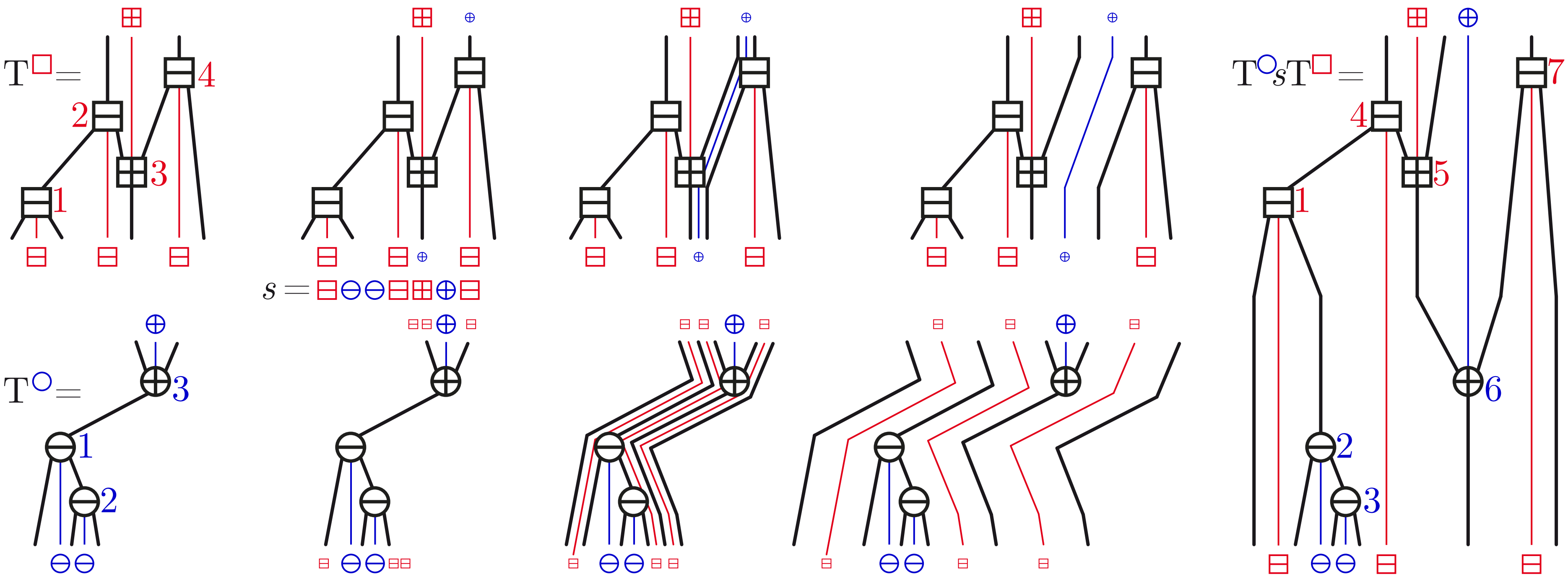
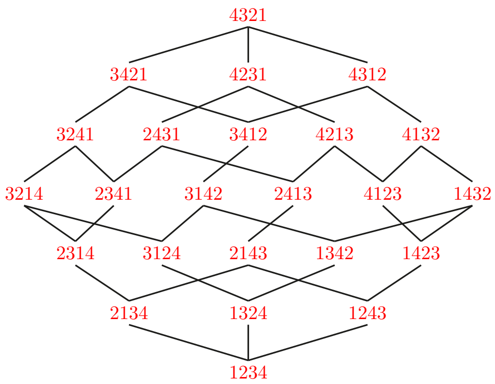
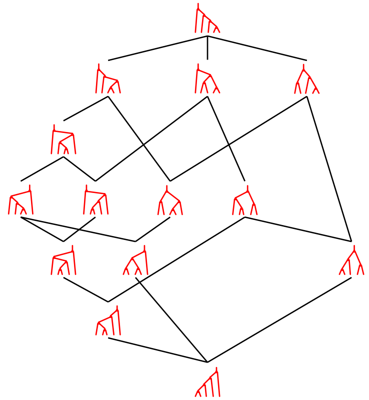
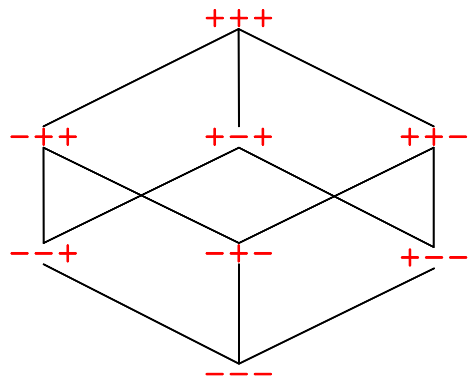
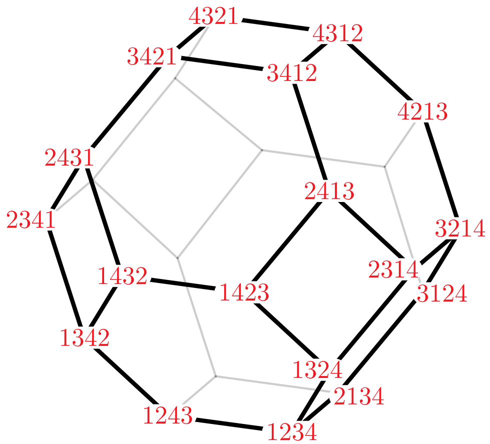
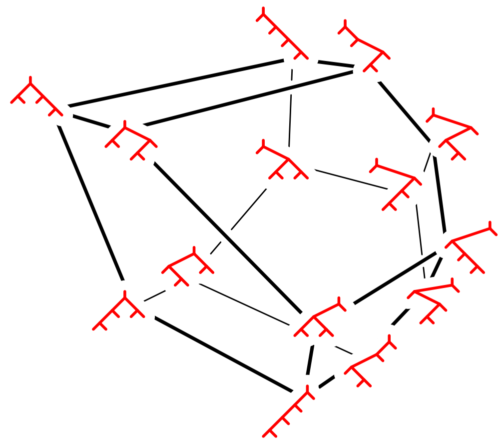
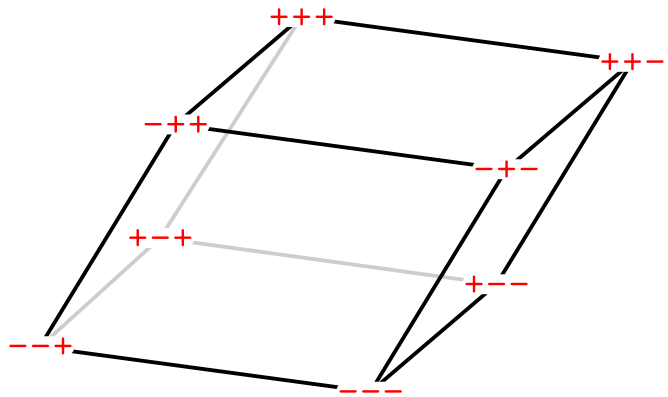
CAMBRIAN HOPF ALGEBRA



Vincent PILAUD
(CNRS & LIX)

Grégory CHATEL
(Univ. MLV)

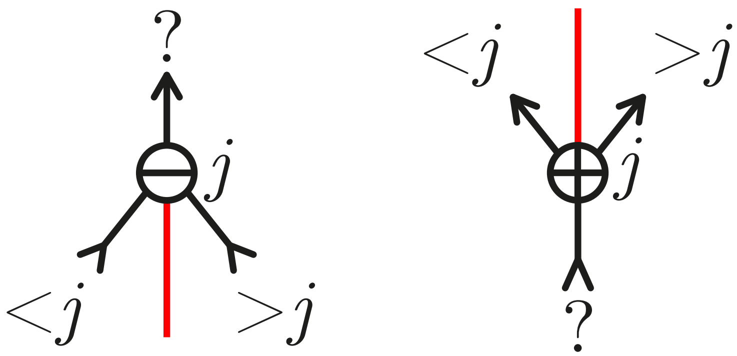
MOTIVATION

	permutations	binary trees	binary sequences
Combinatorics			
Geometry			
Algebra	Malvenuto-Reutenauer algebra	Loday-Ronco algebra	Solomon algebra
	$\mathsf{FQSym} = \mathrm{vect}\langle\, \mathbb{F}_\tau \mid \tau \in \mathfrak{S} \,\rangle$	$\mathsf{PBT} = \mathrm{vect}\langle\, \mathbb{P}_\mathrm{T} \mid \mathrm{T} \in \mathcal{BT} \,\rangle$	$\mathsf{Rec} = \mathrm{vect}\langle\, \mathbb{X}_\eta \mid \eta \in \pm^* \,\rangle$
	$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma$	$\mathbb{P}_T \cdot \mathbb{P}_{T'} = \sum_{T \nearrow T' \leq T'' \leq T \nwarrow T'} \mathbb{P}_{T''}$	$\mathbb{X}_\eta \cdot \mathbb{X}_{\eta'} = \mathbb{X}_{\eta + \eta'} + \mathbb{X}_{\eta - \eta'}$
	$\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$	$\triangle \mathbb{F}_\gamma = \sum_{\gamma \text{ cut}} B(T,\gamma) \otimes A(T,\gamma)$	$\triangle \mathbb{X}_\eta = \sum_{\gamma \text{ cut}} B(\eta,\gamma) \otimes A(\eta,\gamma)$

COMBINATORICS OF CAMBRIAN TREES

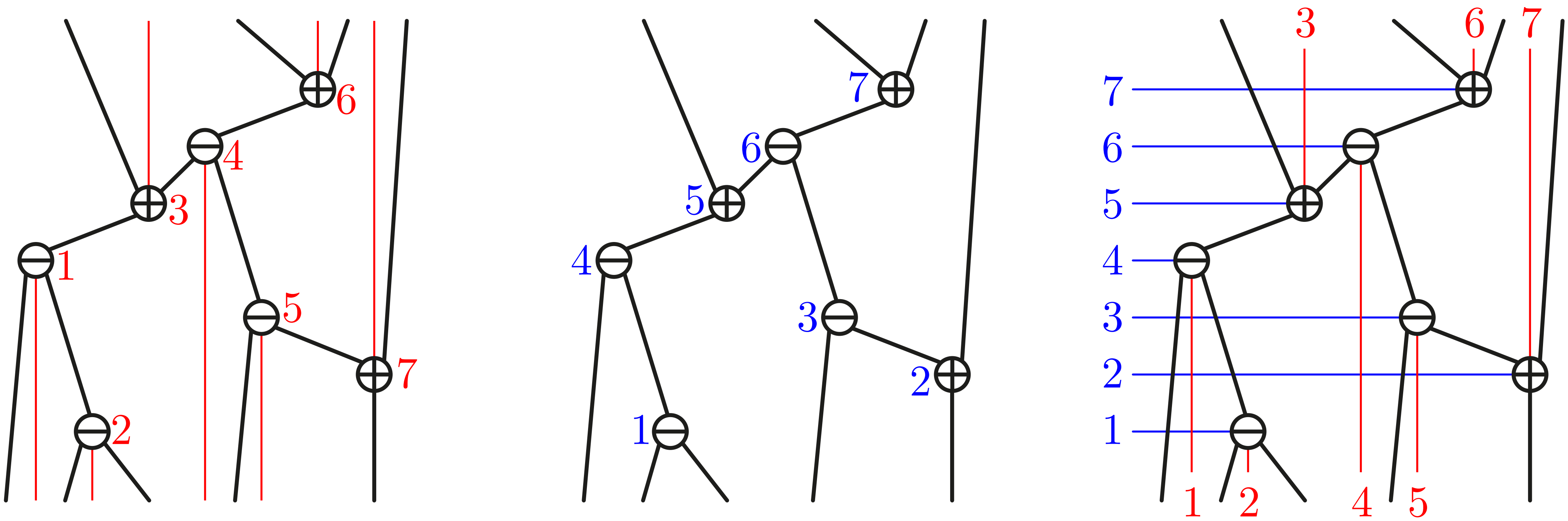
CAMBRIAN TREES

Cambrian tree = directed and labeled tree such that



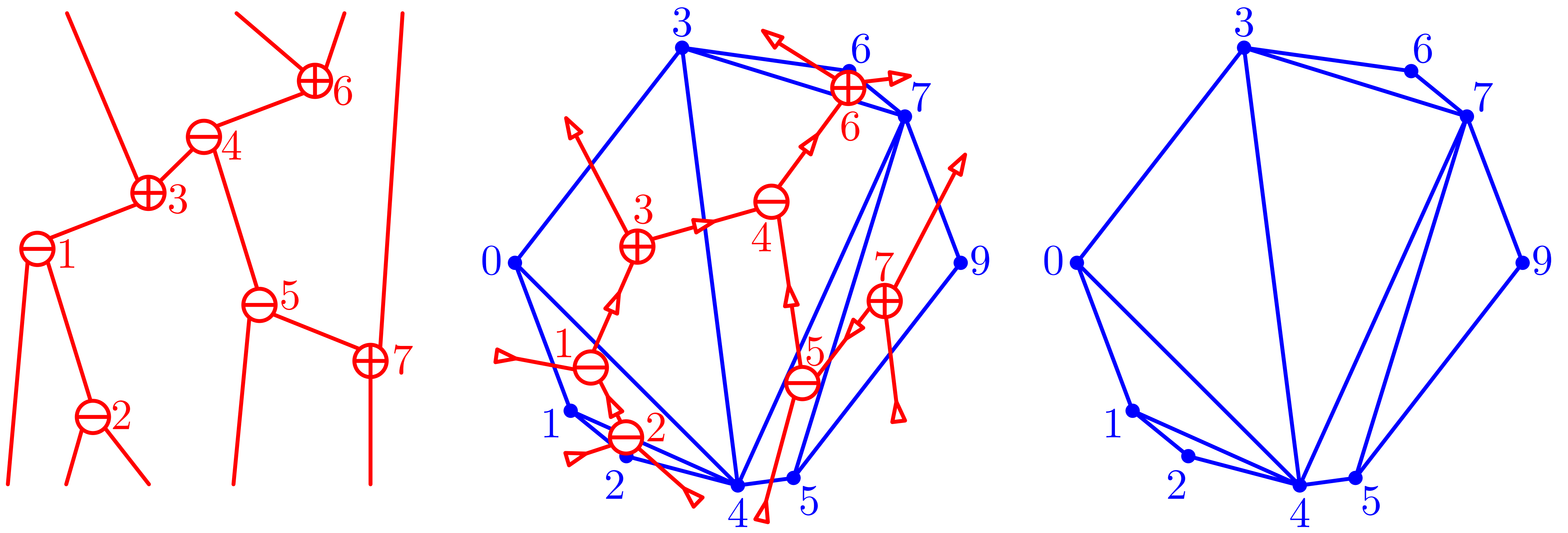
increasing tree = directed and labeled tree such that labels increase along arcs

leveled Cambrian tree = directed tree with a Cambrian labeling and an increasing labeling



