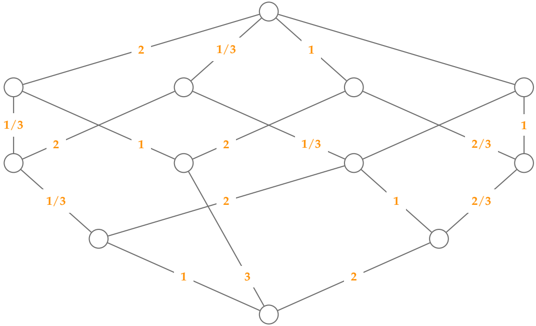

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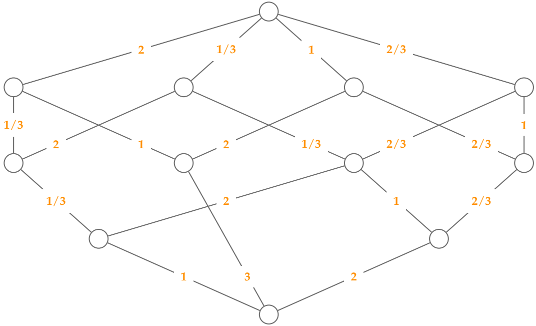

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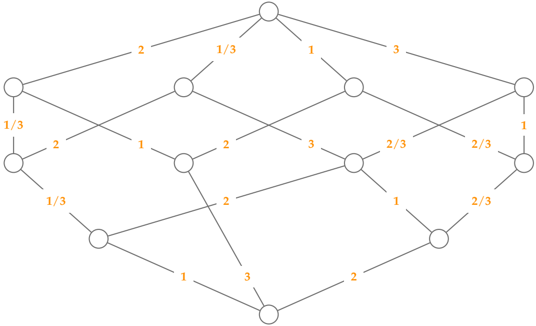

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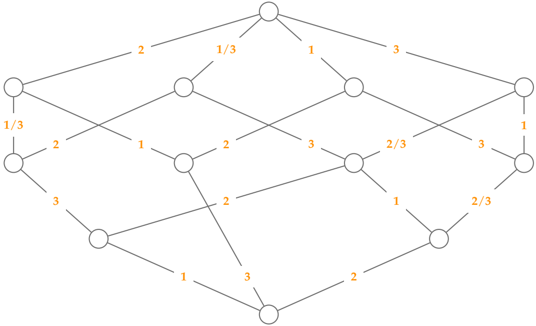

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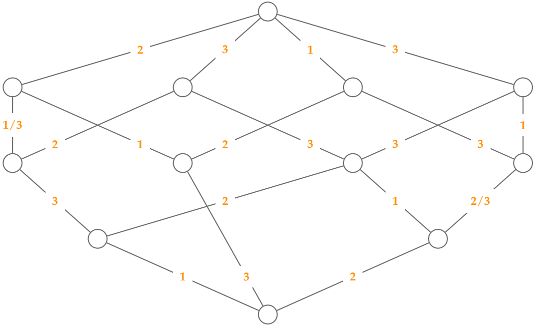

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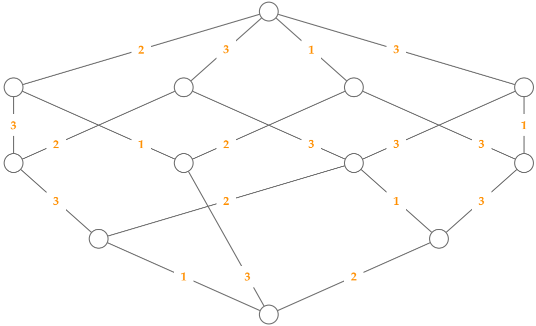

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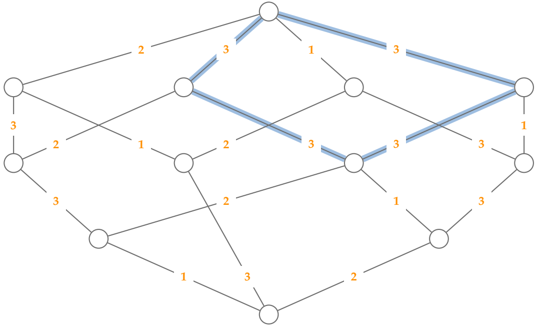


