The Colorful Simplicial Depth Conjecture

Pauline Sarrabezolles
CERMICS, ENPC

LIX, September, 24th

Outline of the talk

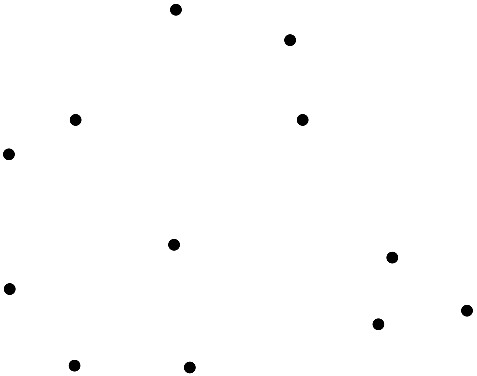
I Colorful Simplicial Depth

II Octahedral Systems

III Proof of the Conjecture

Colorful Simplicial Depth

Simplicial Depth



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s.t. $p \in \mathrm{conv}(\sigma)$

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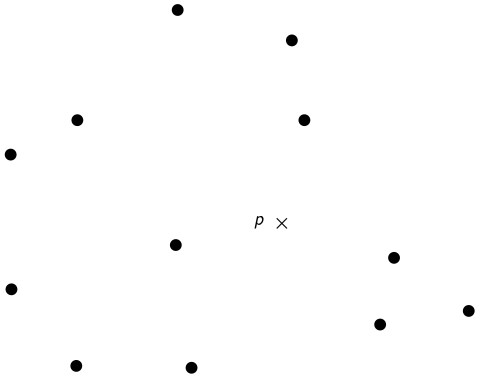

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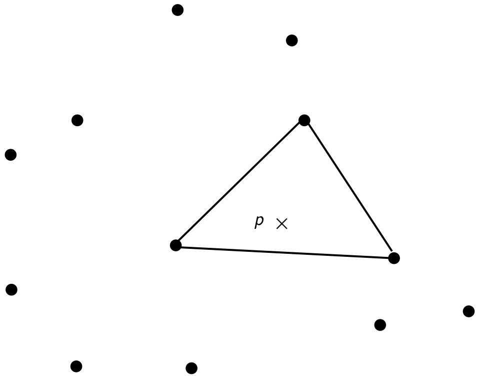

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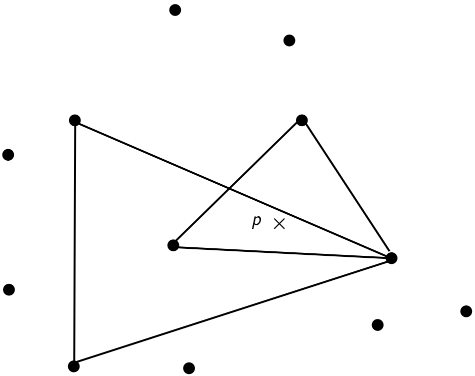

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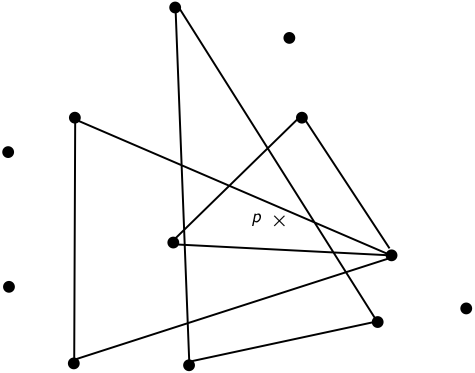

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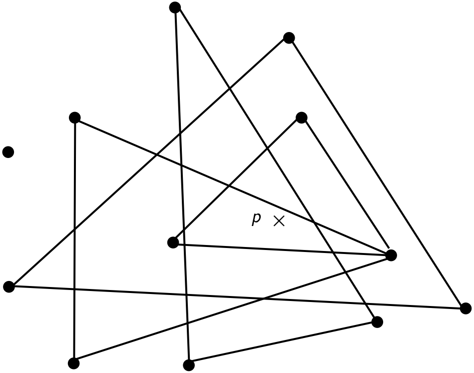