CAMBRIAN TREES AND TRIANGULATIONS

Cambrian trees are dual to triangulations of polygons



signature $\longleftrightarrow$ vertices above or below $[0, 9]$

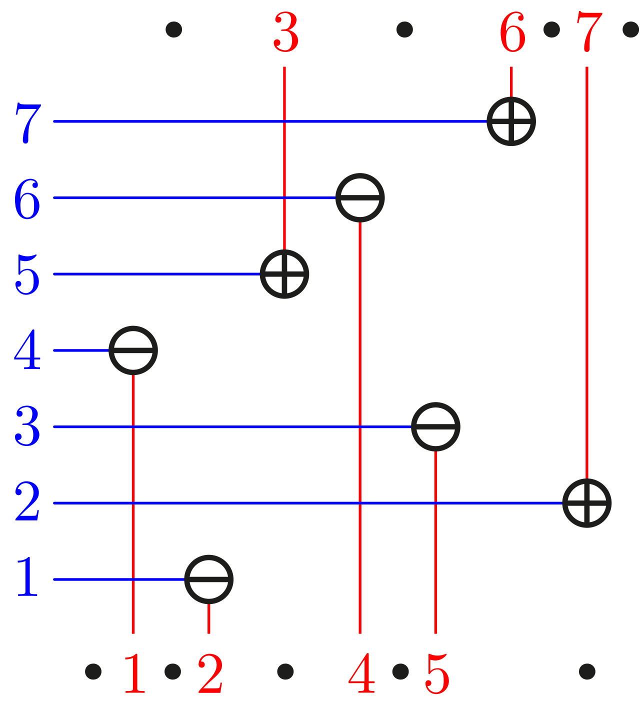
node j $\longleftrightarrow$ triangle $i < j < k$

For any signature ε, there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ ε-Cambrian trees

CAMBRIAN CORRESPONDENCE

Cambrian correspondence = signed permutation $\longmapsto$ leveled Cambrian tree

Exm: signed permutation $\underline{2}\overline{7}\underline{5}\underline{1}\overline{3}\underline{4}\overline{6}$



CAMBRIAN CORRESPONDENCE

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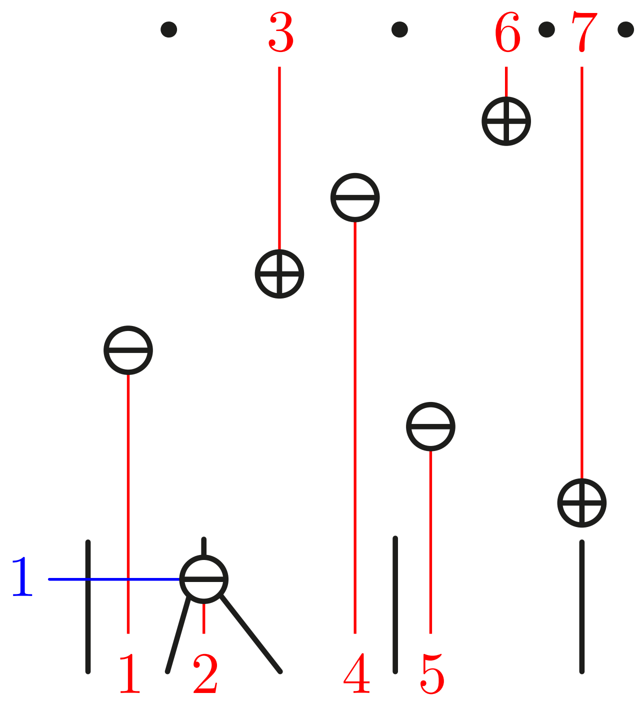

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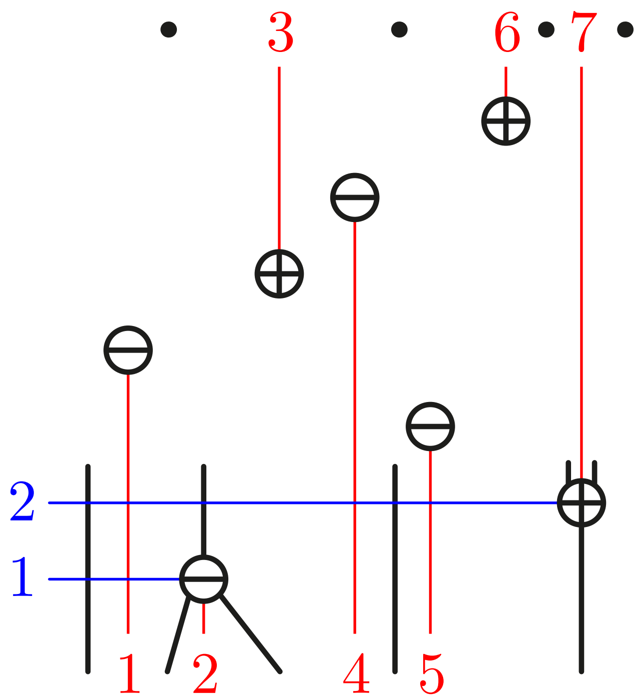

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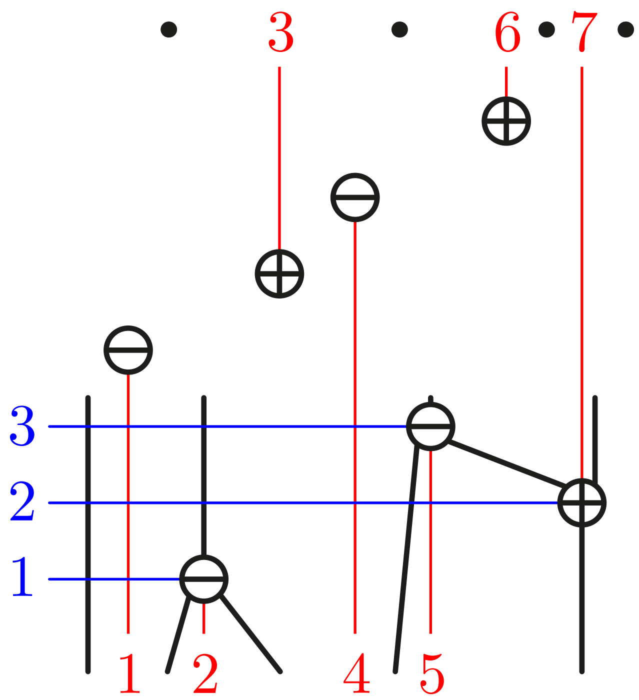

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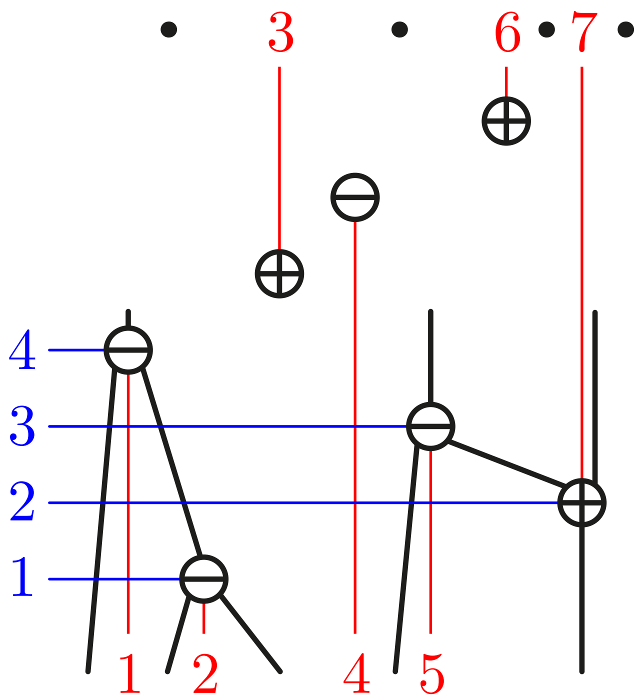

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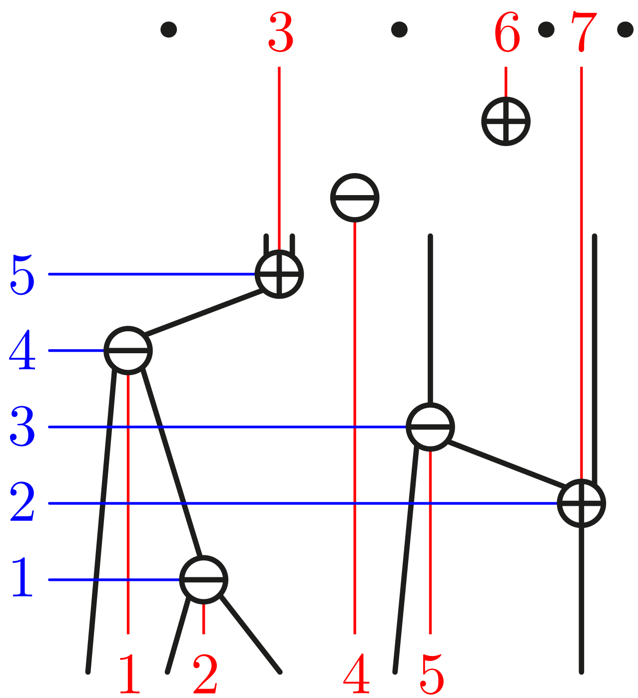

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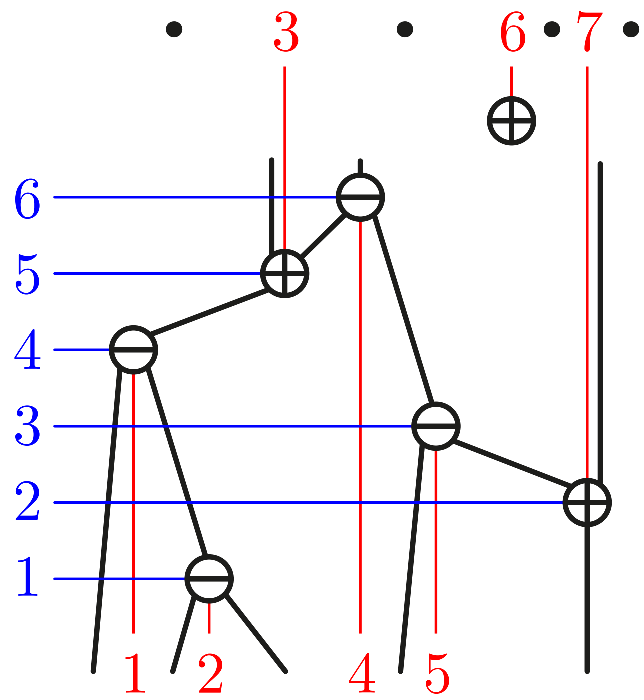

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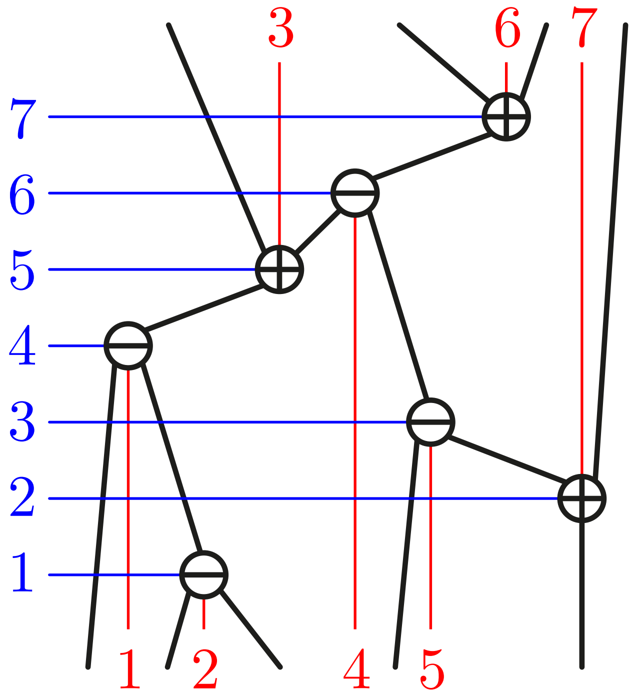

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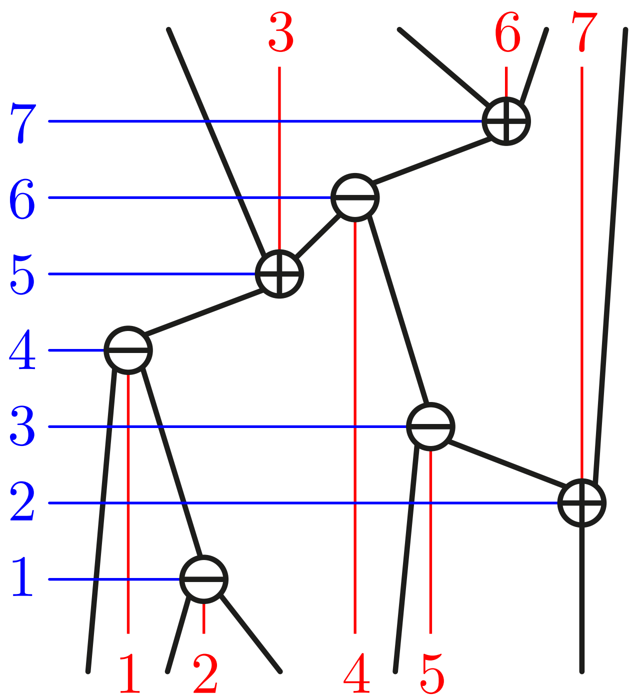


$\mathbf{P}(\tau)$ = $\mathbf{P}$-symbol of τ = Cambrian tree produced by Cambrian correspondence
$\mathbf{Q}(\tau)$ = $\mathbf{Q}$-symbol of τ = increasing tree produced by Cambrian correspondence

(analogy to Robinson-Schensted algorithm)

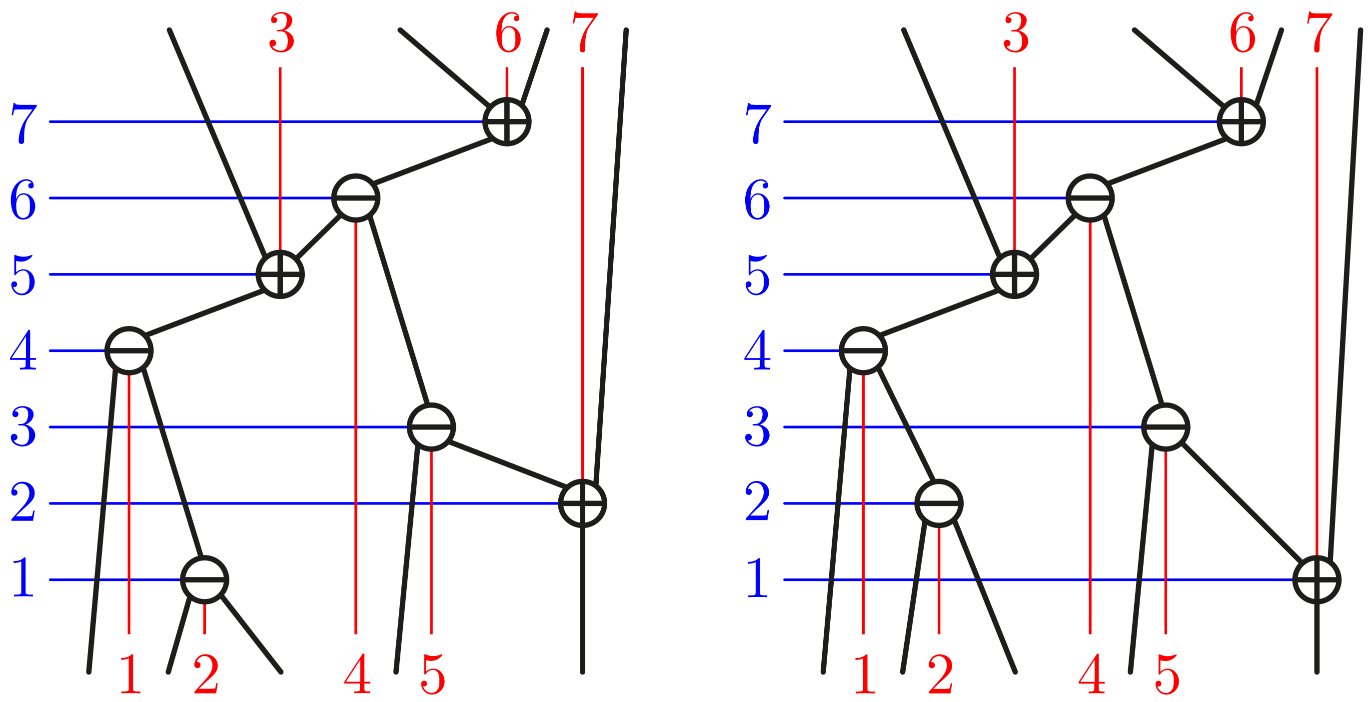
CAMBRIAN CONGRUENCE

ε-Cambrian congruence = transitive closure of the rewriting rules

$$UacVbW \equiv_\varepsilon UcaVbW \quad \text{if } a < b < c \text{ and } \varepsilon_b = -$$
$$UbVacW \equiv_\varepsilon UbVcaW \quad \text{if } a < b < c \text{ and } \varepsilon_b = +$$

where a, b, c are elements of $[n]$ while U, V, W are words on $[n]$

PROP. $\tau \equiv_\varepsilon \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$



$$\underline{2}\overline{7}\underline{51}\overline{3}\underline{4}\overline{6} \equiv \overline{7}\underline{2}\underline{51}\overline{3}\underline{4}\overline{6}$$