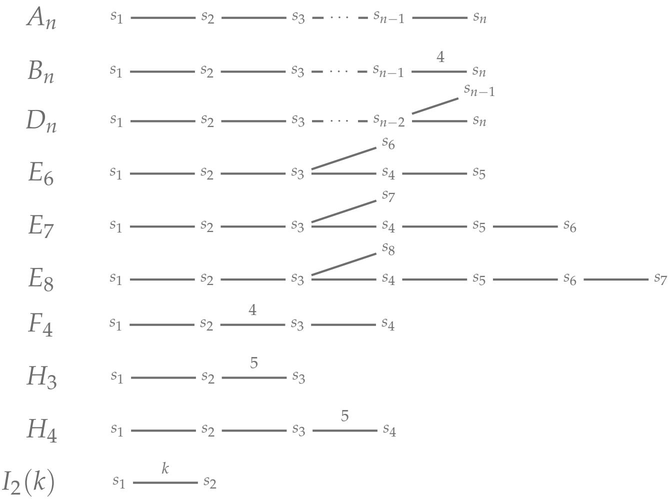
Outline

Coxeter Groups

- **Coxeter group**

Coxeter Groups

- the finite Coxeter groups



Coxeter Groups

- the coincidental types

A_n $\quad s_1$ — s_2 — s_3 - $\cdots$ - s_{n-1} — s_n

B_n $\quad s_1$ — s_2 — s_3 - $\cdots$ - s_{n-1} $\overset{4}{—}$ s_n

H_3 $\quad s_1$ — s_2 $\overset{5}{—}$ s_3

$I_2(k)$ $\quad s_1$ $\overset{k}{—}$ s_2

- **Coxeter element**

$$\gamma = s_4 s_3 s_1 s_2 s_5$$

Sortable Elements

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^{\infty} = s_4s_1s_3s_2s_5 | s_4s_1s_3s_2s_5 | s_4s_1s_3s_2s_5 | s_4s_1s_3s_2s_5$$

Sortable Elements

- **γ-sorting word**

$\mathbf{s_4 s_3 s_1 s_2 s_4 s_1}$	$s_4 s_3 s_1 s_2 s_1 s_4$	$s_4 s_3 s_1 s_4 s_2 s_1$	$s_4 s_3 s_4 s_1 s_2 s_1$
$s_4 s_1 s_3 s_2 s_4 s_1$	$s_4 s_1 s_3 s_2 s_1 s_4$	$s_4 s_1 s_3 s_4 s_2 s_1$	$s_1 s_4 s_3 s_2 s_4 s_1$
	$s_1 s_4 s_3 s_2 s_1 s_4$	$s_1 s_4 s_3 s_4 s_2 s_1$	

$$\gamma^\infty = \mathbf{s_4} s_1 \mathbf{s_3} s_2 s_5 \,|\, s_4 \mathbf{s_1} s_3 \mathbf{s_2} s_5 \,|\, \mathbf{s_4 s_1} s_3 s_2 s_5 \,|\, s_4 s_1 s_3 s_2 s_5$$

Sortable Elements

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$\mathbf{s_4s_3s_1s_2s_1s_4}$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^\infty = \mathbf{s_4}s_1\mathbf{s_3}s_2s_5 | s_4\mathbf{s_1}s_3\mathbf{s_2}s_5 | s_4\mathbf{s_1}s_3s_2s_5 | \mathbf{s_4}s_1s_3s_2s_5$$

Sortable Elements

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$\mathbf{s_4s_3s_1s_4s_2s_1}$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^{\infty} = \mathbf{s_4}s_1\mathbf{s_3}s_2s_5 \,|\, s_4\mathbf{s_1}s_3s_2s_5 \,|\, \mathbf{s_4}s_1s_3\mathbf{s_2}s_5 \,|\, s_4\mathbf{s_1}s_3s_2s_5$$