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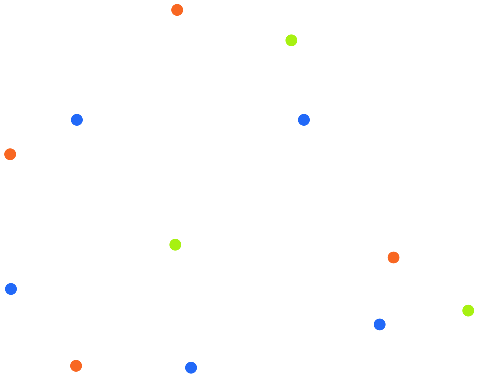

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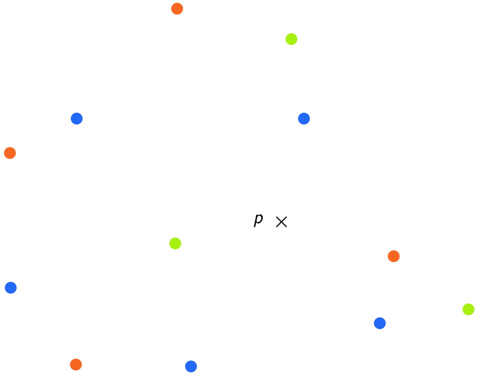

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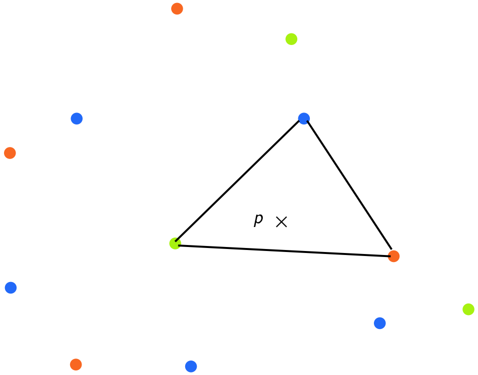

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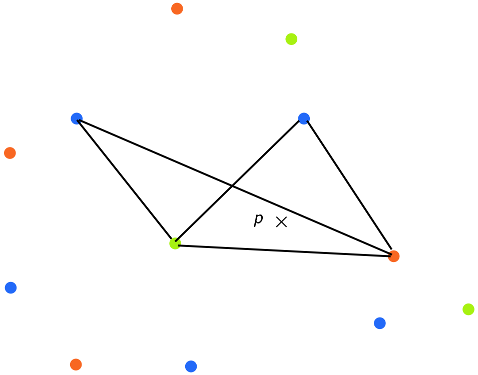

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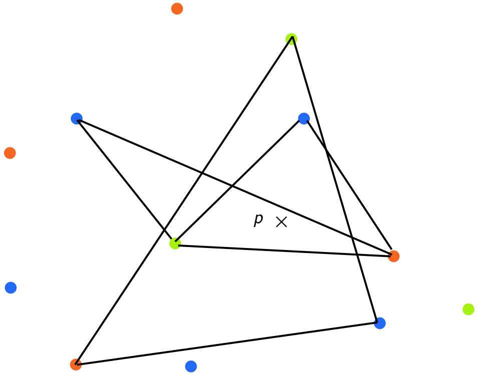

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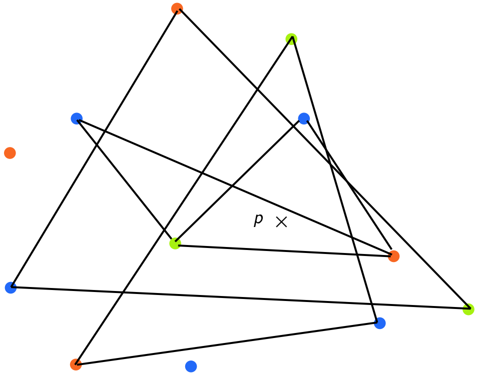

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Related problems

- How deep is the deepest point?