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PROP. $\tau \equiv_\varepsilon \tau' \iff \mathbf{P}(\tau) = \mathbf{P}(\tau')$

PROP. Cambrian congruence class labeled by Cambrian tree T

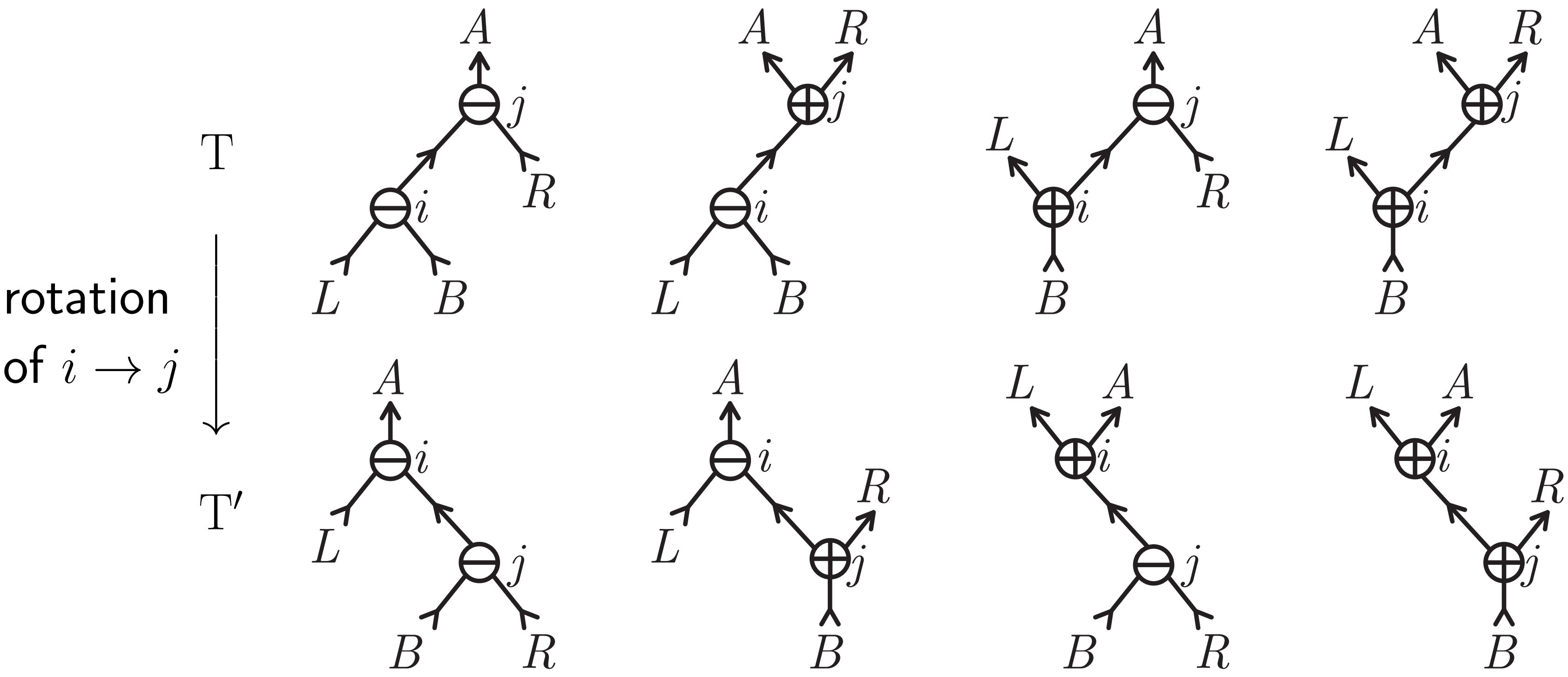
$$\{\tau \in \mathfrak{S}^\varepsilon \mid \mathbf{P}(\tau) = \mathrm{T}\} = \{\text{linear extensions of } \mathrm{T}\}$$

PROP. Cambrian classes are intervals of the weak order
minimums avoid $\overline{2}31$ and $31\underline{2}$ while maximums avoid $\overline{2}13$ and $13\underline{2}$

Reading. Cambrian lattices. 2006

ROTATIONS AND CAMBRIAN LATTICES

Rotation operation preserves Cambrian trees:



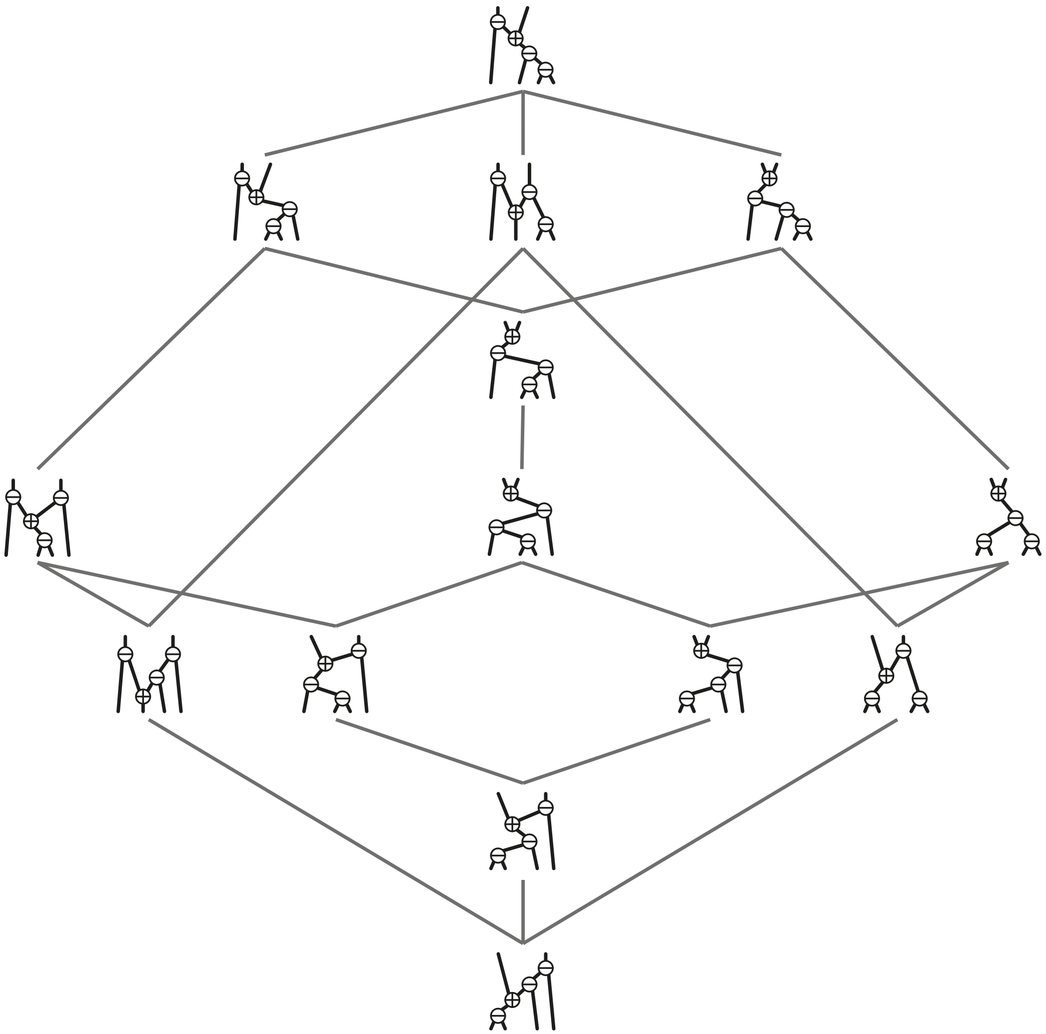
increasing rotation = rotation of edge $i \to j$ where $i < j$

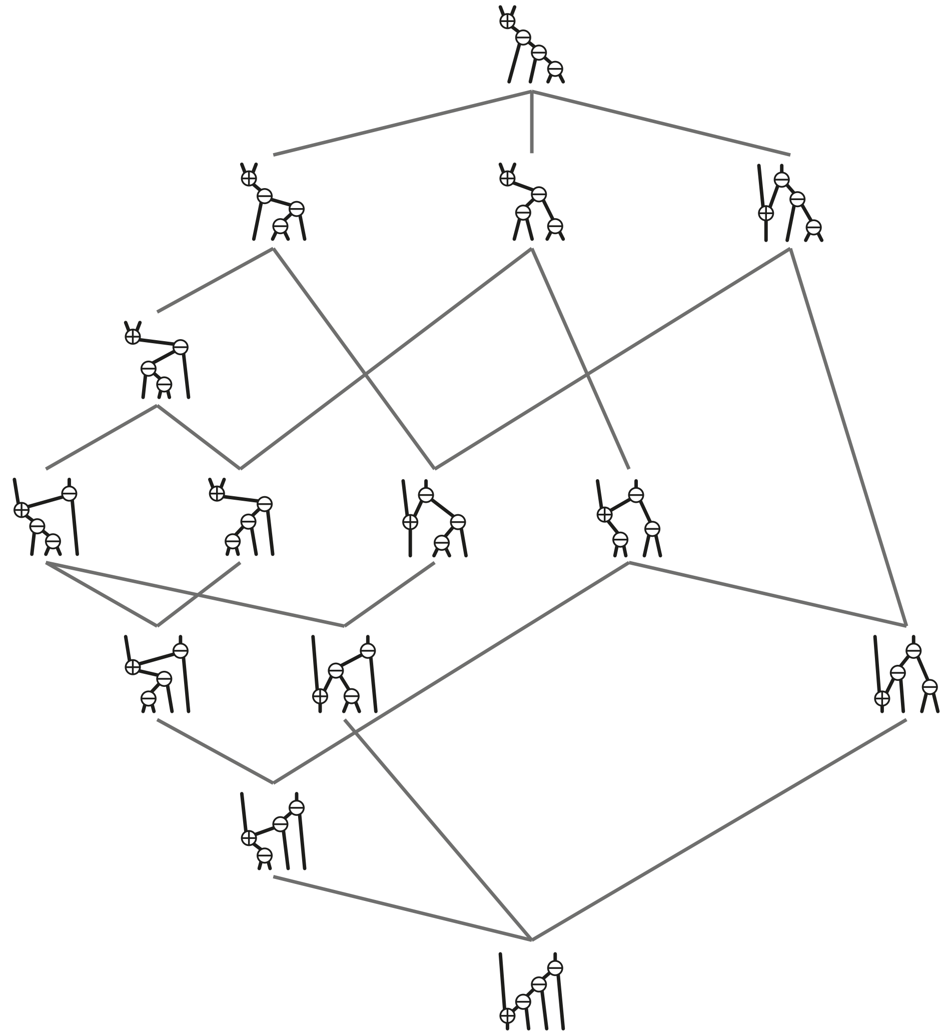
PROP. The transitive closure of the increasing rotation graph is the Cambrian lattice
$\mathbf{P}$ defines a lattice homomorphism from weak order to Cambrian lattice

Reading. Cambrian lattices. 2006

(rotation on Cambrian trees correspond to flips in triangulations)

ROTATIONS AND CAMBRIAN LATTICES





Reading. Cambrian lattices. 2006

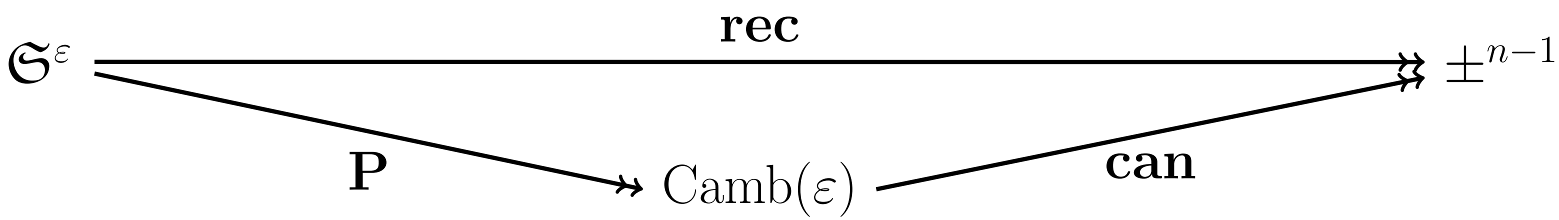
CANOPY

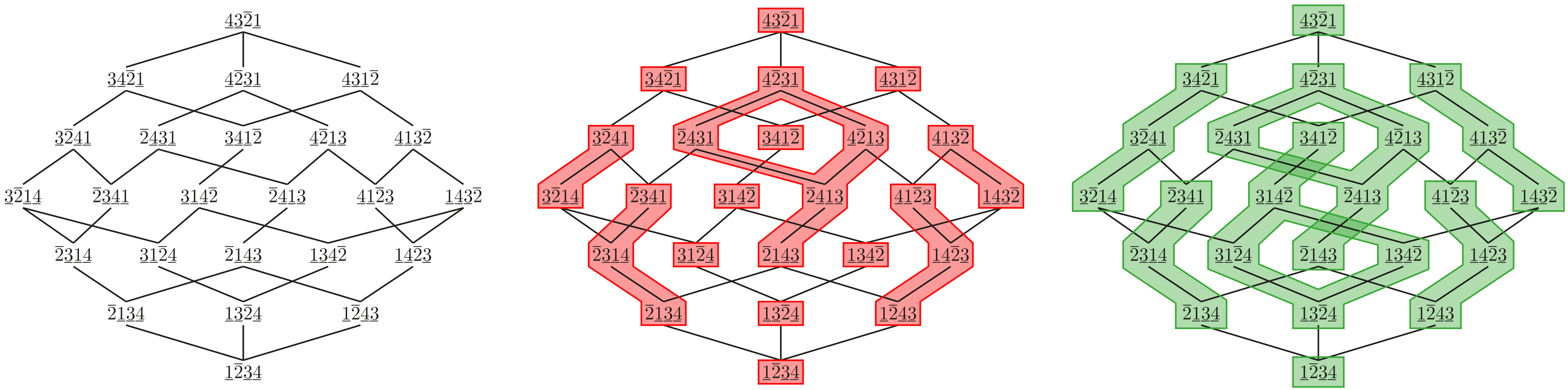
vertices i and $i+1$ are always comparable in a Cambrian tree