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$\mathbf{s_4s_3s_4s_1s_2s_1}$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^{\infty} = \mathbf{s_4}s_1\mathbf{s_3}s_2s_5|\mathbf{s_4}\mathbf{s_1}s_3\mathbf{s_2}s_5|s_4\mathbf{s_1}s_3s_2s_5|s_4s_1s_3s_2s_5$$

Sortable Elements

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$\mathbf{s_4s_1s_3s_2s_4s_1}$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^{\infty} = \mathbf{s_4s_1s_3s_2}s_5 \,|\, \mathbf{s_4s_1}s_3s_2s_5 \,|\, s_4s_1s_3s_2s_5 \,|\, s_4s_1s_3s_2s_5$$

Sortable Elements

- **γ-sorting word**

$$s_1 \xrightarrow{4} s_2 \longleftarrow s_3 \xleftarrow{\infty} s_4 \xrightarrow{5} s_5$$

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$\mathbf{s_4s_1s_3s_2s_1s_4}$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^\infty = \mathbf{s_4s_1s_3s_2}s_5 \mid s_4\mathbf{s_1}s_3s_2s_5 \mid \mathbf{s_4}s_1s_3s_2s_5 \mid s_4s_1s_3s_2s_5$$

Sortable Elements

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$\mathbf{s_4s_1s_3s_4s_2s_1}$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^{\infty} = \mathbf{s_4s_1s_3}s_2s_5 \mid \mathbf{s_4}s_1s_3\mathbf{s_2}s_5 \mid s_4\mathbf{s_1}s_3s_2s_5 \mid s_4s_1s_3s_2s_5$$

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$\mathbf{s_1s_4s_3s_2s_4s_1}$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^{\infty} = s_4\mathbf{s_1}s_3s_2s_5 \,|\, \mathbf{s_4}s_1\mathbf{s_3}\mathbf{s_2}s_5 \,|\, \mathbf{s_4}\mathbf{s_1}s_3s_2s_5 \,|\, s_4s_1s_3s_2s_5$$

Sortable Elements

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$\mathbf{s_1s_4s_3s_2s_1s_4}$	$s_1s_4s_3s_4s_2s_1$	

$$\gamma^{\infty} = s_4\mathbf{s_1}s_3s_2s_5 \,|\, \mathbf{s_4}s_1\mathbf{s_3}\mathbf{s_2}s_5 \,|\, s_4\mathbf{s_1}s_3s_2s_5 \,|\, \mathbf{s_4}s_1s_3s_2s_5$$

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$s_4s_1s_3s_2s_4s_1$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$\mathbf{s_1s_4s_3s_4s_2s_1}$	

$$\gamma^\infty = s_4\mathbf{s_1}s_3s_2s_5 \mid \mathbf{s_4}s_1\mathbf{s_3}s_2s_5 \mid \mathbf{s_4}s_1s_3\mathbf{s_2}s_5 \mid s_4\mathbf{s_1}s_3s_2s_5$$

Sortable Elements

- **γ-sorting word**

$s_4s_3s_1s_2s_4s_1$	$s_4s_3s_1s_2s_1s_4$	$s_4s_3s_1s_4s_2s_1$	$s_4s_3s_4s_1s_2s_1$
$\mathbf{s_4s_1s_3s_2s_4s_1}$	$s_4s_1s_3s_2s_1s_4$	$s_4s_1s_3s_4s_2s_1$	$s_1s_4s_3s_2s_4s_1$
	$s_1s_4s_3s_2s_1s_4$	$s_1s_4s_3s_4s_2s_1$	

Sortable Elements

- **γ-sortable element**

$s_4s_1s_3s_2|s_4s_1 \rightsquigarrow \{s_1,s_2,s_3,s_4\} \supseteq \{s_1,s_4\}$ sortable!

$s_4s_1s_3s_2|s_4s_1s_5 \rightsquigarrow \{s_1,s_2,s_3,s_4\} \not\supseteq \{s_1,s_4,s_5\}$ not sortable!