  Monocolor case: $c_d\binom{n}{d+1}$ [Bárány, Gromov, Karasev, Matoušek, and Wagner]

  Colorful case: $\frac{1}{(d+1)!}n^{d+1}$ [Karasev, Jiang]

- Colored Tverberg (Conjecture by Bárány and Larman)

  Tverberg depth of p = max # disjoint colorful simplices σ, s.t. $p \in \text{conv}(\sigma)$

- **Colorful simplicial depth conjecture** (Colorful depth of a point in the *core*)

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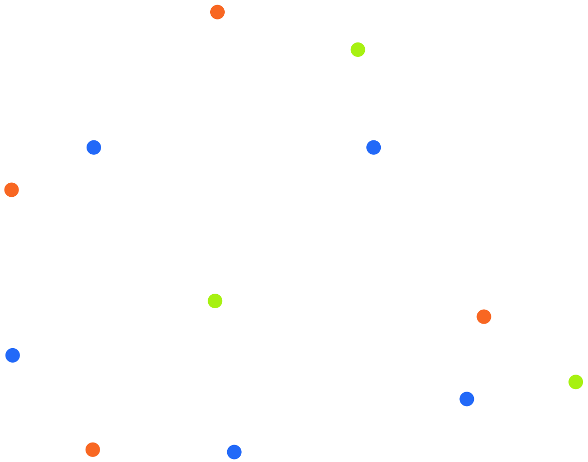
- **Colorful simplicial depth conjecture** (Colorful depth of a point in the *core*)

Colorful Carathéodory Theorem in $\mathbb{R}^2$



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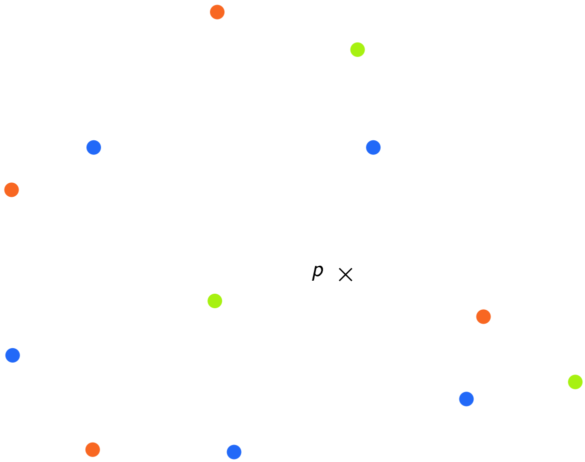

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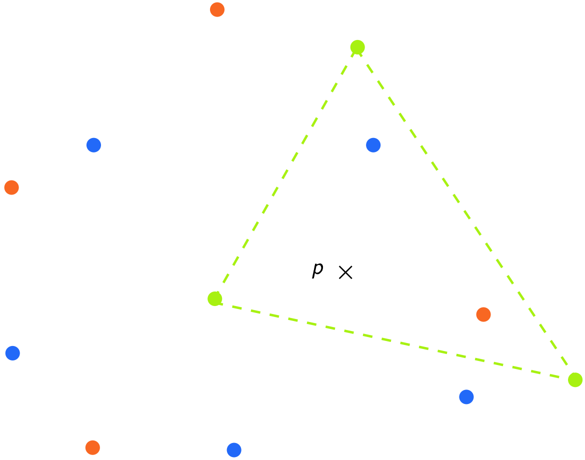

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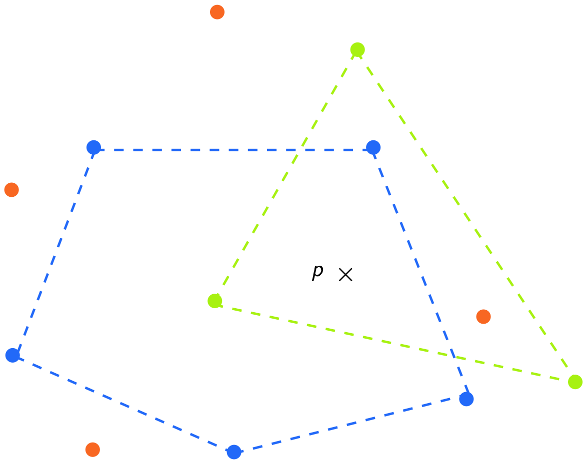