Canopy of a Cambrian tree T = sequence $\mathbf{can}(\mathrm{T}) \in \pm^{n-1}$ defined by

$$\mathbf{can}(\mathrm{T})_i = \begin{cases} - & \text{if } i \text{ above } i+1 \text{ in } \mathrm{T} \\ + & \text{if } i \text{ below } i+1 \text{ in } \mathrm{T} \end{cases}$$

PROP. $\mathbf{P}$, $\mathbf{can}$, and $\mathbf{rec}$ define lattice homomorphisms:





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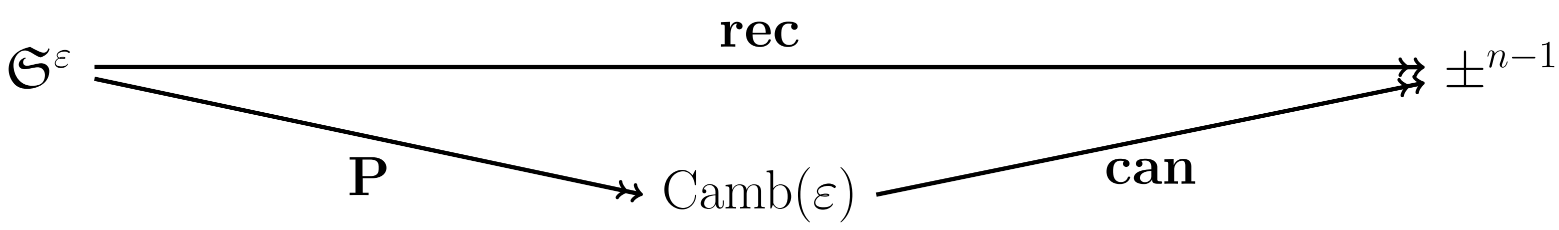
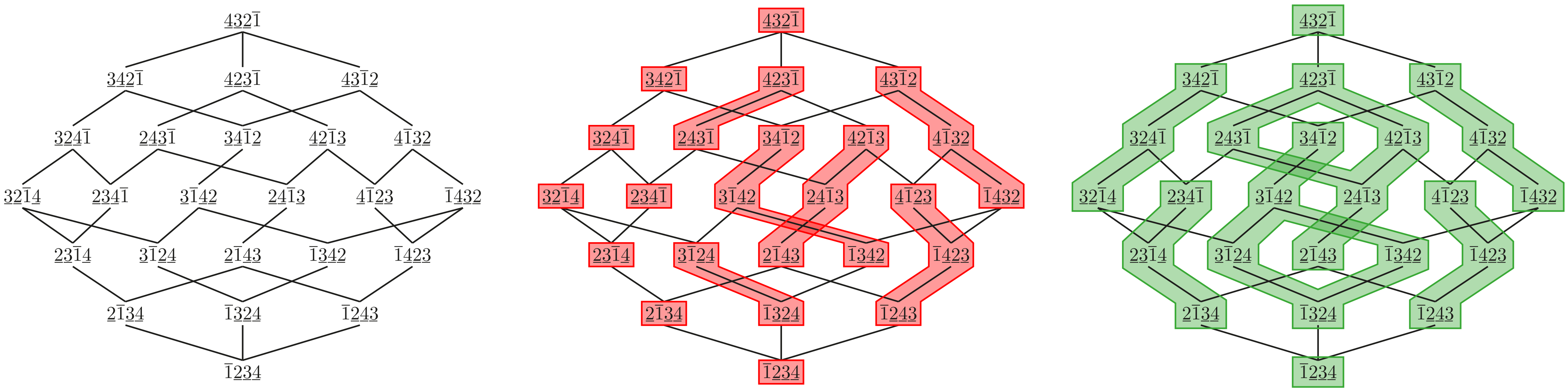




GEOMETRIC REALIZATIONS

Incidence cone $\mathrm{C}(\mathrm{T}) = \mathrm{cone}\,\{e_i - e_j \mid \text{for all } i \to j \text{ in } \mathrm{T}\}$

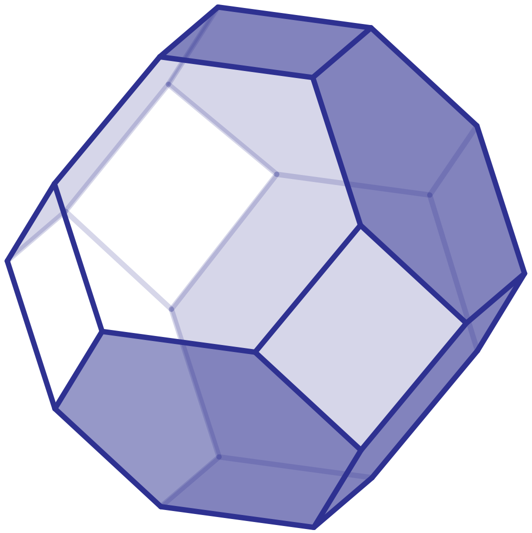
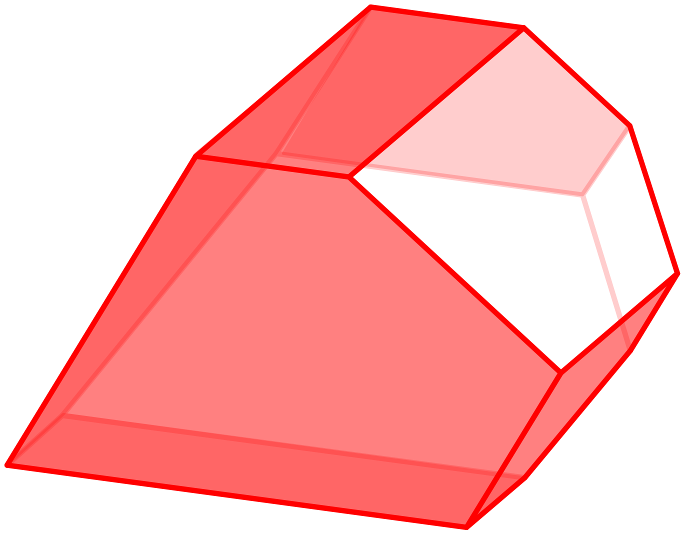
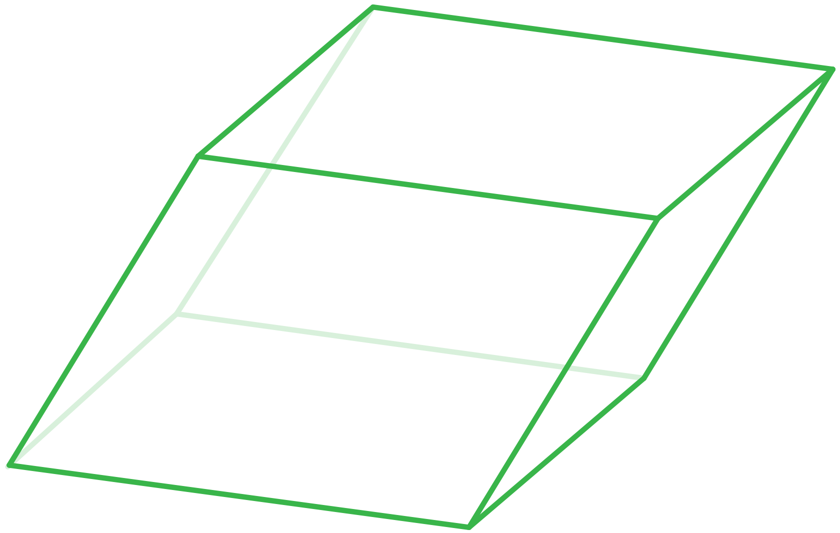
Braid cone $\mathrm{C}^\diamond(\mathrm{T}) = \{\mathbf{x} \in \mathbb{H} \mid x_i \le x_j \text{ for all } i \to j \text{ in } \mathrm{T}\}$

THEO. The cones form complete simplicial fans:

(i) $\{\mathrm{C}^\diamond(\tau) \mid \tau \in \mathfrak{S}_n\}$ = braid fan = normal fan of the permutahedron

(ii) $\{\mathrm{C}^\diamond(\mathrm{T}) \mid \mathrm{T} \in \mathrm{Camb}(\varepsilon)\}$ = ε-Cambrian fan = normal fan of the ε-associahedron

(iii) $\left\{\mathrm{C}^\diamond(\chi) \,\middle|\, \chi \in \pm^{n-1}\right\}$ = boolean fan = normal fan of the parallelepiped

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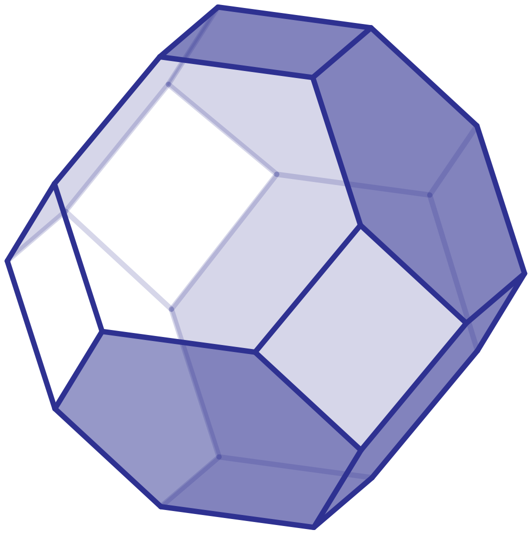
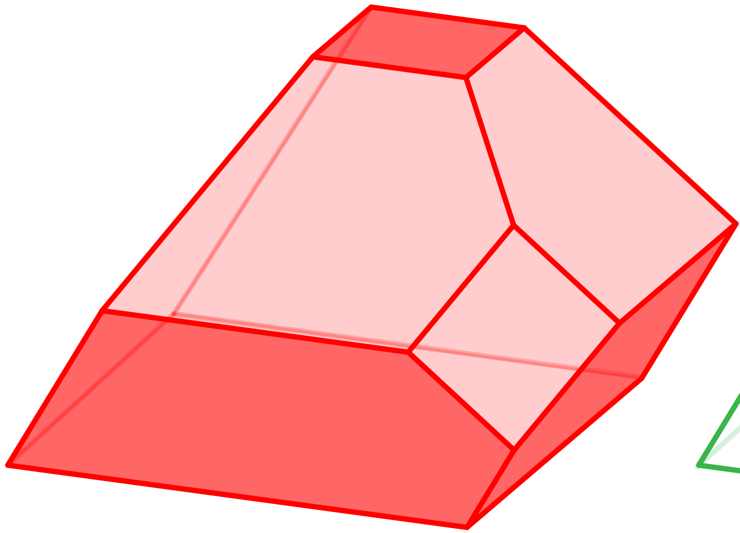
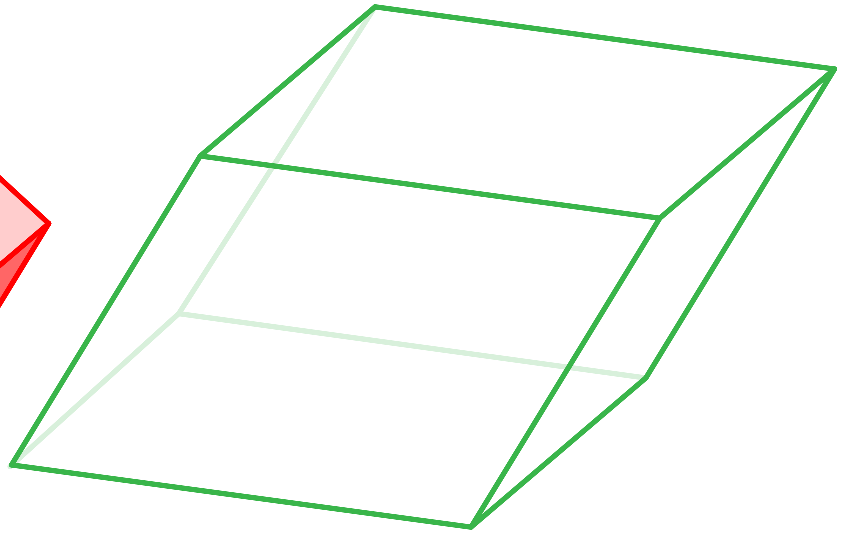

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Characterization of fibers in terms of cones:

$$\mathrm{T} = \mathbf{P}(\tau) \iff \mathrm{C}(\mathrm{T}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\mathrm{T}) \supseteq \mathrm{C}^\diamond(\tau),$$
$$\chi = \mathbf{can}(\mathrm{T}) \iff \mathrm{C}(\chi) \subseteq \mathrm{C}(\mathrm{T}) \iff \mathrm{C}^\diamond(\chi) \supseteq \mathrm{C}^\diamond(\mathrm{T}),$$
$$\chi = \mathbf{rec}(\tau) \iff \mathrm{C}(\chi) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamond(\chi) \supseteq \mathrm{C}^\diamond(\tau).$$

CAMBRIAN HOPF ALGEBRA

SHUFFLE AND CONVOLUTION

For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n:

$$\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n')\}$$

For $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, define

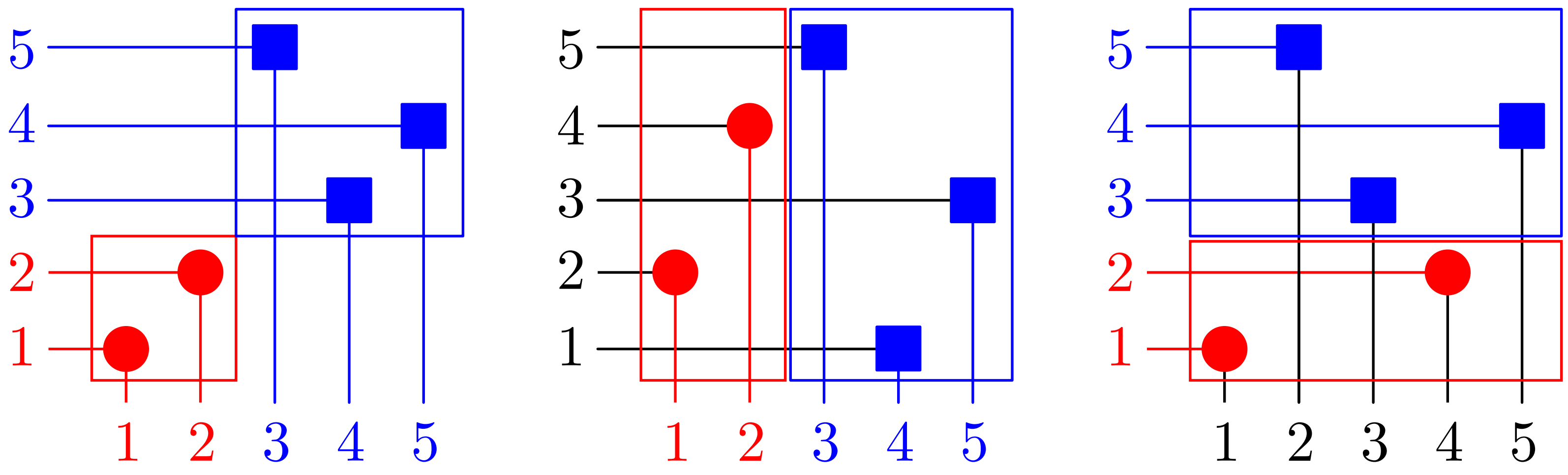
shifted concatenation $\tau\bar{\tau}' = [\tau(1), \dots, \tau(n), \tau'(1)+n, \dots, \tau'(n')+n] \in \mathfrak{S}_{n+n'}$

shifted shuffle product $\tau \,\bar{\sqcup\!\sqcup}\, \tau' = \left\{ (\tau\bar{\tau}') \circ \pi^{-1} \;\middle|\; \pi \in \mathfrak{S}^{(n,n')} \right\} \subset \mathfrak{S}_{n+n'}$

convolution product $\tau \star \tau' = \left\{ \pi \circ (\tau\bar{\tau}') \;\middle|\; \pi \in \mathfrak{S}^{(n,n')} \right\} \subset \mathfrak{S}_{n+n'}$