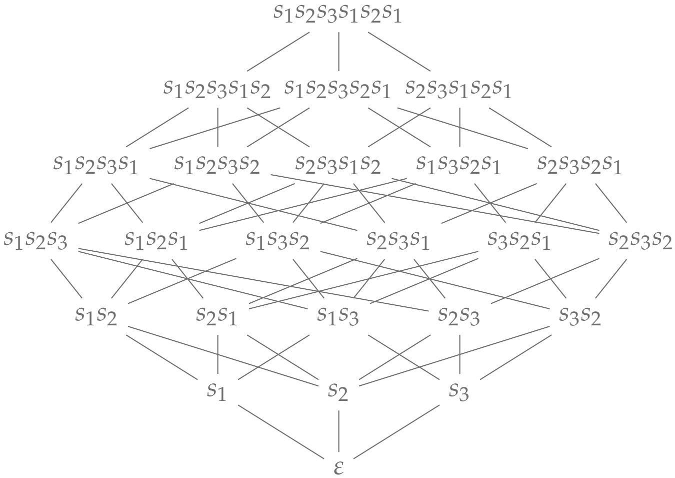
Bruhat Order

- **Bruhat order**

$$s_2s_1s_3s_2 \leq_B s_1s_2s_1s_3s_2$$
$$s_2s_1s_3s_2 \not\leq_B s_3s_2s_1s_2s_3$$