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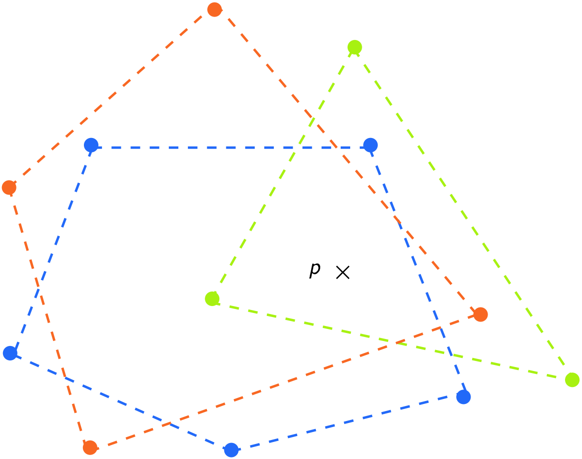

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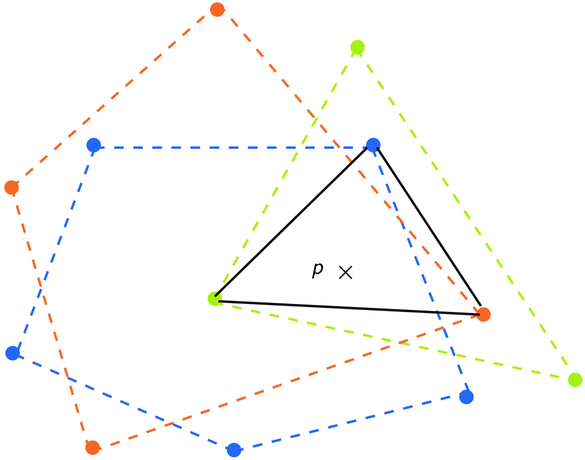


Colorful Carathéodory Theorem, Bárány 1982

Theorem (CCT, 1982)

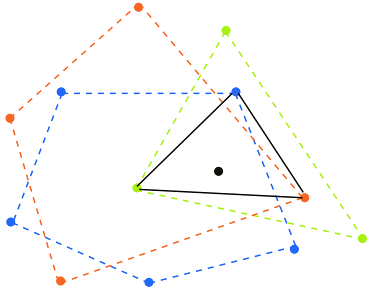
Let $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ *be* $d+1$ *sets of points in* $\mathbb{R}^d$*. If* $p \in \mathrm{conv}(\mathbf{S}_i)$ *for all* i*, there exists a set* $T \subseteq \bigcup_{i=1}^{d+1} \mathbf{S}_i$ *such that*

$$|T \cap \mathbf{S}_i| \leq 1 \text{ for all } i \text{ and } p \in \mathrm{conv}(T).$$

Applications of this theorem:

- First selection lemma.
- Proof of Tverberg's Theorem (generalization of Radon's Theorem).

Colorful simplicial depth conjecture



Theorem (S. 2014)

If $|S_i| \geq d + 1$ *for all* i*, there are at least* $d^2 + 1$ *colorful sets containing* p*.*

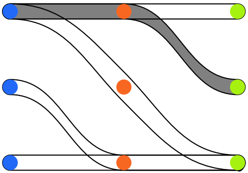
Conjectured by Deza et. al 2006; successive improvements by Bárány, Deza, Matoušek, Stephen, Thomas, Xie, Meunier, and S.

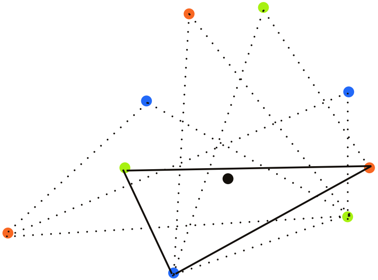
Octahedral systems

A combinatorial counterpart: octahedral systems

An octahedral system is an $(d+1)$-uniform, $(d+1)$-partite hypergraph satisfying the *parity condition:*

The number of edges induced by X, with $|X \cap V_i| = 2$ for all i, is even.