Exm: $12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$



concatenation — shuffle — convolution

MALVENUTO-REUTENAUER ALGEBRA

DEF. Combinatorial Hopf Algebra $=$ combinatorial vector space $\mathcal{B}$ endowed with

$$\text{product } \cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$$

$$\text{coproduct } \triangle : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

which are "compatible", ie.

$$\begin{array}{ccccc}
\mathcal{B} \otimes \mathcal{B} & \xrightarrow{\quad \cdot \quad} & \mathcal{B} & \xrightarrow{\quad \triangle \quad} & \mathcal{B} \otimes \mathcal{B} \\
\triangle \otimes \triangle \downarrow & & & & \uparrow \cdot \otimes \cdot \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & & \xrightarrow[I \otimes \mathrm{swap} \otimes I]{} & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}
\end{array}$$

Malvenuto-Reteunauer algebra $=$ Hopf algebra FQSym with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}}$ and where

$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\overline{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

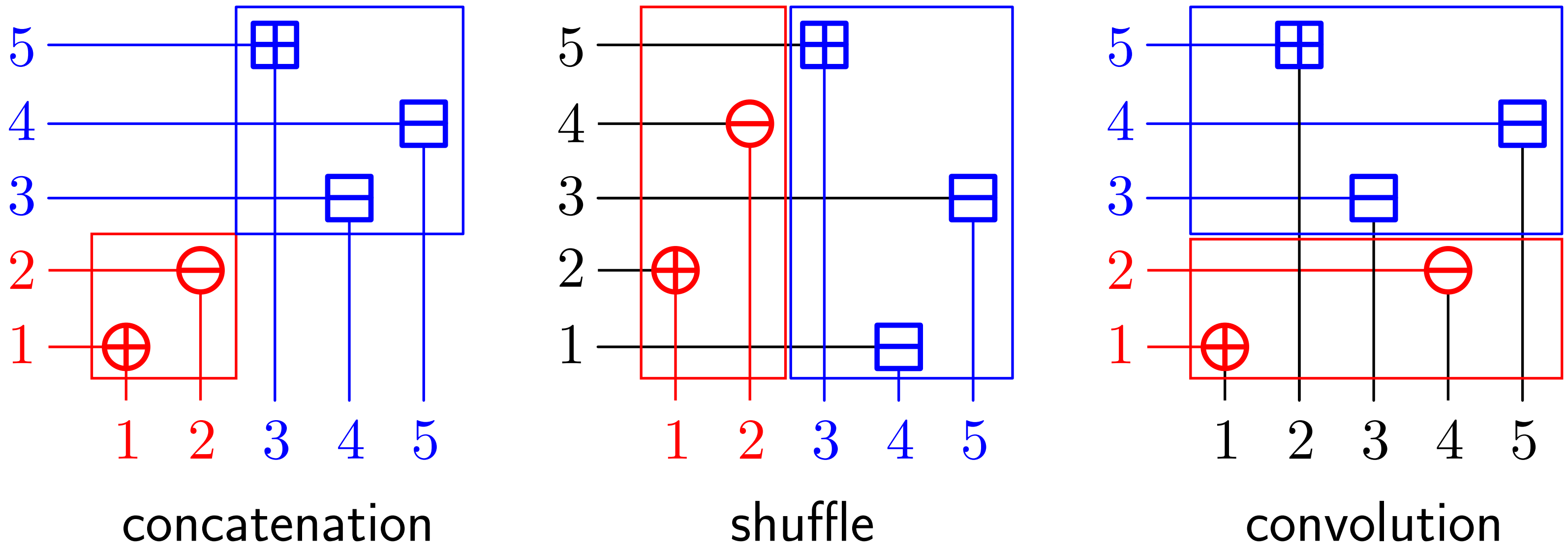
SIGNED VERSION

For signed permutations:

- signs are attached to values in the shuffle product
- signs are attached to positions in the convolution product

Exm: $\bar{1}2 ⧢ 23\bar{1} = \{\bar{1}2453\bar{3}, \bar{1}4253\bar{3}, \bar{1}4523\bar{3}, \bar{1}453\bar{3}2, 4\bar{1}253\bar{3}, 4\bar{1}523\bar{3}, 4\bar{1}53\bar{3}2, 45\bar{1}23\bar{3}, 45\bar{1}\bar{3}2, 45\bar{3}\bar{1}2\}$,

$\bar{1}2 \star 23\bar{1} = \{\bar{1}2453\bar{3}, \bar{1}3452\bar{2}, \bar{1}4352\bar{2}, \bar{1}5342\bar{2}, \bar{2}3451\bar{1}, \bar{2}4351\bar{1}, \bar{2}5341\bar{1}, \bar{3}4251\bar{1}, \bar{3}5241\bar{1}, \bar{4}5231\bar{1}\}$.



$\mathsf{FQSym}_{\pm}$ = Hopf algebra with basis $(\mathbb{F}_\tau)_{\tau \in \mathfrak{S}_\pm}$ and where

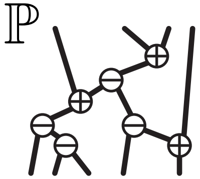
$$\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau ⧢ \tau'} \mathbb{F}_\sigma \qquad \text{and} \qquad \triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$$

CAMBRIAN ALGEBRA AS SUBALGEBRA OF $\mathsf{FQSym}_\pm$

Cambrian algebra = vector subspace Camb of $\mathsf{FQSym}_\pm$ generated by

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S}_\pm \\ \mathbf{P}(\tau) = \mathrm{T}}} \mathbb{F}_\tau = \sum_{\tau \in \mathcal{L}(\mathrm{T})} \mathbb{F}_\tau,$$

for all Cambrian trees T.

Exm: $\mathbb{P}$ 

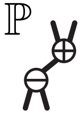
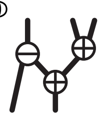
$$= \mathbb{F}_{\underline{21}\overline{37}\underline{54}\overline{6}} + \mathbb{F}_{\underline{21}\overline{73}\underline{54}\overline{6}} + \mathbb{F}_{\underline{21}\overline{7}\underline{5}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{1}\overline{3}\underline{54}\overline{6}} + \mathbb{F}_{\underline{2}\overline{7}\underline{15}\overline{3}\underline{4}\overline{6}}$$
$$+ \mathbb{F}_{\underline{2}\overline{7}\underline{51}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{21}\overline{3}\underline{54}\overline{6}} + \mathbb{F}_{\overline{7}\underline{215}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{251}\overline{3}\underline{4}\overline{6}} + \mathbb{F}_{\overline{7}\underline{521}\overline{3}\underline{4}\overline{6}}$$

THEO. Camb is a subalgebra of $\mathsf{FQSym}_\pm$

(ie. the Cambrian congruence is "compatible" with the product and coproduct in $\mathsf{FQSym}_\pm$)

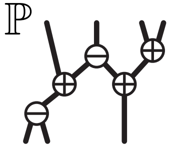
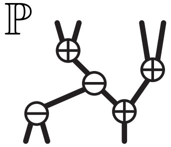
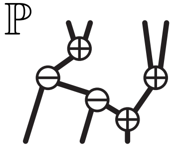
GAME: Explain the product and coproduct directly on the Cambrian trees...

PRODUCT IN CAMBRIAN ALGEBRA

$\mathbb{P}$  $\cdot\, \mathbb{P}$ 

$$= \mathbb{F}_{\underline{1}\overline{2}} \cdot \left(\mathbb{F}_{\overline{2}\underline{1}\overline{3}} + \mathbb{F}_{\overline{2}\overline{3}\underline{1}}\right)$$

$$= \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{F}_{\underline{1}\overline{2}\overline{4}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\underline{3}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\overline{2}\overline{5}\underline{3}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\overline{2}\underline{3}} + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\overline{2}\underline{3}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{2}\overline{5}} + \mathbb{F}_{\underline{1}\overline{4}\underline{3}\overline{5}\overline{2}} \\ + \mathbb{F}_{\underline{1}\overline{4}\overline{5}\underline{3}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{2}\overline{5}} \\ + \mathbb{F}_{\overline{4}\underline{1}\underline{3}\overline{5}\overline{2}} + \mathbb{F}_{\overline{4}\underline{1}\overline{5}\underline{3}\overline{2}} \\ + \mathbb{F}_{\overline{4}\overline{5}\underline{1}\underline{3}\overline{2}} \end{pmatrix} + \begin{pmatrix} \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{2}\overline{5}} + \mathbb{F}_{\overline{4}\underline{3}\underline{1}\overline{5}\overline{2}} \\ + \mathbb{F}_{\overline{4}\underline{3}\overline{5}\underline{1}\overline{2}} + \mathbb{F}_{\overline{4}\overline{5}\underline{3}\underline{1}\overline{2}} \end{pmatrix}$$

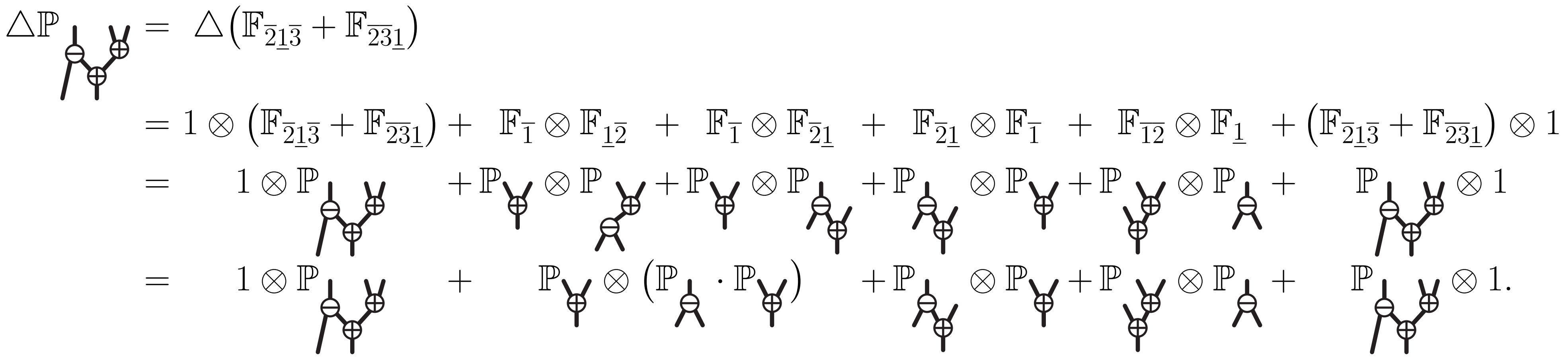
$=$ $\mathbb{P}$  $+$ $\mathbb{P}$  $+$ $\mathbb{P}$ 

PROP. For any Cambrian trees T and T',

$$\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}$$

where S runs over the interval $\left[{}_{\mathrm{T}} \nearrow {}^{\bar{\mathrm{T}}'}, {}^{\mathrm{T}} \nwarrow {}_{\bar{\mathrm{T}}'}\right]$ in the $\varepsilon(\mathrm{T})\varepsilon(\mathrm{T}')$-Cambrian lattice

COPRODUCT IN CAMBRIAN ALGEBRA



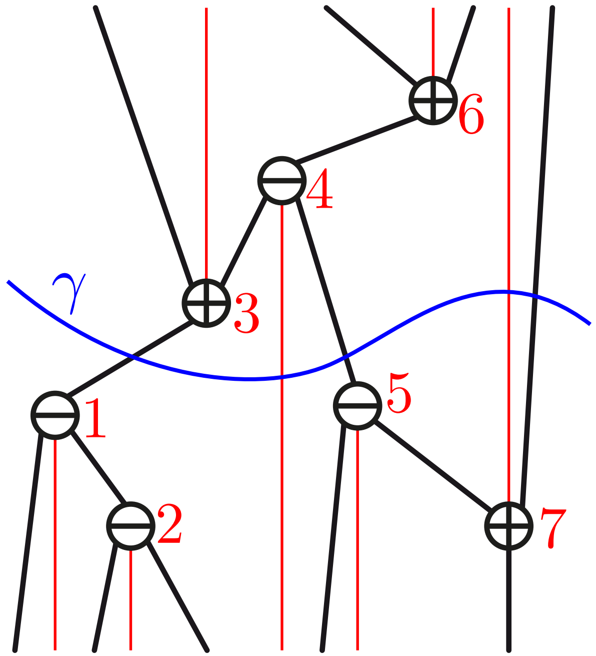
$$\triangle\mathbb{P} = \triangle\left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right)$$

$$= 1 \otimes \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\underline{1}\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\bar{2}\underline{1}} + \mathbb{F}_{\bar{2}\underline{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{\bar{1}\bar{2}} \otimes \mathbb{F}_{\underline{1}} + \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) \otimes 1$$