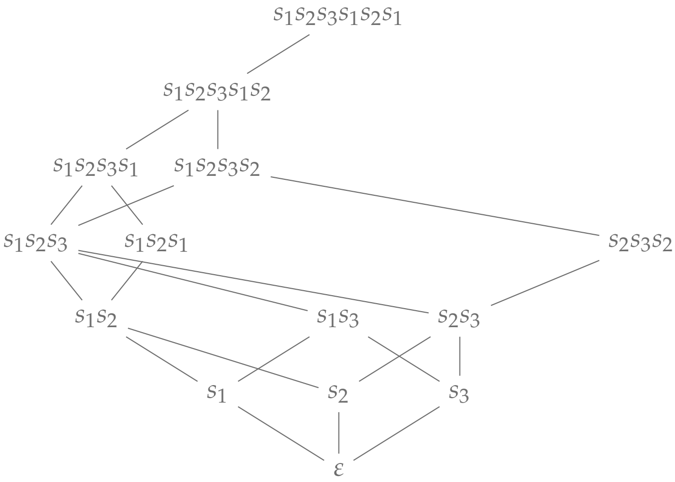
Bruhat Order

- **Bruhat order**: $W = A_3$



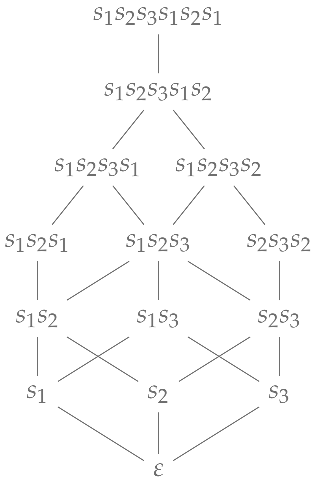
Bruhat Order

- **Bruhat order**: $W = A_3$, $\gamma = s_1s_2s_3$



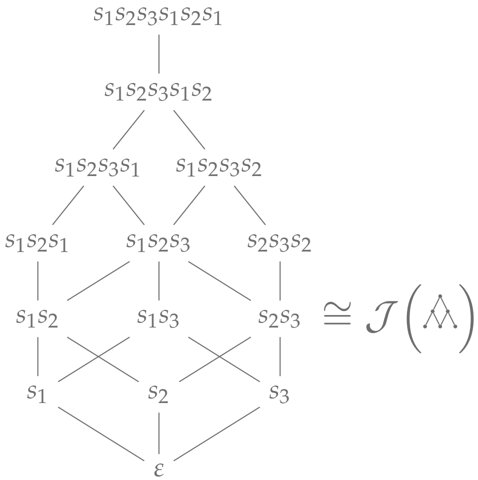
Bruhat Order

- **Bruhat order**: $W = A_3$, $\gamma = s_1s_2s_3$



Bruhat Order

- **Bruhat order**: $W = A_3$, $\gamma = s_1 s_2 s_3$



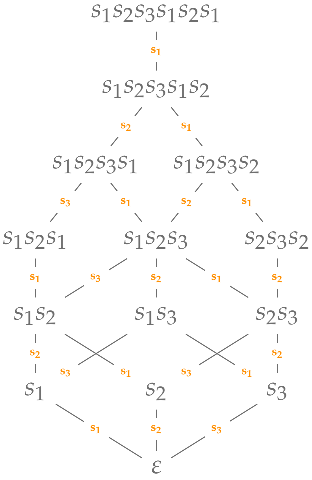
A First Result

Theorem (, 2014)

For any Coxeter group W and any Coxeter element $\gamma \in W$, the Bruhat order on the γ-sortable elements of W admits an SB-labeling.

labeling: $(u, v) \mapsto u^{-1}v$

An Example


$s_1s_2s_3s_1s_2s_1$
s_1
$s_1s_2s_3s_1s_2$
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$s_1s_2s_3s_1$
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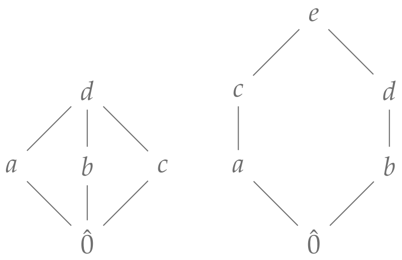
A More General Setting

Theorem (Armstrong, 2009)

For any Coxeter group and any Coxeter element $\gamma \in W$, the Bruhat order on γ-sortable elements is a join-distributive lattice.

A More General Setting

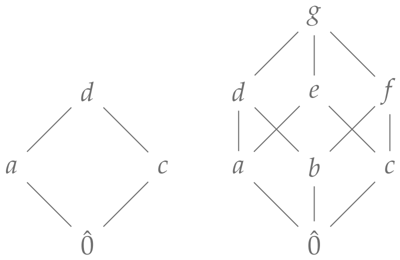
- **join-distributive lattice**



not allowed!

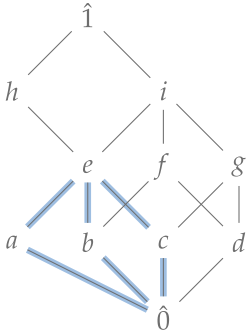
A More General Setting

- **join-distributive lattice**



A More General Setting

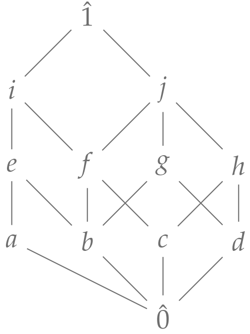
- **join-distributive lattice**



not allowed!

A More General Setting

- **join-distributive lattice**



A More General Setting

- **antimatroid**

a pair $(M, \mathcal{F})$, where

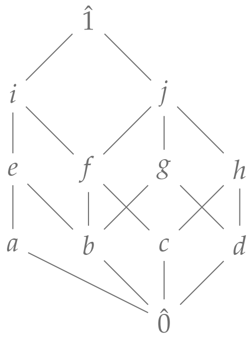
- $\mathcal{F} \subseteq \wp(M)$ ... **feasible sets**
- for $A \in \mathcal{F}$ exists $x \in A$ such that $A \setminus \{x\} \in \mathcal{F}$
- if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$

A More General Setting

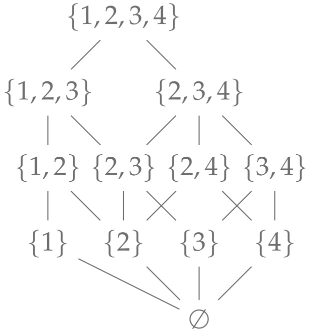
Theorem (Edelman, 1980)

A lattice $\mathcal{P}$ is join-distributive if and only if there exists an antimatroid $(M, \mathcal{F})$ with $\mathcal{P} \cong (\mathcal{F}, \subseteq)$.