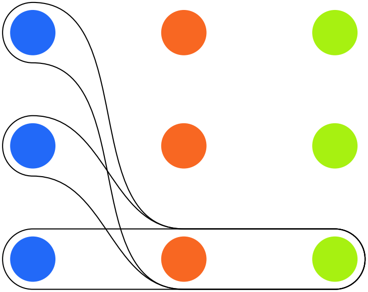


Induction on the number of covered classes

Theorem

If $k \geq 1$ classes are covered, there are at least $k(d-1)+2$ edges.

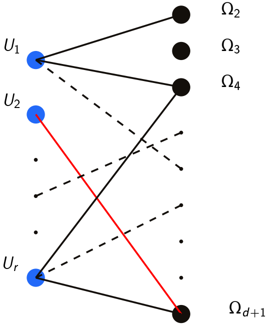
Main idea of the proof:
The octahedral systems are the symmetric differences of elementary octahedral systems: the umbrellas.



Proof via a bipartite graph

$$\Omega = \underbrace{U_1 \triangle U_2 \triangle \cdots \triangle U_r}_{\text{umbrellas of color 1}} \triangle \underbrace{\Omega_2 \triangle \Omega_3 \triangle \cdots \triangle \Omega_{d+1}}_{\text{octahedral systems}},$$

Decomposition graph: $G = (V, E)$



★Edge $U_i\Omega_j$ of decomposition graph

$\updownarrow$

Edge in $U_i \cap \Omega_j$

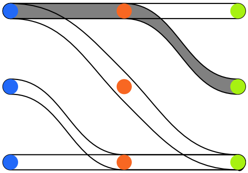
★ $|\Omega| = \sum |U_i| + \sum |\Omega_j| - 2|E|$

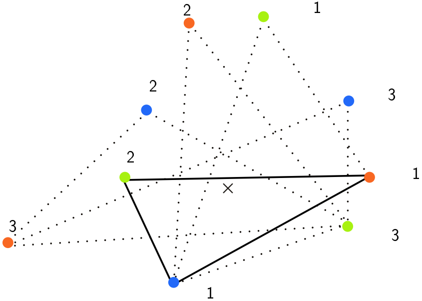
★ Each Ω_j covers less classes (Induction)

★ $|E| = \sum \deg_G(\Omega_j)$

Proof of the Colorful Simplicial Depth Conjecture

A combinatorial counterpart: octahedral systems



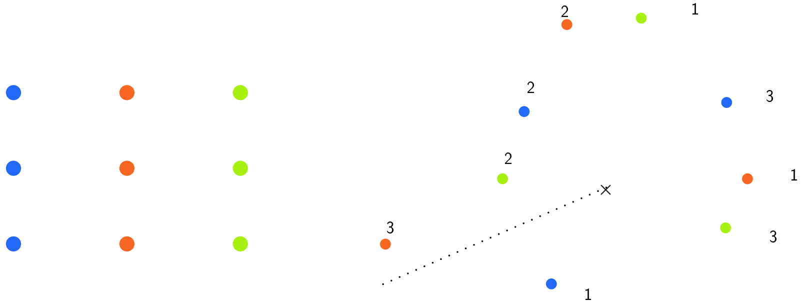


An umbrella is a set $\{x_1\} \times \{x_2\} \times \cdots \times V_i \times \cdots \times \{x_{d+1}\}$.

An octahedral system is a symmetric difference of umbrellas.

A colorful point configuration defines an octahedral system.