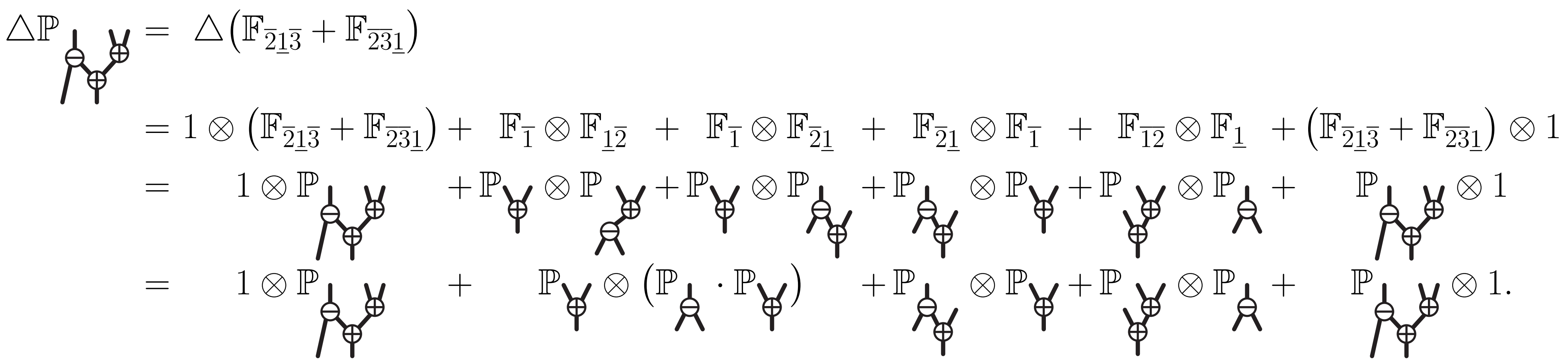
PROP. For any Cambrian tree S,

$$\triangle\mathbb{P}_{\mathrm{S}} = \sum_{\gamma} \left(\prod_{\mathrm{T} \in B(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}} \right) \otimes \left(\prod_{\mathrm{T}' \in A(\mathrm{S},\gamma)} \mathbb{P}_{\mathrm{T}'} \right)$$

where γ runs over all cuts of S, and $A(\mathrm{S},\gamma)$ and $B(\mathrm{S},\gamma)$ denote the Cambrian forests above and below γ respectively



COPRODUCT IN CAMBRIAN ALGEBRA



$$= 1 \otimes \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\underline{1}\bar{2}} + \mathbb{F}_{\bar{1}} \otimes \mathbb{F}_{\bar{2}\underline{1}} + \mathbb{F}_{\bar{2}\underline{1}} \otimes \mathbb{F}_{\bar{1}} + \mathbb{F}_{\underline{1}\bar{2}} \otimes \mathbb{F}_{\underline{1}} + \left(\mathbb{F}_{\bar{2}\underline{1}\bar{3}} + \mathbb{F}_{\bar{2}\bar{3}\underline{1}}\right) \otimes 1$$

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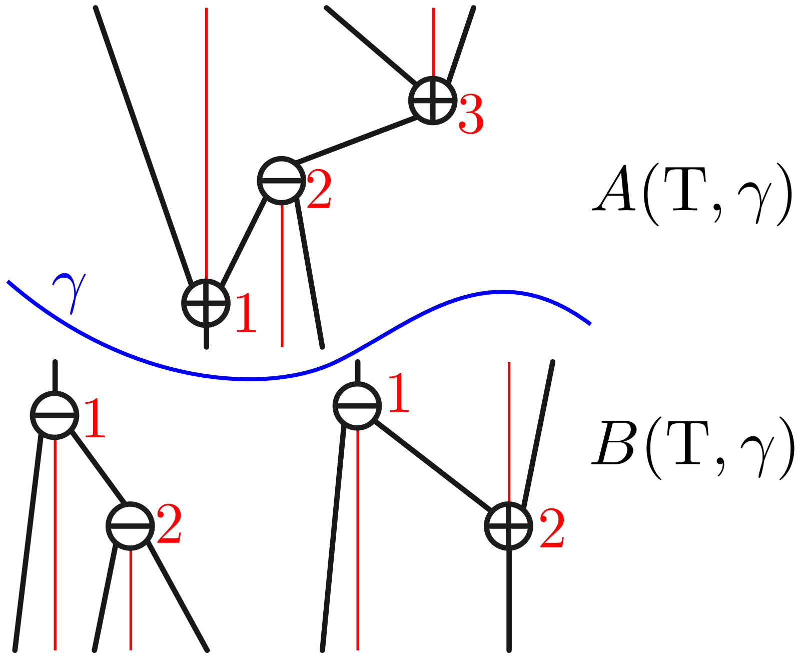


DUAL CAMBRIAN ALGEBRA AS QUOTIENT OF $\mathsf{FQSym}_{\pm}^*$

$\mathsf{FQSym}_{\pm}^*$ = dual Hopf algebra with basis $(\mathbb{G}_\tau)_{\tau \in \mathfrak{S}_\pm}$ and where

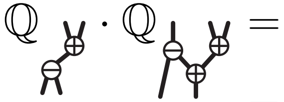
$$\mathbb{G}_\tau \cdot \mathbb{G}_{\tau'} = \sum_{\sigma \in \tau \star \tau'} \mathbb{G}_\sigma \qquad \text{and} \qquad \triangle \mathbb{G}_\sigma = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{G}_\tau \otimes \mathbb{G}_{\tau'}$$

PROP. The graded dual Camb^* of the Cambrian algebra is isomorphic to the image of $\mathsf{FQSym}_{\pm}^*$ under the canonical projection

$$\pi : \mathbb{C}\langle A \rangle \longrightarrow \mathbb{C}\langle A \rangle / \equiv,$$

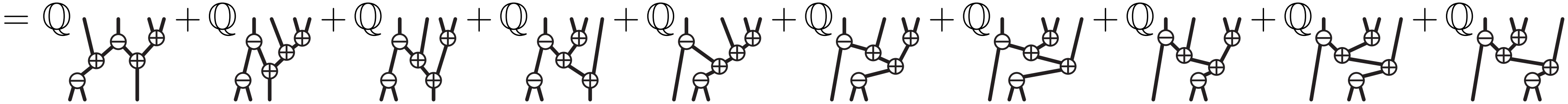
where $\equiv$ denotes the Cambrian congruence. The dual basis $\mathbb{Q}_{\mathrm{T}}$ of $\mathbb{P}_{\mathrm{T}}$ is expressed as $\mathbb{Q}_{\mathrm{T}} = \pi(\mathbb{G}_\tau)$, where τ is any linear extension of T

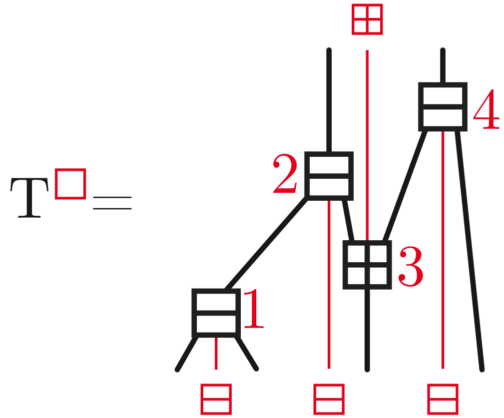
PRODUCT IN DUAL CAMBRIAN ALGEBRA

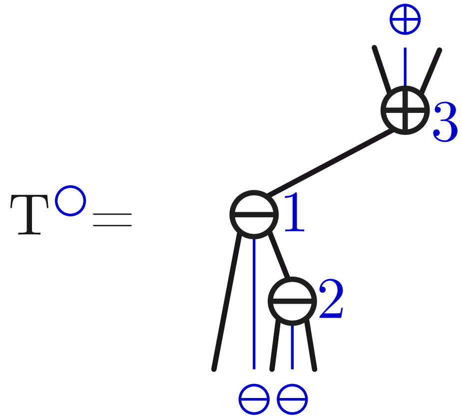


$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

$$= \mathbb{G}_{\underline{1}\overline{2}\overline{4}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{3}\overline{4}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{4}\overline{3}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{5}\overline{3}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{3}\overline{4}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{4}\overline{3}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{5}\overline{3}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{4}\overline{2}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{5}\overline{2}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{5}\overline{2}\underline{1}\overline{3}}$$





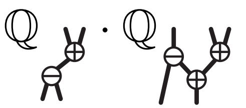


PROP. For any Cambrian trees T and T',

$$\mathbb{Q}_{\mathrm{T}} \cdot \mathbb{Q}_{\mathrm{T}'} = \sum_{s} \mathbb{Q}_{\mathrm{T}s\mathrm{T}'}$$

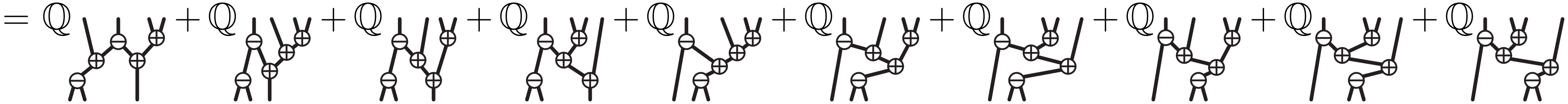
where s runs over all shuffles of $\varepsilon(\mathrm{T})$ and $\varepsilon(\mathrm{T}')$

PRODUCT IN DUAL CAMBRIAN ALGEBRA



$$= \mathbb{G}_{\underline{1}\bar{2}} \cdot \mathbb{G}_{\bar{2}\underline{1}\bar{3}}$$

$$= \mathbb{G}_{\underline{1}\bar{2}\bar{4}\underline{3}\bar{5}} + \mathbb{G}_{\underline{1}\bar{3}\bar{4}\underline{2}\bar{5}} + \mathbb{G}_{\underline{1}\bar{4}\bar{3}\underline{2}\bar{5}} + \mathbb{G}_{\underline{1}\bar{5}\bar{3}\underline{2}\bar{4}} + \mathbb{G}_{\underline{2}\bar{3}\bar{4}\underline{1}\bar{5}} + \mathbb{G}_{\underline{2}\bar{4}\bar{3}\underline{1}\bar{5}} + \mathbb{G}_{\underline{2}\bar{5}\bar{3}\underline{1}\bar{4}} + \mathbb{G}_{\underline{3}\bar{4}\bar{2}\underline{1}\bar{5}} + \mathbb{G}_{\underline{3}\bar{5}\bar{2}\underline{1}\bar{4}} + \mathbb{G}_{\underline{4}\bar{5}\bar{2}\underline{1}\bar{3}}$$



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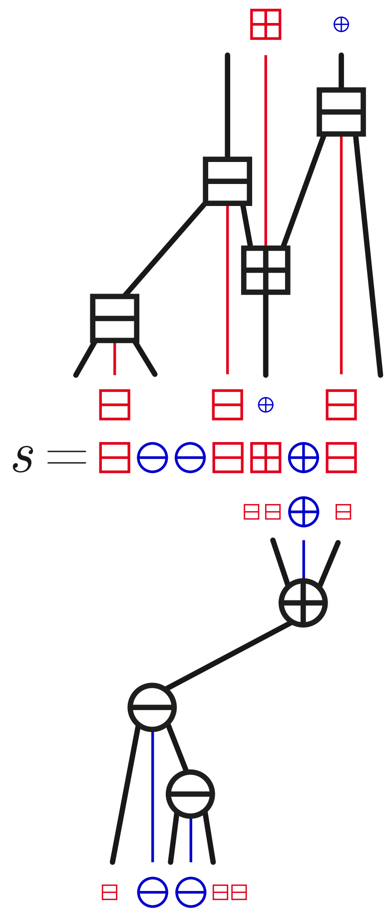
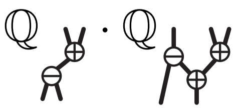
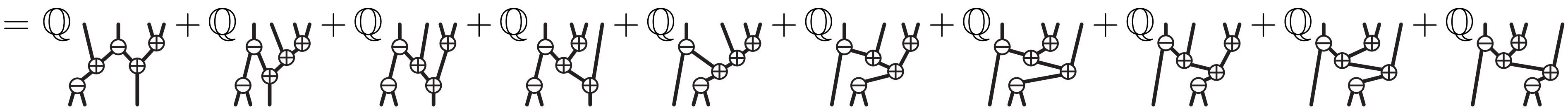


PRODUCT IN DUAL CAMBRIAN ALGEBRA



$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

$$= \mathbb{G}_{\underline{1}\overline{24}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{34}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{43}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{53}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{34}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{43}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{53}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{42}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{52}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{52}\underline{1}\overline{3}}$$



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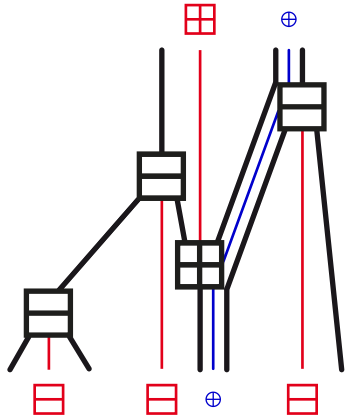

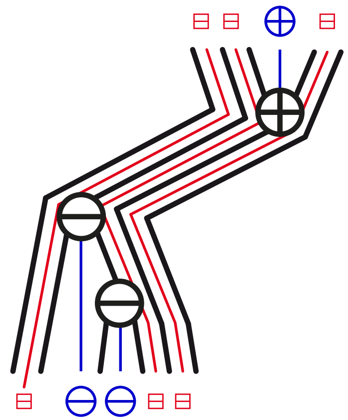

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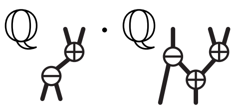
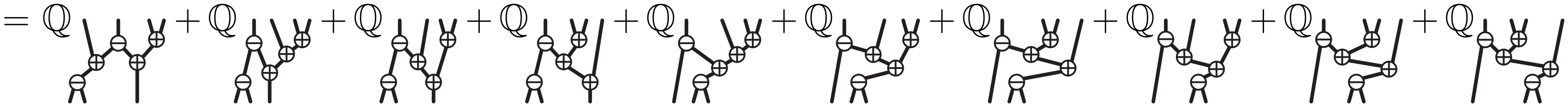
 $= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$

$= \mathbb{G}_{\underline{1}\overline{24}\underline{3}\overline{5}} + \mathbb{G}_{\underline{1}\overline{34}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{43}\underline{2}\overline{5}} + \mathbb{G}_{\underline{1}\overline{53}\underline{2}\overline{4}} + \mathbb{G}_{\underline{2}\overline{34}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{43}\underline{1}\overline{5}} + \mathbb{G}_{\underline{2}\overline{53}\underline{1}\overline{4}} + \mathbb{G}_{\underline{3}\overline{42}\underline{1}\overline{5}} + \mathbb{G}_{\underline{3}\overline{52}\underline{1}\overline{4}} + \mathbb{G}_{\underline{4}\overline{52}\underline{1}\overline{3}}$



PROP. For any Cambrian trees T and T',

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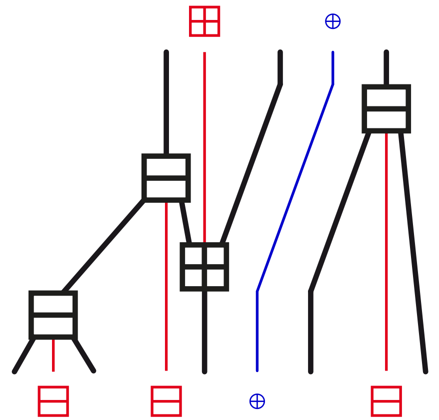

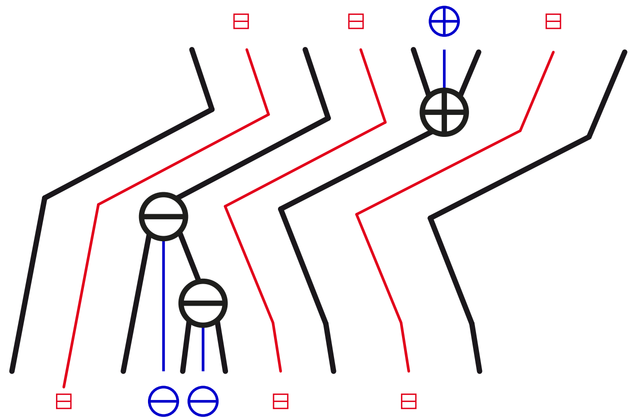

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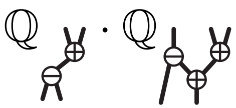

$$= \mathbb{G}_{\underline{1}\overline{2}} \cdot \mathbb{G}_{\overline{2}\underline{1}\overline{3}}$$