A More General Setting



A More General Setting



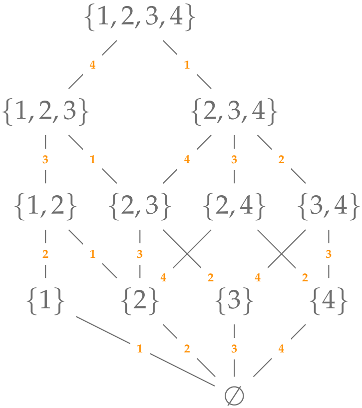
A More General Result

Theorem (, 2014)

Every join-distributive lattice admits an SB-labeling.

labeling: $(A, B) \mapsto B \setminus A$, where A, B are feasible sets

A More General Result



Further Candidates

- semidistributive lattices [Folklore]
- trim lattices [Thomas, 2006]
- Cambrian semilattices Reading [2006], Reading & Speyer [2011]

Recent Progress [McConville, 2014]: Crosscut Simplicial Lattices

- semidistributive lattices [Folklore]
- trim lattices [Thomas, 2006]
- Cambrian semilattices Reading [2006], Reading & Speyer [2011]

Outline

Distributive Bruhat Lattices

- $\mathcal{B}_{s_1s_2s_3}(A_3) \cong \mathcal{J}\Big($  $\Big)$
- $\mathcal{B}_{s_1s_2}(I_2(6)) \cong \mathcal{B}_{s_2s_1}(I_2(6)) \cong \mathcal{J}\Big($ $\Big)$
- $\mathcal{B}_{s_1s_2s_3}(B_3) \cong \mathcal{J}\Big($ $\Big)$
- $\mathcal{B}_{s_1s_2s_3}(H_3) \cong \mathcal{J}\Big($ $\Big)$
- any more?

Distributive Bruhat Lattices

- $\mathcal{B}_{s_1 s_2 s_3}(A_3) \cong \mathcal{J}\Big($  $\Big)$
- $\mathcal{B}_{s_1 s_2}\big(I_2(6)\big) \cong \mathcal{B}_{s_2 s_1}\big(I_2(6)\big) \cong \mathcal{J}\Big($  $\Big)$
- $\mathcal{B}_{s_1 s_2 s_3}(B_3) \cong \mathcal{J}\Big($ $\Big)$
- $\mathcal{B}_{s_1 s_2 s_3}(H_3) \cong \mathcal{J}\Big($ $\Big)$
- any more?

Distributive Bruhat Lattices

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- $\mathcal{B}_{s_1s_2}\big(I_2(6)\big) \cong \mathcal{B}_{s_2s_1}\big(I_2(6)\big) \cong \mathcal{J}\big($  $\big)$
- $\mathcal{B}_{s_1s_2s_3}(B_3) \cong \mathcal{J}\big($  $\big)$
- $\mathcal{B}_{s_1s_2s_3}(H_3) \cong \mathcal{J}\big($ $\big)$
- any more?

Distributive Bruhat Lattices

- $\mathcal{B}_{s_1s_2s_3}(A_3) \cong \mathcal{J}\Bigl($  $\Bigr)$
- $\mathcal{B}_{s_1s_2}\bigl(I_2(6)\bigr) \cong \mathcal{B}_{s_2s_1}\bigl(I_2(6)\bigr) \cong \mathcal{J}\Bigl($  $\Bigr)$
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Distributive Bruhat Lattices

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- any more?

Distributive Bruhat Lattices

- $\mathcal{B}_{s_1s_2s_3}(A_3) \cong \mathcal{J}\Big($  $\Big)$
- $\mathcal{B}_{s_1s_2}\big(I_2(6)\big) \cong \mathcal{B}_{s_2s_1}\big(I_2(6)\big) \cong \mathcal{J}\Big($  $\Big)$
- $\mathcal{B}_{s_1s_2s_3}(B_3) \cong \mathcal{J}\Big($  $\Big)$
- $\mathcal{B}_{s_1s_2s_3}(H_3) \cong \mathcal{J}\Big($  $\Big) \not\cong \mathcal{J}\Big($  $\Big)$
- any more?