A combinatorial counterpart: octahedral systems



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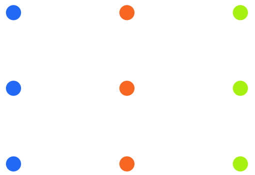

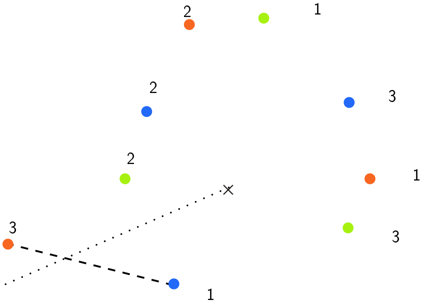

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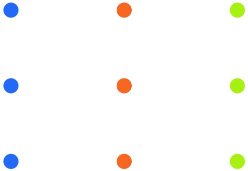

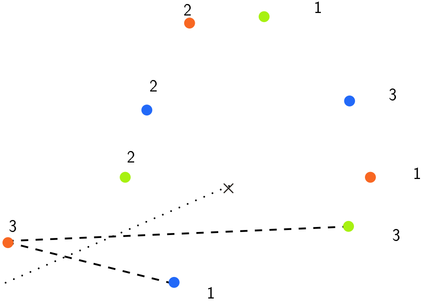

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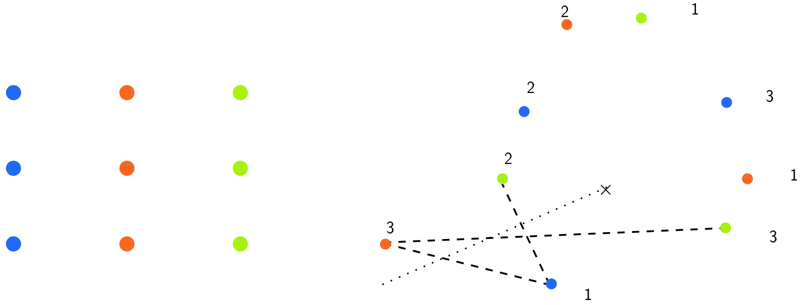

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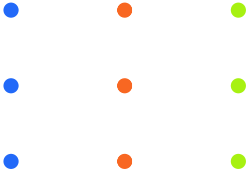

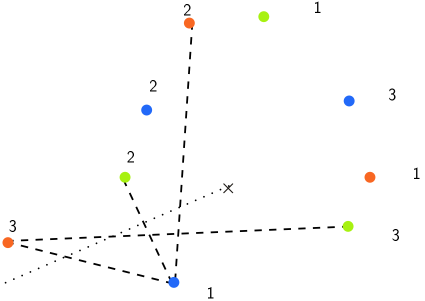

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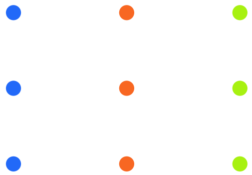

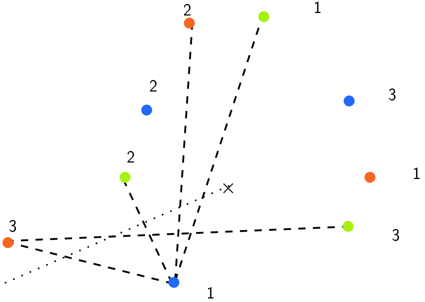

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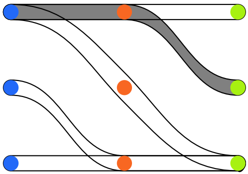

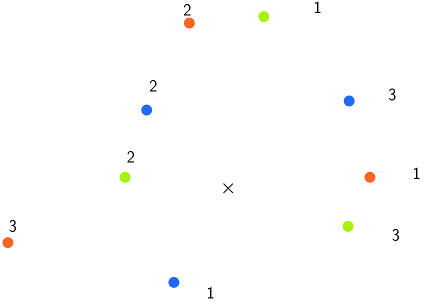

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Colorful simplicial depth conjecture

Theorem (S., 2014)

If $k \geq 1$ classes are covered, there are at least $k(d-1)+2$ edges.

Theorem (Strong Colorful Carathéodory Theorem, Bárány 1982)

Every point in $\bigcup_{i=1}^{d+1} S_i$ is a vertex of some colorful simplex containing p.

Corollary

The corresponding octahedral systems covers all classes, and hence there are at least

$$(d+1)(d-1)+2=d^2+1$$

colorful simplices containing p.

Thank you