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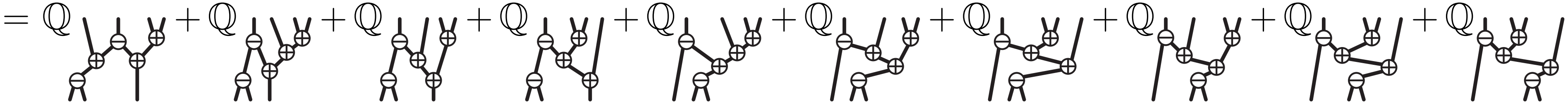

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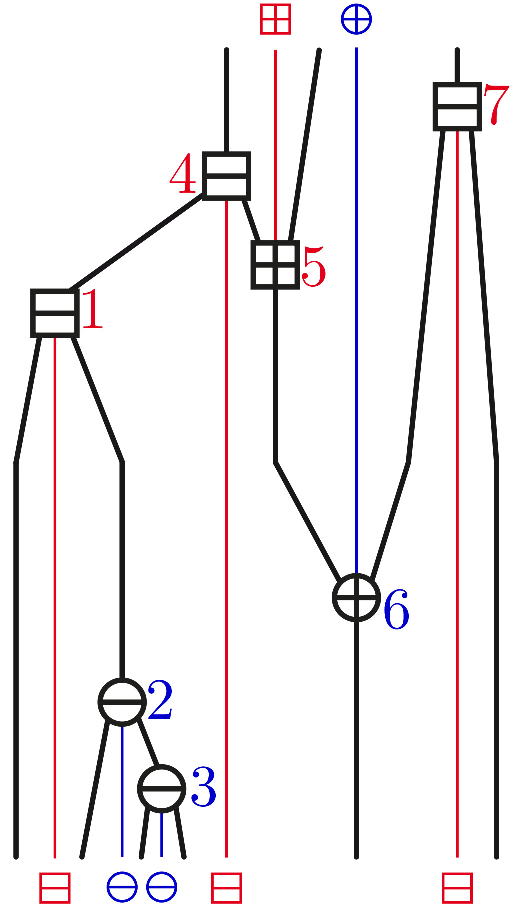
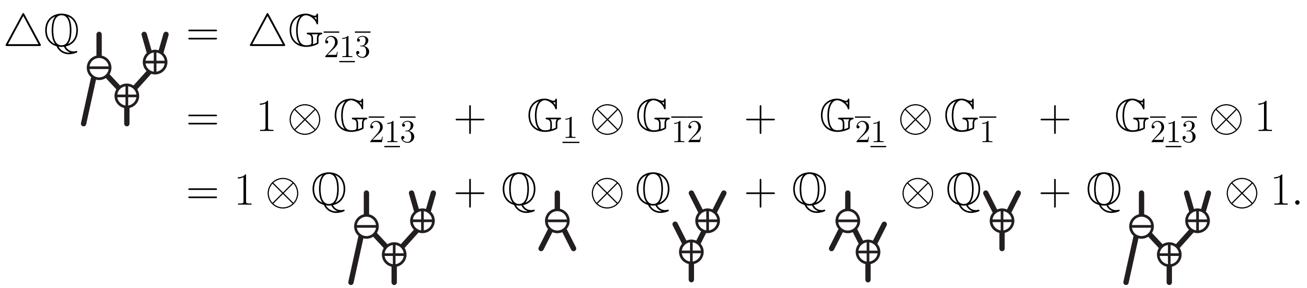


$= \mathrm{T}^{\bigcirc} s \mathrm{T}^{\square}$

COPRODUCT IN DUAL CAMBRIAN ALGEBRA

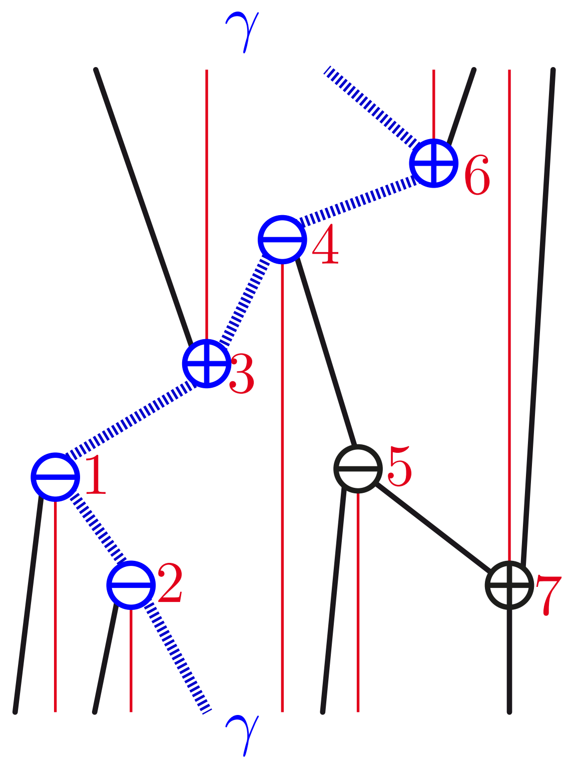


$$\begin{aligned} &= \triangle\mathbb{G}_{\overline{2}\underline{1}\overline{3}} \\ &= 1\otimes\mathbb{G}_{\overline{2}\underline{1}\overline{3}} + \mathbb{G}_{\underline{1}}\otimes\mathbb{G}_{\overline{12}} + \mathbb{G}_{\overline{2}\underline{1}}\otimes\mathbb{G}_{\overline{1}} + \mathbb{G}_{\overline{2}\underline{1}\overline{3}}\otimes 1 \end{aligned}$$

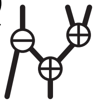
PROP. For any Cambrian tree S,

$$\triangle\mathbb{Q}_{\mathrm{S}} = \sum_{\gamma} \mathbb{Q}_{L(\mathrm{S},\gamma)} \otimes \mathbb{Q}_{R(\mathrm{S},\gamma)}$$

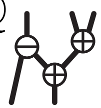
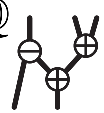
where γ runs over all gaps between vertices of S, and $L(\mathrm{S},\gamma)$ and $R(\mathrm{S},\gamma)$ denote the Cambrian trees left and right to γ respectively



COPRODUCT IN DUAL CAMBRIAN ALGEBRA

$\triangle\mathbb{Q}$  $= \triangle\mathbb{G}_{\bar{2}\underline{1}\bar{3}}$

$= 1\otimes\mathbb{G}_{\bar{2}\underline{1}\bar{3}} + \mathbb{G}_{\underline{1}}\otimes\mathbb{G}_{\overline{12}} + \mathbb{G}_{\bar{2}\underline{1}}\otimes\mathbb{G}_{\bar{1}} + \mathbb{G}_{\bar{2}\underline{1}\bar{3}}\otimes 1$

$= 1\otimes\mathbb{Q}$  $+\ \mathbb{Q}$  $\otimes\ \mathbb{Q}$  $+\ \mathbb{Q}$  $\otimes\ \mathbb{Q}$  $+\ \mathbb{Q}$  $\otimes 1.$

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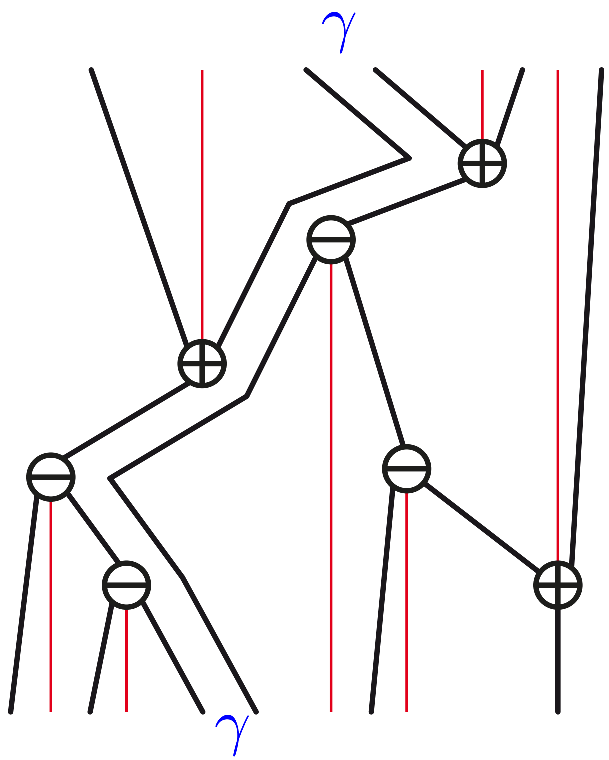
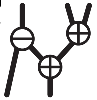
$$\triangle\mathbb{Q}_{\mathrm{S}} = \sum_{\gamma} \mathbb{Q}_{L(\mathrm{S},\gamma)} \otimes \mathbb{Q}_{R(\mathrm{S},\gamma)}$$

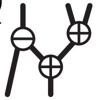
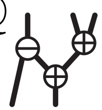
where γ runs over all gaps between vertices of S, and $L(\mathrm{S},\gamma)$ and $R(\mathrm{S},\gamma)$ denote the Cambrian trees left and right to γ respectively



COPRODUCT IN DUAL CAMBRIAN ALGEBRA

$\triangle\mathbb{Q}$  $= \triangle\mathbb{G}_{\overline{2}\underline{1}\overline{3}}$

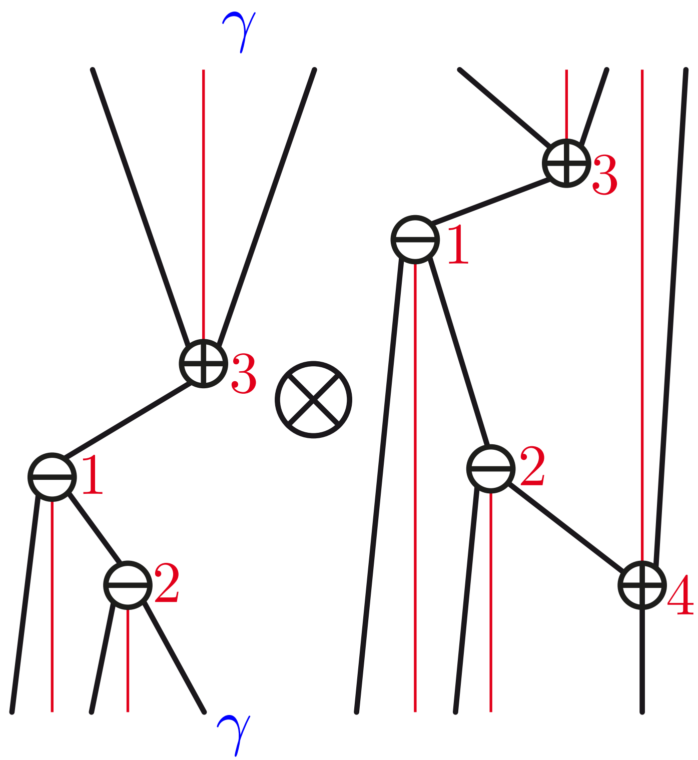
$= 1 \otimes \mathbb{G}_{\overline{2}\underline{1}\overline{3}} + \mathbb{G}_{\underline{1}} \otimes \mathbb{G}_{\overline{1}\overline{2}} + \mathbb{G}_{\overline{2}\underline{1}} \otimes \mathbb{G}_{\overline{1}} + \mathbb{G}_{\overline{2}\underline{1}\overline{3}} \otimes 1$

$= 1 \otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes \mathbb{Q}$  $+ \mathbb{Q}$  $\otimes 1.$

PROP. For any Cambrian tree S,

$$\triangle\mathbb{Q}_{\mathrm{S}} = \sum_{\gamma} \mathbb{Q}_{L(\mathrm{S},\gamma)} \otimes \mathbb{Q}_{R(\mathrm{S},\gamma)}$$

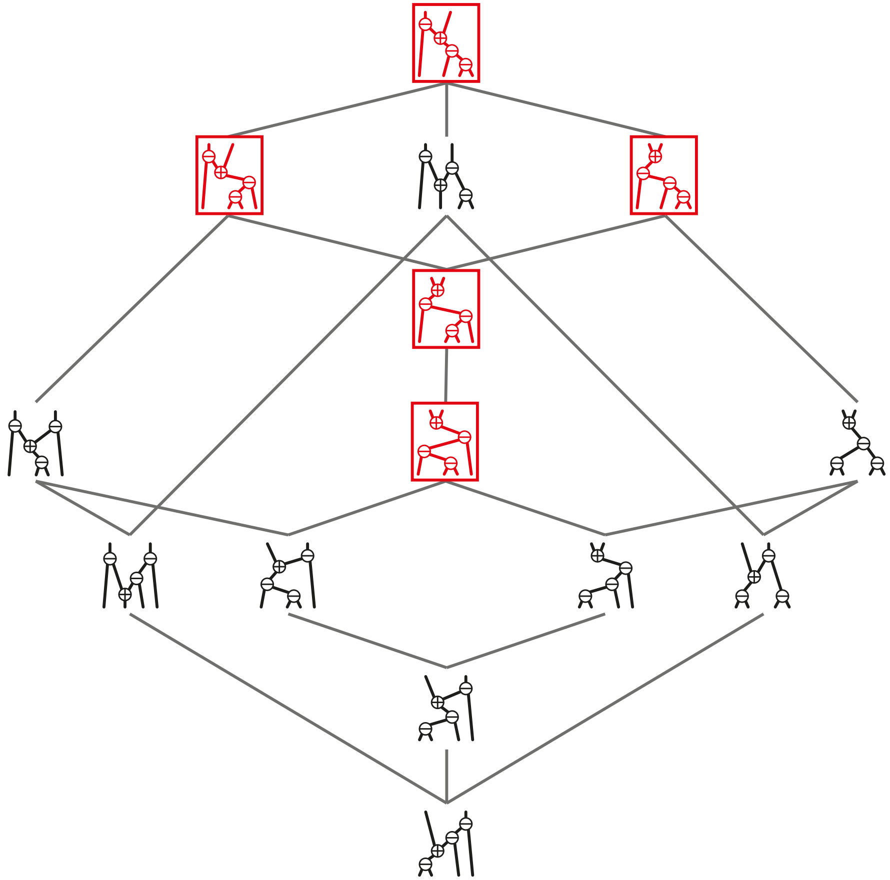
where γ runs over all gaps between vertices of S, and $L(\mathrm{S},\gamma)$ and $R(\mathrm{S},\gamma)$ denote the Cambrian trees left and right to γ respectively