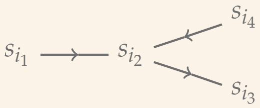
A First Observation ...

Proposition (, 2014)

Let W be a Coxeter group, and let $\gamma \in W$ be a Coxeter element. If the (oriented) Coxeter diagram of W contains one of the following induced subgraphs, then the Bruhat lattice on γ-sortable elements is not distributive:

(i) $s_{i_1} \overset{a}{\longleftarrow} s_{i_2} \overset{b}{\longrightarrow} s_{i_3}$ *for* $a,b \geq 3$

(ii) $s_{i_1} \longleftarrow s_{i_2} \overset{a}{\longleftarrow} s_{i_3}$ *for* $a \geq 4$

(iii) 

A First Observation ...

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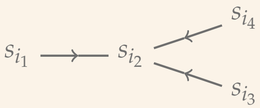
(iv) 

(v) $s_{i_1} \rightarrow s_{i_2} \overset{a}{\rightarrow} s_{i_3} \leftarrow s_{i_4}$ *for* $a \geq 4$,

(vi) $s_{i_1} \rightarrow s_{i_2} \rightarrow s_{i_3} \overset{a}{\rightarrow} s_{i_4}$ *for* $a \geq 5$

(vii) $s_{i_1} \rightarrow s_{i_2} \rightarrow s_{i_3} \overset{a}{\leftarrow} s_{i_4}$ *for* $a \geq 5$.

... and its Consequences

Corollary (, 2014)

If $W = D_n$ for $n \geq 4$, or $W \in \{E_6, E_7, E_8, F_4, H_4\}$, then for any Coxeter element $\gamma \in W$, the Bruhat lattice on the γ-sortable elements of W is not distributive.

... and its Consequences

Corollary (, 2014)

There exists a Coxeter element $\gamma \in W$ such that the Bruhat order on the γ-sortable elements of W is distributive if and only if W is of coincidental type.

A Conjecture

Conjecture (, 2014)

If W is finite, then the list of forbidden subgraphs in the previous proposition is exhaustive, i.e. if the (oriented) Coxeter graph of W does not contain any of the induced subgraphs given there, then the Bruhat order on the γ-sortable elements of W is distributive.

Thank You.

Matroids and Antimatroids

- **closure operator**

a map $\tau : \wp(M) \to \wp(M)$, where

- $A \subseteq \tau(A)$
- $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$
- $\tau\big(\tau(A)\big) = \tau(A)$

Matroids and Antimatroids

- **exchange axiom**

If $x, y \notin \tau(A)$, then $x \in \tau\big(A \cup \{y\}\big)$ implies $y \in \tau\big(A \cup \{x\}\big)$.

("linear span")

Matroids and Antimatroids

- **matroid**

a pair $(M,\mathcal{I})$, where

- $\mathcal{I} \subseteq \wp(M)$ ... **independent sets**
- if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$
- if $A, B \in \mathcal{I}$ with $|B| < |A|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$

Matroids and Antimatroids

- matroids vs. exchange axiom

Theorem (Crapo & Rota, 1970)

A closure operator τ satisfies the exchange axiom if and only if there exists a matroid $(M, \mathcal{I})$ such that the sets closed under τ are precisely the independent sets.

Matroids and Antimatroids

- **antiexchange axiom**

If $x, y \notin \tau(A)$, then $x \in \tau\big(A \cup \{y\}\big)$ implies $y \notin \tau\big(A \cup \{x\}\big)$.

("convex hull")

Matroids and Antimatroids

- **antimatroid**

a pair $(M, \mathcal{F})$, where

- $\mathcal{I} \subseteq \wp(M)$ ... **feasible sets**
- for $A \in \mathcal{F}$ exists $x \in A$ such that $A \setminus \{x\} \in \mathcal{F}$
- if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$

Matroids and Antimatroids

- antimatroids vs. antiexchange axiom

Theorem (Korte, Lovász & Schrader, 1991)

A closure operator τ satisfies the antiexchange axiom if and only if there exists an antimatroid $(M, \mathcal{F})$ such that the sets closed under τ are precisely the complements of the feasible sets.