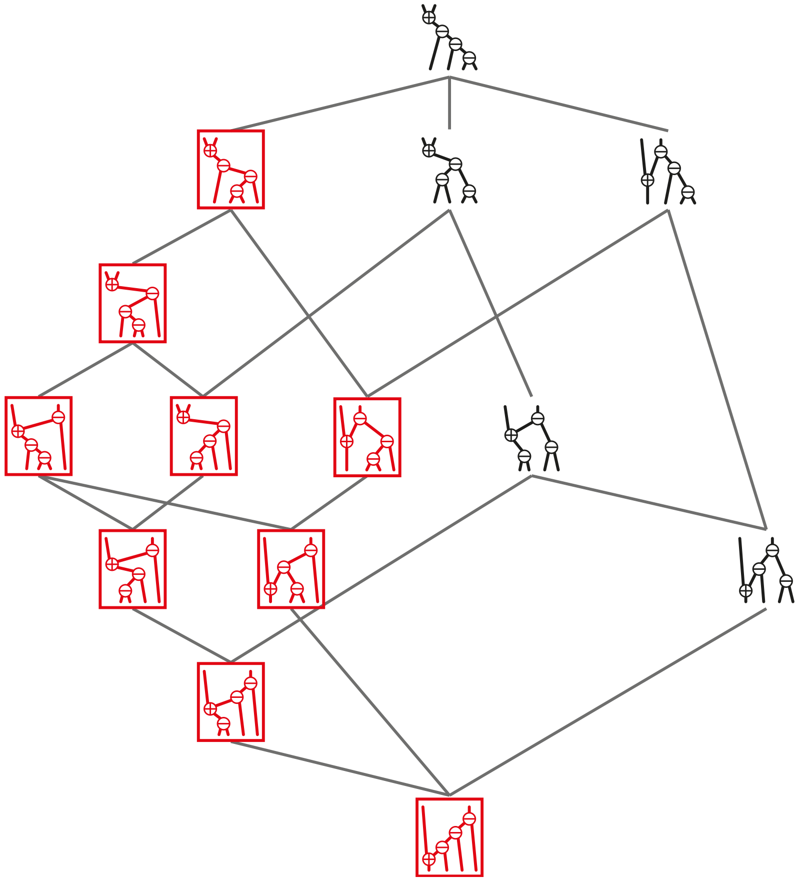


MULTIPLICATIVE BASES
& INDECOMPOSABLE ELEMENTS

MULTIPLICATIVE BASES

Define $$\mathbb{E}^{\mathrm{T}} := \sum_{\mathrm{T} \le \mathrm{T}'} \mathbb{P}_{\mathrm{T}'} \qquad \text{and} \qquad \mathbb{H}^{\mathrm{T}} := \sum_{\mathrm{T}' \le \mathrm{T}} \mathbb{P}_{\mathrm{T}'}.$$





PROP. $(\mathbb{E}^{\mathrm{T}})_{\mathrm{T} \in \mathrm{Camb}}$ and $(\mathbb{H}^{\mathrm{T}})_{\mathrm{T} \in \mathrm{Camb}}$ are multiplicative bases of Camb, ie.

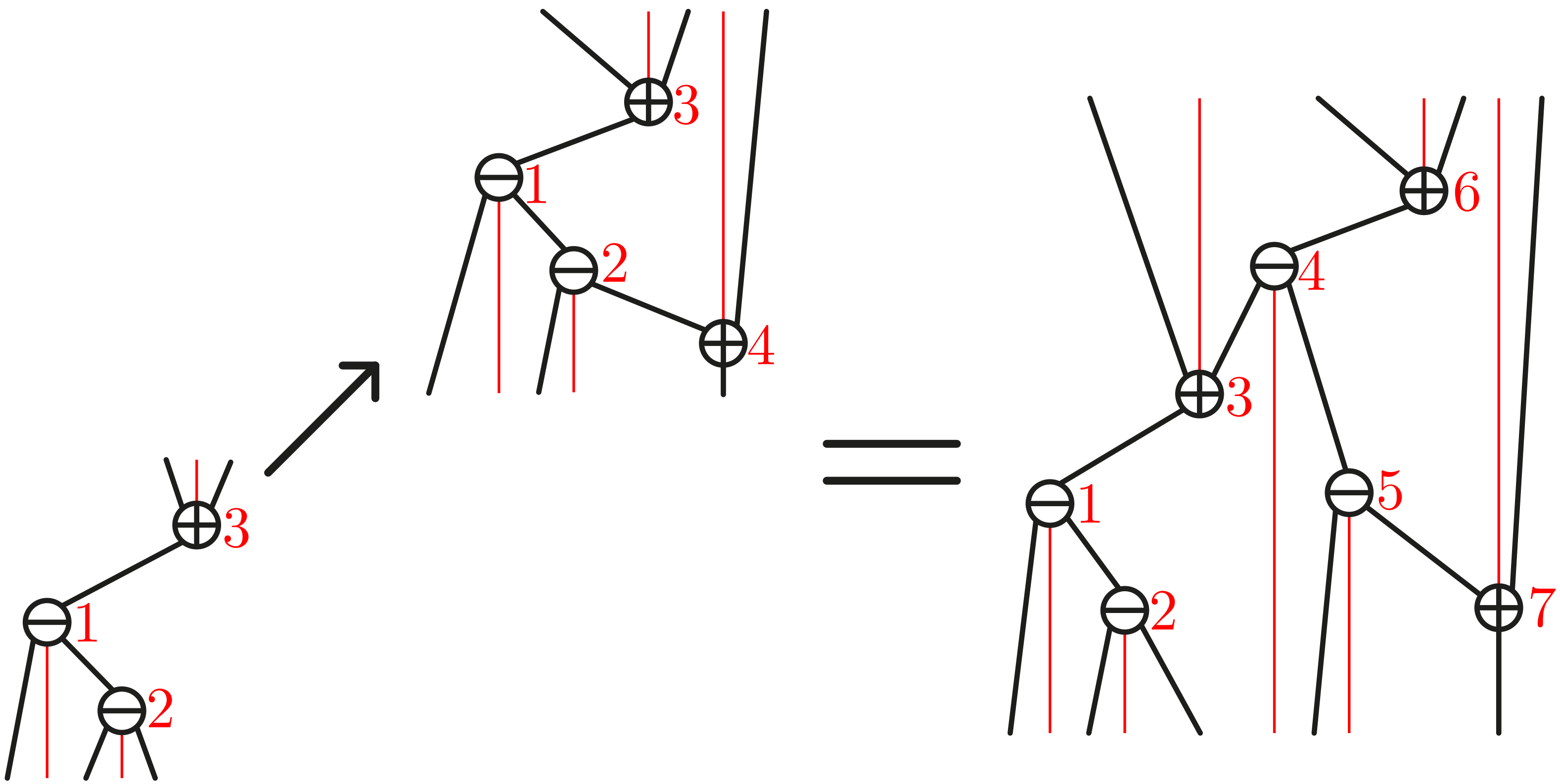
$$\mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'} = \mathbb{E}^{\mathrm{T} \nearrow \bar{\mathrm{T}}'} \qquad \text{and} \qquad \mathbb{H}^{\mathrm{T}} \cdot \mathbb{H}^{\mathrm{T}'} = \mathbb{H}^{\mathrm{T} \nwarrow \bar{\mathrm{T}}'}$$

INDECOMPOSABLE ELEMENTS

PROP. The following properties are equivalent for a Cambrian tree S:

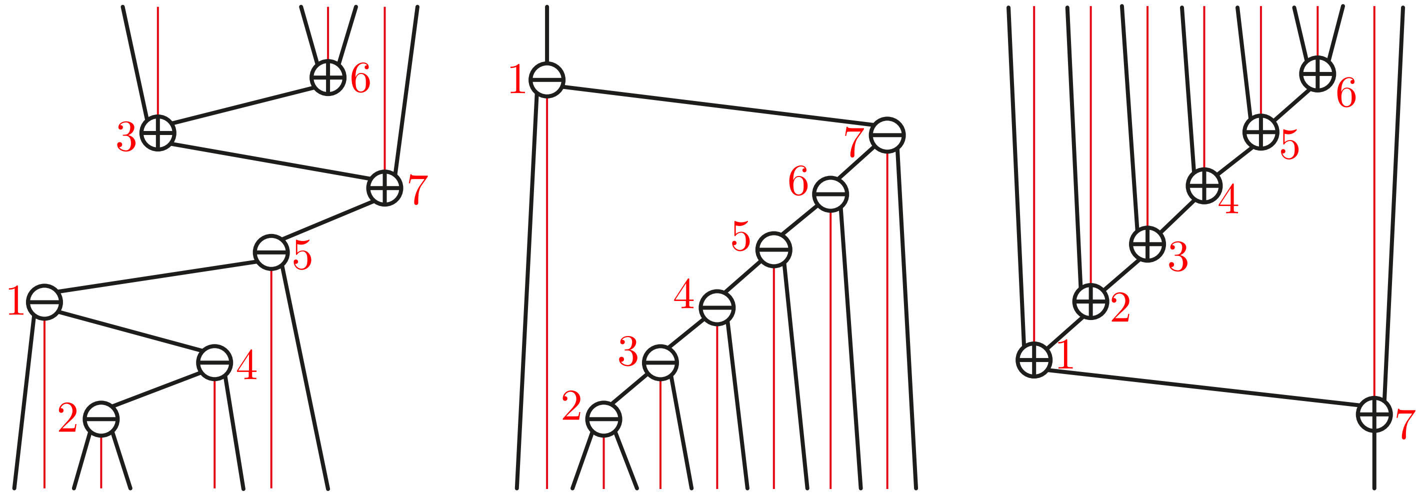
- $\mathbb{E}^{\mathrm{S}}$ can be decomposed into a product $\mathbb{E}^{\mathrm{S}} = \mathbb{E}^{\mathrm{T}} \cdot \mathbb{E}^{\mathrm{T}'}$ for non-empty T, T'
- $([k] \parallel [n] \smallsetminus [k])$ is an edge cut of S for some $k \in [n]$
- at least one linear extension τ of S is decomposable, ie. $\tau([k]) = [k]$ for some $k \in [n]$

The tree S is then called $\mathbb{E}$-decomposable

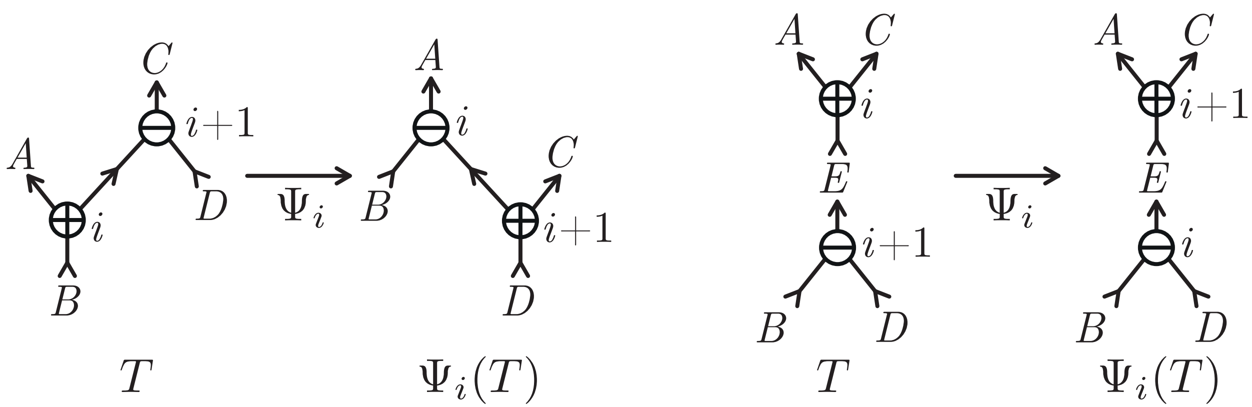


INDECOMPOSABLE ELEMENTS

PROP. For any signature $\varepsilon \in \pm^n$, the set of $\mathbb{E}$-indecomposable ε-Cambrian trees forms a principal upper ideal of the ε-Cambrian lattice



PROP. For any signature $\varepsilon \in \pm^n$, there are C_{n-1} $\mathbb{E}$-indecomposable ε-Cambrian trees. Therefore, there are $2^n C_{n-1}$ $\mathbb{E}$-indecomposable Cambrian trees on n vertices

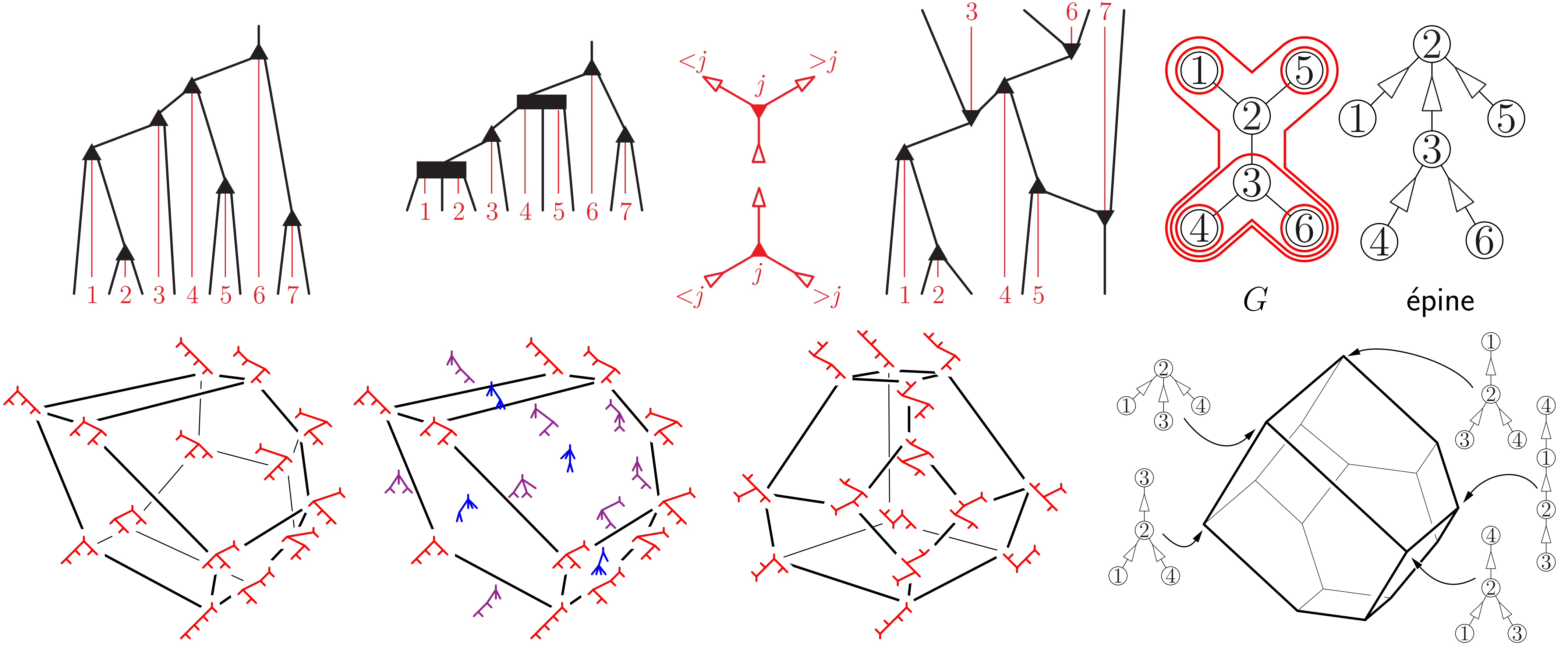


PERSPECTIVES

PERSPECTIVES

Extend combinatorial, geometric and algebraic properties of binary trees to further families of trees...

Binary trees | Schröder trees | Cambrian trees | Spines of a graph



Loday-Ronco algebra | Packed words algebra | Cambrian algebra | Spine algebra ???

THANK YOU